TURBO-STRAIGHTENING FOR DECOMPOSITION INTO STANDARD BASES.

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Abstract. Specht and Hodge have shown that the space generated by products of minors of a matrix admits a linear basis in bijection with Young tableaux. The decomposition of any element into this basis is called *straightening* and corresponds to the iterative use of Plücker relations. Thanks to a well known isomorphism between the space of harmonic polynomials and the space of polynomials modulo the ideal generated by symmetric polynomials, we can now use as a main technical tool the canonical scalar product on this later space. This leads to a different, and possibly better, algorithm for straightening.

1. Introduction.

Young introduced his celebrated tableaux to denote particular elements of the group algebra of the symmetric group $\mathfrak{S}_n$ [**Yo**]. His purpose was to solve certain equations in this algebra, which arose in his study of the theory of Invariants. Moreover, his constructions allowed him to give explicit bases of irreducible representations of the symmetric group.

Using a different approach to describe these representations, Specht considered tableaux as special polynomials [**Sp**]. More explicitely, given any filling t of the shape of a partition with consecutive numbers (that we shall call a *tabloïd*), Specht defines the polynomial $\boxed{t}$ to be the product of Vandermonde determinants corresponding separately to the columns of t. Thus, if

$$t = \begin{array}{|c|cccc} \hline 7 & & & & \\ 4 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{8} & & \\ 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{5} & \\ \hline \end{array} ,$$

$\boxed{t}$ is the polynomial

$$\boxed{t} = \Delta(a_1,a_4,a_7).\Delta(a_2,a_3).\Delta(a_6,a_8).\Delta(a_5).$$

Specht has shown that the space generated by all tabloïds of a given shape is an irreducible representation of $\mathfrak{S}_n$, a basis being given by *tableaux* (i.e. tabloïds with increasing columns and rows).

In fact, Specht polynomials are products of minors of a Vandermonde matrix, and the construction remains valid for any matrix, as was shown by Hodge [**Ho**]. Given a matrix (a_{ij}) and a tabloïd t, the symbol $\boxed{t}$ now means the product of minors corresponding to its successive columns. For instance, the preceeding tableau corresponds to the product of minors:

$$\begin{vmatrix} a_{11} & a_{14} & a_{17} \\ a_{21} & a_{24} & a_{27} \\ a_{31} & a_{34} & a_{37} \end{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} a_{16} & a_{18} \\ a_{26} & a_{28} \end{vmatrix} a_{15}.$$

Hodge obtained that Young tableaux still provide a linear basis of products of minors, and moreover gave an algorithm to decompose any tabloïd into the basis of Young tableaux [**Ho**]. This algorithm is usually named *straightening* and is a direct consequence of a repetitive use of Plücker relations. Various implementations of it have been experimented by White [**Wh**].

A more general point of view of the underlying algebraic structures can be found in the work of De Concini and Procesi, who introduced the Hodge algebra structure [**CP**]. A different approach is given by Gröbner bases, as was shown by Sturmfels and White [**SW**] .

Let us go back to minors. In fact Specht case is generic for what concerns the straightening, and we shall restrict to this case. This was also the point of view of Garnir, who has, independently of Hodge but later, described the explicit relations to straighten tabloïds by means of quadratic syzigies [**Ga**]. His aim was to give explicit matrices for Specht representation.

In the mean time, Rutherford had shown in his exposition of Young's work [**Ru**], that Young's natural representation was intimately associated with a triangular matrix Ξ whose entries are $0, \pm 1$. This matrix Ξ was generalized by Désarménien [**Des**], who established its link with the straightening formula for bitableaux, and proposed a new algorithm.

In this paper, we shall interpret differently the links between these different constructions. Tabloïds still code Specht polynomials. But now, one uses the fact that Specht polynomials are special harmonic polynomials. The space of harmonic polynomials is isomorphic (as a representation of the symmetric group) to the space of polynomials modulo the ideal generated by symmetric polynomials. Tabloïds are sent to special monomials, and Rutherford matrix is now a matrix of scalar products. To straighten an element, one must evaluate its scalar products (but this reduces to mere additions) with the elements of a basis, and locally invert the matrix Ξ. This gives a tree, each level consisting of tableaux, the edges being given by the operation "scalar product & local inversion".

One can combine such an algorithm with other tools more specific to polynomials. As an example, we give the explicit straightening of some tableaux having much symmetries.

2. Harmonic polynomials. The ring $\mathbb{Q}[\mathbb{A}]/\mathcal{S}$ and its scalar product.

Definition 2.1. A harmonic polynomial in $\mathbb{Q}[\mathbb{A}]$ is a polynomial which is annihilated by any symmetric differential operator, *i.e.* by any symmetric polynomial P without constant term in the partial derivatives δ_a, $a \in \mathbb{A}$. Such a differential operator will be denoted by P^δ, or by δ^H when P reduces to a single monomial $a_1^{h_1} \dots a_n^{h_n}$.

