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Factorization of Kazhdan–Lusztig elements for Grassmanians

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Abstract.

We show that the Kazhdan-Lusztig basis elements C_w of the Hecke algebra of the symmetric group, when $w \in S_n$ corresponds to a Schubert subvariety of a Grassmann variety, can be written as a product of factors of the form $T_i + f_j(v)$, where f_j are rational functions.

§1. Notation

In this section, we briefly list the main facts and notations related to Kazhdan–Lusztig polynomials and their parabolic analogues (see [D], [S]). We use the following notations:

$\mathcal{H}$—the Hecke algebra of the symmetric group S_n; we consider it as an algebra over the field $\mathbf{Q}(v)$ (the variable v is related to the variable q used by Kazhdan and Lusztig via $v = q^{1/2}$), and we write the quadratic relation in the form

$$(T_i - v)(T_i + v^{-1}) = 0.$$

C_w—KL basis in $\mathcal{H}$, which we define by the conditions $\overline{C_w} = C_w$, $C_w - T_w \in \oplus v\mathbf{Z}[v]T_y$.

For any subset $J \subset \{1, \dots, n-1\}$, we denote by $W_J \subset S_n$ the corresponding parabolic subgroup, and by W^J the set of minimal length representatives of cosets S_n/W_J. We also denote by M^J the $\mathcal{H}$-module induced from the one-dimensional representation of $\mathcal{H}(W_J)$, given by $T_j m_1 = -v^{-1} m_1, j \in J$. We denote $m_y = T_y m_1, y \in W^J$ the usual basis in M^J.

We define the parabolic KL basis $C^J_y, y \in W^J$ in M^J by $\overline{C^J_y} = C^J_y, C^J_y - m_y \in \oplus_{z \in W^J} v\mathbf{Z}[v] m_z$.

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Denote for brevity $C_J = C_{w_0^J}$ the element of KL basis in $\mathcal{H}$ corresponding to the element of w_0^J of maximal length in W_J. The following result is well-known (see, e.g., [S]).

Lemma 1. (i)

$$C_J = \sum_{w \in W^J} (-v)^{l(w_0^J)-l(w)} T_w.$$

(ii) *Let $w \in W$ be such that it is an element of maximal length in the coset wW_J (which is equivalent to $w = \tau w_0^J$ for some $\tau \in W^J$). Then $C_w = XC_J$ for some $X \in \oplus_{y \in W^J} \mathbf{Z}[v^{\pm 1}]T_y$.*

(iii) *Let $X \in \oplus_{y \in W^J} \mathbf{Z}[v^{\pm 1}]T_y$. Then*

$$Xm_1 = C^J_\tau \iff XC_J = C_{\tau w_0^J}.$$

Let us now consider the special case of the above situation. From now on, fix $k \le n-1$, and let $J = \{1, \dots, k-1, k+1, \dots, n-1\}$ so that $W_J = S_k \times S_{n-k}$ is a maximal parabolic subgroup in S_n. In this case, the module M^J can be described as follows:

$$M = \bigoplus_{\varepsilon \in E} \mathbf{Q}(v)\varepsilon,$$

$$T_i\varepsilon = \begin{cases} s_i\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (+-), \\ -v^{-1}\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (--) \text{ or } (++), \\ s_i\varepsilon + (v - v^{-1})\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (-+), \end{cases} \tag{1}$$

where E is the set of all length n sequences of pluses and minuses which contain exactly k pluses. The relation of this with the previous notation is given by $m_y \leftrightarrow y(\mathbf{1}) = T_y(\mathbf{1})$, where

$$\mathbf{1} = (\underbrace{+\cdots+}_{k}\underbrace{-\cdots-}_{n-k}). \tag{2}$$

In particular, $m_1 \leftrightarrow \mathbf{1}$.

The set of minimal length representatives W^J also admits a description in terms of Young diagrams. Namely, let λ be a Young diagram which fits inside the $k \times (n-k)$ rectangle. Define $w_\lambda \in S_n$ by

$$w_\lambda = \prod_{(i,j) \in \lambda} s_{k+j-i}, \tag{3}$$

where (i,j) stands for the box in the i-th row and j-th column, and the product is taken in the following order: we start with the lower right corner and continue along the row, until we get to the first column; then we repeat the same with the next row, and so on until we reach the upper left corner.

