An Involution for the Gauss Identity

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Abstract. We obtain an involution for a classical identity of Guass on the alternating sum of the Gauss coefficients. It turns out that the refinement of our involution with restrictions on the heights of Ferrers diagrams leads to a generalization of the Gauss identity. Finally, we further extend the Gauss identity in which -1 is replaced by any root of unity.

Keywords: involution, Ferrers diagrams, Gauss identity, Gauss coefficients, Schur function, plethysm.

1. Introduction

We follow the standard notation on q-series [1, 2]. The *q-shifted factorials* $(a;q)_n$ are defined by

$$(a;q)_n = \begin{cases} 1, & n=0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n=1,2,\ldots. \end{cases}$$

The *q-binomial coefficients*, or the *Gauss coefficients*, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

Note that the parameter q is often omitted in the notation of Gauss coefficients when no confusion arises. The following classical identity is due to Gauss:

Theorem 1.1 (Gauss) *We have*

$$\sum_{r=0}^{m}(-1)^r \begin{bmatrix} m \\ r \end{bmatrix} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ (1-q)(1-q^3)\cdots(1-q^{m-1}), & \text{if } m \text{ is even.} \end{cases} \tag{1.1}$$

There have been several proofs of this identity [1, 2]. Goldman and Rota [3] find a proof by using a linear operator. Kupershmidt [4] obtains a generalization of the Gauss identity with an additional variable x. In fact, Gauss' identity is a by-product of results of Littlewood [6] on the evaluation of symmetric functions at roots of unity, and plethysm with power sums (cf. [5]). In this paper, we obtain a combinatorial proof of this identity in terms of pairs of Ferrers diagrams. Based on our involution, we also obtain a generalization of the Gauss identity with an additional parameter n (Theorem 2.2) through a refinement on the heights of the Ferrers diagrams. It turns out that this generalization follows from a further extension of the Gauss identity in which -1 is replaced by any root of unity (Theorem 3.3) .

2. An Involution for the Gauss Identity

Our combinatorial setting for the proof of the Gauss identity is based on the following equivalent form:

$$\sum_{r=0}^{m}(-1)^r \frac{q^r}{(q;q)_r} \cdot \frac{q^{m-r}}{(q;q)_{m-r}} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \dfrac{q^m}{(q^2;q^2)_{m/2}}, & \text{if } m \text{ is even.} \end{cases} \tag{2.1}$$

We proceed to describe our involution for the above identity. First, let us recall the standard notation on partitions as in [7]. The set of nonnegative integers is denoted by $\mathbb{N}$. A partition λ is a sequence of nonnegative integers

$$(\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots) \tag{2.2}$$

in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots$ with only a finite number of nonzero terms. If $\lambda_i = 0$ for all $i > n$, we also write λ in the finite form $(\lambda_1, \ldots, \lambda_n)$. In particular, the partition $(0, 0, \ldots)$ is denoted by 0. The nonzero entries λ_i in (2.2) are called the parts of λ. The number of parts and the sum of parts are called the length and the weight of λ, denoted by $\ell(\lambda)$ and $|\lambda|$, respectively. We also use the exponential notation $\lambda = 1^{m_1}2^{m_2}\cdots r^{m_r}\cdots$ to denote the partition with exactly m_i parts equal to i.

Let $\lambda = (\lambda_1, \ldots, \lambda_i, \ldots)$, $\mu = (\mu_1, \ldots, \mu_i, \ldots)$ be two partitions. We define the addition of λ and μ to be the partition $\lambda + \mu = (\lambda_1 + \mu_1, \ldots, \lambda_i + \mu_i, \ldots)$.

The subtraction of two partitions $\lambda - \mu$ is defined similarly if the resulting sequence is a partition. The Ferrers diagram of a partition $(\lambda_1, \lambda_2, \ldots)$ of n is a left-justified array of squares (also called cells) with λ_i squares in the i-th row.

