SYMMETRIC FUNCTIONS AND COMBINATORIAL OPERATORS ON POLYNOMIALS

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Abstract

We present notes about symmetric functions and their generalizations.

The theory of symmetric functions allows to compute with functions of several variables without seeing the variables. Historically, the "variables" were roots of a polynomial, and becaue it was impossible to explicit them in degree higher than 4, the classics, starting from Girard [10] and Newton [40], gave formulas to express power sums, monomial functions, complete functions in terms of elementary symmetric functions (= the coefficients).

We shall not get involved into change of bases in the ring $\mathfrak{S}ym$ and refer to Macdonald [37] for that topic.

We prefer to stress the functorial aspect: symmetric functions must be considered as operators on the ring of polynomials (in other indeterminates), or as operators on vector spaces or modules.

This is why we adopt the point of view of λ-rings, due to Grothendieck, using Λ^i, S^i, Ψ^i, instead of the more common e_i, h_i, p_i. In fact, Schur functors in place of Schur functions, are unavoidable as soon as one takes several alphabets at the same time.

The avantage of λ-rings, apart from providing compact notations, is to show that all properties of symmetric functions stem from the Cauchy kernel $\prod_{i,j} 1/(1-a_ib_j)$. We should also have recourse to the "plethysm", but our knowledge in this domain is too scanty to permit using it widely.

To explicit the multiplicative structure on $\mathfrak{S}ym$, one relies mostly on Pieri formulas.

With these tools only, it just remain to translate in terms of symmetric functions such classical topics as Euclidean division, continued fractions, Padé approximants, orthogonal polynomials, to see Schur functions appear.

Instead of using only the invariants of the symmetric group, it is more enlightening to directly use the symmetric group itself, and consider the different ways it can be made to act on polynomials. We put a special emphasis on the Newton divided differences which have been neglected during three hundred years.

The basis of Schubert polynomials can be introduced by generalizing Newton interpolation to several variables.

Fixing the number of variables, one has a non-symmetric Cauchy kernel, a quadratic form, and Schubert polynomials appear now as the appropriate generalization of Schur functions, instead of being a linear basis for interpolation.

For the moment, we do not consider extensions to the other classical groups, and end these notes with a non-commutative touch, embedding $\mathfrak{S}ym$ into the non-commutative algebra of Young tableaux.

Symmetric functions

We shall handle functions on different sets of indeterminates (called *alphabets*, though we shall mostly use commutative indeterminates).

A symmetric function of an alphabet $\mathbb{A}$ is a function of the letters which is invariant under permutation of the letters of $\mathbb{A}$.

The simpler symmetric functions are best defined through generating functions. We shall not use the classical notations for symmetric functions (as they can be found in Macdonald's book), because it will become clear in the course of these lectures that we need to consider symmetric functions as *functors*, and connect them with operations on vector spaces and representations. It is a small burden imposed on the reader, but the compact notations that we propose greatly simplifies manipulations of symmetric functions. Notice that exponents are used for products, and that S^J is different from S_J, except if J is of length one (i.e. is an integer).

We need operations on alphabets, the first one being the *addition*, that is the disjoint union that we shall denote by a '+'-sign :

$$\Big(\mathbb{A} = \{a\}\ ,\ \mathbb{B} = \{b\}\Big) \quad \mapsto \quad \mathbb{A} + \mathbb{B} := \{a\} \cup \{b\}$$

More operations will be introduced in the second section.

- **Generating Functions of symmetric functions**

Taking an extra indeterminate z, one has three series

$$\lambda_z(\mathbb{A}) := \prod_{a\in\mathbb{A}} (1 + za)\ ,\ \sigma_z(\mathbb{A}) := \prod_{a\in\mathbb{A}} \frac{1}{1 - za}\ ,\ \Psi_z(\mathbb{A}) := \sum_{i=1}^{\infty} \sum_{a\in\mathbb{A}} z^i a^i / i \quad (1.1)$$

the expansion of which gives the *elementary symmetric functions* $\Lambda^i(\mathbb{A})$ the *complete functions* $S^i(\mathbb{A})$, and the *power sums* $\psi_i(\mathbb{A})$:

$$\lambda_z(\mathbb{A}) = \sum z^i\, \Lambda^i(\mathbb{A})\ ,\ \sigma_z(\mathbb{A}) = \sum z^i\, S^i(\mathbb{A})\ ,\ \Psi_z(\mathbb{A}) = \sum_{i=1}^{\infty} z^i \psi_i(\mathbb{A})/i\ . \quad (1.2)$$

Since $\log(1/(1-a) = \sum_{i>0} a^i/i$, one has

$$\sigma_z(\mathbb{A}) = \exp\left(\Psi_z(\mathbb{A})\right) \quad , \quad \Psi_z(\mathbb{A}) = \log\left(\sigma_z(\mathbb{A})\right) \qquad (1.3)$$

Addition of alphabets implies product of generating series

$$\lambda_z(\mathbb{A} + \mathbb{B}) = \lambda_z(\mathbb{A})\, \lambda_z(\mathbb{B}) \quad , \quad \sigma_z(\mathbb{A} + \mathbb{B}) = \sigma_z(\mathbb{A})\, \sigma_z(\mathbb{B})\ . \qquad (1.4)$$

However, since one can invert formal series beginning by 1, or take any power of them, one can extend (1.1) by

$$\sigma_z(\mathbb{A}-\mathbb{B}) := \frac{\prod_{b\in\mathbb{B}}(1-zb)}{\prod_{a\in\mathbb{A}}(1-za)} \quad , \quad \sigma_z(c\,\mathbb{A}) = (\sigma_z(\mathbb{A})^c \ , \ c\in\mathbb{C} \tag{1.5}$$

In other words, formal series beginning by 1 should be treated as generating series of complete or elementary symmetric functions of some alphabet, that one can manipulate without knowing its elements (as one can use a polynomial without being able to factorize it!).

Having written $\mathbb{A}+\mathbb{B}$ for the disjoint union of alphabets force us to consider a finite alphabet as the sum of its sub-alphabets of cardinality 1, i.e to identify $\mathbb{A}$ and $\sum_{a\in\mathbb{A}} a$.

Given a finite alphabet $\mathbb{A}$, let $\mathfrak{S}ym(\mathbb{A})$ be the ring of symmetric polynomials in $\mathbb{A}$. As a vector space, it has (multiplicative) bases

$$\begin{cases} \Lambda^I(\mathbb{A}) & := & \Lambda^{i_1}(\mathbb{A})\Lambda^{i_2}(\mathbb{A})\cdots \\ S^I(\mathbb{A}) & := & S^{i_1}(\mathbb{A})S^{i_2}(\mathbb{A})\cdots \\ \Psi^I(\mathbb{A}) & := & \Psi^{i_1}(\mathbb{A})\Psi^{i_2}(\mathbb{A})\cdots \end{cases} ,$$

sum over all k, all partitions $I=[i_1,\ldots,i_k]$, $i_k \le card(\mathbb{A})$.

The basis Λ^I is due to Newton. Matrices of change of bases are fairly well known and can be computed by having recourse to some combinatorial objects such as Young tableaux, matrices with fixed row and column sums, etc.

The sum of all elements in the orbits of a monomial a^J under the action of the symmetric group $\mathfrak{S}(\mathbb{A})$ is of course a symmetric function, called *monomial function* and we shall denote it $\Psi_J(\mathbb{A})$, (J partition), rather than m_J.

It has been since long realized that one should use alphabets of infinite cardinality, and thus consider a universal ring $\mathfrak{S}ym$ from which one gets by specialization the rings $\mathfrak{S}ym(\mathbb{A})$, for specific alphabets $\mathbb{A}$ of finite cardinality (more generally, we shall use specializations such that *letters* are no more algebraically independent.