Example 1. Let λ be the diagram shown below, and $k = 7$ (to assist the reader, we put the numbers $k + j - i$ in the diagram).

7	8	9	10	11	12
6	7	8			
5	6	7			
4					
3					

Then $w_\lambda = s_3 \cdot s_4 \cdot s_7 s_6 s_5 \cdot s_8 s_7 s_6 \cdot s_{12} s_{11} s_{10} s_9 s_8 s_7$ (for easier reading, we separated products corresponding to different rows by $\cdot$).

The proof of the following proposition is straightforward.

Proposition 2. *The corespondence $\lambda \mapsto w_\lambda$, where w_λ is defined by* (3)*, is a bijection between the set of all Young diagrams which fit inside the $k \times (n-k)$ rectangle and W^J.*

§2. The main theorem

As before, we fix $k \le n-1$ and let $J = \{1, \dots, k-1, k+1, \dots, n-1\}$. Unless otherwise specified, we only use Young diagrams which fit inside the $k \times (n-k)$ rectangle.

For a Young diagram λ, we define the shifts $r_{i,j} \in \mathbf{Z}_{>0}, (i,j) \in \lambda$ by the following relation

$$r_{ij} = \max(r_{i,j+1}, r_{i+1,j}) + 1, \tag{4}$$

where we let $r_{ij} = 0$ if $(i,j) \notin \lambda$.

Example 2. For the diagram λ from Example 1, the shifts r_{ij} are shown below.

6	5	4	3	2	1
4	3	2			
3	2	1			
2					
1					

Next, let us define for each diagram λ an element $X_\lambda \in \mathcal{H}$ by

$$X_\lambda = \prod_{(i,j)\in\lambda} \left(T_{k+j-i} - \frac{v^{r_{ij}}}{[r_{ij}]} \right) \tag{5}$$

where, as usual, $[r] = (v^r - v^{-r})/(v - v^{-1})$, and the product is taken in the same order as in (3).

The main result of this paper is the following theorem.

Theorem 3. *Let λ be a Young diagram. Then*

$$X_\lambda \mathbf{1} = C^J_{w_\lambda}.$$

Note that by Lemma 1, this is equivalent to

$$X_\lambda C_J = C_{w_\lambda w_0^J}. \tag{6}$$

We remind the reader that the Kazhdan-Lusztig elements $C_{ww_0^J}$, where $w \in W^J$, and W_J is a maximal parabolic in S_n (they are also known as KL elements for Grassmanians), have been studied in a number of papers. A combinatorial description was given in [LS1]; it was interpreted geometrically in [Z], and in terms of representations of quantum $\mathfrak{gl}_m$ in [FKK]. However, it is unclear how these results are related with the factorization given by the theorem above. A similar factorization was given in [L] for those permutations which correspond to nonsingular Schubert varieties—i.e., for those w such that, for any $v \in S_n$, the Kazhdan-Lusztig polynomial $P_{v,w}$ is either 1 or 0.

Note that one can easily check that the elements X_λ are invariant under the Kazhdan–Lusztig involution: $\overline{X_\lambda} = X_\lambda$; thus, all the difficulty is in proving that they are integral and have the right specialization at $v = 0$.

A crucial step in proving this theorem is the following proposition.

Proposition 4. *Theorem 3 holds when λ is the $k \times (n-k)$ rectangle.*

Proof. For any $w \in S_n$, choose a reduced expression $w = s_{i_\ell} \dots s_{i_1}$. Define the element $\nabla_w \in \mathcal{H}$ by

$$\nabla_w = \left(T_{i_\ell} - \frac{v^{r_\ell}}{[r_\ell]} \right) \dots (T_{i_1} - v), \tag{7}$$

where $r_1, \dots, r_\ell \in \mathbf{Z}_+$ are defined as follows: if $s_{i_{m-1}} \dots s_{i_1}(1, \dots, n) = (\dots, a, b, \dots)$ (in i_m-th, (i_m+1)-st places), then $r_m = b - a$. Then $\{\nabla_w, w \in S_n\}$ is a Yang-Baxter basis of the Hecke algebra, and we have (see [DKLLST, §3]):

Lemma 5. (i) *The element ∇_w does not depend on the choice of reduced expression.*