Henceforth, fix a nonnegative integer m. Suppose that $r \in \mathbb{N}$ is such that $r \leq m$. Let P_r be the set of partitions λ with maximal part r. We define the two sets W_r and W as follows:

$$W_r := \{(\lambda, \mu; r) : \lambda \in P_r, \mu \in P_{m-r}\}, \qquad W := \bigcup_{r=0}^{m} W_r.$$

It is easy to see that [1]

$$\sum_{\lambda \in P_r} q^{|\lambda|} = \frac{q^r}{(q;q)_r}.$$

Thus the left hand side of (2.1) can be expressed as

$$\begin{aligned}
&\sum_{r=0}^{m}(-1)^r \left[\left(\sum_{\lambda \in P_r} q^{|\lambda|}\right)\left(\sum_{\mu \in P_{m-r}} q^{|\mu|}\right)\right] \\
&= \sum_{r=0}^{m}(-1)^r \sum_{(\lambda,\mu;r)\in W_r} q^{|\lambda|+|\mu|} \\
&= \sum_{(\lambda,\mu;r)\in W} (-1)^r q^{|\lambda|+|\mu|}. \qquad (2.3)
\end{aligned}$$

For $(\lambda, \mu; r) \in W$, let $s = \ell(\lambda)$ and $t = \ell(\mu)$. We first define an involution $\sigma\colon W \longmapsto W$. Then we show that this involution consists of two parts, one that is sign reversing, the other the identity map. Therefore, the Gauss identity follows from the cancellation in the expansion of the left hand side of (2.1).

Construction of the involution σ. An involution $\sigma\colon W \longmapsto W$ is constructed in accordance with the following subcases for the lengths s and t:

Case 1: $s < t$

Let

$$\lambda' = \lambda + 1^t, \ \mu' = \mu - 1^t, \text{ and } r' = r + 1.$$

Clearly, $\lambda' \in P_{r'}$ and $\mu' \in P_{m-r'}$. Since $t > 0$, r must be less than m and $(\lambda', \mu'; r') \in W$ (The Ferrers diagrams are shown in Figure 1).

Case 2: $s \geq t$

Subcase 2.1: There exists at least one odd number among $\lambda_{t+1}, \ldots, \lambda_s$.

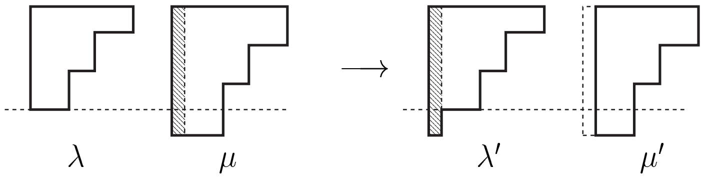

Figure 1: Case $\ell(\lambda) < \ell(\mu)$.

Suppose λ_p is odd, while $\lambda_{p+1}, \ldots, \lambda_s$ are all even. Let

$$\lambda' = \lambda - 1^p,\ \ \mu' = \mu + 1^p,\ \text{ and } r' = r - 1.$$

Since $\lambda_p > 0$, we have $r = \lambda_1 > 0$. Therefore, $(\lambda', \mu'; r') \in W$ (see Figure 2).

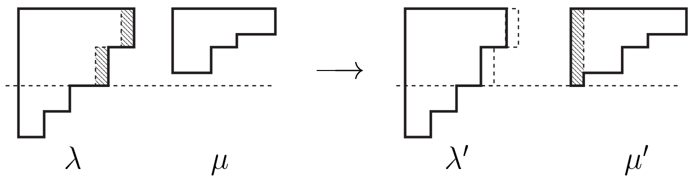

Figure 2: Case λ_p is odd.

Subcase 2.2: $\lambda_{t+1}, \ldots, \lambda_s$ are all even, and $t > 0$.

If λ_t is odd, similar to the above subcase, we define

$$\lambda' = \lambda - 1^t,\ \ \mu' = \mu + 1^t,\ \text{ and } r' = r - 1.$$

Hence, $(\lambda', \mu'; r') \in W$ (see Figure 3).