- **Matrix generating functions**

Let z stands now for the infinite matrix with diagonal $j-i=1$ filled with 1's, all other entries being 0's. Now $\sigma_z(\mathbb{A})$ is an Toeplitz matrix (i.e. a matrix with constant values in each diagonal) that we shall denote by $\mathbb{S}(\mathbb{A})$; similarly $\lambda_z(\mathbb{A})$ is a matrix denoted $\mathbb{L}(\mathbb{A})$:

$$\mathbb{S}(\mathbb{A}) = \Big(S^{j-i}(\mathbb{A})\Big)_{i,j\ge 0} \quad \& \quad \mathbb{L}(\mathbb{A}) = \Big(\Lambda^{j-i}(\mathbb{A})\Big)_{i,j\ge 0} \tag{1.6}$$

Addition or subtraction of alphabets still correspond to product of matrices

$$\mathbb{S}(\mathbb{A} \pm \mathbb{B}) = \mathbb{S}(\mathbb{A})\, \mathbb{S}(\mathbb{B})^{\pm 1} \quad \& \quad \mathbb{L}(\mathbb{A} \pm \mathbb{B}) = \mathbb{L}(\mathbb{A})\, \mathbb{L}(\mathbb{B})^{\pm 1} \ . \tag{1.7}$$

The advantage of matrices, compared to formal series, is that they offer us their minors, that we shall index by (increasing) partitions, or more generally, by vectors with components in $\mathbb{Z}$. More precisely, given $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, $J = (j_1, \ldots, j_n) \in \mathbb{Z}^n$ one defines the *skew Schur function* $S_{J/I}(\mathbb{A})$ to be the minor of $\mathbb{S}(\mathbb{A})$ taken on rows $i_1+1, i_2+2, \ldots, i_n+n$ and columns $j_1+1, \ldots, j_n+n$ (the minor is 0 if one of these numbers is < 0). When $I = 0^n$, the minor is called a *Schur function* and one writes $S_J(\mathbb{A})$ instead of $S_{J/0^n}(\mathbb{A})$.

In other words,

$$S_{J/I}(\mathbb{A}) = \left| S^{j_k - i_h + k - h}(\mathbb{A}) \right|_{1 \leq h,k \leq n} \ . \tag{1.8}$$

It is convenient to also use determinants in elementary symmetric functions :

$$\Lambda_{J/I}(\mathbb{A}) = \left| \Lambda^{j_k - i_h + k - h}(\mathbb{A}) \right|_{1 \leq h,k \leq n} \ . \tag{1.9}$$

Of course, one must not forget that $\Lambda^i(\mathbb{A}) = (-1)^i S^i(-\mathbb{A})$, $i \in \mathbb{Z}$, and thus the $\Lambda_{J/I}(\mathbb{A})$ are also skew Schur functions, but indexed by "column lengths".

We shall also need rectangular sub-matrices of $\mathbb{S}(\mathbb{A})$ that we shall continue to index the same way: $\mathbb{S}_{J/I}(\mathbb{A})$ is the sub-matrix of $\mathbb{S}(\mathbb{A})$ taken on rows $i_1+1, i_2+2, \ldots$, and columns $j_1+1, j_2+2, \ldots$.

Binet-Cauchy theorem for minors of the product of two matrices implies, taking $\mathbb{S}(\mathbb{A}+\mathbb{B})$, the following expansion on skew-Schur functions

$$S_{J/I}(\mathbb{A}+\mathbb{B}) = \sum_K S_{J/K}(\mathbb{A})\, S_{K/I}(\mathbb{B}) \ , \tag{1.10}$$

sum over all partitions (only those $K : I \subseteq K \subseteq J$ give a non-zero contribution).

Jacobi's theorem on minors of the inverse of a matrix implies, denoting by $J^\sim$ the *conjugate partition* (obtained by transposing the Ferrers' diagram of J):

$$\Lambda_{J/I}(\mathbb{A}) = S_{J^\sim/I^\sim}(\mathbb{A}) = (-1)^{|J/I|} S_{J/I}(-\mathbb{A}) \tag{1.11}$$

One needs to enlarge the definition of a Schur function, to be able to play with different alphabets at the same time.

Given n, given two sets of alphabets $\{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_n\}$, $\{\mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_n\}$, and $I, J \in \mathbb{N}^n$, we define the *multi-Schur function*

$$S_{J/I}(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_n - \mathbb{B}_n) := \left| S_{j_k - i_h + k - h}(\mathbb{A}_k - \mathbb{B}_k) \right|_{1 \le h,k \le n} . \tag{1.12}$$

In the case where the alphabets are repeated, we indicate by a semicolon the corresponding block separation : given $H \in \mathbb{Z}^p$, $K \in \mathbb{Z}^q$, then $S_{H;K}(\mathbb{A} - \mathbb{B}; \mathbb{C} - \mathbb{D})$ stands for the multi-Schur function with index the concatenation of H and K, and alphabets $\mathbb{A}_1 = \cdots = \mathbb{A}_p = \mathbb{A}$, $\mathbb{B}_1 = \cdots = \mathbb{B}_p = \mathbb{B}$, $\mathbb{A}_{p+1} = \cdots = \mathbb{A}_{p+q} = \mathbb{C}$, $\mathbb{B}_{p+1} = \cdots = \mathbb{B}_{p+q} = \mathbb{D}$.

These functions are now sufficiently general to allow easy inductions, thanks to the following transformation lemma.

Lemma 1 *Let $S_J(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_n - \mathbb{B}_n)$ be a multi-Schur function, and $\mathbb{D}_0, \mathbb{D}_1, \ldots, \mathbb{D}_{n-1}$ be a family of finite alphabets such that card$(\mathbb{D}_i) \le i$, $0 \le i \le n-1$. Then $S_J(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_n - \mathbb{B}_n)$ is equal to the determinant*

$$\left| S_{j_k - i_h + k - h}(\mathbb{A}_k - \mathbb{B}_k - \mathbb{D}_{n-h}) \right|_{1 \le h,k \le n}$$

In other words, one does not change the value of a multi-Schur function S_J by replacing in row h the difference $\mathbb{A} - \mathbb{B}$ by $\mathbb{A} - \mathbb{B} - \mathbb{D}_{n-h}$. Indeed, thanks to the expansion (1.10) :

$$S_j(\mathbb{A}-\mathbb{B}-\mathbb{D}_h) = S_j(\mathbb{A}-\mathbb{B}) + S_1(-\mathbb{D}_h)\, S_{j-1}(\mathbb{A}-\mathbb{B}) + \cdots + S_h(-\mathbb{D}_h)\, S_{j-h}(\mathbb{A}-\mathbb{B}) \ ,$$

the sum terminating because the $S_k(-\mathbb{D}_h)$ are null for $k > h$, we see that the determinant has been transformed by multiplication by a triangular matrix with 1's in the diagonal, and therefore has kept its value. QED

For example, taking $D_0 = \emptyset$, $\mathbb{D}_1 = \{x\}$, $\mathbb{D}_2 = \{y, z\}$, one has

$$\begin{bmatrix} S_i(\mathbb{A}_1 - y - z) & S_{j+1}(\mathbb{A}_2 - y - z) & S_{h+2}(\mathbb{A}_3 - y - z) \\ S_{i-1}(\mathbb{A}_1 - x) & S_j(\mathbb{A}_2 - x) & S_{h+1}(\mathbb{A}_3 - x) \\ S_{i-2}(\mathbb{A}_1) & S_{j-1}(\mathbb{A}_2) & S_h(\mathbb{A}_3) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & -y-z & yz \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} S_i(\mathbb{A}_1) & S_{j+1}(\mathbb{A}_2) & S_{h+2}(\mathbb{A}_3) \\ S_{i-1}(\mathbb{A}_1) & S_j(\mathbb{A}_2) & S_{h+1}(\mathbb{A}_3) \\ S_{i-2}(\mathbb{A}_1) & S_{j-1}(\mathbb{A}_2) & S_h(\mathbb{A}_3) \end{bmatrix}$$

and the determinant of the left matrix is equal to $S_{ijh}(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3)$.