(ii) *If w_0^J is the longest element in some parabolic subgroup $W_J \subset S_n$, then $\nabla_{w_0^J} = C_J$.*

Now, let us prove our proposition, i.e. that $X_\lambda C_J$ is a KL element for rectangular λ. In this case, w_λ is the longest element in W^J:

$$w_\lambda(\mathbf{1}) = (\underbrace{-\cdots-}_{n-k}\underbrace{+\cdots+}_{k}).$$

Let us choose the following reduced expression for the longest element w_0 in S_n: $w_0 = w_\lambda w_0^J$, where we take for w_λ the reduced expression given by (3). Then one easily sees that definition (7) in this case gives

$$\nabla_{w_0} = X_\lambda \nabla_{w_0^J}.$$

By Lemma 5, we get $C_{w_0} = X_\lambda C_J$, which is exactly the statement of the proposition. Q.E.D.

The proof in the general case is based on the following proposition. Denote

$$O(v^m) = \{f \in \mathbf{Q}(v) | f \text{ has zero of order} \geq m \text{ at } v = 0\}. \tag{8}$$

Proposition 6.

$$X_\lambda \mathbf{1} = w_\lambda(\mathbf{1}) + \sum_{\varepsilon \in E} O(v)\varepsilon.$$

A proof of this proposition is given in Section 3.

Now we can give a proof of the main theorem. First, one easily checks the invariance under the bar involution, since

$$\overline{T_i - \frac{v^r}{[r]}} = T_i - \frac{v^r}{[r]}.$$

Combining this with Proposition 6, we see that it remains to show that $X_\lambda C_J$ are integral, i.e. $X_\lambda C_J \in \oplus \mathbf{Z}[v^{\pm 1}] T_w$ (note that it is not true that X_λ itself is integral.) This will be done by induction.

Let λ be a Young diagram. Then we claim that any such diagram can be presented as a union $\lambda = \lambda' \sqcup \mu$, where μ is a rectangle, and λ' is again a Young diagram such that for $(i,j) \in \lambda'$, the shifts $r^{\lambda'}_{(i,j)} = r^{\lambda}_{(i,j)}$. It can be formally proved as follows: if one writes the successive widths and heights of the stairs of the diagram

$$\infty, (a_1, b_1), (a_2, b_2), \ldots (a_k, b_k), \infty$$

then there is at least one index i for which $a_i \leq b_{i+1}$ and $b_i \leq a_{i+1}$. In that case, the rectangle μ has the lower right corner i.

Example 3. For the diagram λ from Example 1, the sequence (a_k, b_k) is given by $\infty, (1,2), (2,2), (3,1), \infty$, and the subdiagram μ is the shaded 2×2 square, as shown below. As before, we also included the shifts r_{ij} in this diagram. The subsets I^μ, J^μ in this case are given by $I^\mu = \{6,7,8\}, J^\mu = \{6,8\}$.

6	5	4	3	2	1
4	3	2			
3	2	1			
2					
1					

Let us choose for λ the presentation $\lambda = \lambda' \sqcup \mu$, where μ is a rectangle, as above. Then $X_\lambda = X_\mu X_{\lambda'}$.

Define the subsets $I^\mu, J^\mu \subset \{1, \dots, n-1\}$ by $I^\mu = \{k'-a+1, \dots, k'+b-1\}$, $J^\mu = I^\mu \setminus \{k'\}$, where $k' = k - i + j$, (i,j)—coordinates of the UL corner of μ, a and b are numbers of rows and columns in μ respectively.

We need to show that $X_\mu X_{\lambda'} C_J \in \sum \mathbf{Z}[v^{\pm 1}] T_y$. By induction assumption, we may assume that $X_{\lambda'} C_J = C_\sigma$, where we denoted for brevity $\sigma = w_{\lambda'} w_0^J$. It is easy to show that if μ is chosen as before, then σ is the maximal length element in the coset $W_{J^\mu}\sigma$. Thus, by Lemma 1, we can write $C_\sigma = C_{J^\mu} Y$ for some integral $Y \in \mathcal{H}$. Therefore, $X_\mu X_{\lambda'} C_J = X_\mu C_{J^\mu} Y$. Since W_{I^μ} is itself a symmetric group, and W_{J_μ} is a maximal parabolic subgroup in it, we can use Proposition 4, which gives $X_\mu C_{J^\mu} = C_{I^\mu}$, and therefore, $X_\mu X_{\lambda'} C_J = C_{I^\mu} Y \in \sum \mathbf{Z}[v^{\pm 1}] T_w$. Q.E.D.