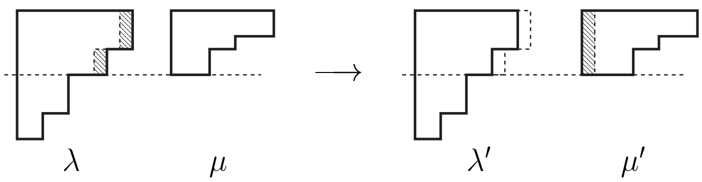

Figure 3: Case λ_t is odd.

If λ_t is even, similar to Case 1, we define

$$\lambda' = \lambda + 1^t,\ \ \mu' = \mu_1 - 1^t,\ \text{ and } r' = r + 1.$$

Then, $(\lambda', \mu'; r') \in W$ (see Figure 4).

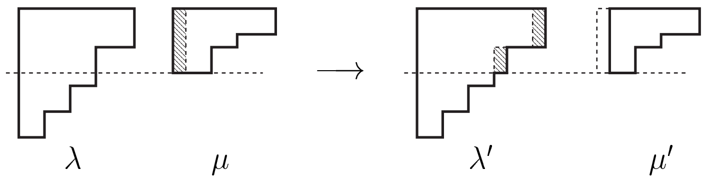

Figure 4: Case λ_t is even.

Subcase 2.3: $\lambda_1, \ldots, \lambda_s$ are all even, and $t = 0$.

Since $t = 0$, the partition μ is the zero partition, whose maximal part is regarded as 0. From the definition of W, we have $m - r = 0$. For this case, we define the image of $(\lambda, 0; m)$ to be itself.

It is easy to verify that σ^2 is the identity map on W:

- In Case 1, we have $\ell(\lambda') = t \geq \ell(\mu')$ and $\lambda'_t = 1$. Hence $\sigma(\lambda', \mu'; r') = (\lambda' - 1^t, \mu' + 1^t; r' - 1) = (\lambda, \mu; r)$.
- In Subcase 2.1, there are two possibilities. (1) $\ell(\lambda') \geq p = \ell(\mu') > 0$ and λ'_i are all even for $i \geq p$. In this case, $\sigma(\lambda', \mu'; r') = (\lambda' + 1^p, \mu' - 1^p; r' + 1) = (\lambda, \mu; r)$. (2) $\ell(\lambda') < p = \ell(\mu')$. We also have $\sigma(\lambda', \mu'; r') = (\lambda' + 1^p, \mu' - 1^p; r' + 1) = (\lambda, \mu; r)$.
- In Subcase 2.2, there are also two possibilities. (1) $\ell(\lambda') = \ell(\lambda)$. It is clear that $\sigma(\lambda', \mu'; r') = (\lambda, \mu; r)$. (2) $\ell(\lambda') < \ell(\lambda)$. Then $s = t$ and $\lambda_s = 1$. Therefore, $\ell(\lambda') < \ell(\mu) = t$ and $\sigma(\lambda', \mu'; r') = (\lambda' + 1^t, \mu' - 1^t; r' + 1) = (\lambda, \mu; r)$
- In Subcase 2.3, σ is the identity map.

Therefore, σ^2 is the identity map on W.

Combinatorial Proof of the Gauss identity. In the above construction of the involution σ, it is clear that $|\lambda'| + |\mu'| = |\lambda| + |\mu|$. Furthermore, except for Subcase 2.3, we have $|r - r'| = 1$, in other words, σ is sign-reversing, which implies that $(-1)^{r'} q^{|\lambda'|+|\mu'|} + (-1)^r q^{|\lambda|+|\mu|} = 0$. It follows that

$$\sum_{(\lambda,\mu;r)\in W} (-1)^r q^{|\lambda|+|\mu|} = \sum_{\substack{(\lambda,\mu;r) \text{ being} \\ \text{fixed point of } \sigma}} (-1)^r q^{|\lambda|+|\mu|} = \sum_{\substack{(\lambda,0;m)\in W \\ \lambda_i \text{ all even}}} (-1)^m q^{|\lambda|}. \quad (2.4)$$

Since $(\lambda, 0; m) \in W$ implies that $m = \lambda_1$, (2.4) reduces to

$$\sum_{(\lambda,\mu;r)\in W} (-1)^r q^{|\lambda|+|\mu|} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \dfrac{q^m}{(q^2;q^2)_{m/2}}, & \text{if } m \text{ is even.} \end{cases}$$

This completes the proof of (2.1). ∎

The above involution, in fact, leads to stronger result which is a generalization of the Gauss identity with an additional parameter n.

