TWO POLYNOMIAL IDENTITIES

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Abstract. We establish two polynomial identities. One of which generalizes a key identity in "A factorization theorem for classical group characters, with applications to plane partitions and rhombus tilings" by Mihai Ciucu and the first author [preprint 2008]. The other shows that the two sides in the identity can be expressed alternatively in form of a Pfaffian.

The purpose of this note is to establish two polynomial identities. The first (see Theorem 1 below) generalizes an identity in [1] which was crucial there to prove certain factorization theorems for classical group characters, and for numbers of rhombus tilings of certain regions. The second (see Theorem 2) provides a surprising alternative expression for the two sides in the identity involving a Pfaffian.

In the statement of our results, we use additive notation for sets of variables ("alphabets") as in [2]. Further notations that we use are: the symbol X_M denotes the alphabet $x_1+x_2+\dots+x_M$; given an alphabet A, we write $V(A)$ for the "Vandermonde product"

$$\prod_{\substack{a,b\in A\\ a<b}}(a-b).$$

The following theorem generalizes Lemma 1 in [1].

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Theorem 1. *Let a, b be two variables. Then there holds the identity*

$$\sum_{\substack{A+B=X_{2N}\\|A|=N}} V(a+A+A^{-1})V(b+B+B^{-1}) = \sum_{A+B=X_{2N}} V(a+A+B^{-1})V(b+B+A^{-1}), \quad (1)$$

where A^{-1} denotes the alphabet $\sum_{\alpha\in A}\alpha^{-1}$, with an analogous meaning for B^{-1}.

Proof. We prove the assertion by induction on N. In fact, the induction proof will prove *two* assertions at the same time, namely (1) and

$$(-1)^N \sum_{\substack{A+B=X_{2N-1}\\|A|=N}} V(A+A^{-1})V(a+b+B+B^{-1})$$
$$= \sum_{A+B=X_{2N-1}} (-1)^{|A|} V(a+A+B^{-1})V(b+B+A^{-1}). \quad (2)$$

For $N=1$, identity (1) reduces to

$$\begin{aligned}(a-x_1)&(a-x_1^{-1})(x_1-x_1^{-1})(b-x_2)(b-x_2^{-1})(x_2-x_2^{-1})\\ &+(a-x_2)(a-x_2^{-1})(x_2-x_2^{-1})(b-x_1)(b-x_1^{-1})(x_1-x_1^{-1})\\ =\;&(a-x_1^{-1})(a-x_2^{-1})(x_1^{-1}-x_2^{-1})(b-x_1)(b-x_2)(x_1-x_2)\\ &+(a-x_1)(a-x_2^{-1})(x_1-x_2^{-1})(b-x_2)(b-x_1^{-1})(x_2-x_1^{-1})\\ &+(a-x_2)(a-x_1^{-1})(x_2-x_1^{-1})(b-x_1)(b-x_2^{-1})(x_1-x_2^{-1})\\ &+(a-x_1)(a-x_2)(x_1-x_2)(b-x_1^{-1})(b-x_2^{-1})(x_1^{-1}-x_2^{-1}),\end{aligned}$$

while (2) reduces to

$$-(x_1-x_1^{-1})(a-b) = -(a-x_1)(b-x_1^{-1})+(a-x_1^{-1})(b-x_1),$$

both of which can be readily verified.

Now let us suppose that we have proved (1) and (2) with N replaced by $N-1$. Below, we shall show that this implies that identities (1) and (2) are true if we specialize x_1 to x_2. Let us for the moment suppose that this is already done. It is easy to see that both sides of (1) are symmetric in the variables $x_1, x_2, \dots, x_{2N}$, while both sides of (2) are symmetric in the variables $x_1, x_2, \dots, x_{2N-1}$. Moreover, both sides of (1) and both sides of (2) remain invariant up to an overall multiplicative sign if we replace x_1 by x_1^{-1} (on the left-hand sides of (1) respectively (2), the latter assertion already applies to each individual summand; on the right-hand sides, one has to group the summands in pairs: the summands corresponding to a pair of alphabets (A, B) and the pair of symmetric differences $(A\triangle\{x_1\}, B\triangle\{x_1\})$ have to be considered together).

