Continued Fractions for Rogers-Szegö polynomials

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Abstract

We evaluate different Hankel determinants of Rogers-Szegö polynomials, and deduce from it continued fraction expansions for the generating function of RS polynomials. We also give an explicit expression of the orthogonal polynomials associated to moments equal to RS polynomials, and a decomposition of the Hankel form with RS polynomials as coefficients.

1 Definitions and Notations

Let $(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. The *q-binomial coefficient*, or the *Gauss polynomial*, is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)},$$

which is equal to the number of k-dimensional vector subspaces of the n-dimensional vector space $V_n(q)$ over the finite field $GF(q)$.

The *Rogers-Szegö polynomials* are defined by

$$h_n(t,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} t^k = (q)_n \sum \frac{t^k}{(q)_k (q)_{n-k}}. \tag{1.1}$$

They are some variants of q-Hermite polynomials which play an important role in the theory of basic hypergeometric series [1]. For $t=1$, Rogers-Szegö polynomials specialize to the Galois numbers of Goldman and Rota [5, p. 77] (cf. also [7]):

$$G_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}.$$

Let us introduce some notations related to symmetric functions (cf. [8]). Given two sets of indeterminates A, B (we say *alphabets*), the complete symmetric functions $S^i(A-B)$, $i \in \mathbb{Z}$, of the formal difference $A-B$ are defined by the following generating function (using another variable z) :

$$\sigma_z(A-B) = \sum_{i \geq 0} S^i(A-B) z^i = \frac{\prod_{b \in B}(1-bz)}{\prod_{a \in A}(1-az)}. \tag{1.2}$$

More generally, for any polynomial $P = \sum_{(c,u)} cu$, $c \in \mathbb{C}$, u monomial, and any extra variable q, we define $S^i(P)$ and $S^i(P/(1-q))$ by the generating functions

$$\sigma_z\Big(\sum_{(c,u)} cu\Big) = \sum_i z^i S^i(P) = \prod_{(c,u)} (1-zu)^{-c} \tag{1.3}$$

$$\sigma_z\left(\frac{P}{1-q}\right) = \sum_i z^i S^i\left(\frac{P}{1-q}\right) = \prod_{(c,u)} \prod_{j=0}^{\infty} (1-zq^j u)^{-c} \tag{1.4}$$

With these notations, a variable x is identified with the alphabet $\{x\}$ of order 1, and $(1-q)^{-1}$ with the infinite alphabet $\{1, q, q^2, \ldots\}$. Any formal series $\sum_{n \geq 0} z^n S^n$, with $S^0 = 1$, can be formally factorized, i.e. identified with $\sigma_z(A)$ for some alphabet A such that $S^n = S^n(A)$, $n \in \mathbb{N}$.

Our basic ring $\mathcal{R}$ is the ring of rational functions in two variables q, t and some alphabets $A, B, \ldots$.

For any element $\theta \in \mathcal{R}$, any $n \in \mathbb{N}$, any $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, the *Schur function* $S_I(\theta)$ is equal to the determinant [9] :

$$S_I(\theta) = \left| S^{i_s+s-r}(\theta) \right|_{1 \leq r,s \leq n}.$$

2 Recursion for Rogers-Szegö Polynomials

Taking coefficients of powers of z in (1.2), one gets

$$S^n(A \pm B) = \sum_{k=0}^{n} S^k(A)\, S^{n-k}(\pm B). \tag{2.1}$$

From (2.1), we see that the Rogers-Szegö Polynomial $h_n(t,q)$ can be written as:

$$(q)_n S^n\left(\frac{1+t}{1-q}\right).$$

Lemma 2.1 *Let B, C be two alphabets of order less than or equal to m. Then $\mathcal{S}_n := S^n\left(\frac{B-C}{1-q}\right)$, $n \in \mathbb{N}$, satisfies the following recurrence:*

$$\sum_{r=0}^{m} \left(S^r(-B) - S^r(-C)q^{n-r}\right) \mathcal{S}_{n-r} = 0. \tag{2.2}$$

Proof. Writing $-B + \frac{B-C}{1-q} = -C + q\frac{B-C}{1-q}$, then from (2.1), together with the fact that $S^n(-B) = 0 = S^n(-C)$, for $n > m$, we get that

$$\sum S^r(-B) S^{n-r}\left(\frac{B-C}{1-q}\right) = \sum S^r(-C) q^{n-r} S^{n-r}\left(\frac{B-C}{1-q}\right).$$

□

Lemma 2.1 implies the following recursion :

