Rational Interpolation and Basic Hypergeometric Series

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Abstract. We give a Newton type rational interpolation formula (Theorem 2.2). It contains as a special case the original Newton interpolation, as well as the recent interpolation formula of Zhi-Guo Liu, which allows to recover many important classical q-series identities. We show in particular that some bibasic identities are a consequence of our formula.

1. Introduction and Notation

As usual, $(a;q)_n$ denotes:

$$\prod_{j=0}^{n-1}(1-aq^j), n=0,1,2,\cdots,\infty,$$

and

$$(a_1,a_2,\cdots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n.$$

The basic hypergeometric series ${}_r\phi_s$ are defined in [3] by:

$${}_r\phi_s\left[\begin{array}{cccc} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{array} ;q,z\right]=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\cdots,a_r;q)_n}{(b_1,b_2,\cdots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n,$$

where $q \neq 0$ when $r > s+1$.

Newton obtained the following interpolation formula:

$$f(x) = f(x_1) + f^{\partial}(x - x_1) + f^{\partial\partial}(x - x_1)(x - x_2) + \cdots ,$$

where $f^{\partial\cdots\partial}$ is the image of $f(x)$ under a product of divided differences which will be defined below.

Special cases of Newton's interpolation are the Taylor formula and the q-Taylor formula (c.f. [5]), with derivatives or q-derivatives instead of divided differences.

Using q-derivatives, Zhi-Guo Liu [7] gave an interpolation formula involving rational functions in x as coefficients, instead of only polynomials in x as in the q-Taylor formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{(1 - aq^{2n})(aq/x; q)_n x^n}{(q, x; q)_n} [D_q f(x)(x; q)_{n-1}]\big|_{x=aq}, \tag{1.1}$$

D_q being defined by

$$D_q f(x) = \frac{f(x) - f(xq)}{x}.$$

Let us remark that Carlitz's q-analog of a special case of the Lagrange inversion formula is the limit for $a \to 0$ of (1.1):

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q, x; q)_n} [D_q f(x)(x; q)_{n-1}]|_{x=0}.$$

Our formula involves two sets of indeterminate $\mathcal{X}$ and $\mathcal{C}$. Newton interpolation is the case when

$$\mathcal{C} = \{0, 0, \cdots\},$$

and Zhi-Guo Liu's expansion is the case when

$$\mathcal{X} = \{aq^1, aq^2, \cdots\},\ \mathcal{C} = \{q^0, q^1, q^2, \cdots\}.$$

2. Rational Interpolation

By convenience, we denote

$$Y_n(x,\mathcal{X}) = (x-x_1)(x-x_2)\cdots(x-x_n)$$

and

$$(x,\mathcal{C})_n = (1-xc_1)(1-xc_2)\cdots(1-xc_n).$$

The divided difference ∂_i (acting on its left), $i=1,2,3,\cdots$ is defined by

$$f(x_1,x_2,\cdots,x_i,x_{i+1},\cdots)\,\partial_i \\ = \frac{f(\cdots,x_i,x_{i+1},\cdots)-f(\cdots,x_{i+1},x_i,\cdots)}{x_i-x_{i+1}} = \frac{f-f^{s_i}}{x_i-x_{i+1}},$$

where s_i denotes the exchange of x_i and x_{i+1}.

Divided differences satisfy a Leibnitz type formula:

$$(f(x_1)g(x_1))\partial_1 = f(x_1)\big(g(x_1)\partial_1\big) + \big(f(x_1)\partial_1\big)\,g(x_2).$$

By induction, one obtains:

$$f(x_1)g(x_1)\,\partial_1\partial_2\cdots\partial_n = \sum_{k=0}^{n}\big(f(x_1)\partial_1\cdots\partial_k\big)\,\big(g(x_{k+1})\partial_{k+1}\cdots\partial_n\big).$$

Lemma 2.1 *Letting i,n be two nonnegative integers, then one has:*

$$Y_n(b_1,\mathcal{X})\,\partial_1\partial_2\cdots\partial_i\big|_{\mathcal{B}=\mathcal{X}} = \begin{cases} 0, & i\neq n; \\ 1, & i=n, \end{cases}$$

where $\big|_{\mathcal{B}=\mathcal{X}}$ denotes the specialization $b_1=x_1,\ b_2=x_2,\cdots$, and the divided differences are relative to $b_1,b_2,\cdots$.