This means that identity (2) holds for *all* specializations of x_1 to one of $x_2, x_2^{-1}, x_3, x_3^{-1}, \dots, x_{2N-1}, x_{2N-1}^{-1}$. These are $4N-4$ specializations. The previously observed fact that the replacement of x_1 by x_1^{-1} changes the sign of both sides of (2) implies that both sides vanish for $x_1=\pm1$. Thus, we have $4N-2$ specializations of x_1 for which (2) holds.

We now specialize $x_1=a$ on both sides of (2). Under this specialization, terms corresponding to alphabets B which contain x_1 vanish on the left-hand side of (2) due to the appearance of the Vandermonde product $V(a+b+B+B^{-1})$, while terms corresponding to alphabets A which contain x_1 vanish on the right-hand side of (2) due to the appearance of the Vandermonde product $V(a+A+B^{-1})$. Therefore, if $x_1=a$, identity (2) reduces to

$$\begin{aligned}(-1)^N(a-a^{-1})(a-b)&\left(\prod_{j=2}^{2N-1}(a-x_j)(a-x_j^{-1})\right)\\&\times\sum_{\substack{A+B=X'\\|A|=N-1}}V(A+a^{-1}+A^{-1})V(b+B+B^{-1})\\&=(a-a^{-1})(b-a)\left(\prod_{j=2}^{2N-1}(a-x_j)(a-x_j^{-1})\right)\\&\qquad\times\sum_{A+B=X'}(-1)^{|A|}V(A+a^{-1}+B^{-1})V(b+B+A^{-1}),\end{aligned}\tag{3}$$

where $X'=x_2+x_3+\cdots+x_{2N-1}$. After clearing the product which is common to both sides, and after little rearrangement, we see that the remaining identity is an instance of (1) with N replaced by $N-1$, the latter being true due to the induction hypothesis.

In summary, we have found $4N-1$ specializations of x_1 for which both sides of (2) agree. On the other hand, as Laurent polynomials in x_1, the degree of both sides of (2) is at most $2N-1$. (That is, the maximal exponent e of a power x_1^e is $e=2N-1$, and the minimal exponent is $e=-(2N-1)$.) Hence, both sides of (2) must be identical.

The earlier arguments show as well that identity (1) holds for *all* specializations of x_1 to one of $x_2,x_2^{-1},x_3,x_3^{-1},\dots,x_{2N},x_{2N}^{-1}$ and ±1. These are $4N$ specializations.

We now specialize $x_1=a$ on both sides of (1). Under this specialization, terms corresponding to alphabets A which contain x_1 vanish on both sides of (1) due to the appearance of the Vandermonde products $V(a+A+A^{-1})$ respectively $V(a+A+B^{-1})$. Therefore, if $x_1=a$, identity (1) reduces to

$$\begin{aligned}(a-a^{-1})(b-a)&\left(\prod_{j=2}^{2N}(a-x_j)(a-x_j^{-1})\right)\\&\times\sum_{\substack{A+B=X''\\|A|=N}}V(A+A^{-1})V(b+B+a^{-1}+B^{-1})\\&=(a-a^{-1})(b-a)\left(\prod_{j=2}^{2N}(a-x_j)(a-x_j^{-1})\right)\\&\qquad\times\sum_{A+B=X''}V(A+a^{-1}+B^{-1})V(b+B+A^{-1}),\end{aligned}\tag{4}$$

where $X'' = x_2 + x_3 + \cdots + x_{2N}$. After clearing the product which is common to both sides, and after little rearrangement, we see that the remaining identity is an instance of (2) which has been just established.

In summary, we have found $4N+1$ specializations of x_1 for which both sides of (1) agree. On the other hand, as Laurent polynomials in x_1, the degree of both sides of (1) is at most $2N$. (That is, the maximal exponent e of a power x_1^e is $e = 2N$, and the minimal exponent is $e = -2N$.) Hence, both sides of (1) must be identical.

It remains to prove that, under the induction hypothesis, Equations (1) and (2) hold for $x_1 = x_2$. We provide the argument establishing this fact for (1). The argument for (2) is completely analogous.