Corollary 2.2 *Let $h_n(t,q)$ be the Rogers-Szegö polynomial. Then*

$$h_n(t,q) = (1+t)h_{n-1}(t,q) - t(q^{n-1}-1)h_{n-2}(t,q). \tag{2.3}$$

Proof. Taking $B = \{1,t\}, C = \emptyset$ in Lemma 2.1, we have

$$(1-q^n)S^n\left(\frac{1+t}{1-q}\right) - (1+t)S^{n-1}\left(\frac{1+t}{1-q}\right) + tS^{n-2}\left(\frac{1+t}{1-q}\right) = 0. \tag{2.4}$$

Substituting $S^n\left(\frac{1+t}{1-q}\right) = h_n(t,q)/(q)_n$ in Equation (2.4), we derive the recursion immediately. □

Remark. Goldman and Rota [5] proved the recursion

$$G_{n+1} = 2G_n + (q^n - 1)G_{n-1}\ ,\ G_0 = 1\ ,\ G_1 = 2\ ,$$

using vector spaces over finite fields. Nijenhuis, Solow and Wilf [10] gave a bijective proof for the same recursion. K. Hikami and B. Basu-Mallick [6] gave also a recursion for generalized Rogers-Szegö polynomials.

3 Determinants of Rogers-Szegö Polynomials

Theorem 3.1 *Let $\mathbb{A}$ be the alphabet defined by $S^i(\mathbb{A}) = h_i(t,q)$, $i \in \mathbb{N}$. We have*

$$\begin{aligned} S_{n^n}(\mathbb{A}) &= t^{\binom{n}{2}} q^{\binom{n+1}{3}} (q)_0 (q)_1 \cdots (q)_{n-1} h_n(t, q^{-1}), && (3.1)\\ S_{n^n,m}(\mathbb{A}) &= (q)_0(q)_1 \cdots (q)_n t^{\binom{n+1}{2}} q^{\binom{n+1}{3}} \begin{bmatrix} m \\ n \end{bmatrix} h_{m-n}(t,q). && (3.2) \end{aligned}$$

In particular,

$$S_{n^{n+1}}(\mathbb{A}) = (q)_0(q)_1 \cdots (q)_n t^{\binom{n+1}{2}} q^{\binom{n+1}{3}}. \qquad (3.3)$$

Proof. Denote $X = \frac{1+t}{1-q}$. By definition,

$$\begin{aligned} S_{n^n}(\mathbb{A}) &= \left| h_{n+j-i}(t,q) \right|_{1 \le i,j \le n} \\ &= \left| (q)_{n+j-i} S^{n+j-i}(X) \right|_{1 \le i,j \le n} \\ &= (q)_1 \cdots (q)_n \cdot \left| (q^{n-i+2})_{j-1} S^{n+j-i}(X) \right|_{1 \le i,j \le n}. && (3.4) \end{aligned}$$

We can write recursion (2.4) as

$$\begin{aligned} &(q^k)_m S^{k+m-1}(X) \\ &= (1+t)(q^k)_{m-1} S^{k+m-2}(X) - t(1-q^{m-1})(q^k)_{m-2} S^{k+m-3}(X) \\ &\qquad -tq^{m-1}(q^{k-1})_{m-1} S^{k+m-3}(X). && (3.5) \end{aligned}$$

Applying identity (3.5) on columns $n, n-1, \ldots, 2$ of the determinant in

(3.4) successively, it becomes

$$(-t)^{n-1}q^{n-2}q^{n-3}\cdots q^0$$
$$\times\begin{vmatrix} S^n(X) & S^{n-1}(X) & (q^n)_1S^n(X) & \cdots & (q^n)_{n-2}S^{2n-3}(X) \\ S^{n-1}(X) & S^{n-2}(X) & (q^{n-1})_1S^{n-1}(X) & \cdots & (q^{n-1})_{n-2}S^{2n-4}(X) \\ \cdots & \cdots & \cdots & \cdots & \\ S^2(X) & S^1(X) & (q^2)_1S^2(X) & \cdots & (q^2)_{n-1}S^{n-1}(X) \\ S^1(X) & S^0(X) & (q)_1S^1(X) & \cdots & (q)_{n-2}S^{n-2}(X) \end{vmatrix}.$$

Repeating the same transformation, we will finally reach

$$(-t)^{\binom{n}{2}}q^{\binom{n}{3}}\begin{vmatrix} S^n(X) & S^{n-1}(X) & \cdots & S^1(X) \\ S^{n-1}(X) & S^{n-2}(X) & \cdots & S^0(X) \\ \cdots & \cdots & \cdots & \\ S^2(X) & S^1(X) & \cdots & 0 \\ S^1(X) & S^0(X) & \cdots & 0 \end{vmatrix} = t^{\binom{n}{2}}q^{\binom{n}{3}}S_{1^n}(X).$$