§3. Proof of regularity at $v = 0$

In this section we give the proof of Proposition 6. Before doing so, let us introduce some notation.

As before, assume that we are given n, k, λ and a collection of positive integers $r_{ij}, (i,j) \in \lambda$ (not necessarily defined as in (4)). Let $\varepsilon \in E$ be a sequence of pluses and minuses. We define the *weight* $r_\lambda(\varepsilon)$ as folllows.

Define $a(i), i = 1 \dots k$ by $a(i) = k + \lambda_i - i + 1$. Equivalently, these numbers can be characterized by saying that $w_\lambda(\mathbf{1})$ has pluses exactly at positions $a(k), \dots, a(1)$.

Define $r_\lambda(\varepsilon) = \sum_{t=1}^n r_t(\varepsilon)$, where $r_t(\varepsilon)$ is defined as follows:

(i) if $t = a(i), \varepsilon_t = -$ then $r_t(\varepsilon) = r_{i,\lambda_i} - 1$

(ii) if $a(i) < t < a(i+1), \varepsilon_t = +$ then $r_t(\varepsilon) = r_{i,j}, k+j-i = t$
(iii) otherwise, $r_t(\varepsilon) = 0$

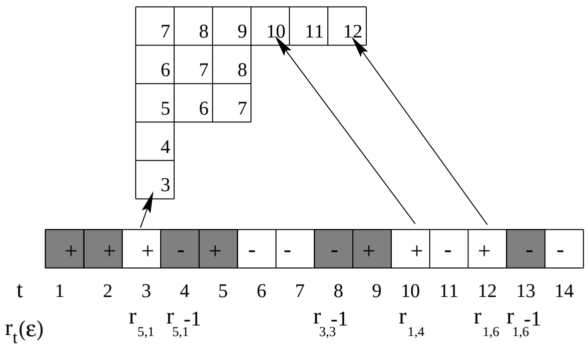
In a sense, $r_\lambda(\varepsilon)$ measures the discrepancy between ε and $w_\lambda(\mathbf{1})$. Indeed, let us denote the numbers of rows and columns in λ by i, j respectively, and let ε be such that

$$\begin{aligned} &\varepsilon_t = + \text{ for } t \leq k - i, \\ &\varepsilon_t = - \text{ for } t > k + j. \end{aligned} \tag{9}$$

Then one easily sees that

$$r_\lambda(\varepsilon) \geq 0, \qquad\qquad r_\lambda(\varepsilon) = 0 \iff \varepsilon = w_\lambda(\mathbf{1}) \tag{10}$$

Example 4. Below we illustrate the calculation of $r_\lambda(\varepsilon)$, where λ is the diagram used in Example 1. The positions $a(i)$ are shaded (thus, the sequence of colors encodes $w_\lambda(\mathbf{1})$, with "shaded" $\leftrightarrow +$, "unshaded" $\leftrightarrow -$), and we connected unshaded pluses with the corresponding box (i, j), defined in (ii) above. For convenience of the reader, we also put the numbers $k + j - i$ (not the shifts r_{ij}!) in the diagram.



Lemma 7. *Let λ be any Young diagram inside the $k \times (n-k)$ rectangle, and let $r_{ij}, (i,j) \in \lambda$, be positive integers satisfying $r_{ij} > r_{i,j+1}, r_{ij} > r_{i+1,j}$. Define $\mathcal{L}_\lambda \subset M^J$ by*

$$\mathcal{L}_\lambda = \sum_{\varepsilon \in E} O(v^{r_\lambda(\varepsilon)})\varepsilon.$$

Then

$$X_\lambda \mathbf{1} \in \mathcal{L}_\lambda.$$

Before proving this lemma note that due to (10), this lemma immediately implies Proposition 6.