Taking $-\mathbb{A}_1, -\mathbb{A}_2, -\mathbb{A}_3$ instead of $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$, and getting rid of signs because of "isobarity", one gets by the same token

$$\Lambda_{i;j;h}(\mathbb{A}_1; \mathbb{A}_2; \mathbb{A}_3) = \begin{bmatrix} \Lambda_i(\mathbb{A}_1 + y + z) & \Lambda_{j+1}(\mathbb{A}_2 + y + z) & \Lambda_{h+2}(\mathbb{A}_3 + y + z) \\ \Lambda_{i-1}(\mathbb{A}_1 + x) & \Lambda_j(\mathbb{A}_2 + x) & \Lambda_{h+1}(\mathbb{A}_3 + x) \\ \Lambda_{i-2}(\mathbb{A}_1) & \Lambda_{j-1}(\mathbb{A}_2) & \Lambda_h(\mathbb{A}_3) \end{bmatrix}$$

In the preceding lemma, we needed only consecutive elements of a column to be complete functions of the same difference of alphabets, of consecutive degrees. A similar transformation can be performed in rows, when alphabets are repeated in some consecutive columns, for partitions having repeated parts.

Lemma 2 *Let j, n be two integers, $\mathbb{D}_0, \ldots, \mathbb{D}_{n-1}$ be a family of alphabets such that $card(\mathbb{D}_i) \leq i$, $0 \leq i \leq n-1$, and let $\mathbb{A}$, $\mathbb{B}$ be two arbitrary alphabets. Let $S_{\diamond; j^n; \circ}(\clubsuit; \mathbb{A}-\mathbb{B}; \spadesuit)$ be a multi-Schur function of which we have specified only n columns.*

Then it is equal to the multi-Schur function

$$S_{\diamond; j, \ldots, j; \circ}(\clubsuit;\ \mathbb{A}-\mathbb{B}-\mathbb{D}_0, \mathbb{A}-\mathbb{B}-\mathbb{D}_1, \ldots, \ \mathbb{A}-\mathbb{B}-\mathbb{D}_{n-1};\ \spadesuit)\ .$$

The notations are a little too heavy for hand-computations. Indeed, one usually play with either indices of functions, or with alphabets, but not with both at the time. For example, in the preceding cases, one transforms only alphabets, the indices of the complete functions do not change. It seems appropriate to use "umbral" notations and write alphabets on the border of the determinant: an entry k in position (i, j) will be interpreted as $S_k(\mathbb{A}\pm\mathbb{B})$ if $\mathbb{A}$ is written in column j and $\pm\mathbb{B}$ in row i. Thus the preceding determinant will be written

$$\begin{array}{ccc|c} i & j+1 & h+2 & -y-z \\ i-1 & j & h+1 & -x \\ i-2 & j-1 & h & \cdot \\ \hline \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{A}_3 & \end{array}$$

The above lemma implies many factorization properties, e.g. for $r \geq 0$,

$$S_J(\mathbb{A}-\mathbb{B}-x)\, x^r = S_{J,r}(\mathbb{A}-\mathbb{B}, x) \tag{1.13}$$

since taking $\mathbb{D}_1 = \mathbb{D}_2 = \cdots = \{x\}$ factorizes the determinant $S_{J,r}(\mathbb{A}-\mathbb{B}, x)$.

More generally, for an alphabet $\mathbb{D}$ of cardinal $\leq r$ and $J \in \mathbb{N}^r$, one has

$$S_I(\mathbb{A}-\mathbb{B}-D)\, S_J(\mathbb{D}) = S_{I,J}(\mathbb{A}-\mathbb{B}\,,\ \mathbb{D})\ . \tag{1.14}$$

Monomial themselves can be written as multi-Schur functions. Given a totally ordered alphabet $\mathbb{A} = \{a_1, a_2, \ldots\}$, denote, for any n, $\mathbb{A}_n := \{a_1, \ldots, a_n\}$. Then, for any $J = [j_1, \ldots, j_n]$, denoting $J^\omega := [j_n, \ldots, j_1]$, one has

$$a^J := a_1^{j_1} \cdots a_n^{j_n} = S_{J^\omega}(\mathbb{A}_n, \ldots, \mathbb{A}_2, \mathbb{A}_1)$$

Indeed, subtract the *flag* $0, \mathbb{A}_1, \mathbb{A}_2, \ldots$ in the successive rows, starting from the bottom one. One sees the monomial appearing in the diagonal, the upper part of the matrix vanishing because it is constituted of $S_k(-(\mathbb{A}_j - A_i))$ for $k > (j-i)$, $j \geq i$.

Given two finite alphabets, the following factorization and vanishing properties implicitly appear in many classical 19-th century texts about elimination theory (modern reference is Berele-Regev ?).

Proposition 3 *Let $\mathbb{A}$, $\mathbb{B}$, be of cardinalities α, β, $p \in \mathbb{N}$, $I \in \mathbb{N}^p$, $J \in \mathbb{N}^\alpha$. Then*

$$S_{i_1,\dots,i_p,\beta+j_1,\dots,\beta+j_\alpha}(\mathbb{A}-\mathbb{B}) = S_I(-\mathbb{B})\, S_J(\mathbb{A}) \prod_{a\in\mathbb{A}, b\in\mathbb{B}} (a-b) \ .$$

Let J be a partition, $J \supseteq (\beta+1)^{\alpha+1}$. Then $S_J(\mathbb{A}-\mathbb{B}) = 0$.

Proof. Subtract $\mathbb{A}$ in the first p rows. One gets the factorization

$$S_I(-\mathbb{B})\, S_{\beta+j_1,\dots,\beta+j_\alpha}(\mathbb{A}-\mathbb{B}) \ .$$

Now, using the partition K conjugate to $[\beta+j_1, \dots, \beta+j_\alpha]$, one gets the factorization of $S_K(\mathbb{B}-\mathbb{A})$ into a Schur function of $\mathbb{A}$ and $S_{\alpha^\beta}(\mathbb{B}-\mathbb{A})$. This last function can be seen equal to the *resultant* $\prod_{a\in\mathbb{A}, b\in\mathbb{B}}(b-a)$ by subtracting the flag $0, \mathbb{B}_1, \mathbb{B}_2, \dots$.

The case J too big can be treated by adding the same letters to $\mathbb{A}$ and $\mathbb{B}$, so that one is reduced to the preceding case. But now the factor $\prod(a-b)$ vanishes.

Given a finite alphabet $\mathbb{A}$ (that one will totally order: $\mathbb{A} = \{a_1, \dots, a_n\}$), Cauchy and Jacobi separately defined the Schur function $S_J(\mathbb{A})$ using the (infinite) *Vandermonde matrix*

$$V(\mathbb{A}) = \left[a_i^j\right]_{1\leq i\leq n; j\geq 0}$$

and the *Vandermonde* $\Delta(\mathbb{A}) := \prod_{i>j}(a_i - a_j)$.

Proposition 4 *Let $J \in \mathbb{N}^n$. Then $S_J(\mathbb{A})\, \Delta(\mathbb{A})$ is equal to the minor of index $(0^n, J)$ of the Vandermonde matrix $V(\mathbb{A})$.*

Proof. Let $\mathbb{S}_J(\mathbb{A})$ denotes the sub-matrix of $\mathbb{S}(A)$ taken on columns $j_1+1, j_2+2, \dots, j_n+n$. Consider the product $V(\mathbb{A})\, \mathbb{S}(-A)\, \mathbb{S}_J(\mathbb{A})$. It can be factorized in two manners, using (1.7):

$$V(\mathbb{A})\, \mathbb{S}(-A)\, \mathbb{S}_J(\mathbb{A}) = [S_j(a_i - \mathbb{A})]\ 1 \leq i \leq n; j \geq 0\, \mathbb{S}_J(\mathbb{A}) = V(\mathbb{A})\, \mathbb{S}(0) \ .$$

However, the $S_j(a_i - \mathbb{A})$ are null for $j \geq n$, because they are the elementary functions (up to sign) of alphabets of cardinality $n-1$. On the other hand,

$S_j(0) = 0$, if $j \neq 0$. In both cases, we have obtained matrices such that only one minor of order n is different from 0. QED

- **Cauchy formula**

The most important formula in the theory of symmetric functions is the following expansion, due to Cauchy.