Noting that

$$S_{1^n}(X) = (-1)^nS^n\left(-\frac{1+t}{1-q}\right) = \frac{q^{\binom{n}{2}}}{(q)_n}h_n(t,q^{-1}),$$

we obtain

$$S_{n^n}(\mathbb{A}) = t^{\binom{n}{2}}q^{\binom{n+1}{3}}(q)_1\cdots(q)_{n-1}h_n(t,q^{-1}).$$

For $m\geq 0$, define

$$W_{n,m} = \frac{S_{n^n,m}(\mathbb{A})}{(q)_0(q)_1\cdots(q)_n}$$
$$= \begin{vmatrix} S^n(X) & (q^{n+1})_1S^{n+1}(X) & \cdots & (q^{n+1})_{n-1}S^{2n-1}(X) & (q^{n+1})_mS^{n+m}(X) \\ S^{n-1}(X) & (q^n)_1S^n(X) & \cdots & (q^n)_{n-1}S^{2n-2}(X) & (q^n)_mS^{n+m-1}(X) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S^1(X) & (q^2)_1S^2(X) & \cdots & (q^2)_{n-1}S^n(X) & (q^2)_mS^{m+1}(X) \\ S^0(X) & (q)_1S^1(X) & \cdots & (q)_{n-1}S^{n-1}(X) & (q)_mS^m(X) \end{vmatrix}.$$

Applying identity (3.5) on the last column of $W_{n,m}$, we transform it into

$$W_{n,m} = (1+t)W_{n,m-1} - t(1-q^{m-1})W_{n,m-2}$$
$$+ \begin{vmatrix} S^n(X) & \cdots & (q^{n+1})_{n-1}S^{2n-1}(X) & (q^n)_{m-1}S^{n+m-2}(X) \\ S^{n-1}(X) & \cdots & (q^n)_{n-1}S^{2n-2}(X) & (q^{n-1})_{m-1}S^{n+m-3}(X) \\ \cdots & \cdots & \cdots & \cdots \\ S^1(X) & \cdots & (q^2)_{n-1}S^n(X) & (q)_{m-1}S^{m-1}(X) \\ S^0(X) & \cdots & (q)_{n-1}S^{n-1}(X) & 0 \end{vmatrix}$$

After successive such transformations, one finally obtains

$$W_{n,m} = (1+t)W_{n,m-1} - t(1-q^{m-1})W_{n,m-2} + t^n q^{m-1} q^{\binom{n-1}{2}} W_{n-1,m-1}. \quad (3.6)$$

Noting that

$$t^{\binom{n+1}{2}} q^{\binom{n+1}{3}} \begin{bmatrix} m \\ n \end{bmatrix} h_{m-n}(t,q)$$

satisfies the recursion (3.6) and coincides with $W_{n,m}$ for $(0,m)$, $(n,0)$ and $(n,1)$, we get (3.2). □

4 Continued Fractions for Rogers-Szegö Polynomial

There are three main types of continued fraction expressions for formal series. They are all obtained by evaluating some Hankel determinants. We refer to Stieltjes [11], Wall [12] and Brezinski [4]. The determinants given by Theorem 3.1 allow us to write the following three continued fractions.

Theorem 4.1 *The generating function of Rogers-Szegö polynomials* $h_n(t,q)$

is equal to each of the three following continued fractions:

$$\sum_{n=0}^{\infty} h_n(t,q)z^n = \cfrac{1}{1-\cfrac{z\gamma_1}{1-\cfrac{z\gamma_2}{1-\cfrac{z\gamma_3}{\cdots}}}}$$

$$= \cfrac{1}{1-\zeta_0 z+\cfrac{\beta_1 z^2}{1-\zeta_1 z+\cfrac{\beta_2 z^2}{1-\zeta_2 z+\cfrac{\beta_3 z^2}{\cdots}}}},$$

$$\sum_{n=0}^{\infty} h_n(t,q)z^{-n-1} = \cfrac{1}{z+\cfrac{1}{-\cfrac{1}{\alpha_1}+\cfrac{1}{\cfrac{z}{\alpha_2}+\cfrac{1}{-\cfrac{1}{\alpha_3}+\cdots}}}},$$