Proof. The proof is by induction. Let (i,j) be a corner of λ, and $\lambda' = \lambda - (i,j)$, so that $X_\lambda = \left(T_{k-i+j} - \frac{v^{r_{ij}}}{[r_{ij}]}\right) X_{\lambda'}$. Since $\frac{v^r}{[r]} \in O(v^{2r-1})$, it suffices to prove that $\left(T_{k-i+j} + O(v^{2r_{ij}-1})\right)\mathcal{L}_{\lambda'} \subset \mathcal{L}_\lambda$. Since this operation only changes $\varepsilon_a, \varepsilon_{a+1}$ $(a = k-i+j)$, we need to consider 4 cases: $(++), (+-), (-+), (--)$. This is done explicitly. For example, for the $(+-)$ case, we have

$$\big(T_m + O(v^{2r_{ij}-1})\big)(\cdots + - \dots) = (\cdots - + \dots) + O(v^{2r_{ij}-1})(\cdots + - \dots)$$

In this case, the first summand has the same weight and comes with the same power of v as the original ε (note that in the original ε, this $(+-)$ didn't contribute to the weight), so it is in $\mathcal{L}_\lambda$. As for the second summand, its weight is increased by $2r_{ij} - 1$ (the plus contributes r and the minus, $r-1$), but it comes with the factor $O(v^{2r_{ij}-1})$, so again, it is in $\mathcal{L}_\lambda$. The other cases are treated similarly.

Q.E.D.

§4. Divided differences and parabolic Kazhdan-Lusztig bases

In this section, we give a factorization for the dual Kazhdan–Lusztig basis for Grassmanians.

To induce a parabolic module, one can start from the 1-dimensional representation $T_j \mapsto v$ instead of $T_j \mapsto -1/v$ which was used in §1. We now denote the corresponding module by M' and its Kazhdan-Lusztig basis by C'^J_y to distinguish from previous case. Note that there exists a natural pairing between M and M', and C^J_y and C'^J_y are dual bases with respect to this pairing (see, e.g., [S], [FKK]). However, we will not use this pairing.

A simple element $T_i - v$ acts now by

$$M' = \bigoplus_{\varepsilon \in E} \mathbf{Q}(v)\varepsilon,$$

$$(T_i - v)\varepsilon = \begin{cases} s_i\varepsilon - v\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (+-), \\ 0, & (\varepsilon_i, \varepsilon_{i+1}) = (--) \text{ or } (++), \\ s_i\varepsilon - v^{-1}\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (-+). \end{cases} \tag{11}$$

Consider the space $\mathcal{P}(k,n)$ of polynomials in $x_1, \dots, x_n$ of total degree $n-k$, and of degree at most 1 in each x_i. For any partition λ, denote by $x^{[\lambda]}$ the monomial $w_\lambda(x_{k+1} \cdots x_n)$, the symmetric group acting now by permutation of the x_i. In other words, if $w_\lambda(\mathbf{1}) = (\varepsilon_1, \dots, \varepsilon_n)$, then $x^{[\lambda]}$ is the product of the x_i's for those i such that $\varepsilon_i = -$.

Consider the isomorphism of vector spaces

$$\begin{aligned} M' &\simeq \mathcal{P}(k,n) \\ w_\lambda(\mathbf{1}) &\mapsto v^{-|\lambda|} x^{[\lambda]}. \end{aligned} \tag{12}$$

Then $T_i - v$ induces the operator ∇_i, acting only on x_i, x_{i+1} as follows:

$$\begin{cases} \nabla_i(x_i) = v x_{i+1} - v^{-1} x_i, \\ \nabla_i(1) = \nabla_i(x_i x_{i+1}) = 0, \\ \nabla_i(x_{i+1}) = -v x_{i+1} + v^{-1} x_i, \end{cases} \tag{13}$$

Therefore ∇_i is the operator

$$f \mapsto (v x_{i+1} - v^{-1} x_i)\, \partial_i(f)$$

denoting by ∂_i the divided difference

$$f \mapsto \frac{f - f^{s_i}}{x_i - x_{i+1}}$$

(for a more general action of the Hecke algebra on the ring of polynomials, see [LS2], [DKLLST]).

We intend to show that divided differences easily furnish the Kazhdan-Lusztig basis of $\mathcal{P}(k,n)$ (i.e. the image of the Kazhdan-Lusztig basis $C'_y, y \in W^J$ of M').

