Determinantal expressions for Macdonald polynomials

L. Lapointe, A. Lascoux, and J. Morse

Institut Gaspard Monge, C.N.R.S.
Université de Marne-la-Vallée, 5 Bd Descartes,
Champs sur Marne, 77454 Marne La Vallée Cedex, France

ABSTRACT: We show that the action of classical operators associated to the Macdonald polynomials on the basis of Schur functions, $S_\lambda\,[X(t-1)/(q-1)]$, can be reduced to addition in $\lambda-$rings. This provides explicit formulas for the Macdonald polynomials expanded in this basis as well as in the ordinary Schur basis, $S_\lambda[X]$, and the monomial basis, $m_\lambda[X]$.

Introduction

Important developments in the theory of symmetric functions rely on the use of the Macdonald polynomial basis, $\{J_\lambda(X;q,t)\}_\lambda$ [M1]. This basis specializes to several fundamental bases including the Schur, Hall-Littlewood, Zonal, and Jack. It has been conjectured that the Macdonald polynomials occur naturally in representation theory of the symmetric group [GH]. It is also known that these polynomials are eigenfunctions of a family of commuting difference operators with significance in many-body physics [RS].

We first reformulate the Macdonald operators in terms of divided differences, operators that act naturally on the Schur function basis. This enables us to show that the action of the Macdonald operators on the modified Schur basis, $S_\lambda[X(t-1)/(q-1)]$, primarily amounts to addition in λ-rings (Theorem 2.1). The action provides a determinantal expression for the Macdonald polynomials expanded in this Schur basis. By involution this expression can be converted into an explicit formula for the Macdonald polynomials in the usual Schur basis, $S_\lambda[X]$, and in the monomial basis, $m_\lambda[X]$ (Theorems 3.1 and 3.3). Further, it is known that the Macdonald polynomials can be built recursively using 'creation operators'. We reformulate these operators in a manner similar to our expression for the Macdonald operators enabling us to give a simple expression for their action on the modified Schur functions (Theorem 4.2).

1. Background

To begin, we clarify some notation. $\mathcal{S}ym$ refers to the ring of symmetric functions.

A partition will be defined as a weakly decreasing sequence of non-negative integers, $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\ell(\lambda)$, the number of parts of λ and λ' the conjugate to λ. To a partition λ, there corresponds a Ferrers' diagram with λ_i lattice squares in the i^{th} row, from the bottom to top. For each square s in the diagram of λ, we define $a(s)$ and $l(s)$ to be the number of squares respectively to the north and east of s. The *order of* λ is the sum of the parts of λ, denoted $|\lambda|$. The *dominance order* on λ is defined such that for $|\lambda| = |\mu|$, $\lambda \le \mu$ when $\lambda_1 + \cdots + \lambda_i \le \mu_1 + \cdots + \mu_i$ for all i.

We shall use λ-rings, needing only the formal ring of symmetric functions $\mathcal{S}ym$ to act on the ring of rational functions in $x_1, \ldots, x_n, q, t$, with coefficients in $\mathbb{R}$. The ring $\mathcal{S}ym$ is generated by power sums Ψ_i, $i = 1, 2, 3 \ldots$. The action of Ψ_i on a rational function $\sum_\alpha c_\alpha u_\alpha / \sum_\beta d_\beta v_\beta$ is by definition

$$\Psi_i \left[\frac{\sum_\alpha c_\alpha u_\alpha}{\sum_\beta d_\beta v_\beta} \right] = \frac{\sum_\alpha c_\alpha u_\alpha^i}{\sum_\beta d_\beta v_\beta^i}, \tag{1.1}$$

with $c_\alpha, d_\beta \in \mathbb{R}$ and u_α, v_β monomials in $x_1, \ldots, x_n, q, t$. Since any symmetric function is uniquely expressed in terms of the power sums, formula 1.1 extends to an action of $\mathcal{S}ym$ on rational functions. In particular, a symmetric function $f(X)$ of $X = \{x_1, \ldots, x_n\}$ can be denoted $f[x_1 + \cdots + x_n]$.

The Schur function S_λ, with $Y, Z \in \mathbb{Q}[x_1, \ldots, x_n, q, t]$, is such that

$$S_\lambda[Y \pm Z] \;=\; \sum_\mu S_{\lambda/\mu}[Y] S_\mu[\pm Z]\ , \tag{1.2}$$

where $S_\mu[-Z] = (-1)^{|\mu|} S_{\mu'}[Z]$, with μ' the partition conjugate to μ. Schur functions will be considered as determinants of complete functions:

$$S_\lambda[X] = \det \begin{vmatrix} S_{\lambda_1}[X] & S_{\lambda_1+1}[X] & \cdots & S_{\lambda_1+n-1}[X] \\ S_{\lambda_2-1}[X] & S_{\lambda_2}[X] & \cdots & S_{\lambda_2+n-2}[X] \\ \vdots & \vdots & \ddots & \vdots \\ S_{\lambda_n-n+1}[X] & S_{\lambda_n-n+2}[X] & \cdots & S_{\lambda_n}[X] \end{vmatrix}, \tag{1.3}$$

as will skew Schur functions:

$$S_{\lambda/\mu}[X] = \det \begin{vmatrix} S_{\lambda_1-\mu_1}[X] & S_{\lambda_1+1-\mu_2}[X] & \cdots & S_{\lambda_1+n-1-\mu_n}[X] \\ S_{\lambda_2-1-\mu_1}[X] & S_{\lambda_2-\mu_2}[X] & \cdots & S_{\lambda_2+n-2-\mu_n}[X] \\ \vdots & \vdots & \ddots & \vdots \\ S_{\lambda_n-n+1-\mu_1}[X] & S_{\lambda_n-n+2-\mu_2}[X] & \cdots & S_{\lambda_n-\mu_n}[X] \end{vmatrix}. \tag{1.4}$$

In these expressions, $S_i = 0$ for $i < 0$.

We denote $X = x_1 + \cdots + x_n$ and a partial alphabet, $X_I = x_{i_1} + \cdots + x_{i_k}$, for $I = \{i_1, \ldots, i_k\}$. The complement of X_I in X will be X_I^c. For $I = \{1, \ldots, k\}$, we have $X_k = x_1 + \cdots + x_k$. Two particular elements, $X^{tq} := X(t-1)/(q-1)$ and $X^t := X(t-1)$ will be used throughout the paper.

We shall need the divided differences ∂_μ, indexed by elements μ of the symmetric group $\mathfrak{S}(n)$. In particular, if μ is the simple transposition σ_i, we have

$$\partial_i : f \longrightarrow (f - \sigma_i f)/(x_i - x_{i+1}) . \qquad 1.5$$

The divided differences satisfy the Coxeter relations [M2]

$$\partial_i \partial_j = \partial_j \partial_i, \quad |i-j| > 1; \qquad \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} , \qquad 1.6$$

which imply that ∂_μ can be written as the product of elementary operators ∂_i corresponding to any reduced decomposition of μ. It should be noted that symmetric functions in x_i, x_{i+1} are scalar with respect to ∂_i and that $\partial_i \cdot 1 = 0$. Consequently, symmetric functions in $x_1, \ldots, x_n$ commute with all ∂_μ.

We have that $\partial_{n-1} \cdots \partial_1$ is the *Lagrange operator* [L1],

$$\partial_{n-1} \cdots \partial_1 : f \in \mathcal{S}ym(1|n-1) \longrightarrow \sum_i \frac{f(x_i, x_i^c)}{R(x_i, x_i^c)} \in \mathcal{S}ym(n) , \qquad 1.7$$

or more generally, $\partial_{(k|n-k)} = (\partial_{n-k} \cdots \partial_1) \cdots (\partial_{n-2} \cdots \partial_{k-1})(\partial_{n-1} \cdots \partial_k)$ is the *Sylvester operator* [L1],

$$\partial_{(k|n-k)} : f \in \mathcal{S}ym(k|n-k) \longrightarrow \sum_{|I|=k} \frac{f(X_I, X_I^c)}{R(X_I, X_I^c)} \in \mathcal{S}ym(n) , \qquad 1.8$$

where $\mathcal{S}ym(k|n-k)$ is the space of functions symmetrical in $x_1, \ldots, x_k$ and in $x_{k+1}, \ldots, x_n$, and where $R(X, Y)$ is the resultant of two alphabets X and Y ;

$$R(X, Y) \;=\; \prod\nolimits_{x \in X, y \in Y} (x - y) . \qquad 1.9$$

These operators may be used to define the Euler-Poincaré characteristic χ. In fact, in his study of Riemann-Roch theorem, Hirzebruch defined a 'χ_y'-characteristic, or 'χ_y'-genus [H],[HBJ]. We shall use the one corresponding to a relative flag manifold, changing y into $1/t$. We define the following operators on polynomials in $x_1, \ldots, x_n$,