Let $\mathbb{A}, \mathbb{B}$ be finite alphabets (that we shall suppose of the same cardinality n). Then

$$K(\mathbb{A}, \mathbb{B}) := 1/\prod_{a\in\mathbb{A}} \prod_{b\in\mathbb{B}} (1-ab) = \sum_J S_J(\mathbb{A}) S_J(\mathbb{B}) ,$$

sum over all partitions $J \in \mathbb{N}^n$.

One will find a proof of it using symmetrizing operators, in the exercises. For the moment, let us recall that this identity is intimately connected to Binet-Cauchy theorem, applied to Cauchy's determinant $|1/(1-ab)|$. This determinant factorizes into the two Vandermonde matrices $V(\mathbb{A})\, V(\mathbb{B})^{tr}$ and the minors of these matrices are exactly Schur functions multiplied by the Vandermondes $\Delta(\mathbb{A})$ and $\Delta(\mathbb{B})$ respectively. Now, the determinant itself is equal to the sum ($\ell(\sigma)$ denoting the *length* of a permutation σ)

$$\sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)} \frac{1}{\Big((1-a_1b_1)\cdots(1-a_nb_n)\Big)^{\sigma}} .$$

Extracting the denominator $\prod 1/(1-ab)$, one has to compute the sum

$$\sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)} \Big(S^{n-1}(1+b_1a_1-b_1\mathbb{A}) \cdots S^{n-1}(1+b_na_n-b_n\mathbb{A}) \Big)^{\sigma}$$

This sum is divisible by $\Delta(\mathbb{A})\,\Delta(\mathbb{B})$, because it vanishes when two of the a's, or two of the b's coincide. Because of degree, the quotient is a constant and one has to check that it is equal to 1.

The last step in the above demonstration miss the crucial fact that what is involved is *Jacobi symmetrizer*

$$\mathbb{C}[a_1, \ldots, a_n] \ni f \mapsto \sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)} f^{\sigma} \frac{1}{\Delta(\mathbb{A})} ,$$

sum over all permutations σ. Jacobi's symmetrizer provides a connection with the theory of characters (and extends to Weyl's character formula for the classical groups). We postpone this point of view to another section.

- **Scalar Product**

There are other decompositions of $K(\mathbb{A},\mathbb{B})$ as a sum of products of symmetric functions in $\mathbb{A}$ and in $\mathbb{B}$. However, there is only one of the type $\sum P(\mathbb{A})P(\mathbb{B})$ over $\mathbb{Z}$: up to signs, the P's are all the Schur functions indexed by partitions in $\mathbb{N}$. Thus $K(\mathbb{A},\mathbb{B})$, that we shall call *Cauchy kernel*, determines the Schur functions. One can interpret differently the kernel, as defining a scalar product on the space of symmetric functions, the Schur functions constituting the only orthogonal basis. Now, any expansion of the type

$$\mathbf{K}(\mathbb{A},\mathbb{B}) = \sum P_J(\mathbb{A})\, Q_J(\mathbb{B})$$

define a pair of adjoint bases $\{P_J\}$, $\{Q_J\}$, with respect to the *canonical scalar product* $(\ ,\ )$ induced by $\mathbf{K}(\mathbb{A},\mathbb{B})$, i.e. the scalar product such that $(S_J, S_J) = 1$, for all partition J.

There are some subtleties for what concerns scalar products when taking finite alphabets, and in the rest of the section, we shall take only infinite alphabets.

The expansion

$$\mathbf{K}(\mathbb{A},\mathbb{B}) = \prod_b (\prod_a \frac{1}{1-ab}) = \prod_b \Big(\sum b^i S^i(\mathbb{A})\Big) = \sum_I \Psi_I(\mathbb{B})\, S^I(\mathbb{A})$$

shows that the basis adjoint to S^I, I partition, is the monomial basis Ψ_I.

The expansion

$$\mathbf{K}(\mathbb{A},\mathbb{B}) = \prod_{a,b} \Big(\sum a^i b^i\Big) = \sum_I \Psi^I(\mathbb{A})\Psi^I(\mathbb{B})/z_I \ ,$$

where $z_I = m_1! 1^{m_1} m_2! 2^{m_2} m_3! 3^{m_3} \cdots$ (using the exponential notation for $I = [1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots])$, shows that the basis of products of power sums is orthogonal, with scalar product $(\Psi_I, \Psi_I) = z_I$.

The name kernel is justified by the following property, which is just another way of stating that $\mathbf{K}(\mathbb{A},\mathbb{B})$ defines a scalar product:

Lemma 5 *Let f be a symmetric function and $(\ ,\ )$ be the scalar product on symmetric functions in $\mathbb{A}$. Then*

$$(\mathbf{K}(\mathbb{A},\mathbb{B})\,,\ f(\mathbb{A})) = f(\mathbb{B}) \ .$$

- **Differential calculus**

Having a scalar product, one can now obtain operators *adjoint* to some simple ones. We did not use the multiplicative structure of $\mathfrak{S}ym$ up to now. Any symmetric function S can be thought as the operator **"multiplication by S"**.

Definition 6 *For $S \in \mathfrak{S}ym$, D_S is the operator adjoint to the multiplication by S, i.e. for every $S', S'' \in \mathfrak{S}ym$ one has*

$$(D_S(S'), S'') = (S', S\,S'') .$$

Schur functions play a special rôle (cf. several proofs in the exercises):

Lemma 7 *For every $I, J \in \mathbb{N}^n$, one has*

$$D_{S_I}(S_J) = S_{J/I} ,$$

$$D_{S_I}(\Psi_J) = \begin{cases} \Psi_K & \text{if } J = K \cup I \\ 0 & \text{otherwise} \end{cases} .$$

The operators D_{Ψ_i} are differential operators. Indeed, for every integer $i > 0$, one has

$$D_{\Psi_i} = i\partial_{\Psi_i}$$

as can be checked by operating on the basis Ψ^J: the scalar product is compatible with the decomposition

$$\mathfrak{S}ym \simeq \mathbb{C}[\Psi_1] \otimes \mathbb{C}[\Psi_2] \otimes \mathbb{C}[\Psi_3] \otimes \cdots ,$$

and the equality

$$(\Psi_i \Psi_i^k , \Psi_i^{m+1}) = \delta_{k,m}\, i^{m+1}\, (m+1)!$$

proves the assertion.

The decomposition of $S_J(\mathbb{A}+\mathbb{B})$ given in (1.10) involves the $S_{J/I}$, i.e. can be written in terms of the D_{S_I}:

$$S_J(\mathbb{A}+\mathbb{B}) = \sum_I S_I(\mathbb{A})\, D_{S_I}\big(S_J(\mathbb{B})\big) .$$

Because Schur functions are a linear basis, this implies a *Taylor formula*

$$\forall S \in \mathfrak{S}ym ,\ S(\mathbb{A}+\mathbb{B}) = \sum_I S_I(\mathbb{A})\, D_{S_I}\big(S(\mathbb{B})\big) .$$

More generally, any pair of adjoint bases $\{P\}$, $\{Q\}$ will give a decomposition

$$\forall S \in \mathfrak{S}ym ,\ S(\mathbb{A}+\mathbb{B}) = \sum P(\mathbb{A})\, D_Q\big(S(\mathbb{B})\big) .$$

For example

$$\forall S \in \mathfrak{S}ym ,\ S(\mathbb{A}+\mathbb{B}) = \sum_I \Psi_I(\mathbb{A})\, D_{S^I}\big(S(\mathbb{B})\big) ,$$

$$\forall S \in \mathfrak{S}ym\,,\ S(\mathbb{A}+\mathbb{B}) = \sum_J \frac{1}{(\Psi^I\,,\,\Psi^I)} \Psi^I(\mathbb{A})\, D_{\Psi^I}\big(S(\mathbb{B})\big)\ .$$

- **Pieri formulas**

The "natural" [1] linear basis of $\mathfrak{S}ym(\mathbb{A})$ consists of the orbits of monomials under the action of the symmetric group. In the present notes, we shall however only use Schur functions, which indeed are the "fundamental" symmetric functions.