Indeed, under this specialization, terms corresponding to alphabets A which contain *both* x_1 and x_2 and to those which contain *neither* x_1 *nor* x_2 vanish on both sides of (1), because of the appearance of the Vandermonde products $V(A)$ respectively $V(B)$. Therefore, if $x_1 = x_2$, identity (1) reduces to

$$2(a-x_1)(a-x_1^{-1})(b-x_1)(b-x_1^{-1})(x_1-x_1^{-1})^2\left(\prod_{j=3}^{2N}(x_1-x_j)(x_1^{-1}-x_j^{-1})(x_1-x_j^{-1})(x_j-x_1^{-1})\right)$$
$$\times \sum_{\substack{A+B=X''' \\ |A|=N-1}} V(a+A+A^{-1})V(b+B+B^{-1})$$
$$= 2(a-x_1)(a-x_1^{-1})(b-x_1)(b-x_1^{-1})(x_1-x_1^{-1})^2\left(\prod_{j=3}^{2N}(x_1-x_j)(x_1^{-1}-x_j^{-1})(x_1-x_j^{-1})(x_j-x_1^{-1})\right)$$
$$\times \sum_{A+B=X'''} V(a+A+B^{-1})V(b+B+A^{-1}), \quad (5)$$

where $X''' = x_3 + x_4 + \cdots + x_{2N}$. After clearing the product which is common to both sides, we see that the remaining identity is equivalent to (1) with N replaced by $N-1$, the latter being true due to the induction hypothesis. This finishes the proof of the theorem. $\square$

Theorem 2. *Let a, b be two variables, and let*

$$\Phi(a,b;x_i,x_j) = \frac{(a-x_i)(a-x_i^{-1})(b-x_j)(b-x_j^{-1}) + (a-x_j)(a-x_j^{-1})(b-x_i)(b-x_i^{-1})}{(x_i + x_i^{-1} - x_j - x_j^{-1})}.$$

Then there holds the identity

$$\prod_{i=1}^{2N}(x_i - x_i^{-1}) \prod_{1\le i<j\le 2N}(x_i + x_i^{-1} - x_j - x_j^{-1}) \operatorname*{Pf}_{1\le i<j\le 2N}\left(\Phi(a,b;x_i,x_j)\right)$$
$$= \sum_{\substack{A+B=X_{2N} \\ |A|=N}} V(a+A+A^{-1})V(b+B+B^{-1}). \quad (6)$$

Proof. We prove the assertion by induction on N. Again, the induction proof will prove *two* assertions at the same time, namely (6) and

$$a^{2N-2}\prod_{i=1}^{2N-1}(x_i-x_i^{-1})\prod_{1\le i<j\le 2N-1}(x_i+x_i^{-1}-x_j-x_j^{-1})\\ \times \underset{1\le i<j\le 2N}{\operatorname{Pf}}\begin{pmatrix} b-a & i=1\\ \Phi(a^{-1},b;x_{i-1},x_{j-1}) & i\ge 2\end{pmatrix}\\ =(-1)^N\sum_{\substack{A+B=X_{2N-1}\\ |A|=N}}V(A+A^{-1})V(a+b+B+B^{-1}). \tag{7}$$

For $N=1$, both (6) and (7) reduce to tautologies.

Now let us suppose that we have proved (6) and (7) with N replaced by $N-1$. Below, we shall show that this implies that identities (6) and (7) are true if we specialize x_1 to x_2. Let us for the moment suppose that this is already done. It is easy to see that both sides of (6) are symmetric in the variables $x_1,x_2,\dots,x_{2N}$, while both sides of (7) are symmetric in the variables $x_1,x_2,\dots,x_{2N-1}$. Moreover, both sides of (6) and both sides of (7) remain invariant up to an overall multiplicative sign if we replace x_1 by x_1^{-1}.

This means that identity (7) holds for *all* specializations of x_1 to one of $x_2,x_2^{-1},x_3,x_3^{-1}$, $\dots,x_{2N-1},x_{2N-1}^{-1}$. These are $4N-4$ specializations. The previously observed fact that the replacement of x_1 by x_1^{-1} changes the sign of both sides of (7) implies that both sides vanish for $x_1=\pm1$. Thus, we have $4N-2$ specializations of x_1 for which (7) holds.