$$\chi^{(k)}_{(1|n-k)} \;=\; \partial_{n-1} \cdots \partial_k \, R(x_k, X_k^c/t) , \qquad 1.10$$

considered as a composition of multiplication by the resultant followed by a sequence of divided differences. Note that the image of 1 is $(1+\cdots+t^{-n+k})$ and that the superscript refers to the variable at which the symmetrization begins. We shall also use

$$\chi_{(k|n-k)} \;=\; \partial_{(k|n-k)}\, R(X_k, X_k^c/t) \tag{1.11}$$

and

$$\chi_{\omega(k)} \;=\; \partial_{\omega(k)} \prod_{1\le i<j\le k} (x_i - x_j/t)\;, \tag{1.12}$$

where $\omega(k)$ is the longest permutation in $\mathfrak{S}(k)$ and $\chi_{\omega(k)}\cdot 1 = k_t! = t^{-k(k-1)/2}(t)_k/(1-t)^k$, where $(t)_k = (1-t)\cdots(1-t^k)$. To compute with $\partial_{\omega(k)}$, we shall use the factorization $\partial_{\omega(k)} = (\partial_{k-1})(\partial_{k-2}\partial_{k-1})\cdots(\partial_1\cdots\partial_{k-1})$.

When operating on elements of $\mathcal{S}ym(\ell|k-\ell)$, one has

$$\chi_{\omega(k)}\, f \;=\; \ell_t!\,(k-\ell)_t!\,\chi_{(\ell|k-\ell)}\, f \qquad \forall\, f\in Sym(\ell|k-\ell)\,. \tag{1.13}$$

We shall also need the factorization

$$\chi_{(k|n-k)}\chi^{(1)}_{(1|k-1)}\cdots\chi^{(k-2)}_{(1|2)}\chi^{(k-1)}_{(1|1)} \;=\; \chi^{(1)}_{(1|n-1)}\cdots\chi^{(k-1)}_{(1|n-k+1)}\chi^{(k)}_{(1|n-k)} \tag{1.14}$$

and the following lemma (proved in the appendix).

Lemma 1.1 *For an alphabet C, symmetric in $x_1,\ldots,x_n$, and $c(k) = t^{-1}+\cdots+t^{-n+k}$, one has*

$$\chi^{(k)}_{(1|n-k)}\, S_j[C+X_k^t] \;=\; c(k)\, S_j[C+X_{k-1}^t] + S_j[C+X^t]\,. \tag{1.15}$$

2. Macdonald Operators

This section presents the action of the Macdonald operators on $S_\lambda[X^{tq}]$ as a sum of determinants. The Macdonald operators [M1] are defined as

$$M_k \;=\; t^{\binom{k}{2}} \sum_{|I|=k} R(tX_I, X_I^c)\,\Theta_{i_k}\cdots\Theta_{i_1} \qquad\qquad k=1,2,3,\ldots\,, \tag{2.1}$$

where the sum is over all k-element subsets $I=(i_1,\ldots,i_k)$ of $\{1,\ldots,n\}$ and

$$\Theta_l : f[X] \longrightarrow f[X+(q-1)x_l]\,. \tag{2.2}$$

It is more convenient to use a reformulation for these operators in terms of divided differences,

$$\begin{aligned} M_k \;&=\; t^{\binom{k}{2}+k(n-k)}\chi_{(k|n-k)}\Theta_k\cdots\Theta_1 \\ &=\; t^{\binom{k}{2}}\,\partial_{(k|n-k)}\, R(tX_k, X_k^c)\,\Theta_k\cdots\Theta_1\,, \end{aligned} \tag{2.3}$$

thanks to 1.8.

We shall express the action of M_k on $S_\lambda[X^{tq}]$ with formal operators, $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{bmatrix}$, acting on the columns of an $n\times n$ matrix of Schur functions such that for all elements in any column ℓ , $S_\beta[Y] \to S_{\beta+a_\ell}[Y+b_\ell]$. For example;

$$\begin{bmatrix} 0 & 30 & -5 \\ b_1 & Y & 0 \end{bmatrix} : \begin{pmatrix} S_{11}[A] & S_4[B] & S_8[C] \\ S_0[A] & S_2[B] & S_{13}[C] \\ S_1[A] & S_2[B] & S_3[C] \end{pmatrix} \to \begin{pmatrix} S_{11}[A+b_1] & S_{34}[B+Y] & S_3[C] \\ S_0[A+b_1] & S_{32}[B+Y] & S_8[C] \\ S_1[A+b_1] & S_{32}[B+Y] & S_{-2}[C] \end{pmatrix} . \tag{2.4}$$

Theorem 2.1 *The action of the Macdonald operators on $S_\lambda[X^{tq}]$, with $\ell(\lambda)\le n$, can be expressed as*

$$M_k S_\lambda[X^{tq}] \;=\; \sum_{1\le i_1<\cdots<i_k\le n} t^{(n-i_1)+\cdots+(n-i_k)} \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & X^t & \cdots & X^t & \cdots & 0 \end{bmatrix} S_\lambda[X^{tq}] \,, \tag{2.5}$$

where X^t occurs in columns $i_1,\ldots,i_k$.

For example, if $n=2$,

$$M_1 S_{(5,2)} \;=\; t\begin{vmatrix} S_5[X^{tq}+X^t] & S_6[X^{tq}] \\ S_1[X^{tq}+X^t] & S_2[X^{tq}] \end{vmatrix} \;+\; \begin{vmatrix} S_5[X^{tq}] & S_6[X^{tq}+X^t] \\ S_1[X^{tq}] & S_2[X^{tq}+X^t] \end{vmatrix} , \tag{2.6}$$

or more concisely since $X^{tq}+X^t = qX^{tq}$, using S_i for $S_i[X^{tq}]$,

$$M_1 S_{(5,2)} \;=\; t\begin{vmatrix} q^5 S_5 & S_6 \\ qS_1 & S_2 \end{vmatrix} \;+\; \begin{vmatrix} S_5 & q^6 S_6 \\ S_1 & q^2 S_2 \end{vmatrix} . \tag{2.7}$$

Proof of Theorem 2.1 The Macdonald operators,

$$M_k \;=\; t^{\binom{k}{2}}\, \partial_{(k|n-k)}\, R(tX_k, X_k^c)\, \Theta_k \cdots \Theta_1 \,, \tag{2.8}$$

act first with a product of Θ_i, defined to replace x_i by qx_i. However, this q-deformation is actually a t-deformation when acting on $f[X^{tq}]$ since

$$\Theta_\ell : f[X^{tq}] \to f\left[\left(X+(q-1)x_\ell\right)\cdot\frac{t-1}{q-1}\right] = f[X^{tq}+(t-1)x_\ell] \,. \tag{2.9}$$

Now for $Y = X^{tq}-X$, if we define a formal operator Θ_ℓ^t that sends $x_i\to x_i$ and $x_\ell\to tx_\ell$ while leaving Y invariant; i.e. $Y+X\to Y+X+(t-1)x_\ell$, we obtain

$$R(tX_k, X_k^c)\,\Theta_k\cdots\Theta_1\, S_\lambda[Y+X] \;=\; \Theta_k^t\cdots\Theta_1^t\, R(X_k,X_k^c)\, S_\lambda[Y+X] \,, \tag{2.10}$$

where Θ_i acts on both Y and X on the left hand side of this expression. Further, we have the following determinantal expression for $R(X_k, X_k^c)\, S_\lambda[Y+X]$;

$$R(X_k, X_k^c)\, S_\lambda[Y+X] \;=\; \Big| S_{\alpha_i-k+n}[Y+X_k]\cdots S_{\alpha_i-1+n}[Y+X_k] S_{\alpha_i+k}[Y+X_k^c]\cdots S_{\alpha_i+n-1}[Y+X_k^c] \Big|_{i=1\cdots n}, \qquad 2.11$$

where $\alpha_i \to \lambda_i - i + 1$ (to obtain this expression see the appendix). Since $\Theta_k^t \cdots \Theta_1^t$ acts exclusively on the first k columns of this determinant, we have that

$$t^{-\binom{k}{2}} M_k S_\lambda[Y+X] \;=\; \partial_{(k|n-k)} \Big| S_{\alpha_i-k+n}[Y+tX_k]\cdots S_{\alpha_i-1+n}[Y+tX_k] S_{\alpha_i+k}[Y+X_k^c]\cdots S_{\alpha_i+n-1}[Y+X_k^c] \Big|_{i=1..n}. \qquad 2.12$$

As it remains to show that this is equivalent to expression 2.5, we proceed by giving the action of $\partial_{(k|n-k)}$ on products of arbitrary minors of order k on the first k columns and of order $n-k$ on the last $n-k$ columns of this determinant. The action of $\partial_{(k|n-k)}$ on the whole determinant will then follow by linearity.