Proof. If $i\le n$, using Leibnitz formula, we have:

$$\begin{aligned}
&Y_n(b_1,\mathcal{X})\,\partial_1\partial_2\cdots\partial_i\big|_{\mathcal{B}=\mathcal{X}} \\
&= \prod_{k=2}^{n}(b_1-x_k)\,\partial_1\partial_2\cdots\partial_i(b_1-x_1)\big|_{\mathcal{B}=\mathcal{X}} + \prod_{k=2}^{n}(b_2-x_k)\,\partial_2\cdots\partial_i(b_1-x_1)\,\partial_1\big|_{\mathcal{B}=\mathcal{X}} \\
&= \prod_{k=2}^{n}(b_2-x_k)\,\partial_2\cdots\partial_i\big|_{\mathcal{B}=\mathcal{X}} = \cdots = \prod_{k=i}^{n}(b_i-x_k)\,\partial_i\big|_{\mathcal{B}=\mathcal{X}} \\
&= \begin{cases} \prod_{k=i+1}^{n}(b_{i+1}-x_k)\big|_{\mathcal{B}=\mathcal{X}} = 0, & i<n; \\ (b_n-x_n)\,\partial_n\big|_{\mathcal{B}=\mathcal{X}} = 1, & i=n. \end{cases}
\end{aligned}$$

In the case $i > n$, nullity comes from the fact that each ∂_i decreases degree by 1. ∎

Theorem 2.2 *For any formal series $f(x)$ in x, we have the following identity in the ring of formal series in $x, x_1, x_2, \cdots$:*

$$\begin{aligned} f(x) &= f(x_1) + f(x_1)\,\partial_1(1-x_2c_1)\frac{Y_1(x,\mathcal{X})}{(x,\mathcal{C})_1} \\ &+ f(x_1)(1-x_1c_1)\,\partial_1\partial_2(1-x_3c_2)\frac{Y_2(x,\mathcal{X})}{(x,\mathcal{C})_2} \\ &+ \cdots + f(x_1)(x_1,\mathcal{C})_{n-1}\,\partial_1\cdots\partial_n(1-x_{n+1}c_n)\frac{Y_n(x,\mathcal{X})}{(x,\mathcal{C})_n} + \cdots . \end{aligned} \tag{2.2}$$

Proof. Let

$$f(b) = \sum_{n=0}^{\infty} A_n \frac{Y_n(b,\mathcal{X})}{(b,\mathcal{C})_n}.$$

Specializing b to x_1 or x_2, one gets the following coefficients:

$$A_0 = f(x_1), A_1 = f(x_1)\,\partial_1(1-x_2c_1).$$

Now we have to check,

$$\frac{Y_n(b_1,\mathcal{X})}{(b_1,\mathcal{C})_n}(b_1,\mathcal{C})_{k-1}\,\partial_1\partial_2\cdots\partial_k\big|_{\mathcal{B}=\mathcal{X}} = \begin{cases} 0, & k \neq n; \\ \frac{1}{1-x_{n+1}c_n}, & k = n. \end{cases} \tag{2.3}$$

If $k > n$, $\dfrac{Y_n(b,\mathcal{X})}{(b,\mathcal{C})_n}(b,\mathcal{C})_{k-1}$ is a polynomial of degree $k-1$, and therefore annihilated by a product of k divided differences.

If $k < n$, from Leibnitz formula, we get:

$$\begin{aligned} &\frac{Y_n(b_1,\mathcal{X})}{(b_1,\mathcal{C})_n}(b_1,\mathcal{C})_{k-1}\,\partial_1\partial_2\cdots\partial_k\big|_{\mathcal{B}=\mathcal{X}} \\ &= \frac{Y_n(b_1,\mathcal{X})}{\prod_{p=k}^{n}(1-b_1c_p)}\,\partial_1\partial_2\cdots\partial_k\big|_{\mathcal{B}=\mathcal{X}} \\ &= \sum_{i=0}^{k}\frac{1}{\prod_{p=k}^{n}(1-b_{i+1}c_p)}\,\partial_{i+1}\cdots\partial_k Y_n(b_1,\mathcal{X})\,\partial_1\cdots\partial_i\big|_{\mathcal{B}=\mathcal{X}}, \end{aligned}$$

and Lemma 2.1 shows that this function is equal to 0.