Example 2.2. Let Δ be the Vandermonde determinant on the alphabet $\mathbb{A}$. For any symmetrical polynomial P without constant term, $P^\delta(\Delta)$ is an antisymmetric polynomial of degree smaller than Δ, hence is null. Therefore Δ is harmonic. Since partial derivatives commute, $\delta^H(\Delta)$ is also harmonic for all $H \in \mathbb{N}^n$.

Proposition 2.3. *The linear space $\mathcal{H}(\mathbb{A})$ of harmonic polynomials is generated by the derivatives $\delta^H(\Delta)$ of the Vandermonde.*

Specht polynomials are particular cases of polynomials $\delta^H(\Delta)$. To be more explicit, we shall need some definitions.

Definition 2.4. Let t be a tabloïd of shape I and weight n, and let $J = j_1 \geq j_2 \geq \dots \geq j_k$ denote the conjugate partition of I. Define $H \in \mathbb{N}^n$ by

$$H = (\underbrace{0,\dots,0}_{j_1}, \underbrace{j_1,\dots,j_1}_{j_2}, \underbrace{j_1+j_2,\dots,j_1+j_2}_{j_3}, \dots, \underbrace{j_1+\dots+j_{k-1},\dots,j_1+\dots+j_{k-1}}_{j_k})$$

Now, let σ be the permutation of $1, 2, \dots, n$ obtained by reading the columns of t from bottom to top and from left to right. We set

$$Col(t) = \sigma^{-1}(H) := (h_{\sigma^{-1}(1)}, \dots, h_{\sigma^{-1}(n)}), \quad sgn(t) = sgn(\sigma)\ .$$

In other words the column-indicator $Col(t)$ is obtained by attaching to each column of the diagram of the partition I, the weight of the columns situated to its left, and by reading the sequence of columns corresponding to the tabloïd t.

Example 2.5. Let $n = 8$ and t be the tabloïd

$$t = \begin{array}{|c|ccc}\cline{1-1} 7 & & & \\ \cline{1-3} 4 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{8} & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{cccc} \cline{1-1} \multicolumn{1}{|c|}{\ } & & & \\ \cline{2-3} \multicolumn{1}{|c|}{\ } & \multicolumn{1}{c|}{\ } & \multicolumn{1}{c|}{\ } & \\ \cline{4-4} \multicolumn{1}{|c|}{\ } & \multicolumn{1}{c|}{\ } & \multicolumn{1}{c|}{\ } & \multicolumn{1}{c|}{\ } \\ \hline 0 & 3 & 5 & 7 \end{array}$$

Then $I = 134$, $J = 3221$, $H = 00033557$, $\sigma = 14723685$, $\sigma^{-1} = 14528637$, so that $sgn(t) = +1$ and $Col(t) = 03307505$.

Now for any tabloïd t, the determinant $\delta^{Col(t)}(\Delta)$ is easily seen to be equal, up to a constant factor, to the Specht polynomial $\boxed{t}$. Thus, for the tableau t considered above, one has

$$\boxed{t} \;=\; \delta^{03307505}(\Delta)/7!\ 6!\ 5!\ 4!\ 3!.$$

We shall now describe the canonical isomorphism which will enable us to replace computations on tabloïds by computations on monomials.

Let $\mathcal{S}$ be the ideal generated in $\mathbb{Q}[\mathbb{A}]$ by symmetric polynomials without constant term. Denote by a^H the monomial $a_1^{h_1}\ldots a_n^{h_n}$. We have the following result (cf. for example [**La**]).

PROPOSITION 2.6. *The morphism defined by $\phi(a^H)=\delta^H(\Delta)$ induces an isomorphism Φ between $\mathbb{Q}[\mathbb{A}]/\mathcal{S}$ and the space $\mathcal{H}(\mathbb{A})$ of harmonic polynomials.*

Thus, any identity between Specht polynomials (that is, any identity between tabloïds considered as products of minors) may be transformed by means of Φ^{-1} into an identity between monomials modulo $\mathcal{S}$.

EXAMPLE 2.7. Let $n=4$. The simplest case of Plücker's relations reads:

$$\begin{array}{|c|c|}\hline 2 & 4\\ 1 & 3\\ \hline\end{array} \;-\; \begin{array}{|c|c|}\hline 3 & 4\\ 1 & 2\\ \hline\end{array} \;+\; \begin{array}{|c|c|}\hline 4 & 3\\ 1 & 2\\ \hline\end{array} \;=\; 0$$

In $\mathbb{Q}[a_1,a_2,a_3,a_4]/\mathcal{S}$ this identity becomes:

$$a^{0022}+a^{0202}+a^{0220}=0.$$

Let us consider now the straightening of a tabloïd t of shape I. Let H be the vector of $\mathbb{N}^n$ associated with I as shown in 2.4. The image in $\mathbb{Q}[\mathbb{A}]/\mathcal{S}$ of the straightening of t is now an identity between monomials of degree $\sigma(H)$, $\sigma\in\mathfrak{S}_n$. We shall denote by $\mathcal{W}(H)$ the subspace generated in $\mathbb{Q}[\mathbb{A}]/\mathcal{S}$ by all monomials whose degree is a permutation of H. Such a subspace is called a *weight space*. Let us describe the monomials which are images of tableaux under the isomorphism Φ^{-1}. We shall need a definition.