Lemma 2.2 *For $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_{n-k})$, and A, B invariant under $\mathfrak{S}(n)$, we have*

$$\partial_{(k|n-k)}\Big(S_\alpha[A+tX_k]S_\beta[B+X_k^c]\Big) \;=\; t^{k(n-k)}\sum_I (-1/t)^{|I|} \times \left(\begin{bmatrix} -n+k+i_k & \cdots & -n+k+i_1 \\ tX & \cdots & tX \end{bmatrix} S_\alpha[A]\right) \left(\begin{bmatrix} -i'_1 & \cdots & -i'_{n-k} \\ X & \cdots & X \end{bmatrix} S_\beta[B]\right), \qquad 2.13$$

where $I = (i_1, \ldots, i_k)$ and $(i'_1, \ldots, i'_{n-k})$ are pairs of conjugate partitions.

Proof of lemma Define the elements $A' = A + tX$ and $B' = B + X$. The product $S_\alpha[A+tX_k]S_\beta[B+X_k^c]$ can be expanded, thanks to 1.2, into products of Schur functions in $-X_k^c$ and X_k (with coefficients in A' and B'),

$$\begin{aligned} S_\alpha[A+tX_k]\, S_\beta[B+X_k^c] &= S_\alpha[A'-tX_k^c]\, S_\beta[B'-X_k] \\ &= \sum_{I,J} S_{\alpha/I}[A']t^{|I|}S_I[-X_k^c]S_{\beta/J}[B'](-1)^{|J|}S_{J'}[X_k], \end{aligned} \qquad 2.14$$

where I and J are partitions and J' is conjugate to J. Since $\ell(I), \ell(J') \le k$ and $\ell(I'), \ell(J) \le n-k$, where I' is conjugate to I, we have $I, J' \subseteq (n-k, \ldots, n-k)$. Thus the following property describing two adjoint bases of $\mathcal{S}ym(k, n-k)$ as a free module over $\mathcal{S}ym(n)$ ([L2], Corollary 2.2),

$$\partial_{(k|n-k)}\Big(S_I[-X_k^c]\, S_J[X_k]\Big) = \begin{cases} 1 & \text{if } I = (n-k-J_k, \ldots, n-k-J_1) \\ 0 & \text{else} \end{cases}, \qquad 2.15$$

gives

$$\partial_{(k|n-k)}\Big(S_\alpha[A+tX_k]\,S_\beta[B+X_k^c]\Big)=t^{k(n-k)}\sum_J(-1/t)^{|J|}S_{\alpha/(n-k-J'_k,\dots,n-k-J'_1)}[A']S_{\beta/J}[B']. \tag{2.16}$$

The lemma is proved by letting $J\to I'$ in the last expression and using the definition 1.4 of skew Schur functions. □

Now applying Lemma 2.2 directly to expression 2.12, we have

$$\begin{aligned} &M_kS_\lambda[Y+X]\\ &=\sum_I(-1/t)^{|I|}t^{k(n-k)+k(k-1)/2}\begin{bmatrix}-n+k+i_k & \cdots & -n+k+i_1 & -i'_1 & \cdots & -i'_{n-k}\\ tX & \cdots & tX & X & \cdots & X\end{bmatrix}\\ &\qquad\times\Big|S_{\alpha_i-k+n}[Y]\cdots S_{\alpha_i-1+n}[Y]S_{\alpha_i+k}[Y]\cdots S_{\alpha_i+n-1}[Y]\Big|_{i=1..n}\\ &=\sum_I(-1/t)^{|I|}t^{k(n-k)+k(k-1)/2}\begin{bmatrix}i_k & \cdots & i_1+k-1 & -i'_1+k & \cdots & -i'_{n-k}+n-1\\ tX & \cdots & tX & X & \cdots & X\end{bmatrix}\\ &\qquad\times\Big|S_{\alpha_i}[Y]\cdots S_{\alpha_i}[Y]S_{\alpha_i}[Y]\cdots S_{\alpha_i}[Y]\Big|_{i=1..n}. \end{aligned} \tag{2.17}$$

For $\rho=(0,\dots,n-1)$ and λ^ω the reverse reading of a partition λ, we have from [M1] that $\rho+\lambda^\omega\cup-\lambda'=\sigma\rho$ for some permutation σ. The top row of the formal operator in 2.17, $\rho+(i_k,\dots,i_1,-i'_1,\dots,-i'_{n-k})$, may thus be rearranged to ρ. The rearrangement sends column $\ell+1$, containing tX and corresponding to index $i_{k-\ell}+\ell$, to column $i_{k-\ell}+\ell+1$. This requires $i_{k-\ell}$ commutations with X-columns inducing a $(-1)^{i_{k-\ell}}$. Proceeding on all tX-columns yields

$$M_k\,S_\lambda[Y+X]\;=\;\sum_I t^{k(n-k)+k(k-1)/2-|I|}\begin{bmatrix}0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\ X & \cdots & tX & \cdots & tX & \cdots & X\end{bmatrix}S_\lambda[Y]\,, \tag{2.18}$$

where the tX are in columns $i_k+1,\dots,i_1+k$. Finally, the substitution $Y=X^{tq}-X$ with the shift $i_k+1\to i_k,\dots,i_1+k\to i_1$ gives Theorem 2.1 . □

Remark: The image of $S_\lambda[X^{tq}]$ under the Macdonald operator M_k in the basis of products of complete functions $S^\mu[X^{tq}]=S_{\mu_1}[X^{tq}]S_{\mu_2}[X^{tq}]\dots$ can be derived from 2.5. However, there exists a scalar product for which the $S^\mu[X^{tq}]$ basis is adjoint to the $m_\mu[X]$ basis, the $S_\lambda[X^{tq}]$ basis is adjoint to the $S_{\lambda'}[X]$ basis, and M_k is self-adjoint. This allows that formula 2.5 be connected to the action of the Macdonald operators on monomial functions given in [M1],VI 3.6.

3. Macdonald Polynomials

This section contains determinantal expressions for the Macdonald polynomials in the bases, $S_\mu[X^{tq}]\,, S_\mu[X]\,,$ and $m_\mu[X]\,,$ obtained from the action of the operator M_1 on $S_\mu[X^{tq}]$.

The Macdonald polynomials [M1] can be defined to be the eigenfunctions of the Macdonald operators. In particular we have , for any partition λ,

$$M_1\, J_\lambda(X;q,t) \;\;=\;\; [|\lambda|]\, J_\lambda(X;q,t)\;, \qquad 3.1$$

where the symbol

$$[|\alpha|] \;:=\; q^{\alpha_1}t^{n-1}+q^{\alpha_2}t^{n-2}+\cdots+q^{\alpha_n}\;, \qquad 3.2$$

will be used for any composition $\alpha\in\mathbb{N}^n$. Since the eigenvalues are distinct for each λ, any polynomial that satisfies 3.1 must be proportional to $J_\lambda(X;q,t)$. It is with this in mind that we proceed to construct such polynomials using the action of M_1 on $S_\lambda[X^{tq}]$.

First, we have that the ordered expansion of S_μ in terms of complete functions is given, for $S^\alpha = S_{\alpha_1}\cdots S_{\alpha_n}$, by

$$S_\mu[X] \;\;=\; \sum_{\sigma\in\mathfrak{S}(n)} (-1)^{\ell(\sigma)}\, S^{\sigma(\mu+\rho)-\rho}[X] \;:=\; \sum_\alpha \epsilon(\mu,\alpha)\, S^\alpha[X]\;, \qquad 3.3$$

summed over all vectors $\alpha\in\mathbb{N}^n$, where $\epsilon(\mu,\alpha)\in\{0,+1,-1\}$. The set of all α with a corresponding nonzero $\epsilon(\mu,\alpha)$ is included in the set $\{\bar\sigma\mu \,|\, \bar\sigma\in\mathfrak{S}(\ell(\mu))\}$ where we define $\bar\sigma_i\mu = -(\ldots,\mu_{i+1}-1,\mu_i+1,\ldots)$. For example, if $n=3$,

$$\mu=(3,1,1)\;:\qquad \begin{tikzcd}[column sep=small] & -(0,4,1) \arrow[r, "\bar\sigma_2"] & (0,0,5) \arrow[dr, "\bar\sigma_1"] & \\ (3,1,1) \arrow[ur, "\bar\sigma_1"] \arrow[dr, "\bar\sigma_2"'] & & & -(-1,1,5) \\ & -(3,0,2) \arrow[r, "\bar\sigma_1"] & (-1,4,2) \arrow[ur, "\bar\sigma_2"'] & \end{tikzcd}\;. \qquad 3.4$$

For $\mu=(3,1,1)$, $\epsilon(\mu,\alpha)\,\alpha = \{(3,1,1),-(0,4,1),(0,0,5),-(3,0,2)\}$, since $\alpha\in\mathbb{N}^n$ cannot have negative components.