Still, $\mathfrak{S}ym$ is a ring, and to recover its multiplicative structure, one needs to describe the product of two Schur functions. This is given by the so-called *Littlewood-Richardson Rule*, which is better stated, and proved, in terms of non-commutative symmetric functions (cf. last section). Since products Λ^I or S^I also constitute linear bases, it will be sufficient to describe the product of general Schur functions by a complete or elementary symmetric function to determine the multiplicative structure. The products $S^r S_J$ and $\Lambda^r S_J$, $r \in \mathbb{N}$, J partition, were in fact determined by the Italian geometer Pieri in 1873. They have the remarkable property of being multiplicity free.

First, let us introduce two notations for family of partitions deduced from a given one $I = [i_1, \ldots, i_n]$: $\{I \otimes k\}$ and $\{I \otimes 1^k\}$ respectively denote

$$\{I \otimes k\} := \{J = [j_0, \ldots, j_n], 0 \le j_0 \le i_1 \le j_1 \le \cdots \le i_n \le j_n\},\ |J| = |I| + k\}\,,$$
$$\{I \otimes 1^k\} := \{J : J^\sim \in \{I^\sim \otimes k\}\}\ .$$

Lemma 8 *Let k be a positive integer, I be a partition. Then one has the following decompositions :*

$$\begin{aligned} S^k\, S_I &= \sum\nolimits_{J \in \{I \otimes k\}} S_J\ , \\ \Lambda^k\, S_I &= \sum\nolimits_{J \in \{I \otimes 1^k\}} S_J\ . \end{aligned} \tag{1}$$

Replacing I by $[0^k, I]$, the last sum can be written $\sum_H S_H$, sum over all $H \in \mathbb{N}^{n+k}$ such that $h_1 - i_1, \ldots, h_{n+k} - i_{n+k}$ are 0 or 1, and $|H| = |I| + k$, because the extra terms such that H is not a partition index null determinants.

The involution $\mathbb{A} \mapsto -\mathbb{A}$ exchanges the two Pieri formulas. Graphically, the first one consists in adding *horizontal strips* (no two boxes in the same vertical) to the diagram of I, the second one, *vertical strips*.

[1] For the classics, monomial functions were the *symmetric functions*. Alphabets being defined as sets of roots of polynomials, the problem was to express the symmetric functions in terms of the $\Lambda^I(\mathbb{A})$, which were the data. Vandermonde solved this problem, without explaining his method, and published tables in the *Mémoires de l'Académie* for degree up to 10 – with no mistake, as controlled by D.Knuth.

These rules also occur in the non commutative world, as shows the following proposition.

Proposition 9 *Let $\mathcal{R}$ be a ring containing a family of elements s_J, J = all partitions, satisfying $(Pieri1)$ for all r, J. Then the $\mathbb{Z}$-module generated by the s_J is a subring, which is a quotient of the ring of symmetric polynomials with coefficient in $\mathbb{Z}$.*

The proof follows from the fact that there is a natural order on partitions of the same weight, such that the leading term in a product $S_r S_J$ is $S_{(r,J)}$, with (r, J) := increasing reordering of $[r, j_1, j_2, \dots]$. By recursion, it allows to obtain all Schur functions, starting from the S_r's, it proves that they commute and can be expressed as determinants in the S_r's .

In the commutative case, the proposition is due to Giambelli, who used it to characterize the cohomology ring of a Grassmann manifold as a quotient of $\mathfrak{S}ym$. For what concerns me, Pieri's rules are the reason why I met non-commutativity, combinatorics and Schützenberger [20] seeing that Schensted construction satisfied the property [2] $S_2 S_J = \sum S_K$, and that it implied the general Pieri rules, I realized that it was the only natural way of obtaining intersection of Schubert cycles.

[2] In the non commutative case, because multiplication is an associative operation, one needs only to describe products $xy\,t$, x, y letters, t tableau, to get all products of tableaux by increasing or decreasing words.

Symmetric Functions as Operators and λ-Rings

We have used powers of generating functions. Passing from $\sigma_z(\mathbb{A})$ to $\sigma_z(2\mathbb{A}) := (\sigma_z(\mathbb{A}))^2$ can be interpreted as doubling the alphabet $\mathbb{A} = \{a\} \mapsto 2\mathbb{A} := \{a'\} \cup \{a''\}$, computing the symmetric functions in $2\mathbb{A}$, and eventually erasing the diacritics on the letters a's.

However, the inverse operation $\mathbb{A} = \{a\} \mapsto \frac{1}{2}\mathbb{A}$ cannot be described in the same way, which means that one must not stick to considering alphabets as composed of letters.

More explicitely, at the level of power sums, multiplication by 2 is just

$$\psi_i(2\mathbb{A}) = \sum (a')^i + \sum (a'')^i = 2\,\psi_i(\mathbb{A}) \ .$$

Therefore the alphabet $\mathbb{B} = k\,\mathbb{A}$, $k \in \mathbb{C}$ will be defined by

$$\psi_i(\mathbb{B}) := k\,\psi_i(\mathbb{A}) \ , \ i \geq 1 \ . \tag{2.1}$$

Instead of only multiplying alphabets by constants, one can go one step further and realize that every polynomial can play the rôle of an alphabet, i.e. that *the ψ_i are operators on polynomials* ($\alpha \in \mathbb{C}$, u monôme):

$$P = \sum_{\alpha,u} \alpha\, u \Rightarrow \psi_i(P) = \sum_{\alpha,u} \alpha\, u^i \ . \tag{2.2}$$

The ring $\mathfrak{S}ym$, being generated by the ψ_i, $i = 1, 2, \ldots$, formulas (2.2) extend to an action of $\mathfrak{S}ym$ on the ring of polynomials.

Thus, the generating functions of elementary and symmetric functions become operators, their explicit action being

$$P = \sum_{\alpha,u} \alpha\, u \mapsto \lambda_z(P) = \prod (1 + zu)^{\alpha} \ , \tag{2.3}$$

$$P = \sum_{\alpha,u} \alpha\, u \mapsto \sigma_z(P) = \prod (1 - zu)^{-\alpha} \ . \tag{2.4}$$

Each of formulas (2.2), (2.3) or (2.4) can be chosen at will to define the action of symmetric polynomials and implies the two others, and the ring of polynomials, with $\mathfrak{S}ym$ operating on it, is called a *λ-ring*.

Grothendieck choose *lambda operations*, that is, the *exterior powers* of vector spaces, or more generally, fiber bundles, to introduce λ-rings. In the same interpretation, the S^i are the symmetric powers. Algebraic topologists prefers the *Adams operations* ψ_i.

Having taken polynomials as objects, rather than more sophisticated mathematical entities, we needed only one axiom (2.2). The general theory of λ-rings require three axioms: the compatibility of the λ-operations with addition, product and composition.

Let us remark the different rôle played by constants α and monomials u :

$$\begin{cases} \psi_i(\alpha) = \alpha\ , \quad S^i(\alpha) = \binom{\alpha+i-1}{i}\ , \qquad \Lambda^i(\alpha) = \binom{\alpha}{i} \\ \psi_i(u) = u^i\ , \qquad S^i(u) = u^i\ , \qquad \Lambda^i(u) = 0,\ i > 1, \quad \Lambda^1(u) = u \end{cases} \tag{2.5}$$

When implementing λ-rings, one must distinguish between indeterminate coefficients α and variables u. The preceding terminology is not satisfactory; it is preferable, instead of using the term "monomial" to say *element of rank 1* (i.e. non zero element x such that $\Lambda^i(x) = 0\ \forall i > 1$), and avoid the term "constant" for the elements invariant under the ψ_i, but rather say *binomial element* as a tribute to Rota, because these elements are such that their images $\Lambda^i(\alpha) = \binom{\alpha}{i}$ are binomial coefficients.