DEFINITION 2.8. Given a totally ordered alphabet $\mathbb{X}=\{x,y,z\ldots\}$, a word on $\mathbb{X}$ is a *Yamanouchi word*, iff any of its left factors contains more x's than y's, more y's than z's, etc.

Thus $xxyzyxyz$ is a Yamanouchi word, but $xyyxz$, $xyxzzxyy$ are not since xyy has 2 y's and only one x, and $xyxzz$ has 2 z's and only one y.

Now we may state the following easy proposition.

PROPOSITION 2.9. *A tabloïd t whose columns are increasing from bottom to top is a tableau iff its columns–indicator $Col(t)$ is a Yamanouchi word.*

EXAMPLE 2.10. Let us consider the Specht module $\mathcal{M}(I)$ generated by all polynomials $\boxed{t}$ associated to tabloïds t of shape $I = 1122$. This module is sent by Φ^{-1} on the weight space $\mathcal{W}(000044)$. There are 9 tableaux of shape I corresponding to the 9 following Yamanouchi words:

000044, 000404, 004004, 040004, 000440, 004040, 040040, 004400, 040400.

The straightening formula shows that tableaux of shape I constitute a basis of $\mathcal{M}(I)$, from which it is deduced that $\mathcal{M}(I)$ is the irreducible representation of $\mathfrak{S}_n$ corresponding to the partition I. This theorem lifted to $\mathbb{Q}[\mathbb{A}]/\mathcal{S}$ by means of Φ^{-1} becomes:

THEOREM 2.11. *Let $\mathcal{W}(H)$ be the image of the Specht module $\mathcal{M}(I)$. The monomials a^K, where K runs through all permutations of H satisfying the Yamanouchi condition, form a basis of $\mathcal{W}(H)$ that we shall denote by $\mathcal{Y}(H)$. $\mathcal{W}(H)$ is the irreducible representation of $\mathfrak{S}_n$ corresponding to the partition I, for the action by permutation of variables and multiplication by the signature.*

Note however that only a few weight spaces $\mathcal{W}(H)$ are images of Specht modules. For instance, in $\mathbb{Q}[a_1,a_2,a_3,a_4]/\mathcal{S}$ these weight spaces are $\mathcal{W}(0000)$, $\mathcal{W}(0003)$, $\mathcal{W}(0022)$, $\mathcal{W}(0023)$, $\mathcal{W}(0123)$, and correspond respectively to the irreducible representations indexed by partitions 1111, 112, 22, 13, 4.

The imbedding of Specht modules into $\mathbb{Q}[\mathbb{A}]/\mathcal{S}$ will make it possible to straighten tableaux using the natural scalar product in this space.

DEFINITION 2.12. Let P, Q be polynomials in $\mathbb{Q}[\mathbb{A}]$ and let $\overline{P}$, $\overline{Q}$ denote their class modulo $\mathcal{S}$. The scalar product $< \overline{P} \,|\, \overline{Q} >$ is defined as the constant term of the symmetric polynomial

$$\sum_{\sigma \in \mathfrak{S}_n} (P.Q/\Delta)^{\sigma} \ .$$

Thus, denoting in the same way monomials and their classes modulo $\mathcal{S}$, we have the following easy rule:

$$< a^H \mid a^K > = \begin{cases} sgn(\sigma) & \text{if } H + K = \sigma(\rho), \\ 0 & \text{else,} \end{cases}$$

where ρ denotes the staircase partition $\rho = 0, 1, 2, \ldots, n-1$.

From this definition results immediately the following proposition:

PROPOSITION 2.13. *Let $\mathcal{W}(H)$ be the image of a Specht module. The only monomials which are not orthogonal to $\mathcal{W}(H)$ belong to the weight space $\mathcal{W}(\rho - H)$.*

We now proceed to describe the fundamental correspondance between the Yamanouchi monomials of $\mathcal{W}(H)$ and $\mathcal{W}(\rho - H)$. For that purpose, it will be convenient to define, by analogy with 2.4., the row–indicator of a tabloïd.

Definition 2.14. Let t be a tabloïd of shape I. Number the rows of the diagram of I from bottom to top (the bottom row has number 0). $Row(t)$ is the sequence of levels of the successive boxes of I, enumerating them in the order indicated by t.