If $k = n$, we have:

$$\begin{aligned}
&\frac{Y_n(b_1, \mathcal{X})}{1 - b_1 c_n} \, \partial_1 \partial_2 \cdots \partial_n \big|_{\mathcal{B}=\mathcal{X}} \\
&= \sum_{i=0}^{n} \frac{1}{1 - b_{i+1} c_n} \, \partial_{i+1} \cdots \partial_n Y_n(b_1, \mathcal{X}) \, \partial_1 \cdots \partial_i \big|_{\mathcal{B}=\mathcal{X}} \\
&= \frac{1}{1 - b_{n+1} c_n} Y_n(b_1, \mathcal{X}) \, \partial_1 \cdots \partial_n \big|_{\mathcal{B}=\mathcal{X}} \\
&= \frac{1}{1 - x_{n+1} c_n}.
\end{aligned}$$

Formula (2.3) thus implies

$$A_n = f(x_1)(x_1, \mathcal{C})_{n-1} \partial_1 \cdots \partial_n (1 - x_{n+1} c_n),$$

and the theorem. ∎

3. Application to Basic summation formulas

Taking

$$f(x) = (x; q)_n,$$

and

$$\mathcal{X} = \{aq^1, aq^2, \cdots\}, \; \mathcal{C} = \{0, 0, \cdots\},$$

we have:

$$\begin{aligned}
(x; q)_n &= \sum_{k=0}^{n} (x - aq) \cdots (x - aq^k)(x_1; q)_n \partial_1 \cdots \partial_k \\
&= \sum_{k=0}^{n} (x - aq) \cdots (x - aq^k)(-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} (aq^{k+1}; q)_{n-k} \\
&= (aq; q)_n \sum_{k=0}^{n} \frac{(q^{-n}, aq/x; q)_k}{(q, aq; q)_k} (xq^n)^k.
\end{aligned}$$

Replacing aq by a, x by a/c, we get a q-analogue of Vandermonde's formula:

$$\frac{(a/c;q)_n}{(a;q)_n} = \sum_{k=0}^{n} \frac{(q^{-n},c;q)_k}{(q,a;q)_k}(aq^n/c)^k = {}_2\phi_1\left[\begin{matrix} q^{-n}, & c \\ & a \end{matrix} ;q, aq^n/c\right].$$

Recall another q-analogue of Vandermonde's formula:

$$\frac{(a/c;q)_n}{(a;q)_n}c^n = \sum_{k=0}^{n} \frac{(q^{-n},c;q)_k}{(q,a;q)_k}q^k. \tag{3.4}$$

Taking

$$f(x) = \frac{1}{(a\beta x;q)_\infty}$$

and

$$\mathcal{X} = \{q/a, q^2/a, \cdots\}, \mathcal{C} = \{a\beta, a\beta q, \cdots\},$$

gives:

$$f(x_1)(x_1;q)_{n-1}\partial_1 \cdots \partial_n = \frac{1}{(a\beta x_1 q^{n-1};q)_\infty}\partial_1 \cdots \partial_n = \frac{(a\beta)^n q^{n(n-1)}}{(q;q)_n(\beta q^n;q)_\infty}.$$

From Theorem 2.2, we have:

$$\frac{1}{(a\beta x;q)_\infty} = \frac{1}{(\beta;q)_\infty}\sum_{n=0}^{\infty} \frac{(1-\beta q^{2n})(q/ax,\beta;q)_n(a\beta x)^n q^{n(n-1)}}{(q;q)_n(a\beta x;q)_n},$$

and we get Jackson's formula [4]:

$$\frac{(\beta;q)_\infty}{(a\beta x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(1-\beta q^{2n})(q/ax,\beta;q)_n(a\beta x q^{n-1})^n}{(q;q)_n(a\beta x;q)_n}. \tag{3.5}$$

Setting $x \to 0$, replacing β by βq, we derive Sylvester's formula:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \beta^n q^{n(3n+1)/2}(1-\beta q^{2n+1})}{(q;q)_n(\beta q^{n+1};q)_\infty} = 1.$$

In (3.5), letting $a=1$, $x=q^N$, replacing β by $-\beta q$, we get Andrews' formula [1]:

$$(-\beta q;q)_{2N} = \sum_{n=0}^{N}(-\beta q;q)_{n-1}(1+\beta q^{2n})q^{n(3n-1)/2}\begin{bmatrix} N \\ n \end{bmatrix}(-\beta q^{n+N+1};q)_{N-n}.$$

Taking

$$f(x) = {}_3\phi_2 \left[\begin{matrix} q^{-n}, & a, & x/b \\ x, & aq^{1-n}/e & \end{matrix} ; q, q \right],$$

and

$$\mathcal{X} = \{0, 0, \cdots\},\ \mathcal{C} = \{q^0, q^1, q^2, \cdots\},$$

we have:

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} \frac{x^i}{(x;q)_i} f(x_1)(x_1;q)_{i-1}\, \partial_1 \cdots \partial_i \\ &= \sum_{i=0}^{\infty} \frac{x^i}{(x;q)_i} \sum_{k=i}^{n} \frac{(q^{-n}, a; q)_k q^k}{(q, aq^{1-n}/e; q)_k} \frac{(x_1/b;q)_k}{(x_1 q^{i-1}; q)_{k-i+1}}\, \partial_1 \cdots \partial_i \\ &= \sum_{i=0}^{\infty} \frac{(q^{-n}, a, q^{1-i}/b; q)_i (xq^i)^i}{(q, x, aq^{1-n}/e; q)_i} \sum_{k=i}^{n} \frac{(q^{-n+i}, aq^i; q)_{k-i} q^{k-i}}{(q, aq^{1-n+i}/e; q)_{k-i}} \\ &= \sum_{i=0}^{\infty} \frac{(q^{-n}, a, q^{1-i}/b; q)_i (xq^i)^i}{(q, x, aq^{1-n}/e; q)_i} \frac{(q^{1-n}/e; q)_{n-i}}{(a/eq^{1-n+i}; q)_{n-i}} (aq^i)^{n-i} \quad \text{(from 3.4)} \\ &= \frac{(q^{1-n}/e;q)_n}{(a/eq^{1-n};q)_n} a^n \sum_{i=0}^{n} \frac{(q^{-n}, a, b; q)_i}{(q, x, e; q)_i} \left(\frac{xeq^n}{ab} \right)^i. \end{aligned}$$

Raplacing x by d, we obtain Sears' formula ([3] (3.2.5)):

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, & a, & b \\ d, & e & \end{matrix} ; q, deq^n/ab \right] = \frac{(e/a;q)_n}{(e;q)_n} {}_3\phi_2 \left[\begin{matrix} q^{-n}, & a, & d/b \\ d, & aq^{1-n}/e & \end{matrix} ; q, q \right].$$

4. Bibasic summation formula

Proposition 4.3 *Taking*

$$f(x) = \frac{1-ux}{1-vx}$$

and

$$\mathcal{X} = \{x_1, x_2, \cdots\},\ \mathcal{C} = \{c_1, c_2, \cdots\},$$

we have:

$$f(x) = \frac{1-ux_1}{1-vx_1} + \sum_{k=1}^{\infty} \frac{(v-u) Y_{k-1}(v, \mathcal{C})(1 - x_{k+1}c_k)}{(v, \mathcal{X})_{k+1}} \frac{Y_k(x, \mathcal{X})}{(x, \mathcal{C})_k} \qquad (4.6)$$

The proposition is a direct application of Theorem 2.2 and of the following lemma.

Lemma 4.4

$$\frac{1-ux_1}{1-vx_1}(x_1,\mathcal{C})_{k-1}\partial_1\partial_2\cdots\partial_k = (v-u)Y_{k-1}(v,\mathcal{C})_k/(v,\mathcal{X})_{k+1}. \tag{4.7}$$

We first need to recall some facts about symmetric functions [8]. The generating function for the elementary symmetric functions $e_i(x_1,x_2,\cdots)$, and the complete functions $S_i(x_1,x_2,\cdots)$ are

$$\sum_{i\geq 0} e_i(x_1,x_2,\cdots)t^i = \prod_{i\geq 0}(1+x_it),$$

and

$$\sum_{i\geq 0} S_i(x_1,x_2,\cdots)t^i = \prod_{i\geq 0} 1/(1-x_it).$$

We shall need a slightly more general notion than usual, for a Schur function. Given $\lambda\in\mathbb{N}^n$, given n sets of variable $A_1,\ldots,A_n$, then the Schur function $S_\lambda(A_1,\ldots,A_n)$ is equal to $\left|S_{\lambda_j+j-i}(A_j)\right|_{1\leq i,j\leq n}$.