Polynomials in one indeterminate are conveniently coded in a λ-ring. Let $\mathbb{A}$ be an alphabet of cardinality n. Then

$$\prod_{a\in\mathbb{A}} (x-a) = S^n(x-\mathbb{A}) = \sum x^{n-i} S^i(-\mathbb{A})$$

Now, expanding $S^{n+k}(x-\mathbb{A})$, $k \in \mathbb{N}$, one sees that

$$S^{n+k}(x-\mathbb{A}) = x^k\, S^n(x-\mathbb{A})\ . \tag{2.6}$$

On the other hand, $S^{n-k}(x-\mathbb{A})$ is the component of positive degree of the Laurent polynomial $x^{-k}\, S^n(x-\mathbb{A})$.

Derivating with respect to x the generating function $\sigma_z(x-\mathbb{A})$, one sees that for any $k \in \mathbb{N}$, the successive derivatives of $S^k(x-\mathbb{A})$ are

$$S^{k-1}(2x-\mathbb{A})\,,\ 2!\,S^{k-2}(3x-\mathbb{A})\,,\ 3!\,S^{k-3}(4x-\mathbb{A})\,, \ldots\ , j!\,S^{k-j}((j+1)x-\mathbb{A})\ldots$$

There are other codings. For example, given any $\mathbb{A}$, any $n \in \mathbb{N}$, and an element x of rank 1,

$$\Lambda^n(nx+\mathbb{A}) = \sum \binom{n}{i} x^{n-i}\, \Lambda^i(\mathbb{A}) \tag{2.7}$$

is a polynomial of degree n.

Derivating with respect to x the generating function $\lambda_z(nx + \mathbb{A}) = (1 + zx)^n \lambda_z(\mathbb{A})$, one sees that the successive derivatives are now

$$n\,\Lambda^{n-1}((n-1)x + \mathbb{A})\,,\; n(n-1)\,\Lambda^{n-2}((n-2)x + \mathbb{A})\,,\;\ldots$$

Therefore, the integral of $\Lambda^n(nx + \mathbb{A})$ is $\frac{1}{n+1}\Lambda^{n+1}((n+1)x + \mathbb{A})$, up to the addition of a constant, and more generally, the k-th integral will be

$$\frac{1}{(n+1)\cdots(n+k)}\Lambda^{n+k}((n+k)x + \mathbb{A})\ .$$

We shall see in another section that orthogonal polynomials are also conveniently expressed in λ-rings.

One can extends the action of symmetric polynomials to rational functions or formal power series :

$$\psi_i\left(\frac{\sum \alpha u}{\sum \beta v}\right) = \frac{\sum \alpha u^i}{\sum \beta v^i}\ , \tag{2.8}$$

If q is of rank 1, then one has now $\frac{1}{1-q} = 1 + q + q^2 + \cdots$, i.e. $\frac{1}{1-q}$ is the infinite alphabet $\{1, q, q^2, \ldots\}$ and $\psi_i(\frac{1}{1-q}) = \frac{1}{1-q^i}$, $i \geq 1$. Cauchy had obtained that :

$$S_i(\frac{1}{1-q}) = \frac{1}{(1-q)\cdots(1-q^i)} \tag{2.9}$$

Therefore

$$\sigma_x(\frac{1}{1-q}) = \sum \frac{x^i}{(1-q)\cdots(1-q^i)} \tag{2.10}$$

is the *q-exponential*, equality $\frac{1}{1-q} = 1+q+q^2+\cdots$ implying the factorization

$$\sigma_x(\frac{1}{1-q}) = \prod_{i=1}^{\infty} \frac{1}{1-xq^i} \tag{2.11}$$

Such a factorization renders the q-exponential sometimes easier to use than the classical exponential.

Some classical identities on q-series come from addition in λ-rings. For example, the *q-binomial identity*

$$\sum_{i\geq 0} z^i \frac{(1-a)(1-aq)\cdots(1-aq^{i-1})}{(1-q)(1-q^2)\cdots(1-q^i)} = \prod_{i\geq 0} \frac{1-zaq^i}{1-zq^i} \tag{2.12}$$

is considering z, a, q to be of rank 1, writing $z\frac{1-a}{1-q} = \frac{z}{1-q} + \frac{-za}{1-q}$ and recognizing the left hand side to be $\sigma_1(z\frac{1-a}{1-q})$, the right one being $\sigma_1(\frac{z}{1-q})\,\sigma_1(\frac{-za}{1-q})$.

Up to now, we have only used binomial or rank 1 elements. One can "mix" these two fundamental types. Take for example x binomial, q of rank 1, and let y be such that $y(1-q)$ is of rank 1. The *q-Charlier polynomials* are defined by :

$$Ch_n(x) := n!\,\Lambda^n(x-y)\ . \tag{2.13}$$

To obtain their explicit expression, it is safer to introduce $z := y(1-q)$. Then one can expand

$$\Lambda^n(x - \frac{z}{1-q}) = \sum \Lambda^{n-i}(x)\Lambda^i(\frac{-z}{1-q}) = \sum \Lambda^{n-i}(x)(-z)^i S^i(\frac{1}{1-q})$$

$$Ch_n(x) = n! \sum \frac{x\cdots(x-n+i+1)}{(n-i)!}\frac{(-y)^i(1-q)^i}{(1-q)\cdots(1-q^i)}\ . \tag{2.14}$$

For $y=1$, $q\to 1$, on finds back the classical Charlier polynomials ([43], p.64).

Addition, or subtraction, of a rank-1 element q must be considered as operators. When applying them to Schur functions indexed by rectangular partitions, they become appropriate tools in the theory of Padé approximants. For example, we shall need the following lemmas :

Lemma 10 *Let $\mathbb{A}$, $\mathbb{B}$ be two elements of a λ-ring, q be a rank 1-element, n, k be two integers. Then*

$$S_{k^n}(\mathbb{A}-q)S_{k^{n+1}}(\mathbb{A}) - S_{k^n}(\mathbb{A})S_{k^{n+1}}(\mathbb{A}-q) = qS_{(k+1)^n}(\mathbb{A}-q)S_{(k-1)^{n+1}}(\mathbb{A})\ ,$$

$$S_{n^k}(\mathbb{B})S_{(n+1)^k}(\mathbb{B}-q) - S_{n^k}(\mathbb{B}-q)S_{(n+1)^k}(\mathbb{B}-q) = qS_{n^{k+1}}(\mathbb{B})S_{(n+1)^{k-1}}(\mathbb{B}-q)\ .$$

Proof. Because of homogeneity, one can put $q=1$ without loss of information. To simplify indices, take $n=2, k=5$. Take the matrix $\mathbb{S}_{555}(\mathbb{A}-1)$, doubling the first column, this time with complete functions of $\mathbb{A}$, and add a column $[0,0,1]$. Writing the alphabets outside, and the indices of complete functions inside, the matrix to be considered is

$$\begin{array}{c|ccccc}
\textit{alphabets} & 0 & \mathbb{A} & \mathbb{A}-1 & \mathbb{A}-1 & \mathbb{A}-1 \\
\hline
 & \cdot & 5 & 5 & 6 & 7 \\
 & \cdot & 4 & 4 & 5 & 6 \\
 & 0 & 3 & 3 & 4 & 5 \\
\hline
\textit{indices} & 0 & 4 & 5 & 6 & 7
\end{array}\ .$$

The identity to be shown is just, in terms of minors of this matrix, a Plücker relation :

$$[056]\,[467] - [046][567] = [456][067]\ ,$$

using that $[456] = S_{5,4,4}(\mathbb{A}, \mathbb{A}-1, \mathbb{A}-1) = S_{5,4,4}(\mathbb{A}, \mathbb{A}, \mathbb{A}-1) - S_{5,3,4}(\mathbb{A}, \mathbb{A}, \mathbb{A}-1) = S_{4,4,4}(\mathbb{A}, \mathbb{A}, \mathbb{A}-1) = S_{444}(\mathbb{A})$.