Example 2.15. Take again the tabloïd t considered in 2.5:

$$t = \begin{array}{|c|cccc} \hline 7 & \multicolumn{1}{|c}{} & & & \\ \cline{2-3} 4 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{8} & & \\ \cline{4-4} 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{5} & \\ \hline \end{array} \qquad\qquad \begin{array}{c} \begin{array}{|c|} \hline 2 \\ 1 \\ 0 \\ \hline \end{array} \\ \textit{Row level} \end{array}$$

Its row–indicator is $Row(t) = 00110021$.

If t is a tabloïd whose columns are increasing, $Row(t)$ is evidently a Yamanouchi word, and the correspondance between $Col(t)$ and $Row(t)$ is worth noting. We first set the following definition:

Definition 2.16. Let $H \in \mathbb{N}^n$. Given a permutation K of H, read the word K from left to right and replace the first occurences of any letter by 0's, the second occurences by 1's, etc.. We denote by $\mathbf{p}(K)$ the word obtained thus.

For example, $\mathbf{p}(30050300) = 00102134$. We have now the following results:

Proposition 2.17. *Let I be a partition of n and define H as in* 2.4. *A permutation of H is called a word of evaluation H.*

(i) *The map $\mathbf{p}$ is a surjection from the set of words of evaluation H onto the set of Yamanouchi words of evaluation $\rho - H$.*

(ii) *The restriction of $\mathbf{p}$ to the set of Yamanouchi words of evaluation H is a bijection $\mathbf{b}$ decreasing for the lexicographic order.*

(iii) *For any tabloïd t whose columns are increasing from bottom to top, one has $Row(t) = \mathbf{p}(Col(t))$ and $Col(t) + Row(t) = \sigma^{-1}(\rho)$, where σ is the permutation obtained by reading the columns of t from bottom to top and from left to right.*

(iv) *If t and u are two tabloids whose columns are increasing from bottom to top, and such that $Col(t) + Row(u)$ is a permutation of ρ, then t and u have the same shape and $Col(t)$ is before $Col(u)$ in the lexicographic order.*

This combinatorial lemma may now be translated in terms of monomials modulo $\mathcal{S}$.

Proposition 2.18. *Let $\mathcal{W}(H)$ be the image of a Specht module in $\mathbb{Q}[\mathbb{A}]/\mathcal{S}$.*

(i) *The map $a^K \longrightarrow a^{\mathbf{p}(K)}$ is a canonical surjection from the set of monomials in $\mathcal{W}(H)$ onto the set $\mathcal{Y}(\rho - H)$ of Yamanouchi monomials in $W(\rho - H)$.*

(ii) *The restriction of this map to $\mathcal{Y}(H)$ is a bijection.*

(iii) *The matrix Ξ of scalar products $< a^K \mid a^L >$, where a^K, a^L run through the sets $\mathcal{Y}(H)$, lexically ordered, and $\mathcal{Y}(\rho-H)$, anti-lexically ordered, is triangular with entries $0, +1, -1$.*

Example. Take $I = 1122$, so that $\mathcal{W}(H) = \mathcal{W}(000044)$ and $\mathcal{W}(\rho-H) = \mathcal{W}(001123)$. Writing the word K in place of the monomial a^K, one has for example:

$$\mathbf{p}(440000) = \mathbf{p}(404000) = 001123, \mathbf{p}(400040) = 001213, \mathbf{b}(000440) = 012013.$$

The image under $\mathbf{b}$ of the basis $\mathcal{Y}(000044)$ given above is

$$\mathcal{Y}(001123) = (012301,012031,012013,010231,010213,010123,001231,001213,001123).$$

Finally the matrix Ξ is

$$\Xi = \begin{pmatrix} 1 & . & . & . & . & . & . & . & . \\ . & -1 & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & -1 & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & -1 & . & . \\ -1 & . & . & . & . & . & . & 1 & . \\ . & 1 & -1 & . & . & . & . & . & -1 \end{pmatrix}.$$

Now, we can easily describe an algorithm for the decomposition on the basis $\mathcal{Y}(H)$ of any polynomial belonging to $\mathcal{W}(H)$, or, what is equivalent as we have just shown, for the straightening of any linear combination of tableaux of shape I. Indeed, let P be the polynomial to be decomposed, and let

$$P = \sum_i \lambda_K a^K$$

denote its decomposition. The coefficients λ_K are the solutions of the linear system obtained by computing the scalar products $\mu_L =< P \mid a^L >$, $a^L \in \mathcal{Y}(\rho - H)$. The matrix of this system is Ξ, and it remains only to compute its inverse Ξ^{-1}, which is made possible, despite of its hugeness, by its being unitriangular with lots of entries equal to 0.

We conclude this section by emphasizing that the matrix Ξ is already well-known in representation theory of $\mathfrak{S}_n$. Up to sign factors this matrix is exactly the matrix Ξ used by Rutherford to obtain the so-called *natural representation* [**Ru**]. The fact that it may also be used to compute the coefficients of the straightening formula was shown by Désarménien, by means of Capelli operators [**Des**].