Theorem 2.1 *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{r=0}^{m} (-1)^r \begin{bmatrix} n+r \\ n \end{bmatrix} \begin{bmatrix} n+m-r \\ n \end{bmatrix} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \begin{bmatrix} n+\frac{m}{2} \\ n \end{bmatrix}_{q^2}, & \text{if } m \text{ is even.} \end{cases} \tag{2.5}$$

Proof. Note that the above involution σ preserves the maximum value of the lengths of λ and μ. Therefore, this involution can be restricted to partitions with length not greater than $n+1$, and the arguments can be effected as before. Let $P_{n,r}$ be the set of partitions with maximal component r and length not greater than $n+1$. Note that the generating function of partitions in $P_{n,r}$ is (see [8, Proposition 1.3.19])

$$\sum_{\lambda \in \mathcal{P}_{n,r}} q^{|\lambda|} = q^r \begin{bmatrix} n+r \\ n \end{bmatrix}. \tag{2.6}$$

Thus we obtain (2.1). ∎

The identity (1.1) is the limiting case of (2.5) by taking $n \to \infty$. We now reformulate Theorem 2.1 into a symmetric form:

Theorem 2.2 *We have*

$$\sum_{r=0}^{m} (-1)^r \frac{(a;q)_r}{(q;q)_r} \frac{(a;q)_{m-r}}{(q;q)_{m-r}} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \dfrac{(a^2;q^2)_{m/2}}{(q^2;q^2)_{m/2}}, & \text{if } m \text{ is even.} \end{cases} \tag{2.7}$$

Proof. We can rewrite the identity (2.5) as

$$\sum_{r=0}^{m} (-1)^r \frac{(q^n;q)_r}{(q;q)_r} \frac{(q^n;q)_{m-r}}{(q;q)_{m-r}} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \dfrac{(q^{2n};q^2)_{m/2}}{(q^2;q^2)_{m/2}}, & \text{if } m \text{ is even.} \end{cases} \tag{2.8}$$

Setting $a = q^n$, by the continuation argument we obtain (2.5). ∎

Note that the case of $a = 0$ in (2.7) specializes to the Gauss identity and the case $a = \infty$ reduces to the Gauss identity with parameter q replaced by q^{-1}.

3. Generalization to a p-th root of unity

In this section, we consider a further extension of the Gauss identity to the p-th root of the unit. This generalization reduces to Theorem 2.1 or (2.8) when $p = 2$. We first give a bijective proof of this extension that is a refinement of the involution in the previous section. For completeness, we also present an algebraic proof.

Theorem 3.1 *Let $\zeta = e^{\frac{2\pi i}{p}}$ be the p-th root of unity. Then we have*

$$\sum_{r_1+\cdots+r_p=m} \zeta^{r_1+2r_2+\cdots+pr_p} \begin{bmatrix} n+r_1 \\ n \end{bmatrix} \begin{bmatrix} n+r_2 \\ n \end{bmatrix} \cdots \begin{bmatrix} n+r_p \\ n \end{bmatrix}$$

$$= \begin{cases} 0, & \text{if } p \nmid m, \\ \begin{bmatrix} n+\frac{m}{p} \\ n \end{bmatrix}_{q^p}, & \text{if } p \mid m. \end{cases} \tag{3.1}$$