Now, recall that Theorem 2.1 gives explicitly

$$M_1\, S_\mu[X^{tq}] \;\;=\; \sum_{1\le i\le n} t^{n-i} \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & X^t & \cdots & 0 \end{bmatrix} S_\mu[X^{tq}]\;. \qquad 3.5$$

The expansion of the right hand side, taking into account that $S_k[X^{tq}+X^t]=q^kS_k[X^{tq}]$, is

$$M_1\, S_\mu[X^{tq}] = \sum_{\sigma\in\mathfrak{S}(n)} (-1)^{\ell(\sigma)}\,[|\sigma(\mu+\rho)-\rho|]\, S^{\sigma(\mu+\rho)-\rho}[X^{tq}] = \sum_\alpha \epsilon(\mu,\alpha)\,[|\alpha|]\, S^\alpha[X^{tq}]\;. \; 3.6$$

The action of the Macdonald operator M_1 is triangular on the basis, $S_\lambda[X^{tq}]$, giving that

$$M_1\, S_\mu[X^{tq}] \;=\; [|\mu|]\, S^\mu[X^{tq}] + \sum_\alpha \epsilon(\mu,\alpha)\, S^\nu[X^{tq}]\,, \qquad 3.7$$

where the sum is over all α a permutation of partitions $\nu > \mu$. Formula 3.7 demonstrates that the eigenspaces of M_1 are 1-dimensional and reveals the eigenvalues.

For any pair of partitions $\mu, \nu \in \mathbb{N}^n$ of the same weight, let us write

$$[\lambda]_{\mu\nu}(q,t) \;=\; \sum_\alpha \epsilon(\mu,\alpha)\left([|\lambda|] - [|\alpha|]\right)\,, \qquad 3.8$$

where α runs over all distinct permutations of $\nu \in \mathbb{N}^n$. Note that, in the following, $[\lambda]_{\mu\nu}$ shall stand for $[\lambda]_{\mu\nu}(q,t)$.

Theorem 3.1 *Expanded in terms of $S_\lambda[X^{tq}]$, the Macdonald polynomials are*

$$J_\lambda(X;q,t) = \frac{c_{\lambda'}(t,q)}{v_\lambda(q,t)} \det \begin{vmatrix} S_\lambda[X^{tq}] & \dots & S_\mu[X^{tq}] & \dots \\ \vdots & & \vdots & \\ [\lambda]_{\lambda\nu} & \dots & [\lambda]_{\mu\nu} & \dots \\ \vdots & & \vdots & \end{vmatrix}_{\substack{\nu>\lambda \\ \mu\geq\lambda}} . \qquad 3.9$$

Columns are indexed by all partitions $\mu \geq \lambda$. Aside from the first row, the entries are the polynomials in q, t, $[\lambda]_{\mu\nu}$, $\nu > \lambda$. The normalization factor consists of

$$v_\lambda(q,t) = \prod_{\mu>\lambda}\Big([|\lambda|] - [|\mu|]\Big) = \prod_{\mu>\lambda}[\lambda]_{\mu\mu}\,, \qquad 3.10$$

and

$$c_\lambda(q,t) \;=\; \prod_{s\in\lambda}\Big(1 - q^{a(s)}t^{l(s)+1}\Big)\,. \qquad 3.11$$

For example, up to a scalar,

$$J_{2,2,1}(X;q,t) \doteq \begin{array}{|ccccc|l} S_{2,2,1}[X^{tq}] & S_{3,1,1}[X^{tq}] & S_{3,2}[X^{tq}] & S_{4,1}[X^{tq}] & S_5[X^{tq}] & \\ -[1,3,1] & [3,1,1] & 0 & 0 & 0 & {\scriptstyle (3,1,1)} \\ -[2,0,3] & -[3,0,2] & [3,2,0] & 0 & 0 & {\scriptstyle (3,2,0)} \\ [1,0,4] & -[0,4,1] & -[1,4,0] & [4,1,0] & 0 & {\scriptstyle (4,1,0)} \\ 0 & [0,0,5] & 0 & -[0,5,0] & [5,0,0] & {\scriptstyle (5,0,0)} \end{array} \qquad 3.12$$

where an entry $\pm[\alpha]$ stands for $\pm\big([|2,2,1|] - [|\alpha|]\big)$. Each column of such determinant is obtained from the ordered expansion of a Schur function. For example, the second column is given by the expansion of $S_{3,1,1}$ computed in 3.4. Notice we can limit ourselves to vectors of length $\ell(\lambda)$ since each entry is the difference of two vectors of which the last $n - \ell(\lambda)$ components are equal.

Proof of Theorem 3.1 It suffices to check that $\Big([|\lambda|] - M_1\Big) J'_\lambda(X;q,t) = 0$, where $J'_\lambda(X;q,t)$ is the determinantal expression 3.9, given that the eigenvalues are distinct for each λ. The operator acts only on the first row of the determinant allowing us to use formula 3.6 for the action of M_1 on $S_\mu[X^{tq}]$. This gives

$$\big([|\lambda|] - M_1\big) J'_\lambda(X;q,t) \quad = \quad \det \begin{vmatrix} \sum_{\beta\geq\lambda}[\lambda]_{\lambda\beta}\, S^\beta & \dots & \sum_{\beta\geq\mu}[\lambda]_{\mu\beta}\, S^\beta & \dots \\ \vdots & & \vdots & \\ [\lambda]_{\lambda\nu} & \dots & [\lambda]_{\mu\nu} & \dots \\ \vdots & & \vdots & \end{vmatrix} . \tag{3.13}$$

The determinant can be shown to vanish by examining the coefficient of all S^β's. The coefficient of S^λ is proportional to $[\lambda]_{\lambda\lambda} = 0$. For $\beta > \lambda$, the coefficient of S^β is equal to the determinant

$$\det \begin{vmatrix} [\lambda]_{\lambda\beta} & \dots & [\lambda]_{\mu\beta} & \dots \\ \vdots & & \vdots & \\ [\lambda]_{\lambda\beta} & \dots & [\lambda]_{\mu\beta} & \dots \\ \vdots & & \vdots & \end{vmatrix} \begin{matrix} \text{row } 1 \\ \\ \\ \text{row } \beta \\ \\ \end{matrix} \quad , \tag{3.14}$$

which has two identical rows, and thus also vanishes.

To verify the normalization, we recall [M1]

$$J_\lambda(X;q,t) \quad = \quad c_{\lambda'}(t,q)\, S_\lambda[X^{tq}] + \sum_{\mu>\lambda} c_{\lambda\mu} S_\mu[X^{tq}] \tag{3.15}$$

and observe that because, from 3.7, $[\lambda]_{\mu\nu} = 0$ for $\mu > \nu$, the sub-determinant giving the coefficient of $S_\lambda[X^{tq}]$ in $J'_\lambda(X;q,t)$ is triangular and equals $v_\lambda(q,t)$. $\square$

The formula for $J_\lambda(X;q,t)$ in terms of $S_\lambda[X^{tq}]$ can be converted into a similar expression in terms of the $S_\mu[X]$ basis by applying to expression 3.9, the involution $\omega_{q,t}$ satisfying properties [M1]:

$$\omega_{q,t} S_\lambda[X^{tq}] = S_{\lambda'}[X] \qquad \text{and} \qquad \omega_{q,t} J_\lambda(X;q,t) \doteq J_{\lambda'}(X;t,q), \quad \text{up to a scalar} .$$

Renormalizing using

$$J_\lambda(X;q,t) \quad = \quad c_\lambda(q,t)\, S_\lambda[X] + \sum_{\mu<\lambda} c_{\lambda\mu}\, S_\mu[X] \, , \tag{3.16}$$

we have

Corollary 3.2

$$J_\lambda(X;q,t) \;=\; \frac{c_\lambda(q,t)}{u_\lambda(t,q)} \det \left| \begin{matrix} S_\lambda[X] & \dots & S_\mu[X] & \dots \\ \vdots & & \vdots & \\ [\lambda']_{\lambda'\nu'}(t,q) & \dots & [\lambda']_{\mu'\nu'}(t,q) & \dots \\ \vdots & & \vdots & \end{matrix} \right|_{\substack{\nu<\lambda \\ \mu\leq\lambda}} , \qquad 3.17$$

where $u_\lambda(q,t) = \prod_{\mu<\lambda}\Big([|\lambda'|] - [|\mu'|]\Big)$.