3. The turbo-straightening algorithm.

The basic routine of this algorithm generates the rows of the matrix Ξ, that is, computes the monomials in $\mathcal{Y}(\rho - H)$ whose scalar product with a given monomial in $\mathcal{W}(H)$ is non-zero. It is based on the following proposition:

PROPOSITION 3.1. *Let a^K be a monomial in $\mathcal{W}(H)$ and let $a^L = a^{\mathbf{p}(K)}$ denote its image in $\mathcal{Y}(\rho - H)$ under the surjection $\mathbf{p}$. From the two words $K = (k_1, \dots k_n)$ and $L = (l_1 \dots l_n)$, form the biword $\binom{K}{L} = \binom{k_1}{l_1} \dots \binom{k_n}{l_n}$. Let a^M belong to $\mathcal{Y}(\rho - H)$. Then $< a^M \mid a^K > \neq 0$ iff the biword $\binom{K}{M}$ is a reordering of $\binom{K}{L}$.*

EXAMPLE 3.2. Take $K = 047040740$, so that $L = 000112123$, and

$$\binom{K}{L} = \binom{0}{0}\binom{4}{0}\binom{7}{0}\binom{0}{1}\binom{4}{1}\binom{0}{2}\binom{7}{1}\binom{4}{2}\binom{0}{3}.$$

The routine enumerates all Yamanouchi biwords $\binom{K}{L'}$, which are obtained by permuting between themselves biletters having the same upper letter. In this case, the list of all words L' such that $\binom{K}{L'}$ is Yamanouchi is the following, where the permutations operated are indicated by the symbols $\sharp$, $\flat$, $\natural$.

$$\begin{array}{ccccccccc}
0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 2^\flat & 3^\natural & 1 & 1^\flat & 2^\natural \\
0 & 0 & 1^\sharp & 1 & 2^\flat & 2 & 0^\sharp & 1^\flat & 3 \\
0 & 0 & 1^\sharp & 1 & 2^\flat & 3^\natural & 0^\sharp & 1^\flat & 2^\natural \\
0 & 0 & 1^\sharp & 2^\natural & 1 & 3^\natural & 0^\sharp & 2 & 1^\natural \\
0 & 1^\flat & 0 & 1 & 0^\flat & 2 & 1 & 2 & 3 \\
0 & 1^\flat & 0 & 2^\natural & 0^\flat & 1^\natural & 1 & 2 & 3 \\
0 & 1^\flat & 0 & 2^\natural & 0^\flat & 3^\natural & 1 & 2 & 1^\natural
\end{array}$$

Now the straightening of a monomial a^K of $\mathcal{W}(H)$ is performed by sequential application of the routine described above. Indeed, we saw in section 1 that this straightening is equivalent to the resolution of a unitriangular linear system. It is easily seen that the computations to be done may be conveniently represented by the construction of a tree of biwords, whose root is the word $\binom{K}{\mathbf{p}(K)}$. The sons of a given word $\binom{S}{T}$ in this tree are all words $\binom{U}{V}$ different from $\binom{S}{T}$ satisfying the conditions:

(i) $< a^S \mid a^V > \neq 0$,

(ii) a^V belongs to $\mathcal{Y}(\rho - H)$, and $U = \mathbf{b}^{-1}(V)$.

Each edge $\binom{S}{T} \longrightarrow \binom{U}{V}$ bears the sign $\varepsilon =< a^S \mid a^V >< a^U \mid a^V >$. The result of the straightening is the sum of the signed monomials $\varepsilon(S)a^S$ corresponding to all words $\binom{S}{T}$ of the tree (except the root), the sign $\varepsilon(S)$ being the product of the two following signs:

(a) the product of the signs beared by the edges going from $\binom{S}{T}$ to the root of the tree;

(b) $(-)^{k-1}$ where k is the level of the tree to which the word $\binom{S}{T}$ belongs (the root is at level 0).

Example 3.3. The straightening tree of the monomial $a^{030663306}$ is

030663306 001011222	030636036 001011222	030603366 001021212	000336366 012010212
			000333666 012012012
		000363636 012001122	000333666 012012012
	030630636 001012122	030306366 001120212	000336366 012010212
		030303666 001122012	000333666 012012012
		000363636 012001122	000333666 012012012
	030366036 001101222	000336636 012010122	
		030063366 001201212	
		000363636 012001122	000333666 012012012
	030360636 001102122	030036366 001210212	
		030033666 001212012	
		000363636 012001122	000333666 012012012
	000336366 012010212		
	000333666 012012012		

The result is, writing K in place of a^K:

$$030663306 = \underbrace{030636036 + 030630636 + 030366036 + 030360636 - 000336366 - 000333666}_{first\ level}$$

$$\underbrace{-030603366 - 030306366 - 030303666 - 000336636 - 030063366 - 030036366 - 030033666}_{second\ level}$$

Note that in this case four words of the second level and all words of the third level are annihilated in the summation. We shall see in Prop.3.5, 3.6 that some of these cancellations could be foreseen, and therefore the corresponding computations avoided.