Proof. Let $P_{n,r}$ be the set of partitions with maximal part r and length not greater than $n+1$. Define the set W as

$$W := \{(\lambda^1, \ldots, \lambda^p; r_1, \ldots, r_p) \colon r_1 + \cdots + r_p = m \text{ and } \lambda^k \in \mathcal{P}_{n,r_k}\},$$

and define the weight of $x = (\lambda^1, \ldots, \lambda^p; r_1, \ldots, r_p) \in W$ as

$$w(x) := \zeta^{r_1+2r_2+\cdots+pr_p} q^{|\lambda^1|+\cdots+|\lambda^p|}.$$

It follows from the generating function (2.6) that (3.1) is equivalent to the following relation:

$$\sum_{x\in W} w(x) = \begin{cases} 0, & \text{if } p \nmid m, \\ \displaystyle\sum_{\lambda\in P_{n,m/p}} q^{p|\lambda|}, & \text{if } p \mid m. \end{cases}$$

Let $\overline{a}$ $(0 \le \overline{a} < p)$ denote the remainder of a modulo p. Define

$$W_h := \{(\lambda^1, \ldots, \lambda^p; r_1, \ldots, r_p) \in W_m \colon \\ h \text{ is the smallest integer such that } \overline{\lambda^1_k} = \lambda^2_k = \cdots = \lambda^p_k = 0,\ \forall\, k > h\},$$

and

$$W_{h,s} := \{(\lambda^1, \ldots, \lambda^p; r_1, \ldots, r_p) \in W_h \colon \\ s \text{ is the smallest integer such that } \overline{\lambda^1_h} + \lambda^2_h + \cdots + \lambda^s_h \ge p - s + 1\},$$

where λ^i_j denote the j-th part of the partition λ^i. Noting that $\overline{\lambda^1_h} < p$, $W_{h,1} = \emptyset$ for any $h > 0$, we have

$$W = \bigcup_{h=0}^{n+1} W_h = W_0 \bigcup \left(\bigcup_{h=1}^{n+1} \bigcup_{s=2}^{p} W_{h,s} \right),$$

where $\bigcup$ denotes the disjoint union.

We now focus on $W_{h,s}$ with $h > 0$ and $s \geq 2$. For $x = (\lambda^1, \dots, \lambda^p; r_1, \dots, r_p) \in W_{h,s}$, let

$$\begin{aligned} d_x &:= (p - s + 1) - (\overline{\lambda^1_h} + \lambda^2_h + \cdots + \lambda^{s-1}_h), \\ I_x &:= (\overline{\lambda^1_h}, \lambda^2_h, \dots, \lambda^{s-1}_h, d_x). \end{aligned}$$

From the definition of $W_{h,s}$, we have

$$\overline{\lambda^1_h} + \lambda^2_h + \cdots + \lambda^{s-1}_h < p - (s-1) + 1 = p - s + 2,$$

which implies $I_x \in \mathbb{N}^s$. Moreover, the following relation holds:

$$\lambda^s_h - d_x = (\overline{\lambda^1_h} + \lambda^2_h + \cdots + \lambda^s_h) - (p - s + 1) \geq 0.$$

Next, for each $I = (i_1, \dots, i_s) \in \mathbb{N}^s$ with $|I| = i_1 + \cdots + i_s = p - s + 1$, define

$$W_{h,s,I} := \{x \in W_{h,s} \colon I_x = I\}.$$

Since I_x is uniquely determined by x, we find that $W_{h,s}$ is the disjoint union of $W_{h,s,I}$:

$$W_{h,s} = \bigcup_{\substack{I \in \mathbb{N}^s \\ |I| = p-s+1}} W_{h,s,I}.$$

For $I = (i_1, \dots, i_s), J = (j_1, \dots, j_s) \in \mathbb{N}^s$ with $|I| = |J| = p - s + 1$, there is a bijection $\sigma_{I,J}$ from $W_{h,s,I}$ to $W_{h,s,J}$ defined by

$$\begin{array}{rccc} \sigma_{I,J} : & W_{h,s,I} & \to & W_{h,s,J} \\ & x & \mapsto & y, \end{array}$$

where

$$x = (\lambda^1, \dots, \lambda^p; r_1, \dots, r_p), \quad y = (\mu^1, \dots, \mu^p; r'_1, \dots, r'_p),$$

$$\mu^k = \begin{cases} \lambda^k - (i_k)^h + (j_k)^h, & \text{if } 1 \leq k \leq s, \\ \lambda^k, & s < k \leq p, \end{cases}$$

$$r'_k = \begin{cases} r_k - i_k + j_k, & \text{if } 1 \leq k \leq s, \\ r_k, & s < k \leq p. \end{cases}$$