The first row is the list of Schur fonctions S_μ, $\mu \leq \lambda$, the other entries are the polynomials $[\lambda']_{\mu'\nu'}(t,q)$ for $\nu < \lambda$.

A determinantal formula for the Macdonald polynomials in terms of the monomial basis may now be obtained from formula 3.17 . The interpretation of the entries of the matrix in 3.17 (aside from the top row) as scalar products will allow that we pass to other bases of symmetric functions.

We use the space of polynomials in $x_1, \dots, x_n$ as a free module over the ring of symmetric polynomials. Under the scalar product, $\langle f, g\rangle = \partial_\omega(fg)$, this space has the following two adjoint bases

$$\big\{\bar{x}^I = (-x_1)^{I_1}\cdots(-x_m)^{I_m}\big\}_{0^m\subseteq I\subseteq\rho} \quad \text{and} \quad \big\{E_I = e_{I_1}[0]e_{I_2}[X_1]\cdots e_{I_m}[X_{m-1}]\big\}_{0^m\subseteq I\subseteq\rho}, \qquad 3.18$$

for $\rho = (0,\dots,m-1)$ and $0^m = (0,\dots,0) \in \mathbb{N}^m$. That is,

$$\langle \bar{x}^I, E_{\rho-J}\rangle = \partial_\omega\Big(\bar{x}^I E_{\rho-J}\Big) = \delta_{IJ}, \qquad I, J \subseteq \rho\,. \qquad 3.19$$

This result can be used to determine the coefficients in the following matrix;

$$S_{\lambda'}[X_n] = \det \left| \begin{matrix} e_{\lambda_1}[X_n] & e_{\lambda_1+1}[X_{n+1}] & \cdots & e_{\lambda_1+n-1}[X_{2n-1}] \\ e_{\lambda_2-1}[X_n] & e_{\lambda_2}[X_{n+1}] & \cdots & e_{\lambda_2+n-2}[X_{2n-1}] \\ \vdots & \vdots & \ddots & \vdots \\ e_{\lambda_n-n+1}[X_n] & e_{\lambda_n-n+2}[X_{n+1}] & \cdots & e_{\lambda_n}[X_{2n-1}] \end{matrix} \right| , \qquad 3.20$$

which is obtained by applying the ω involution, $\omega S_k[X] = e_k[X]$, $\omega S_\lambda[X] = S_{\lambda'}[X]$, to 1.3 and increasing the alphabets in each column with the relation, $e_k[X_l + x_{l+1} + \cdots + x_{2n-1}] = e_k[X_l] + \sum_{j=1}^{2n-l-1} e_{k-j}[X_l]\, e_j[x_{l+1} + \cdots + x_{2n-1}]$. Note that this relation does not change the value of the determinant since it corresponds to multiplication by a unitriangular matrix. Now using 3.19 with $m = 2n$, the coefficients in the expansion of this matrix are

$$S_{\lambda'}[X_n] = \sum_{\alpha' \in \rho'} \langle \bar{x}^{\rho-\alpha'}, S_{\lambda'}[X_n] \rangle \, E_{\alpha'} \,, \qquad 3.21$$

where $\alpha' = (0, \ldots, 0, \alpha_1, \ldots, \alpha_n)$ and $\rho' = (0, \ldots, 0, n, \ldots, 2n-1)$ for $\alpha', \rho' \in \mathbb{N}^{2n}$. Since the ordered expansion of 3.20, and thus 3.21, must coincide with the ordered expansion 3.3 of $S_\lambda[X]$ in terms of complete functions, we have

$$\epsilon(\lambda, \alpha) = \langle \bar{x}^{\rho-\alpha'}, S_{\lambda'}[X_n] \rangle \,. \qquad 3.22$$

This implies that 3.17 can be rewritten, using

$$\langle f \rangle_{\lambda\nu} \;=\; \sum_{\alpha} \langle \bar{x}^{\rho-\alpha'}, f \rangle \left([|\lambda|]_{t,q} - [|\alpha|]_{t,q} \right) \,, \qquad 3.23$$

where the sum is over all distinct permutations α of $\nu \in \mathbb{N}^n$, as

$$J_\lambda(X_n; q,t) \;=\; \frac{c_\lambda(q,t)}{u_\lambda(t,q)} \det \left| \begin{matrix} S_\lambda[X_n] & \ldots & S_\mu[X_n] & \ldots \\ \vdots & & \vdots & \\ \langle S_\lambda[X_n] \rangle_{\lambda'\nu'} & \ldots & \langle S_\mu[X_n] \rangle_{\lambda'\nu'} & \ldots \\ \vdots & & \vdots & \end{matrix} \right|_{\substack{\mu \leq \lambda \\ \nu < \lambda}} \,. \qquad 3.24$$

With $\epsilon(\mu, \alpha)$ now defined as a scalar product, we can write Macdonald polynomials in any linear basis of the space generated by $\{S_\mu\}_{\mu \leq \lambda}$. In particular, we have the following theorem:

Theorem 3.3 *The Macdonald polynomials in terms of monomial symmetric functions are*

$$J_\lambda(X_n; q,t) \;=\; \frac{c_\lambda(q,t)}{u_\lambda(t,q)} \det \left| \begin{matrix} m_\lambda[X_n] & \ldots & m_\mu[X_n] & \ldots \\ \vdots & & \vdots & \\ \langle m_\lambda[X_n] \rangle_{\lambda'\nu'} & \ldots & \langle m_\mu[X_n] \rangle_{\lambda'\nu'} & \ldots \\ \vdots & & \vdots & \end{matrix} \right|_{\substack{\mu \leq \lambda \\ \nu < \lambda}} \,. \qquad 3.25$$

For example, $J_{2,2}(X; q,t)$ is given by

$$J_{2,2}(X_4; q,t) = \frac{c_{2,2}(q,t)}{u_{2,2}(t,q)} \det \left| \begin{matrix} m_{2,2}[X_4] & m_{2,1,1}[X_4] & m_{1,1,1,1}[X_4] \\ -[3,1] - [1,3] & [3,1] & 0 \\ [4,0] + [0,4] & -3[4,0] - [0,4] & [4,0] \end{matrix} \right| , \qquad 3.26$$

with $[\alpha] = \left([|2,2|]_{t,q} - [|\alpha|]_{t,q} \right)$. In this expression, the term $-3[4,0]$ appears, for instance, because $\langle \bar{x}^{(0,1,2,3,0,5)}, m_{2,1,1}[X_4] \rangle = -3$. Again, we work with vectors of length $\ell(\lambda)$, since adding zeros to α does not change the result. We skip the problem of computing efficiently all the scalar products in the matrix.

4. Creation Operators

A Macdonald polynomial associated to any partition can be constructed by repeated application of creation operators, $B_k^{(n)}$. Specifically,

$$B_k^{(n)} \, J_\lambda(X;q,t) \;=\; J_{\lambda+1^k}(X;q,t)\,. \tag{4.1}$$

The creation operators were defined originally to be [LV],[KN]

$$B_k^{(n)} \;=\; \sum_{|I|=k} \sum_{l=0}^{k} (-t)^l x^I \, \frac{R(X_I, X_I^c/t)}{R(X_I, X_I^c)} \, M_l^{(I)}\,, \tag{4.2}$$

where I is a k subset of $\{1,\ldots,n\}$ and $M^{(I)}$ acts only on x_i for all $i \in I$. We will see that the action of these operators on the modified Schur function basis may be expressed in a manner similar to that of the Macdonald operators on this basis, though the former increases degrees. We begin by giving a new expression for the creation operators.

Proposition 4.1 *The creation operators acting on the space of symmetric functions are*

$$B_k^{(n)} \;=\; \frac{1}{k_t!}\, \chi_{(k|n-k)}\, \chi_{\omega(k)}\, x_1 \cdots x_k \,(1-t\Omega_k)\cdots(1-t^k\Omega_k) \tag{4.3}$$

where $\Omega_k = \sigma_1 \cdots \sigma_{k-1}\Theta_k$.