It will perhaps be helpful to restate this example using the classical tableau notation. The monomial $a^{030663306}$ corresponds to the tableau

$$t = \begin{array}{|c|c|c|} \hline 8 & 7 & 9 \\ 3 & 6 & 5 \\ 1 & 2 & 4 \\ \hline \end{array}$$

The straightening tree of t is:

$$\begin{array}{|c|c|c|}\hline 8&7&9\\ 3&6&5\\ 1&2&4\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 7&8&9\\ 3&5&6\\ 1&2&4\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 6&8&9\\ 3&5&7\\ 1&2&4\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 7&8&9\\ 3&4&6\\ 1&2&5\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 6&8&9\\ 3&4&7\\ 1&2&5\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&7&9\\ 2&5&8\\ 1&4&6\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&6&9\\ 2&5&8\\ 1&4&7\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 5&7&9\\ 3&6&8\\ 1&2&4\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&8&9\\ 2&6&7\\ 1&4&5\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 5&7&9\\ 3&4&8\\ 1&2&6\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 5&6&9\\ 3&4&8\\ 1&2&7\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&8&9\\ 2&6&7\\ 1&4&5\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&8&9\\ 2&5&7\\ 1&4&6\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 4&7&9\\ 3&6&8\\ 1&2&5\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&8&9\\ 2&6&7\\ 1&4&5\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 4&7&9\\ 3&5&8\\ 1&2&6\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 4&6&9\\ 3&5&8\\ 1&2&7\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&8&9\\ 2&6&7\\ 1&4&5\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 3&7&9\\ 2&5&8\\ 1&4&6\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&6&9\\ 2&5&8\\ 1&4&7\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&6&9\\ 2&5&8\\ 1&4&7\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&7&9\\ 2&5&8\\ 1&4&6\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&6&9\\ 2&5&8\\ 1&4&7\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&6&9\\ 2&5&8\\ 1&4&7\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&6&9\\ 2&5&8\\ 1&4&7\\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&6&9\\ 2&5&8\\ 1&4&7\\ \hline\end{array}$$

The construction of this tree may be easily described in terms of tableaux only. The root is the tableau to be straightened. The sons of a given tableau u are all standard tableaux s different from u which are obtained by permuting the columns of u and reordering the lines. Thus, the transposition of 4 and 5 in the last column of the tableau t straightened above, leads to a son in the first level of the tree:

$$\begin{array}{|c|c|c|} 8 & 7 & 9 \\ 3 & 6 & 5 \\ 1 & 2 & 4 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} 8 & 7 & 9 \\ 3 & 6 & 4 \\ 1 & 2 & 5 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} 7 & 8 & 9 \\ 3 & 4 & 6 \\ 1 & 2 & 5 \\ \hline \end{array}$$

But the transposition of 2 and 6 in the second column of t leads to a tableau whose columns are not increasing from bottom to top, and therefore must be rejected:

$$\begin{array}{|c|c|c|} 8 & 7 & 9 \\ 3 & 6 & 5 \\ 1 & 2 & 4 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} 8 & 7 & 9 \\ 3 & 2 & 5 \\ 1 & 6 & 4 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} 7 & 8 & 9 \\ 2 & 3 & 5 \\ 1 & 4 & 6 \\ \hline \end{array}$$

The sign beared by an edge $u \longrightarrow s$ of the tree, is now the signature of the permutation of the columns of u which is operated to get s. The sign of s in the straightening of t is, as above, the product of the signs of the edges going from s to the root of the tree, multiplied by $(-)^{k-1}$, where k denotes the level of the tree to which s belongs, the root being at level 0.

We proceed to describe some symmetries of the straightening tree of a monomial (or of a tabloïd). These symmetries are not surprising, since the ordinary action of the symmetric group (by permutation of the variables and multiplication by the signature) is compatible with the scalar product in $\mathbb{Q}[\mathbb{A}]$. But it must be noted that the set of Yamanouchi monomials is not stable for this action. We have only the following property:

PROPOSITION 3.4. *Let a^S be a monomial in $\mathcal{Y}(H)$ and let $a^T = a^{\mathbf{b}(S)}$ be its image in $\mathcal{Y}(\rho - H)$ under the bijection* $\mathbf{b}$. *Suppose that two adjacent biletters $\binom{s_i}{t_i}$, $\binom{s_{i+1}}{t_{i+1}}$ of the word $\binom{S}{T}$ satisfy $s_i \neq s_{i+1}$ and $t_i \neq t_{i+1}$. Then the words obtained by permuting these letters in S and T are still Yamanouchi words.*

In terms of tableaux this is the classical lemma which states that when two consecutive numbers of a tableau lie in different columns and rows, the tabloïd obtained by permuting these numbers is still a tableau.

From this we deduce at once the following proposition.

PROPOSITION 3.5. *A polynomial symmetrical in two consecutive letters a_i, a_{i+1} will have a decomposition in the basis $\mathcal{Y}(H)$ presenting the same symmetry.*

More precisely, the following proposition shows that the algorithm respects step by step such a symmetry.