We now specialize $x_1=a$ on both sides of (7). Since

$$\Phi(a^{-1},b;a,x_j)=a^{-1}(b-a)(b-a^{-1}),$$

under this specialization, the left-hand side of (7) becomes

$$a^{2N-2}(a-a^{-1})\prod_{i=2}^{2N-1}(x_i-x_i^{-1})\prod_{j=2}^{2N-1}(a+a^{-1}-x_j-x_j^{-1})\prod_{2\le i<j\le 2N-1}(x_i+x_i^{-1}-x_j-x_j^{-1})\\ \times \underset{1\le i<j\le 2N}{\operatorname{Pf}}\begin{pmatrix} b-a & i=1\\ a^{-1}(b-a)(b-a^{-1}) & i=2\\ \Phi(a^{-1},b;x_{i-1},x_{j-1}) & i\ge 3\end{pmatrix}\\ =a^{2N-2}(a-a^{-1})(b-a)\\ \times\prod_{i=2}^{2N-1}(x_i-x_i^{-1})\prod_{j=2}^{2N-1}(a+a^{-1}-x_j-x_j^{-1})\prod_{2\le i<j\le 2N-1}(x_i+x_i^{-1}-x_j-x_j^{-1})\\ \times \underset{1\le i<j\le 2N}{\operatorname{Pf}}\begin{pmatrix} 1 & i=1,j=2\\ b-a & i=1,j>2\\ a^{-1}(b-a^{-1}) & i=2\\ \Phi(a^{-1},b;x_{i-1},x_{j-1}) & i\ge 3\end{pmatrix}.$$

Here, the Pfaffian can be simplified. To see this most transparently, it is useful to display the skew symmetric matrix of which the Pfaffian is taken:

$$\begin{pmatrix} 0 & 1 & b-a & b-a & \dots & b-a \\ -1 & 0 & a^{-1}(b-a^{-1}) & a^{-1}(b-a^{-1}) & \dots & a^{-1}(b-a^{-1}) \\ -(b-a) & -a^{-1}(b-a^{-1}) & & & & \\ \vdots & \vdots & & \ast & & \\ -(b-a) & -a^{-1}(b-a^{-1}) & & & & \end{pmatrix}.$$

If we subtract $a(b-a)/b-a^{-1}$ times the second row from the first row (and subsequently do the same operation with the corresponding columns), then this matrix is transformed to

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & a^{-1}(b-a^{-1}) & a^{-1}(b-a^{-1}) & \dots & a^{-1}(b-a^{-1}) \\ 0 & -a^{-1}(b-a^{-1}) & & & & \\ \vdots & \vdots & & \ast & & \\ 0 & -a^{-1}(b-a^{-1}) & & & & \end{pmatrix},$$

without changing the Pfaffian. The Pfaffian of the last matrix however is $\operatorname{Pf}(\ast)$. Therefore, under the specialization $x_1 = a$, the left-hand side of (7) is equal to

$$= a^{2N-2}(a-a^{-1})(b-a)\prod_{i=2}^{2N-1}(x_i - x_i^{-1})\prod_{j=2}^{2N-1}(a+a^{-1}-x_j-x_j^{-1}) \\ \times \prod_{2\le i<j\le 2N-1}(x_i+x_i^{-1}-x_j-x_j^{-1}) \underset{1\le i<j\le 2N-2}{\operatorname{Pf}}\left(\Phi(a^{-1},b;x_{i+1},x_{j+1})\right).$$

On the other hand, the specialization of the right-hand side of (7) to $x_1 = a$ is exactly the left-hand side in (3). Hence, after clearing common factors, we see that for $x_1 = a$ identity (7) reduces to

$$\prod_{i=2}^{2N-1}(x_i - x_i^{-1})\prod_{2\le i<j\le 2N-1}(x_i+x_i^{-1}-x_j-x_j^{-1}) \underset{1\le i<j\le 2N-2}{\operatorname{Pf}}\left(\Phi(a^{-1},b;x_{i+1},x_{j+1})\right) \\ = \sum_{\substack{A+B=X' \\ |A|=N-1}} V(a^{-1}+A+A^{-1})V(b+B+B^{-1}),$$

where $X' = x_2 + x_3 + \cdots + x_{2N-1}$. Clearly, this is an instance of (6) with N replaced by $N-1$, the latter being true due to the induction hypothesis.

In summary, we have found $4N-1$ specializations of x_1 for which both sides of (7) agree. On the other hand, as Laurent polynomials in x_1 (the left-hand side of (7) is indeed a Laurent polynomial in the x_i's since the denominators in the Pfaffian cancel with the product over $1 \le i < j \le 2N-1$!), the degree of both sides of (7) is at most $2N-1$. Hence, both sides of (7) must be identical.