where

$$\alpha_m = \begin{cases} (-1)^{n-1}\dfrac{q^{\binom{n}{2}}h_n(t,q^{-1})h_{n-1}(t,q^{-1})}{t^{n-1}(q)_{n-1}}, & m=2n-1, \\ (-1)^n\dfrac{t^n(q)_n}{q^{\binom{n}{2}}h_n^2(t,q^{-1})}, & m=2n, \end{cases}$$

$$\gamma_m = \begin{cases} \dfrac{q^{n-1}h_n(t,q^{-1})}{h_{n-1}(t,q^{-1})}, & m=2n-1, \\ -t(1-q^n)\dfrac{h_{n-1}(t,q^{-1})}{h_n(t,q^{-1})}, & m=2n, \end{cases}$$

$$\beta_n = q^{n-1}t(1-q^n),$$

$$\zeta_n = q^n(1+t).$$

Remark. In our notations, Stieltjes' parameters, for the series $\sum z^n S^n(\mathbb{A})$, are [8]

$$\alpha_m = \begin{cases} (-1)^{n-1} \dfrac{S_{n^n}(\mathbb{A}) S_{(n-1)^{n-1}}(\mathbb{A})}{S_{(n-1)^n}(\mathbb{A})^2}, & m = 2n-1, \\ (-1)^n \dfrac{S_{n^{n+1}}(\mathbb{A}) S_{(n-1)^n}(\mathbb{A})}{S_{n^n}(\mathbb{A})^2}, & m = 2n, \end{cases}$$

$$\beta_n = \frac{S_{n^{n+1}}(\mathbb{A}) S_{(n-2)^{n-1}}(\mathbb{A})}{S_{(n-1)^n}(\mathbb{A})^2} = -\gamma_{2n}\gamma_{2n-1},$$

$$\gamma_n = \alpha_n \alpha_{n-1},$$

$$\zeta_n = \frac{S_{n^n,n+1}(\mathbb{A})}{S_{n^{n+1}}(\mathbb{A})} - \frac{S_{(n-1)^{(n-1)},n}(\mathbb{A})}{S_{(n-1)^n}(\mathbb{A})}.$$

5 Hankel Forms

To any alphabet $\mathbb{A}$ is associated a Hankel form

$$H(\mathbb{A};x) = \sum_{i,j=0}^{\infty} S_{i+j}(\mathbb{A})\, x_i x_j \ .$$

The problem of decomposing Hankel forms into sums of squares has also been solved by Stieltjes [3]. He found the coefficients, that we denote by $\star$, such that

$$H(\mathbb{A};x) = (x_0 + \star x_1 + \cdots)^2 + \star(x_1 + \star x_2 + \cdots)^2 + \star(x_2 + \star x_3 + \cdots)^2 + \cdots \ .$$

Theorem 5.1 (Stieltjes) *Let $\mathbb{A}$ be a generic alphabet. With $\mathbb{A}^0 = \mathbb{A}$, define the alphabets $\mathbb{A}^1, \mathbb{A}^2, \ldots$ by the generating series*

$$\sigma_z(\mathbb{A}^k) = S_{k^{k+1}}(\mathbb{A})^{-1} \sum_{j=0}^{\infty} S_{k^k,\, k+j}(\mathbb{A}) z^j \ .$$

Then the form $K(\mathbb{A};\, x, y) = \sum_{i,j=0}^{\infty} S_{i+j}(\mathbb{A}) x^i y^j$ decomposes as

$$\begin{aligned} K(A;\, x,y) &= \sigma_1\Big((x+y)\mathbb{A}^0\Big) - xy S_{11}(\mathbb{A})\, \sigma_1\Big((x+y)\mathbb{A}^1\Big) + \cdots \\ &\quad + \cdots + \ (-xy)^n \frac{S_{n^{n+1}}(\mathbb{A})}{S_{(n-1)^n}(\mathbb{A})} \sigma_1\Big((x+y)\mathbb{A}^n\Big) + \cdots \ . \end{aligned}$$

Substituting Equation (3.2) into the above identity, we instantly derive that

Corollary 5.2 *For Rogers-Szegö polynomials $h_n(t,q)$, we have*

$$\sum_{i,j=0}^{\infty} h_{i+j}(t,q)x^i y^j = \sum_{n=0}^{\infty}(-t)^n (q)_n q^{\binom{n}{2}} \times \left(\sum_{u=0}^{\infty} h_u(t,q)\begin{bmatrix} n+u \\ u \end{bmatrix} x^{n+u}\right)\left(\sum_{v=0}^{\infty} h_v(t,q)\begin{bmatrix} n+v \\ v \end{bmatrix} y^{n+v}\right)$$