PROPOSITION 3.6. *Let $\binom{S}{T}$ be a vertex of the straightening tree of a^K.*

(i) If the i^{th} and $(i+1)^{th}$ letters of S are equal, the sum of all signed monomials $\pm a^U$ coming from the subtree of root $\binom{S}{T}$ is a polynomial symmetric in the two letters a_i and a_{i+1}.

(ii) If the word $\binom{S'}{T'}$ obtained by permuting the i^{th} and $(i+1)^{th}$ letters of $\binom{S}{T}$ is another vertex in the tree, the sum of all signed monomials $\pm a^U$ coming from the subtrees of root $\binom{S}{T}$ and $\binom{S'}{T'}$ is a polynomial symmetric in the two letters a_i and a_{i+1}.

PROOF. The proof is by induction on the number of levels of the subtrees. Suppose that the sum of monomials coming from level k is the polynomial $P = \sum_U c_U a^U$, symmetrical in a_i, a_{i+1}. The sum coming from level $k+1$ will be

$$Q = \sum_{U,V} c_U < a^U | a^{\mathbf{b}(V)} >< a^V | a^{\mathbf{b}(V)} > a^V =: \sum_{U,V} d_{U,V} a^V,$$

where a^V runs through the basis $\mathcal{Y}(H)$, and $\mathbf{b}$ is the bijection described in 2.17. Let σ denote the transposition of i and $i+1$. If $< a^U | a^{\mathbf{b}(V)} > \neq 0$ and a^V is not symmetric in a_i, a_{i+1}, there are two cases:

(1) $a^{\sigma(V)} \in \mathcal{Y}(H)$, in which case the compatibility of the scalar product with the action of σ and the equality $c_U = c_{\sigma(U)}$ show that $d_{U,V} = d_{\sigma(U),\sigma(V)}$

(2) $a^{\sigma(V)} \notin \mathcal{Y}(H)$, which implies that $a^{\mathbf{b}(V)}$ is symmetric in a_i, a_{i+1} (Prop.3.4.), so that $d_{U,V} = -d_{\sigma(U),V}$ and the total coefficient of a^V in Q is zero.

Hence Q is symmetric in a_i, a_{i+1}, and the proof follows by induction. □

EXAMPLE 3.7. In the staightening tree of example 3.3 there are several subtrees which may illustrate the proposition. For example, the subtree of root

$$\begin{matrix} 030366036 \\ 001101222 \end{matrix}$$

and the whole tree for case (i); and the couples of subtrees of roots

$$\begin{matrix} 030636036 \\ 001011222 \end{matrix} \quad \begin{matrix} 030366036 \\ 001101222 \end{matrix} ; \qquad \begin{matrix} 030630636 \\ 001012122 \end{matrix} \quad \begin{matrix} 030360636 \\ 001102122 \end{matrix}$$

for case (ii).

This proposition has an important consequence concerning the straightening of tableaux with repeated letters. Indeed, these tableaux may always be treated as tableaux without repetition, by merely polarizing the equal letters $a = a = a < b = b < \ldots$, that is replacing them by $a < a' < a" < b < b' \ldots$, straightening the tableaux obtained, and specializing $a = a' = a" < b = b' < \ldots$ in the result. But this method is not very effective, since a lot of useless computations are made. Proposition 3.6. shows that specializations corresponding to equal letters may be used at every level of the construction of the straightening tree.

4. Final comments.

Much of what has been given here is not new, but we hope it clarifies the subject to see, for example, that the matrix introduced by Rutherford is a matrix of scalar products. Moreover in the classical presentation, one has to test if, given a pair of tableaux, there exists a horizontal permutation of the entries of the first one and a vertical permutation of the entries of the second one which generate the same tabloïd. This looks much heavier than to test if the sum of two vectors is a permutation of $0, 1, 2, \ldots, n-1$ or not, though it is strictly equivalent. By transfering to the ring of polynomials, we can use much more tools than that is available for tableaux or minors of matrices (change of variables, partial symmetries, divided differences, etc.).

Let us give an example, also known to White, which shows how to use symmetric functions to straighten a particular monomial. We first give a definition.

Definition 4.1. Let $J = j_1,\ j_2,\ \ldots,\ j_k$ be a partition of n, and $L = (l_1, l_2, \ldots, l_n)$ be a vector of integers. Set $A_1 = \{a_1, \ldots, a_{j_1}\},\ \ldots,\ A_k = \{a_{n-j_k+1}, \ldots, a_n\}$, and $L_1 = (l_1, \ldots, l_{j_1}),\ \ldots,\ L_k = (l_{n-j_k+1}, \ldots, l_n)$. We define the multi–monomial function $\Psi_L(A_1|A_2|\ldots|A_k)$ by setting

$$\Psi_L(A_1|A_2|\ldots|A_k) = \Psi_{L_1}(A_1)\ldots\Psi_{L_k}(A_k),$$

where $\Psi_T(X)$ denotes the ordinary monomial symmetric function of degree T on the set of variables X (cf. **[LS]** or **[Mcd]**).