The earlier arguments show as well that identity (6) holds for *all* specializations of x_1 to one of $x_2, x_2^{-1}, x_3, x_3^{-1}, \dots, x_{2N}, x_{2N}^{-1}$ and ± 1. These are $4N$ specializations.

We now specialize $x_1 = a$ on both sides of (6). Since

$$\Phi(a,b;a,x_j) = a(b-a)(b-a^{-1}),$$

under this specialization, the left-hand side of (6) becomes

$$(a-a^{-1})a(b-a)\prod_{i=2}^{2N}(x_i - x_i^{-1})\prod_{j=2}^{2N}(a+a^{-1}-x_j-x_j^{-1}) \\ \times \prod_{2\le i<j\le 2N}(x_i+x_i^{-1}-x_j-x_j^{-1}) \operatorname*{Pf}_{1\le i<j\le 2N}\begin{pmatrix} b-a^{-1} & i=1 \\ \Phi(a,b;x_i,x_j) & i\ge 2\end{pmatrix}.$$

On the other hand, the specialization of the right-hand side of (6) to $x_1 = a$ is exactly the left-hand side in (4). Hence, after clearing common factors, we see that for $x_1 = a$ identity (6) reduces to

$$a^{2-2N}\prod_{i=2}^{2N}(x_i - x_i^{-1})\prod_{2\le i<j\le 2N}(x_i+x_i^{-1}-x_j-x_j^{-1}) \operatorname*{Pf}_{1\le i<j\le 2N}\begin{pmatrix} b-a^{-1} & i=1 \\ \Phi(a,b;x_i,x_j) & i\ge 2\end{pmatrix} \\ = (-1)^N \sum_{\substack{A+B=X'' \\ |A|=N}} V(A+A^{-1})V(a^{-1}+b+B+B^{-1}),$$

where $X'' = x_2 + x_3 + \dots + x_{2N}$. This is an instance of (7) which has been just established.

In summary, we have found $4N+1$ specializations of x_1 for which both sides of (6) agree. On the other hand, as Laurent polynomials in x_1, the degree of both sides of (6) is at most $2N$. Hence, both sides of (6) must be identical.

It remains to prove that, under the induction hypothesis, Equations (6) and (7) hold for $x_1 = x_2$. We provide the argument establishing this fact for (6). The argument for (7) is completely analogous.

Indeed, under this specialization, due to the term

$$\prod_{1\le i<j\le 2N}(x_i+x_i^{-1}-x_j-x_j^{-1})$$

on the left hand side of (6), in the expansion of the Pfaffian only terms survive which contain the $(1,2)$-entry of the Pfaffian. Thus, for $x_1 = x_2$, the left-hand side of (6) reduces to

$$2(a-x_1)(a-x_1^{-1})(b-x_1)(b-x_1^{-1})(x_1-x_1^{-1})^2\prod_{i=3}^{2N}(x_i-x_i^{-1})\left(\prod_{j=3}^{2N}(x_1+x_1^{-1}-x_j-x_j^{-1})^2\right) \\ \times \prod_{3\le i<j\le 2N}(x_i+x_i^{-1}-x_j-x_j^{-1}) \operatorname*{Pf}_{3\le i<j\le 2N}\left(\Phi(a,b;x_i,x_j)\right).$$

On the other hand, the specialization of the right-hand side of (6) to $x_1 = a$ is exactly the left-hand side of (5). Hence, after clearing common factors, we see that for $x_1 = a$ identity (6) reduces to

$$\prod_{i=3}^{2N}(x_i - x_i^{-1}) \prod_{3\le i<j\le 2N} (x_i + x_i^{-1} - x_j - x_j^{-1}) \operatorname*{Pf}_{3\le i<j\le 2N} \left(\Phi(a,b;x_i,x_j)\right) \\ = \sum_{\substack{A+B=X'''\\ |A|=N-1}} V(a+A+A^{-1})V(b+B+B^{-1}),$$

where $X''' = x_3 + x_4 + \cdots + x_{2N}$. This identity is equivalent to (6) with N replaced by $N-1$, the latter being true due to the induction hypothesis. This finishes the proof of the theorem. □

References

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