To any element $\varepsilon := w_\lambda(\mathbf{1})$ of E one associates a polynomial Q_ε as follows

1) pair recursively $-,+$ (as one pairs opening and closing parentheses)

2) replace each pair $(-,+)$, where $-$ is in position i and $+$ in position j, with a $x_i - v^{j+1-i} x_j$

3) replace each single $-$, in position i, by x_i

The product of all these factors by $v^{-|\lambda|}$, where $|\lambda| = \lambda_1 + \lambda_2 + \cdots$, is by definition Q_ε.

Theorem 8. *Let E be the set of sequences of $(+,-)$ of length n with k pluses. Then the collection of polynomials Q_ε, $\varepsilon \in E$, is the Kazhdan-Lusztig basis of the space $\mathcal{P}(k,n)$.*

Proof. We shall show that

$$Q_\varepsilon = \nabla_j \cdots \nabla_h (x_1 \cdots x_k)$$

when $\varepsilon = w_\lambda(\mathbf{1})$, and when $s_j \cdots s_h$ is a reduced decomposition of w_λ. Now, it is clear that the inverse image of Q_ε in M' is invariant under involution, and it is easy to check the powers of v to get that for $v = 0$, it specializes to ε.

Assume by induction that we already know Q_ε. Let us add on the right of ε sufficiently many pluses, so that all minuses are now paired (the original polynomial is recovered from the new one by specializing $x_{n+1}, x_{n+2}, \ldots$ to 0). Take now any simple transposition s_i such that $\varepsilon_i = +, \varepsilon_{i+1} = -$. The variables x_i, x_{i+1} involve two or one factor in P_ε, depending whether ε_i is paired or not. The only possible cases for those factors and their images under ∇_i are

$$(x_{i-a} - v^{a+1}x_i)(x_{i+1} - v^{b+1}x_{i+b+1}) \mapsto (x_{i-a} - v^{a+b+2}x_{i+b+1})(v^{-1}x_i - vx_{i+1})$$
$$(x_{i+1} - v^{b+1}x_{i+b+1}) \mapsto (v^{-1}x_i - vx_{i+1})$$

but now the new pairing of $-, +$ differs from the previous one exactly in the places described by the factors on the right. Q.E.D.

Corollary 9. *Let $\sigma_j \cdots \sigma_h$ be a reduced decomposition of $w \in W^J$. Then the corresponding Kazhdan-Lusztig element $C'^J_w \in M'$ is equal to $(T_j - v) \cdots (T_h - v)(\mathbf{1})$.*

This factorization is equivalent to the one given in [FKK, Theorem 3.1]. One can check on examples that this factorization is compatible, via the duality between the two modules M and M', with the factorization given by Theorem 3. However, deducing Theorem 3 from Theorem 8 seems more intricate than proving the two factorization properties directly.

Example 5. Let $\lambda = [5, 3, 2]$ and $\mu = [5, 3, 3]$. Then one has

places	1	2	3	4	5	6	7	8	9
$w_\lambda(\mathbf{1})$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$-$	$+$
	$+$	$-$					$-$		
pairing			$-$	$+$	$-$	$+$		$-$	$+$
polynomial		x_2					x_7		
			$(x_3$	$-v^2x_4)$	$(x_5$	$-v^2x_6)$		$(x_8$	$-v^2x_9)$
$w_\mu(\mathbf{1})$	$+$	$-$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
	$+$	$-$					$-$		
pairing			$-$			$+$			
				$-$	$+$			$-$	$+$
polynomial		x_2					x_7		
			x_3			$-v^4x_6$			
				$(x_4$	$-v^2x_5)$			$(x_8$	$-v^2x_9)$

and thus

$$
\begin{aligned}
Q_{w_\lambda(\mathbf{1})} &= v^{-10}\, x_2x_7\,(x_3 - v^2x_4)\,(x_5 - v^2x_6)\,(x_8 - v^2x_9) \\
Q_{w_\mu(\mathbf{1})} &= v^{-11}\, x_2x_7\,(x_3 - v^4x_6)\,(x_4 - v^2x_5)\,(x_8 - v^2x_9).
\end{aligned} \tag{14}
$$

Note that the pairing between $-, +$, which was a key point in the description of Kazhdan-Lusztig polynomials for Grassmannians in [LS1], is provided by divided differences, starting from the monomial $x_{k+1}\cdots x_n$.

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