The second statement is obtained by putting $\mathbb{A} = -\mathbb{B} + q$ in the first one. QED

Lemma 11 *Let $\mathbb{A}$ be an element of a λ-ring, q be a rank 1-element, n, k be two integers. Then*

$$S_{k^n}(\mathbb{A}+q)S_{k^{n-1}}(\mathbb{A}-q) - qS_{(k-1)^n}(\mathbb{A}+q)S_{k+1^{n-1}}(\mathbb{A}-q) = S_{k^n}(\mathbb{A})S_{k^{n-1}}(\mathbb{A}) \ ,$$

and

$$S_{n^k}(\mathbb{A}-q)S_{(n-1)^k}(\mathbb{A}+q) - qS_{n^{k-1}}(\mathbb{A}-q)S_{(n-1)^{k+1}}(\mathbb{A}+q) = S_{n^k}(\mathbb{A})S_{(n-1)^k}(\mathbb{A}) \ .$$

Proof. We have to check a three-terms quadratic relation; once more, it can be no other than Plücker relation. The extra factor q should not puzzle the reader, because one can specialize it to 1, by homogeneity, or, in a more informative way, make it disappear by noticing that

$$\begin{aligned} S_{k,k-1,J}(\mathbb{A}, \mathbb{A}+q, \mathbb{B}) &= S_{k,k-1,J}((\mathbb{A}+q)-q, \mathbb{A}+q, \mathbb{B}) = \\ &S_{k\,k-1,J}(\mathbb{A}+q, \mathbb{B}) - qS_{k-1\ k-1,J}(\mathbb{A}+q, \mathbb{B}) = -qS_{k-1\ k-1,J}(\mathbb{A}+q, \mathbb{B}) \ . \end{aligned}$$

Take now, for simplicity $n = 3, k = 5$, and the matrix obtained by adding two columns to $\mathbb{S}_{5^3}(\mathbb{A})$ as follows:

$$\begin{array}{r} \\ \\ \\ \scriptstyle alphabets \\ \scriptstyle indexing \end{array} \begin{bmatrix} 5 & 5 & 6 & 7 & 2 \\ 4 & 4 & 5 & 6 & 1 \\ 3 & 3 & 4 & 5 & 0 \\ \mathbb{A}+q & \mathbb{A} & \mathbb{A} & \mathbb{A} & q \\ \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 & \scriptstyle 7 & \scriptstyle q \end{bmatrix} .$$

Then one has the relation

$$[467]\,[56q] - [456]\,[67q] = [567]\,[46q] \ ,$$

which translates into

$$S_{555}(\mathbb{A}+q)S_{55}(\mathbb{A}-q) - qS_{444}(\mathbb{A}+q)S_{66,0}(\mathbb{A}, q) = S_{5^3}(\mathbb{A})S_{5,5,0}(\mathbb{A}+q, \mathbb{A}, q) \ .$$

This is what is needed, because $S_{5,5,0}(\mathbb{A}+q, \mathbb{A}, q) = S_{55,0}(\mathbb{A}+q, q) = S_{55}(\mathbb{A}-q)$.

The second statement of the lemma is obtained from the first one under the involution $\mathbb{A} \mapsto -\mathbb{A}$. QED

Euclidean Division

Let $P(x)$ and $Q(x)$ be two polynomials in one variable, with respective roots $\mathbb{B}$, $\mathbb{A}$: $P(x) = R(x, \mathbb{B})$, $Q(x) = R(x, \mathbb{A})$.

Algebraic operations on these two polynomials preserve the symmetry in the roots of each of the two polynomials, and thus produce symmetric functions in $\mathbb{A}$ and $\mathbb{B}$ separately.

Let us look at the iterated division of P and Q, i.e.

$$\begin{aligned} P &= \star Q + \mathcal{R}_1 \ , \\ Q &= \star \mathcal{R}_1 + \mathcal{R}_2 \ , \\ \mathcal{R}_1 &= \star \mathcal{R}_2 + \mathcal{R}_3 \ , \ \ldots \end{aligned} \tag{2}$$

the polynomials $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \ldots$ being the successive remainders, the $\star$'s being some polynomial coefficients, of degree 1 in the generic case, except possibly for the first step.

Euler found that the first remainder $\mathcal{R}_1$ can be characterized by the fact that it is, up to a scalar (we mean a function independent of x, but depending on $\mathbb{A}$ and $\mathbb{B}$), the only polynomial in the space linearly generated by $P(x), Q(x), xQ(x), \ldots, x^{m-n}Q(x)$ of degree $\leq n$–1, with $card(\mathbb{A}) = n$, $card(\mathbb{B}) = m$.

In other words, for any integer N, let $\mathcal{P}ol_N(\mathbb{A}, \mathbb{B})$ be the linear span of

$$S^m(x - \mathbb{B}), \ldots, S^N(x - \mathbb{B})\,;\ S^n(x - \mathbb{A}), \ldots, S^N(x - \mathbb{A}) \ .$$

Then $\mathcal{R}_1$ is the only polynomial of degree $\leq n - 1$ in $\mathcal{P}ol_m(\mathbb{A}, \mathbb{B})$.

This result is clear when one eliminates the powers $x^m, x^{m-1}, \ldots, x^n$ in $S^m(x - \mathbb{B})$ by subtracting scalar multiples of $S^m(x - \mathbb{A}), \ldots, S^n(x - \mathbb{A})$: the process is triangular and the scalar by which to multiply the polynomials $S^m(x - \mathbb{A}), \ldots, S^n(x - \mathbb{A})$ are fixed at each step.

This process can conveniently be represented by a determinant

$$\mathcal{D}_1(\mathbb{A}, \mathbb{B}) := \begin{vmatrix} S_0(-\mathbb{B}) & \cdots & S_{m-n}(-\mathbb{B}) & S_m(x - \mathbb{B}) \\ S_0(-\mathbb{A}) & \cdots & S_{m-n}(-\mathbb{A}) & S_m(x - \mathbb{A}) \\ S_{-1}(-\mathbb{A}) & \cdots & S_{m-n-1}(-\mathbb{A}) & S_{m-1}(x - \mathbb{A}) \\ \vdots & & \vdots & \vdots \\ S_{n-m}(-\mathbb{A}) & \cdots & S_0(-\mathbb{A}) & S_n(x - \mathbb{A}) \end{vmatrix} \ . \tag{$Div1$}$$

Proposition 12 *The determinant $\mathcal{D}_1(\mathbb{A},\mathbb{B})$ is the first remainder of the division of $S^m(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$, with $card(\mathbb{A})=n$, $card(\mathbb{B})=m$.*

Proof. If we expand the determinant according to its last column, we see that it belongs to the space $\mathcal{P}ol_m$. To find the degree of $\mathcal{D}_1$ in x, one expands each polynomial in the last column as follows (this expression is valid for any $k \le m$; the reader should not be perplexed by the fact that we have introduced complete functions of negative index, which are 0):

$$S^k(x-C) = x^m S_{k-m}(-C) + x^{m-1}S_{k-m+1}(-C) + \cdots + x^0 S_k(-C) \ .$$

Now, subtracting column $1,2,\ldots$, multiplied by the respective factors $x^m, x^{m-1},\ldots$ to the last column, one sees that all terms of degree $> n-1$ disappear. QED

It is now easy to infer the general theorem. We shall not repeat its proof which is exactly the same as for the first remainder, apart from different degrees.

Euler's characterization remains valid: the r-th remainder is, up to a scalar, the unique polynomial of degree $\le n-r$, in the space $\mathcal{P}ol_{m+r-1}$. It is easy to write a determinant which satisfy this characterization.