We can show that $\sigma_{I,J}$ is well defined, that is, $y \in W_{h,s,J}$. Since $I_x = I$, we have

$$\lambda_h^1 \geq \overline{\lambda_h^1} = i_1, \quad \lambda_h^k = i_k \text{ for } k = 2, \ldots, s-1, \quad \lambda_h^s \geq d_x = i_s.$$

Since $\overline{\lambda_k^1} = \lambda_k^2 = \cdots = \lambda_k^p = 0$ for $k > h$, the $\lambda^k - (i_k)^h$ is well defined, as well as $\mu^k = \lambda^k - (i_k)^h + (j_k)^h$. Furthermore, the maximal part of μ^k is $r_k - i_k + j_k = r'_k$. Therefore, $y \in W$. It is easy to see that

$$(\overline{\mu_h^1}, \mu_h^2, \ldots, \mu_h^s) = (j_1, \ldots, j_{s-1}, (\lambda_h^s - d_s) + j_s). \tag{3.2}$$

Since $J \neq 0$, $(\overline{\mu_h^1}, \mu_h^2, \ldots, \mu_h^s) \neq 0$, it follows that $y \in W_h$. Noting that $\overline{\mu_h^1} + \mu_h^2 + \cdots + \mu_h^t \leq |J| < p - t + 1$ for $t < s$ and $\overline{\mu_h^1} + \mu_h^2 + \cdots + \mu_h^s = \overline{\lambda_h^1} + \lambda_h^2 + \cdots + \lambda_h^s \geq p - s + 1$, we have $y \in W_{h,s}$. Moreover, from (3.2) and $|I_y| = |J| = p - s + 1$, it follows that $I_y = J$, that is, $y \in W_{h,s,J}$. Thus, we have shown that $\sigma_{I,J}$ is a well defined map from $W_{h,s,I}$ to $W_{h,s,J}$.

It is a routine to verify that $\sigma_{I,J} \circ \sigma_{J,I}$ and $\sigma_{J,I} \circ \sigma_{I,J}$ are the identity maps on $W_{h,s,I}$ and $W_{h,s,J}$, respectively. Then it follows that $\sigma_{I,J}$ is a bijection from $W_{h,s,I}$ to $W_{h,s,J}$. Moreover, we have that

$$\begin{aligned} w(\sigma_{I,J}(x)) &= w(y) \\ &= \zeta^{r'_1 + 2r'_2 + \cdots + pr'_p} q^{|\mu^1| + \cdots + |\mu^p|} \\ &= \zeta^{r_1 + 2r_2 + \cdots + pr_p} \zeta^{-(i_1 + 2i_2 + \cdots + si_s)} \zeta^{j_1 + 2j_2 + \cdots + sj_s} q^{|\lambda^1| + \cdots + |\lambda^p|} \\ &= \zeta^{-(i_1 + 2i_2 + \cdots + si_s)} w(x) \zeta^{j_1 + 2j_2 + \cdots + sj_s}. \end{aligned}$$