Proof The binomial expansion of the right hand side of expression 4.3 becomes

$$rhs \;=\; \frac{1}{k_t!} \sum_{l=0}^{k} \frac{(-1)^l t^{\binom{l+1}{2}} (t)_k}{(t)_{k-l}(t)_l}\, \chi_{(k|n-k)}\, x_1\cdots x_k\, \chi_{\omega(k)}\, \Theta_l \cdots \Theta_1\,, \tag{4.4}$$

since $\Omega_k^l = \Theta_l \cdots \Theta_1$ on symmetric functions. Further, the image of such a function under Ω_k^l belongs to $\mathcal{S}ym(l|k-l)$ allowing that we use property 1.12 of $\chi_{\omega(k)}$ to obtain

$$rhs \;=\; \frac{1}{k_t!} \sum_{l=0}^{k} \frac{(-1)^l t^{\binom{l+1}{2}+l(k-l)} (t)_k}{(t)_{k-l}(t)_l}\, l_t!(k-l)_t! \chi_{(k|n-k)} x_1\cdots x_k\, \chi_{(l|k-l)}\, \Theta_l\cdots\Theta_1\,. \tag{4.5}$$

The definition of the Macdonald operators and $\chi_{(k|n-k)}$ then imply that

$$rhs \;=\; \sum_{l=0}^{k} (-t)^l \partial_{(k|n-k)}\, R(X_k, X_k^c/t)\, x_1\cdots x_k\, M_l^{(k)}\,. \tag{4.6}$$

Finally, the action of $\partial_{(k|n-k)}$ as described in 1.8 confirms that this expression, and therefore 4.3, are in fact equivalent to the original definition for the creation operators. □

Theorem 4.2 *The action of $B_k^{(n)}$ on $S_\lambda[X^{tq}]$, for $\ell(\lambda) \le k$, can be expressed as*

$$B_k^{(n)}\, S_\lambda[X^{tq}] \;=\; \prod_{i=1}^{k} \left(\begin{bmatrix} \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \end{bmatrix} - t^{k-i} \begin{bmatrix} \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & X^t & 0 & \cdots \end{bmatrix} \right) S_\lambda[X^{tq}], \quad 4.7$$

where the i^{th} column is the only non-zero column of the operators.
The relation, $S_j[X^{tq} + X^t] = S_j[qX^{tq}] = q^j S_j[X^{tq}]$, yields immediately

Corollary 4.3 *Let $f(i,j) = \lambda_i + (j - i + 1)$. For any $\ell(\lambda) \le k$ we have*

$$B_k^{(n)}\, S_\lambda[X^{tq}] \;=\; \left| (1 - q^{f(i,j)} t^{k-j}) S_{f(i,j)}[X^{tq}] \right|_{i,j\le k} . \qquad 4.8$$

For example, when $k = 2$, we have

$$B_2^{(n)}\, S_{\lambda_1,\lambda_2}[X^{tq}] \;=\; \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - t \begin{bmatrix} 1 & 0 \\ X^t & 0 \end{bmatrix} \right) \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - t^0 \begin{bmatrix} 0 & 1 \\ 0 & X^t \end{bmatrix} \right) S_{\lambda_1,\lambda_2}[X^{tq}]$$

$$= \begin{vmatrix} (1 - q^{\lambda_1+1} t) S_{\lambda_1+1}[X^{tq}] & (1 - q^{\lambda_1+2}) S_{\lambda_1+2}[X^{tq}] \\ (1 - q^{\lambda_2} t) S_{\lambda_2}[X^{tq}] & (1 - q^{\lambda_2+1}) S_{\lambda_2+1}[X^{tq}] \end{vmatrix} .$$

Proof of Theorem 4.2 In what follows, as we have done in the case of the Macdonald operators, we shall use Θ^t instead of Θ in the operator B_k. This will allow us to use the action of M_k on $S_\lambda[Y + X]$ (with Y an alphabet invariant under Θ^t) obtained in the previous section. Letting $Y = X^{tq} - X$ in the final result will prove the theorem.

First, notice from 4.3 that $B_k^{(n)} = \chi_{(k|n-k)} B_k^{(k)}$. This given, we will first describe the action of $B_k^{(k)}$ on $S_\lambda[Y + X]$.

Claim 4.4 *For $\ell(\lambda) \le k$ and $\alpha_i \to \lambda_i - i + 1$, we have*

$$\begin{aligned} B_k^{(k)}\, S_\lambda[Y + X] \;&=\; \prod_{i=1}^{k} \left(\begin{bmatrix} \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \end{bmatrix} - t^{k-i} \begin{bmatrix} \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & X_k^t & 0 & \cdots \end{bmatrix} \right) S_\lambda[Y + X] \\ &=\; \left| G_{\alpha_i+1}(t^{k-1}, X_k) G_{\alpha_i+2}(t^{k-2}, X_k) \cdots G_{\alpha_i+k}(t^0, X_k) \right|_{i=1..k} . \end{aligned} \qquad 4.9$$

where $G_h(t^r, X_k) = S_h[Y + X] - t^r S_h[Y + X + X_k^t]$.

Proof of Claim With $Y' = Y + X_k^c$, expression 4.2, where $n = k$, yields

$$B_k^{(k)}\, S_\lambda[Y + X] = x_1 \cdots x_k \sum_{l=0}^{k} (-t)^l M_l^{(k)} S_\lambda[Y' + X_k], \qquad 4.10$$

which, using the Macdonald operator action on $S_\lambda[Y'+X_k]$, gives

$$B_k^{(k)}\, S_\lambda[Y+X] = x_1\cdots x_k \sum_{l=0}^{k}(-t)^l \sum_{1\le i_1<\cdots<i_l\le k} t^{(n-i_1)+\cdots+(n-i_l)} \times \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & X_k^t & \cdots & X_k^t & \cdots & 0 \end{bmatrix} S_\lambda[Y'+X_k]\,. \tag{4.11}$$

In the last expression, for a fixed l, X^t occurs in positions $i_1, ..., i_l$. Comparing such terms, we obtain

$$B_k^{(k)}\, S_\lambda[Y+X] = x_1\cdots x_k \prod_{i=1}^{k}\left(\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0\end{bmatrix} - t^{k+1-i}\begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & X_k^t & 0 & \cdots & 0\end{bmatrix}\right) S_\lambda[Y+X]\,, \tag{4.12}$$

or more compactly

$$B_k^{(k)}\, S_\lambda[Y+X] \;=\; x_1\cdots x_k \left| G_{\alpha_i}(t^k, X_k) G_{\alpha_i+1}(t^{k-1}, X_k)\cdots G_{\alpha_i+k-1}(t^1, X_k)\right|_{i=1..k}. \tag{4.13}$$

The proof of the claim is completed using the following lemma(proved in the appendix):

Lemma 4.5 *For $F_j(t^r, X_k) = S_j[C+X_k] - t^r S_j[C+tX_k]$, we have*

$$\left|F_{j+1}(t^{k-1}, X_k)F_{j+2}(t^{k-2}, X_k)\ldots F_{j+k}(1, X_k)\right| = x_1\cdots x_k \left|F_j(t^k, X_k)F_{j+1}(t^{k-1}, X_k)\ldots F_{j+k-1}(t, X_k)\right|, \tag{4.14}$$

where the expressions between brackets must be understood as $k\times k$ determinants.
In effect, letting $C\to Y'$, we have $F_j(t^k, X_k)\to G_j(t^k, X_k)$, and thus using 4.14 in 4.13 proves the claim. □

Expression 4.9 gives that the action of $B_k^{(n)} = \chi_{(k|n-k)}\, B_k^{(k)}$ on $S_\lambda[Y+X]$ is

$$B_k^{(n)}\, S_\lambda[Y+X] \;=\; \chi_{(k|n-k)}\left|G_{\alpha_i+1}(t^{k-1}, X_k)\ldots G_{\alpha_i+k}(1, X_k)\right|_{i=1..k}. \tag{4.15}$$

The proof of Theorem 4.2 is now equivalent to showing that $\chi_{(k|n-k)}$ acts on this determinant by extending the alphabets from X_k to X. This will be achieved with the following lemma (proved in the appendix);

Lemma 4.6 *For any $l\ge k$, we have*

$$\left|G_{j+1}(t^{k-1}, X_l)G_{j+2}(t^{k-2}, X_l)\ldots G_{j+k}(1, X_l)\right| = \chi^{(1)}_{(1|l-1)}\cdots\chi^{(k-1)}_{(1|l-k+1)}\chi^{(k)}_{(1|l-k)}\left|G_{j+1}(t^{k-1}, X_k)G_{j+2}(t^{k-2}, X_{k-1})\ldots G_{j+k}(1, X_1)\right|, \tag{4.16}$$

where, again, the expression between brackets are $k \times k$ determinants.