Proposition 4.2. *Let $\mathcal{W}(h_n \ldots h_1)$, with $h_1 \geq \ldots \geq h_n$, be the weight space image of the Specht module $\mathcal{M}(I)$, and let $J = j_1 \leq j_2 \leq \ldots \leq j_k$ denote the conjugate partition of I. The straightening of the monomial a^H is given by an explicit formula:*

$$a^H = \pm\Psi_L(A_1|A_2|\ldots|A_k),$$

where $L = \mathbf{b}^{-1} \circ \mathbf{p}(H)$.

Example. Let $I = 11233$ so that $J = 235$ and $H = 8855500000$. Then, $\mathbf{p}(H) = 0101201234$, $L = \mathbf{b}^{-1}(\mathbf{p}(H)) = 00\ 550\ 88500$, $L_1 = 00$, $L_2 = 550$, $L_3 = 88500$. The straightening of a^H reads

$$a^{8855500000} = -\Psi_{0055088500}(A_1|A_2|A_3) := -\Psi_{00}(a_1,a_2)\Psi_{550}(a_3,a_4,a_5)\Psi_{88500}(a_6, \ldots a_{10}).$$

PROOF. By proposition 3.5, we know that the straightening decomposition of a^H is symmetric in each subset $\{a_1, \ldots, a_{j_1}\} \ldots \{a_{n-j_k+1}, \ldots, a_n\}$. Thus, its decomposition on $\mathcal{Y}(H)$ will be of the form

$$a^H = \sum_M c_M \Psi_M(A_1|A_2|\ldots|A_k),$$

where the summation is over a certain set of words M of evaluation H. Now, all monomials obtained by expanding the product $\Psi_M(A_1|A_2|\ldots|A_k)$ must belong to $\mathcal{Y}(H)$, from which we deduce that, if the letters of each subword $M_1, \ldots, M_k$ are rearranged in decreasing order, the word obtained (that we may continue to denote by $M = M_1 \ldots M_k$) will be a Yamanouchi word. It turns out that this condition is satisfied by a single word of evaluation H, which is $S = \mathbf{b}^{-1}(\mathbf{p}(H))$. Indeed, the condition on M is equivalent to the condition, that every subword $N_1, \ldots, N_k$ of $N := \mathbf{b}(M)$ be strictly increasing. In view of the evaluation $\rho - H$ of the word N, we see that the only possibility is

$$N = 01\ldots j_1 \; 01\ldots j_2 \; \ldots \; 01\ldots j_k = \mathbf{p}(H).$$

We conclude by identifying the coefficient $c_L =< a^H|a^{\mathbf{b}(L)} >< a^L|a^{\mathbf{b}(L)} >= \pm 1$, since by proposition 2.18 $< a^K|a^{\mathbf{b}(L)} >= 0$ for all monomials $a^K \neq a^L$ appearing in $\Psi_L(A_1|A_2|\ldots|A_k)$. □

The result of Proposition 4.2. may be translated in terms of tableaux. For instance the example given above is equivalent to the following identity:

$$\begin{array}{|c|ccc}\cline{1-1} 10 & & & \\ 9 & & & \\ \cline{2-2} 8 & \multicolumn{1}{|c|}{5} & & \\ \cline{3-3} 7 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{2} & \\ 6 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{1} & \\ \hline \end{array} \; = \; \sum \pm \begin{array}{|c|ccc}\cline{1-1} 10^\sharp & & & \\ 9^\sharp & & & \\ \cline{2-2} 5^\flat & \multicolumn{1}{|c|}{8^\sharp} & & \\ \cline{3-3} 2 & \multicolumn{1}{|c|}{4^\flat} & \multicolumn{1}{c|}{7^\sharp} & \\ 1 & \multicolumn{1}{|c|}{3^\flat} & \multicolumn{1}{c|}{6^\sharp} & \\ \hline \end{array},$$

where the sum on the right side is the alternated sum of all $\binom{3}{2,1}\binom{5}{2,1,2} = 90$ tableaux obtained by permutation of the letters of type $\flat$ between themselves, and of the letters of type $\sharp$ between themselves.

In this article we have only used minors of a very special matrix, namely the Vandermonde matrix of powers of indeterminates. However, this case is sufficiently generic for what concerns the straightening through Plücker relations. The relations used to describe Specht modules or Hodge algebras are the same, though in the second case, one uses minors of generic matrices. What we did in this paper was to translate these relations into simpler computations in the quotient ring $\mathbb{Q}[\mathbb{A}]/\mathcal{S}$. More general computations on minors could also be interpreted in this ring. This is the case for example of Turnbull formula which is, to our knowledge, the most general identity between minors of a generic matrix (cf. [**Le**]).

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