Denote $I_0 = (p - s + 1, 0, \ldots, 0) \in \mathbb{N}^s$. Using the bijection $\sigma_{I_0,J}$, we obtain the following sequence of identities:

$$\begin{aligned} \sum_{x \in W_{h,s}} w(x) &= \sum_{\substack{J \in \mathbb{N}^s \\ |J| = p-s+1}} \sum_{x \in W_{h,s,J}} w(x) \\ &= \sum_{\substack{J \in \mathbb{N}^s \\ |J| = p-s+1}} \sum_{x \in W_{h,s,I_0}} w(\sigma_{I_0,J}(x)) \\ &= \sum_{\substack{J \in \mathbb{N}^s \\ |J| = p-s+1}} \sum_{x \in W_{h,s,I_0}} \zeta^{-(p-s+1)} w(x) \zeta^{j_1 + 2j_2 + \cdots + sj_s} \\ &= \sum_{x \in W_{h,s,I_0}} \zeta^{-(p-s+1)} w(x) \sum_{\substack{J \in \mathbb{N}^s \\ |J| = p-s+1}} \zeta^{j_1 + 2j_2 + \cdots + sj_s}. \end{aligned}$$

The second summation can be computed as follows:

$$\begin{aligned}
\sum_{J\in\mathbb{N}^s, |J|=p-s+1} t^{j_1+2j_2+\cdots+sj_s} &= \sum_{\substack{\lambda=1^{j_1}2^{j_2}\cdots s^{j_s}\\ j_1+\cdots+j_s=p-s+1}} t^{|\lambda|} \\
&= \sum_{\substack{\ell(\lambda)=p-s+1\\ \text{each part } \leq s}} t^{|\lambda|} \\
&= \sum_{\substack{\ell(\lambda)\leq s\\ \text{maximal part is } p-s+1}} t^{|\lambda|} \\
&= \sum_{\lambda\in\mathcal{P}_{s-1,p-s+1}} t^{|\lambda|} \\
&= t^{p-s+1}\frac{(t;t)_p}{(t;t)_{s-1}(t;t)_{p-s+1}}.
\end{aligned}$$

Since $1-\zeta^p = 0$ and $1-\zeta^k \neq 0$ for $k = 1, \ldots, p-1$, we obtain

$$\sum_{\substack{J\in\mathbb{N}^s\\ |J|=p-s+1}} \zeta^{j_1+2j_2+\cdots+sj_s} = 0, \quad \forall\, 2 \leq s \leq p,$$

which implies that $\sum_{x\in W_{h,s}} w(x)$ vanishes for all $h > 0$ and $2 \leq s \leq p$. Therefore,

$$\sum_{x\in W} w(x) = \sum_{x\in W_0} w(x).$$

By definition,

$$W_0 = \{(\lambda, 0, \ldots, 0; \lambda_1, 0, \ldots, 0) \in W \colon \lambda_1 = m \text{ and } p \mid \lambda_i \ \forall\, i \geq 1\}.$$

If $p \nmid m$, W_0 is the empty set. Otherwise, there is a bijection between W_0 and $\mathcal{P}_{n,m/p}$, obtained by dividing each part of λ by p. Hence,

$$\sum_{x\in W_0} w(x) = \begin{cases} 0, & \text{if } p \nmid m, \\ \displaystyle\sum_{\lambda\in\mathcal{P}_{n,m/p}} q^{p|\lambda|}, & \text{if } p \mid m. \end{cases}$$

This completes the combinatorial proof. ∎

Setting $n \to \infty$, we obtain a further generalization of the Gauss identity.

Corollary 3.2 *Let $\zeta = e^{\frac{2\pi i}{p}}$ be the p-th root of unity. Then we have*

$$\sum_{r_1+\cdots+r_p=m} \zeta^{r_1+2r_2+\cdots+pr_p} \begin{bmatrix} m \\ r_1, \ldots, r_p \end{bmatrix} = \begin{cases} 0, & \textit{if } p \nmid m, \\ \displaystyle\prod_{\substack{1\leq k\leq m\\ p\nmid k}} (1-q^k), & \textit{if } p \mid m, \end{cases}$$

where

$$\begin{bmatrix} m \\ r_1, \ldots, r_p \end{bmatrix} = \frac{(q;q)_m}{(q;q)_{r_1} \cdots (q;q)_{r_p}}$$

is a q-multinomial coefficient.