Application of this lemma to expression 4.15, with $l = k$, gives

$$
\begin{aligned}
B_k^{(n)}\, S_\lambda[Y+X] \;&=\; \\
&\chi_{(k|n-k)}\, \chi^{(1)}_{(1|k-1)} \cdots \chi^{(k-1)}_{(1|1)} \Big| G_{\alpha_i+1}(t^{k-1}, X_k) G_{\alpha_i+2}(t^{k-2}, X_{k-1}) \ldots G_{\alpha_i+k}(1, X_1) \Big|_{i=1,\ldots,k} .
\end{aligned}
\qquad 4.17
$$

Property 1.14 allows a refactorization of χ that transforms 4.17 into

$$
\begin{aligned}
B_k^{(n)}\, S_\lambda[Y+X] \;&=\; \\
&\chi^{(1)}_{(1|n-1)} \cdots \chi^{(k)}_{(1|n-k)} \left| G_{\alpha_i+1}(t^{k-1}, X_k) G_{\alpha_i+2}(t^{k-2}, X_{k-1}) \ldots G_{\alpha_i+k}(1, X_1) \right|_{i=1,\ldots,k} .
\end{aligned}
\qquad 4.18
$$

Lemma 4.6 may be applied again, with $l = n$, giving

$$
B_k^{(n)}\, S_\lambda[Y+X] \;=\; \left| G_{\alpha_i+1}(t^{k-1}, X) G_{\alpha_i+2}(t^{k-2}, X) \ldots G_{\alpha_i+k}(1, X) \right|_{i=1,\ldots,k} , \qquad 4.19
$$

which proves Theorem 4.2. □

Appendix

Proof of Lemma 1.1 Using formula 1.2, we have

$$
S_j[X^{tq} + X_k^t] \;=\; S_j[D + x_k^t] \;=\; \sum_{l=0}^{j} S_{j-l}[D]\, S_l[x_k^t] , \qquad A.1
$$

where $D = X^{tq} + X_{k-1}^t$. Further properties of Schur functions yield, as $x_k^t = tx_k - x_k$,

$$
\begin{aligned}
S_j[X^{tq} + X_k^t] \;&=\; S_j[D] \;+ \sum_{l=1}^{j} (1 - 1/t) S_{j-l}[D]\, (tx_k)^l \\
&=\; S_j[D] \;+ \sum_{l=0}^{j-1} t(x_k - x_k/t) S_{j-l-1}[D]\, (tx_k)^l .
\end{aligned}
\qquad A.2
$$

The alphabet D is invariant under permutations of $x_k, \ldots, x_n$, and thus the first term is invariant up to a constant under $\chi^{(k)}_{(1|n-k)}$. Thus, using 1.10, we have

$$
\chi^{(k)}_{(1|n-k)} S_j[X^{tq} + X_k^t] \;=\; (1 + t^{-1} + \cdots + t^{-n+k}) S_j[D] + \chi^{(k)}_{(1|n-k)} \sum_{l=0}^{j-1} t(x_k - x_k/t) S_{j-l-1}[D]\, (tx_k)^l , \qquad A.3
$$

and are left only to show

$$\chi^{(k)}_{(1|n-k)} \sum_{l=0}^{j-1} t(x_k - x_k/t) S_{j-l-1}[D]\,(tx_k)^l \quad = \quad -S_j[D] + S_j[X^{tq} + X^t]\,. \qquad A.4$$

The definition of $\chi^{(k)}_{(1|n-k)}$ and $(x_k - x_k/t)R(x_k, X^c_k/t) = R(x_k, X^c_{k-1}/t)$ convert the left hand side of this expression into

$$lhs \quad = \quad \partial_{n-1}\cdots\partial_k\, R(x_k, X^c_{k-1}/t) \sum_{l=0}^{j-1} tS_{j-l-1}[D](tx_k)^l\,. \qquad A.5$$

The identity $S_n[x - X/t] = R(x, X/t)$ may be used to eliminate the resultant, and we obtain

$$lhs \quad = \quad \partial_{n-1}\cdots\partial_k \sum_{l=0}^{j-1} tS_{j-l-1}[D](tx_k)^l S_{n-k}[x_k - X^c_{k-1}/t]\,. \qquad A.6$$

A further Schur function property allows that the factor x^l_k be used to increase the index of S_{n-k};

$$lhs = \partial_{n-1}\cdots\partial_k \sum_{l=0}^{j-1} t^{l+1} S_{j-l-1}[D] S_{n-k+l}[x_k - X^c_{k-1}/t]\,. \qquad A.7$$

Now we can let the divided differences act (see [L1] for similar computations). Using the property $\partial_k S_l[x_k - X^c_{k-1}/t] = S_{l-1}[x_k + x_{k+1} - X^c_{k-1}/t]$, we arrive at the expression,

$$\begin{aligned} lhs = \sum_{l=0}^{j-1} t^{l+1} S_{j-l-1}[D] S_{l+1}[x_k + \cdots + x_n - X^c_{k-1}/t] &= \sum_{l=0}^{j-1} S_{j-l-1}[D] S_{l+1}[(X^c_{k-1})^t] \\ &= -S_j[D] + S_j[D + (X^c_{k-1})^t]\,. \end{aligned} \qquad A.8$$

Clearly $D + (X^c_{k-1})^t = X^{tq} + X^t$, which implies that expression A.4 holds. □

Proof of formula 2.11 We want to show that

$$\begin{aligned} &S_\lambda[Y + X] R(X_k, X^c_k) \\ &= \Big| S_{\alpha_i - k + n}[Y + X_k] \cdots S_{\alpha_i - 1 + n}[Y + X_k] S_{\alpha_i + k}[Y + X^c_k] \cdots S_{\alpha_i + n - 1}[Y + X^c_k] \Big|_{i=1..n}\,, \end{aligned} \qquad A.9$$

where $\alpha_i \to \lambda_i - i + 1$. Let f_i, $i = 1, \ldots, n$, denote arbitrary one variable polynomials in the space of polynomials in $x_1, \ldots, x_n$ taken as a free module over the ring of symmetric polynomials. From the usual Newton's interpolation in one variable ([L1], Lemma Ni5), given such f_i, we have

$$\det\, \big|\partial_{j-1}\ldots\partial_1 f_i(x_1)\big|_{1\le i,j\le n} \quad = \quad \frac{1}{\Delta(X)}\, \det\, \big|f_i(x_j)\big|_{1\le i,j\le n}, \qquad A.10$$

where $\Delta(X)=\prod_{i<j}(x_j-x_i)$ is the Vandermonde determinant. If in this expression we let $f_i(x)=S_{\alpha_i+n-1}[Y+x]$, using the identity $\partial_k S_j[Y+X_k]=S_{j-1}[Y+X_{k+1}]$, we obtain

$$\begin{aligned}\frac{1}{\Delta(X)}\ \det\left|S_{\alpha_i+n-1}[Y+x_j]\right|_{1\le i,j\le n} &= \det\left|\partial_{j-1}\ldots\partial_1 S_{\alpha_i+n-1}[Y+x_1]\right|_{1\le i,j\le n} \\ &= \det\left|S_{\alpha_i+n-j}[Y+X_j]\right|_{1\le i,j\le n}.\end{aligned} \qquad A.11$$

The alphabets can be increased in each column of the determinant on the right hand side of this expression by using the relation

$$S_{\alpha_i+n-j}[Y+X_j]=S_{\alpha_i+n-j}[Y+X]+\sum_{\ell=1}^{n-j}S_{\alpha_i+n-j-\ell}[Y+X]S_\ell[-X_j^c]\ ,$$

giving that

$$\det\left|S_{\alpha_i+n-j}[Y+X_j]\right|_{1\le i,j\le n}=\det\left|S_{\alpha_i+n-j}[Y+X]\right|_{1\le i,j\le n}\ . \qquad A.12$$

The right hand side may thus be substituted into formula $A.11$ implying that

$$S_\lambda[Y+X]\Delta(X)=\left|S_{\alpha_i+n-1}[Y+x_n]S_{\alpha_i+n-1}[Y+x_{n-1}]\cdots S_{\alpha_i+n-1}[Y+x_1]\right|_{i=1..n}\ . \qquad A.13$$