One has the following identity [6]:

$$S_\lambda(x_2,x_3,\cdots)x_1^r = S_{\lambda,r}(\mathcal{X},x_1). \tag{4.8}$$

Proof of Lemma 4.4 Multiply the denominator of $\dfrac{1-ux_1}{1-vx_1}(x_1,\mathcal{C})_{k-1}$ by the symmetrical factor $(v,\mathcal{X})_{k+1}$, which commutes with $\partial_1\cdots\partial_k$. Let $\mathcal{X}_k = \{x_1,x_2,\ldots x_{k+1}\}$. One has:

$$\begin{aligned}
&(1-ux_1)\prod_{i=1}^{k-1}(1-x_1c_i)\prod_{j=2}^{k+1}(1-vx_j)\\
&=\sum_{i=0}^{k}\sum_{j=0}^{k}(-1)^i(-v)^je_i(u,c_1,\cdots,c_{k-1})e_j(x_2,x_3,\cdots,x_{k+1})x_1^i\\
&=\sum_{i=0}^{k}\sum_{j=0}^{k}(-1)^i(-v)^je_i(u,c_1,\cdots,c_{k-1})S_{1^j,i}(\underbrace{\mathcal{X}_k,\ldots,\mathcal{X}_k}_{j},x_1),
\end{aligned}$$

thanks to (4.8), and to the fact that for every j, $e_j(\mathcal{X}) = S_{1^j}(\underbrace{\mathcal{X},\dots,\mathcal{X}}_{j})$.

The image of a power of x_1 under $\partial_1\cdots\partial_k$ is a complete function in $\mathcal{X}$ [6]. Therefore,

$$S_{1^j,i}(\underbrace{\mathcal{X}_k,\dots,\mathcal{X}_k}_{j}, x_1)\,\partial_1\cdots\partial_k = S_{1^j,i-k}(\underbrace{\mathcal{X}_k,\dots,\mathcal{X}_k}_{j+1}).$$

This determinant is equal to 0 (because it has two identical columns), except for $i+j=k$, in which case it is equal to $S_{0^{j+1}}(\mathcal{X}) = (-1)^j$.

Now:

$$\frac{1-ux_1}{1-vx_1}(x_1,\mathcal{C})_{k-1}(v,\mathcal{X})_{k+1}\,\partial_1\partial_2\cdots\partial_k$$
$$= \sum_{i+j=n}(-1)^i v^j e_i(u,c_1,\cdots,c_{k-1}) = (v-u)Y_{k-1}(v,\mathcal{C})_k,$$

thus (4.7) is true. ∎

In [2], Gasper obtained the following identity:

$$\sum_{k=0}^{\infty}\frac{1-ap^kq^k}{1-a}\frac{(a;p)_k(b^{-1};q)_kb^k}{(q;q)_k(abp;p)_k} = 0. \tag{4.9}$$

We also prove an identity due to Gosper (c.f. [3]):

$$\sum_{k=0}^{n}\frac{1-ap^kq^k}{1-a}\frac{(a;p)_k(c;q)_kc^{-k}}{(q;q)_k,(ap/c;p)_k} = \frac{(ap;p)_n(cq;q)_nc^{-n}}{(q;q)_n(ap/c;p)_n},$$

or equivalently,

$$\sum_{k=0}^{n}\frac{(1-ap^{n-k}q^{n-k})(q^{n-k+1};q)_k(ap^{n-k+1}/c;p)_k}{(cq^{n-k};q)_{k+1}(ap^{n-k};p)_{k+1}}c^k = \frac{1}{1-c}. \tag{4.10}$$

In fact, (4.9) and (4.10) are special cases of Proposition 4.3.

Taking $u=0$ in (4.6), we get:

$$\frac{1}{1-vx} = \frac{1}{1-vx_1} + \sum_{k=1}^{\infty}\frac{vY_{k-1}(v,\mathcal{C})(1-x_{k+1}c_k)}{(v,\mathcal{X})_{k+1}}\frac{Y_k(x,\mathcal{X})}{(x,\mathcal{C})_k}. \tag{4.11}$$

Multiplying both sides of (4.11) by $(1-vx_1)$, one has:

$$\frac{1-vx_1}{1-vx} = 1 + \sum_{k=1}^{\infty} \frac{vY_{k-1}(v,\mathcal{C})(1-x_{k+1}c_k)}{(v,\mathcal{X}/x_1)_k} \frac{Y_k(x,\mathcal{X})}{(x,\mathcal{C})_k},$$

where $\mathcal{X}/x_1 = \{x_2, x_3, \cdots\}$.

Taking

$$\mathcal{X} = \{q^0, q^1, q^2, \cdots\},\ \mathcal{C} = \{ap^1, ap^2, \cdots\},\ v=1,\ x=b,$$

we get (4.9).

In (4.6), taking

$$\mathcal{X} = \{p^{-n}/a, p^{-n+1}/a, \cdots\},\ \mathcal{C} = \{q^{-n+1}, q^{-n+2}, \cdots\},$$

$u = q^{-n}$, $v=1$, $x = c^{-1}$, we get (4.10).

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