As a further generalization of Theorem 2.2, we have

Theorem 3.3 *Let $\zeta = e^{\frac{2\pi i}{p}}$ be the p-th root of unity. Then we have*

$$\sum_{r_1+\cdots+r_p=m} \zeta^{r_1+2r_2+\cdots+pr_p} \frac{(a;q)_{r_1}}{(q;q)_{r_1}} \cdots \frac{(a;q)_{r_p}}{(q;q)_{r_p}} = \begin{cases} 0, & \text{if } p \nmid m, \\ \dfrac{(a^p;q^p)_{m/p}}{(q^p;q^p)_{m/p}}, & \text{if } p \mid m. \end{cases}$$

To conclude this paper, we present an algebraic proof of Theorem 3.3 from the Cauchy identity (q-binomial theorem):

$$\sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} t^r = \prod_{r=0}^{\infty} \frac{(1-atq^r)}{(1-tq^r)}$$

Algebraic Proof of Theorem 3.3. Let $\zeta = e^{\frac{2\pi i}{p}}$ be the p-th root of unity. From the Cauchy identity it follows that

$$\begin{aligned} & \prod_{r=0}^{\infty} \frac{(1-atq^r)}{(1-tq^r)} \cdot \prod_{r=0}^{\infty} \frac{(1-a\zeta tq^r)}{(1-\zeta tq^r)} \cdots \prod_{r=0}^{\infty} \frac{(1-a\zeta^{p-1}tq^r)}{(1-\zeta^{p-1}tq^r)} \\ = & \sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} t^r \cdot \sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} (\zeta t)^r \cdots \sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} (\zeta^{p-1} t)^r \\ = & \sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} (\zeta t)^r \cdot \sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} (\zeta^2 t)^r \cdots \sum_{r=0}^{\infty} \frac{(a;q)_r}{(q;q)_r} (\zeta^p t)^r \\ = & \sum_{m=0}^{\infty} t^m \sum_{r_1+\cdots+r_p=m} \frac{(a;q)_{r_1}}{(q;q)_{r_1}} \cdots \frac{(a;q)_{r_p}}{(q;q)_{r_p}} \zeta^{r_1+2r_2+\cdots+pr_p}. \end{aligned}$$

On the other hand, from the relation $1-x^p = (1-x)(1-\zeta x)\cdots(1-\zeta^{p-1}x)$ and the Cauchy identity, we obtain

$$\begin{aligned}
&\prod_{r=0}^{\infty}\frac{(1-atq^r)}{(1-tq^r)}\cdot\prod_{r=0}^{\infty}\frac{(1-a\zeta tq^r)}{(1-\zeta tq^r)}\cdots\prod_{r=0}^{\infty}\frac{(1-a\zeta^{p-1}tq^r)}{(1-\zeta^{p-1}tq^r)}\\
&\quad=\prod_{r=0}^{\infty}\frac{(1-atq^r)(1-\zeta atq^r)\cdots(1-\zeta^{p-1}atq^r)}{(1-tq^r)(1-\zeta tq^r)\cdots(1-\zeta^{p-1}tq^r)}\\
&\quad=\prod_{r=0}^{\infty}\frac{1-(atq^r)^p}{1-(tq^r)^p}\\
&\quad=\prod_{r=0}^{\infty}\frac{1-a^pt^p(q^p)^r}{1-t^p(q^p)^r}\\
&\quad=\sum_{r=0}^{\infty}\frac{(a^p;q^p)_r}{(q^p;q^p)_r}(t^p)^r.
\end{aligned}$$

Comparing the coefficients of t^m, we arrive at Theorem 3.3. ∎

More generally, Littlewood [6] obtained the image of any Schur function under the morphism

$$\mathbf{p}_k \to \begin{cases} p\,\mathbf{p}_k, & \text{if } p \text{ divides } k,\\ 0, & \text{otherwise,}\end{cases}$$

where $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k, \ldots$ are the power sums.

The case treated in Theorem 3.3 corresponds to the specialization of a complete function under

$$\mathbf{p}_k \to \begin{cases} p\,(1-a^k)(1-q^k)^{-1}, & \text{if } p \text{ divides } k,\\ 0, & \text{otherwise.}\end{cases}$$

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