This result applied to arbitrary minors of order $n-k$ in the first $n-k$ columns and of order k in the last k columns of the right hand side of $A.13$, gives by linearity,

$$\begin{aligned}&S_\lambda[Y+X]\Delta(X)\\ &\quad=\left|S_{\alpha_i+n-1}[Y+x_n]\cdots S_{\alpha_i+n-1}[Y+x_{k+1}]S_{\alpha_i+n-1}[Y+x_k]\cdots S_{\alpha_i+n-1}[Y+x_1]\right|_{i=1..n}\\ &\quad=\left|S_{\alpha_i+k}[Y+X_k^c]\cdots S_{\alpha_i+n-1}[Y+X_k^c]S_{\alpha_i-k+n}[Y+X_k]\cdots S_{\alpha_i+n-1}[Y+X_k]\right|\\ &\qquad\qquad\times\Delta(X_k^c)\Delta(X_k).\end{aligned} \qquad A.14$$

Finally, with $\Delta(X)/\big(\Delta(X_k)\Delta(X_k^c)\big)=R(X_k,X_k^c)(-1)^{k(n-k)}$, one sees that $A.14$ is equivalent to $A.9$, which proves the assertion. □

Proof of Lemma 4.5 As a result of $\partial_i S_j[A+b_i]=S_{j-1}[A+b_i+b_{i+1}]$, we have

$$\partial_i\, F_j(t^r,A_i)\ =\ F_{j-1}(t^{r+1},A_{i+1})\ , \qquad A.15$$

allowing that the left hand side of 4.14 be rewritten as

$$lhs\ =\ \left|\partial_{k-1}..\partial_1F_{j+k}(1,X_1),\partial_{k-1}..\partial_2F_{j+k}(1,X_2),\ldots,\partial_{k-1}F_{j+k}(1,X_{k-1}),F_{j+k}(1,X_k)\right|\ . \qquad A.16$$

We can factor out $\partial_{k-1}\cdots\partial_1$, since all the columns, except the first one are symmetrical in $x_1,\ldots,x_k$. Similarly we can factor out successively $(\partial_{k-1}\cdots\partial_2),\ldots,\partial_{k-1}$ to get

$$lhs \;=\; (\partial_{k-1}..\partial_1)(\partial_{k-1}..\partial_2)\cdots(\partial_{k-1})\Big|F_{j+k}(1,X_1),\ldots,F_{j+k}(1,X_{k-1}),F_{j+k}(1,X_k)\Big| \,. \quad A.17$$

With $F_j(t^r,X_k) = x_kF_{j-1}(t^{r+1},X_k) + F_j(t^r,X_{k-1})$, we can substitute $x_kF_{j+k-1}(t,X_k) + F_{j+k}(1,X_{k-1})$ in the last column of the matrix. This produces a sum of two determinants, of which the second vanishes thanks to column $k-1$. We thus obtain

$$lhs \;=\; (\partial_{k-1}..\partial_1)(\partial_{k-1}..\partial_2)\cdots(\partial_{k-1})\Big|F_{j+k}(t^0,X_1),\ldots,F_{j+k}(t^0,X_{k-1}),x_kF_{j+k-1}(t,X_k)\Big| \,, \quad A.18$$

and by iteration;

$$lhs = (\partial_{k-1}..\partial_1)\cdots(\partial_{k-1})\Big|x_1F_{j+k-1}(t,X_1),\ldots,x_{k-1}F_{j+k-1}(t,X_{k-1}),x_kF_{j+k-1}(t,X_k)\Big| \,. \quad A.19$$

The symmetric factor $x_1\cdots x_k$ can be factored out of the determinant and the divided differences may be reintroduced;

$$lhs \;=\; x_1\cdots x_k\Big|\partial_{k-1}\cdots\partial_1F_{j+k-1}(t,X_1),\ldots,\partial_{k-1}F_{j+k-1}(t,X_{k-1}),F_{j+k-1}(t,X_k)\Big| \,. \quad A.20$$

Transformation on columns of this determinant using relation A.15 completes the proof. □

Proof of Lemma 4.6 This identity, as it is a consequence of a sucession of transformations performed on the columns, can be proven by induction on k given that

$$G_j(t^r,X_l) \;=\; \chi^{(k)}_{(1|l-k)}\,G_j(t^r,X_k) - c(k)\,G_j(t^r,X_{k-1})\,, \quad A.21$$

which is a direct consequence of lemma 1.1. If $k=1$, $G_{j+1}(t^0,X_0)=0$ implies that

$$G_{j+1}(1,X_l) \;=\; \chi^{(1)}_{(1|l-1)}\,G_{j+1}(1,X_1)\,, \quad A.22$$

which is the case $k=1$ of the lemma. Assuming that the lemma holds for a $k-1$ determinant, we get

$$\begin{aligned}&\Big|G_{j+1}(t^{k-1},X_l)G_{j+2}(t^{k-2},X_l)\ldots G_{j+k}(1,X_l)\Big| \\ &\quad= \chi^{(1)}_{(1|l-1)}\cdots\chi^{(k-1)}_{(1|l-k+1)}\Big|G_{j+1}(t^{k-1},X_l)G_{j+2}(t^{k-2},X_{k-1})\ldots G_{j+k}(1,X_1)\Big| \,.\end{aligned} \quad A.23$$

The first column becomes $\chi^{(k)}_{(1|l-k)}G_{j+1}(t^{k-1},X_k) - c(k)\,G_{j+1}(t^{k-1},X_{k-1})$, by relation A.21, producing a sum of two determinants. As the first of these is exactly the desired result, the second must be shown to vanish. Specifically, it must be that

$$\chi^{(1)}_{(1|l-1)}\cdots\chi^{(k-1)}_{(1|l-k+1)}\Big|G_{j+1}(t^{k-1},X_{k-1})G_{j+2}(t^{k-2},X_{k-1})\ldots G_{j+k}(1,X_1)\Big| = 0\,. \quad A.24$$

The left hand side of this expression may be transformed first by using property 1.14 to refactor $\chi^{(1)}_{(1|l-1)}\cdots\chi^{(k-1)}_{(1|l-k+1)}$ into the form $\chi_{(k-1|n-k+1)}\chi^{(1)}_{(1|k-2)}\cdots\chi^{(k-2)}_{(1|1)}$, and then by using the induction hypothesis, where $l=k-1$;

$$\begin{aligned} lhs &= \chi_{(k-1|n-k+1)}\chi^{(1)}_{(1|k-2)}\cdots\chi^{(k-2)}_{(1|1)}\Big|G_{j+1}(t^{k-1},X_{k-1})G_{j+2}(t^{k-2},X_{k-1})\ldots G_{j+k}(1,X_1)\Big| \\ &= \chi_{(k-1|n-k+1)}\Big|G_{j+1}(t^{k-1},X_{k-1})G_{j+2}(t^{k-2},X_{k-1})\ldots G_{j+k}(1,X_{k-1})\Big|\,. \end{aligned} \qquad A.25$$

We now have $\chi_{(k-1|n-k+1)}$ applied to a determinant with k columns that satisfies the special case of $x_k=0$ in Lemma 4.5. Thus, the determinant vanishes, proving the lemma. □

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References

[GH] A.M. Garsia and M. Haiman, *A graded representation module for Macdonald's polynomials*, Proc. Natl. Acad. Sci. USA V **90** (1993) 3607–3610.

[H] F. Hirzebruch, *Topological methods in algebraic geometry*, 3rd ed., Springer, 1966.

[HBJ] F. Hirzebruch, T. Berger and R. Jung, *Manifolds and Modular Forms*, Aspects of Mathematics, Vieweg, Braunschweig/Wiesbaden, 1992.

[KN] A. Kirillov and M. Noumi, *Affine Hecke algebras and raising operators for Macdonald polynomials*, Duke Math. J., to appear.

[L1] A. Lascoux, *Notes on interpolation in one and several variables*, http://phalanstere.univ-mlv.fr/~al/.

[L2] A. Lascoux, *Fonctions de Schur et grasmanniennes*, C.R. Acad. Sc. Paris **281** Série A (1975), 813–815.

[LS] A. Lascoux and M.-P. Schützenberger, *Polynômes de Schubert*, C.R. Acad. Sc. Paris **294** Série I (1982), 447–450.

[LV] L. Lapointe and L. Vinet, A short proof of the integrality of the Macdonald (q,t)-Kostka coefficients, Duke Math. J. **91** (1998), 205–214.

[M1] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2d ed., Oxford Math. Monographs, Clarendon Press, New York, 1995.

[M2] I.G. Macdonald, *Notes on Schubert Polynomials*, Publ. du LACIM, Montréal, 1991.

[RS] S.N.M. Ruijsenaars and H. Schneider, *A new class of integrable systems and its relation to solitons*, Ann. Phys. **170** (1986), 370–405.

[V] S. Veigneau, *ACE, an Algebraic Combinatorics Environment for the computer algebra system MAPLE, Version 3.0*, Université de Marne-la-Vallée, 1998, http://phalanstere.univ-mlv.fr/~ace/.