Comparing the coefficients of $t^m s^n$ of both sides, we get the following formula due to Askey and Ismail [2]:

$$h_{m+n}(t,q) = \sum_{k=0}^{\min\{m,n\}} \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}(-t)^k (q)_k q^{\binom{k}{2}} h_{m-k}(t,q) h_{n-k}(t,q).$$

6 Orthogonal Polynomials

Let $\int$ be a linear functional on polynomials in x such that there exists a family of *orthogonal polynomials* $\{P_n(x)\}$:

$$\int P_m(x)P_n(x) = 0 \text{ if } m \neq n \ , \quad \int P_n(x)P_n(x) = c_n \neq 0 \ .$$

c_n is called the normalization constant.

Identifying the *moments* $\int x^n$ to the complete functions $S^n(A)$ of some alphabet A, we can rewrite the classical expressions of orthogonal polynomials in term of symmetric functions:

$$P_n(x) = S_{n^n}(A-x), \ c_n = (-1)^n S_{(n-1)^n}(A)\, S_{n^{n+1}}(A), \quad n \geq 0. \qquad (6.1)$$

Theorem 6.1 *Let $\int_{\mathbb{A}}$ be the linear functional with moments equal to the Rogers-Szegö polynomials: $\int_{\mathbb{A}} x^n = h_n(t,q)$. Then*

$$P_n(x) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} h_k(t,q^{-1})x^{n-k} = (q)_n S^n\left(\frac{x-1-t}{1-q}\right),$$

are its associated orthogonal polynomials, with normalization constant

$$c_n = t^n q^{\binom{n}{2}}(q-1)(q^2-1)\cdots(q^n-1).$$

Proof. Start with

$$S_{n^n}(\mathbb{A}-x)=\begin{vmatrix} S^n(\mathbb{A}) & S^{n+1}(\mathbb{A}) & \cdots & S^{2n-1}(\mathbb{A}) & x^n \\ S^{n-1}(\mathbb{A}) & S^n(\mathbb{A}) & \cdots & S^{2n-2}(\mathbb{A}) & x^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S^1(\mathbb{A}) & S^2(\mathbb{A}) & \cdots & S^n(\mathbb{A}) & x^1 \\ S^0(\mathbb{A}) & S^1(\mathbb{A}) & \cdots & S^{n-1}(\mathbb{A}) & 1 \end{vmatrix},$$

and write $X=\frac{1+t}{1-q}$. The same kind of transformation as for Theorem 3.1 allows us to write the preceding determinant as

$$(q)_0(q)_1\cdots(q)_n\begin{vmatrix} S^n(X) & \cdots & (q^{n+1})_{n-1}S^{2n-1}(X) & x^n/(q)_n \\ S^{n-1}(X) & \cdots & (q^n)_{n-1}S^{2n-2}(X) & x^{n-1}/(q)_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ S^1(X) & \cdots & (q^2)_{n-1}S^n(X) & x^1/(q)_1 \\ S^0(X) & \cdots & (q)_{n-1}S^{n-1}(X) & 1 \end{vmatrix}$$

$$\begin{aligned} &= (q)_0(q)_1\cdots(q)_n(-t)^{\binom{n}{2}}q^{\binom{n}{3}} \\ &\quad \begin{vmatrix} S^n(X) & S^{n-1}(X) & \cdots & S^1(X) & x^n/(q)_n \\ S^{n-1}(X) & S^{n-2}(X) & \cdots & S^0(X) & x^{n-1}/(q)_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S^1(X) & S^0(X) & \cdots & 0 & x^1/(q)_1 \\ S^0(X) & 0 & \cdots & 0 & 1 \end{vmatrix} \\ &= (q)_1\cdots(q)_n t^{\binom{n}{2}}q^{\binom{n}{3}}\sum_{k=0}^{n}(-1)^{n-k}\frac{x^{n-k}}{(q)_{n-k}}S_{1^k}(X) \\ &= (-1)^n(q)_1\cdots(q)_{n-1}t^{\binom{n}{2}}q^{\binom{n}{3}}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}q^{\binom{k}{2}}h_k(t,q^{-1})x^{n-k}, \end{aligned}$$

After normalization, we finally get the polynomials

$$P_n(x)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}q^{\binom{k}{2}}h_k(t,q^{-1})x^{n-k}=(q)_nS^n\left(\frac{x-1-t}{1-q}\right),$$

with normalization constant

$$c_n=t^nq^{\binom{n}{2}}(q-1)(q^2-1)\cdots(q^n-1).$$

□

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