Let $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ be the following determinant of order $m-n+2r$, with $k=m-n+2r-2$:

$$\mathcal{D}_r(\mathbb{A},\mathbb{B}) := \begin{vmatrix} S_0(-\mathbb{B}) & \cdots & S_k(-\mathbb{B}) & S_{m+r-1}(x-\mathbb{B}) \\ \vdots & & \vdots & \vdots \\ S_{-r+1}(-\mathbb{B}) & \cdots & S_{k-r+1}(-\mathbb{B}) & S_m(x-\mathbb{B}) \\ S_0(-\mathbb{A}) & \cdots & S_k(-\mathbb{A}) & S_{m+r-1}(x-\mathbb{A}) \\ \vdots & & \vdots & \vdots \\ S_{n+1-m-r}(-\mathbb{A}) & \cdots & S_{r-1}(-\mathbb{A}) & S_n(x-\mathbb{A}) \end{vmatrix} \ . \qquad (Div2)$$

Theorem 13 *The determinant $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ is, up to non-zero scalar, equal to the r-remainder in the Euclidean division of $S^m(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$.*

For example, with $m=5$, $n=4$, $S^5(x-\mathbb{B}) = x^5+2x^4+3x^3+4x^2+3x+2$, $S^4(x-\mathbb{A}) = (x^5-1)/(x-1)$, one gets the following successive remainders, up to sign :

$$\mathcal{R}_1 = \begin{vmatrix} 1 & 2 & 3x^3+4x^2+3x+2 \\ 1 & 1 & x^3+\ x^2+\ x \\ 0 & 1 & x^3+\ x^2+\ x+1 \end{vmatrix} = x^3+2x^2+x+1,$$

$$\mathcal{R}_2 = \begin{vmatrix} 1 & 2 & 3 & 4 & 3x^2+2x \\ 0 & 1 & 2 & 3 & 4x^2+3x+2 \\ 1 & 1 & 1 & 1 & x^2 \\ 0 & 1 & 1 & 1 & x^2+\ x \\ 0 & 0 & 1 & 1 & x^2+\ x+1 \end{vmatrix} = 2x^2+x+2,$$

$$\mathcal{R}_3 = \begin{vmatrix} 1&2&3&4&3&2&0 \\ 0&1&2&3&4&3&2x \\ 0&0&1&2&3&4&3x+2 \\ 1&1&1&1&1&0&0 \\ 0&1&1&1&1&1&0 \\ 0&0&1&1&1&1&x \\ 0&0&0&1&1&1&x+1 \end{vmatrix} = 3x+2,\ \mathcal{R}_4 = \begin{vmatrix} 1&2&3&4&3&2&0&0&0 \\ 0&1&2&3&4&3&2&0&0 \\ 0&0&1&2&3&4&3&2&0 \\ 0&0&0&1&2&3&4&3&2 \\ 1&1&1&1&1&0&0&0&0 \\ 0&1&1&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1&0&0 \\ 0&0&0&1&1&1&1&1&0 \\ 0&0&0&0&1&1&1&1&1 \end{vmatrix} = 5\ .$$

Remark If $S^m(x-\mathbb{B})$ and $S^n(x-\mathbb{A})$ have a greatest common divisor G of order $n-r$, then the successive remainders will be divisible by G. This implies that r-th remainder is equal to G (up to a scalar), and that the remainder of higher order are null. Conversely, if $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ is different from 0, and the determinants $\mathcal{D}_p(\mathbb{A},\mathbb{B})$ are all 0 for $p>r$, then the greatest common divisor of $S^m(x-\mathbb{B})$ and $S^n(x-\mathbb{A})$ is of degree $n-r$ and equal to $\mathcal{D}_r(\mathbb{A},\mathbb{B})$.

Let us notice that the determinant $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ also furnishes Euler's multiplicators, i.e. the polynomials $C_{\mathbb{A}}$, $C_{\mathbb{B}}$ such that

$$\mathcal{D}_r(\mathbb{A},\mathbb{B}) = C_{\mathbb{A}}\, R(x,\mathbb{A}) + C_{\mathbb{B}}\, R(x,\mathbb{B})\ .$$

Indeed, evaluating $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ modulo $R(x,\mathbb{A})$ consists in changing the last column into $[S_{m+r-1}(x-\mathbb{B}),\dots,S_m(x-\mathbb{B}),0,\dots,0]$. Subtracting x to the alphabets in the first $r-1$ rows, one gets, as a last column,

$$[\,S_{m+r-1}(-\mathbb{B}),\dots,S_{m-1}(-\mathbb{B}),\ S_m(x-\mathbb{B}),0,\dots,0\,]$$

that is, $[\,0,\dots,0,R(x-\mathbb{B}),0,\dots,0\,]$, because the $S_k(-\mathbb{B})$ are $\pm$ the elementary symmetric functions of an alphabet of cardinality m.

Now the cofactor of $R(x-\mathbb{B})$ is the determinant

$$\begin{vmatrix} S_0(-x-\mathbb{B}) & \cdots & S_k(-x-\mathbb{B}) \\ \vdots & & \vdots \\ S_{-r+2}(-x-\mathbb{B}) & \cdots & S_{k-r+2}(-x-\mathbb{B}) \\ S_0(-\mathbb{A}) & \cdots & S_k(-\mathbb{A}) \\ \vdots & & \vdots \\ S_{n+1-m-r}(-\mathbb{A}) & \cdots & S_{r-1}(-\mathbb{A}) \end{vmatrix}$$

Expanding this last determinant according to the first $r-1$ rows, one recognize that it is equal to $S_{\square}(\mathbb{A}-x-\mathbb{B})$, with $\square=(r-1)^{m-n+r}$, $k=m-n+2r-2$.

By symmetry changing $\mathbb{A},\mathbb{B}$, one therefore gets

$$\mathcal{D}_r(\mathbb{A},\mathbb{B}) = \pm S_{(m-n+r)^{r-1}}(\mathbb{A}-x-\mathbb{B})R(x,\mathbb{B}) \pm S_{r^{m-n+r-1}}(\mathbb{B}-x-\mathbb{A})R(x,\mathbb{A}) \ , \tag{$Div3$}$$

with signs that specialists will know how to write.

Let us parameterize the remainder by its degree d, to emphasize symmetry between $\mathbb{A}$ and $\mathbb{B}$. Then $(Div3)$ shows that $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ can be interpreted as a first remainder :

Lemma 14 *Let $\mathbb{A}$, $\mathbb{B}$ be two alphabets, $card(\mathbb{A})=n$, $card(\mathbb{B})=m$, d be an integer : $d<m,n$. Then the remainder of degree d in the Euclidean division of $R(x,\mathbb{B})$ by $R(x,\mathbb{A})$ is equal, up to a constant, to the remainder of*

$$S_{(m-d)^{n-d-1}}(\mathbb{A}-x-\mathbb{B})\, R(x,\mathbb{B}) = S_{(m-d)^{n-d-1},m}(\mathbb{A}-\mathbb{B}, x-\mathbb{B}) \tag{$Div4$}$$

modulo $R(x,\mathbb{A})$, and to the remainder of

$$S_{(n-d)^{m-d-1}}(\mathbb{B}-x-\mathbb{A})\, R(x,\mathbb{A}) = S_{(n-d)^{m-d-1},n}(\mathbb{B}-\mathbb{A}, x-\mathbb{A}) \tag{$Div5$}$$

modulo $R(x,\mathbb{B})$.

One has a more compact expression of the remainders, thanks to Schur functions. Let us begin with the first remainder.

Proposition 15 *The first remainder in the division of $S^m(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$ is equal to*

$$S_{1^{n-1},m-n+1}(\mathbb{A}-x,\mathbb{A}-\mathbb{B}) \ . \tag{$Div6$}$$

Proof. The degree (in x) of $S_{1^{n-1},m-n+1}(\mathbb{A}-x,\mathbb{A}-\mathbb{B})$ is at most $n-1$, since x appears in degree 1 in the $n-1$ first columns. To check the second criterium, that it belongs to the space $\mathcal{P}ol_m(\mathbb{A},\mathbb{B})$ we write

$$S_{1^n-1,m-n+1}(\mathbb{A}-x,\mathbb{A}-\mathbb{B}) = S_{1^n-1,m-n+1}\big(\mathbb{A}-x,(\mathbb{A}-x)+(x-\mathbb{B})\big)$$

and expand, according to $S^k(\big(\mathbb{A}-x,(\mathbb{A}-x)+(x-\mathbb{B})\big) = \sum_i S^{k-i}(\mathbb{A}-x)S^i(x-\mathbb{B})$. One has

$$S_{1^n-1,m-n+1}\big(\mathbb{A}-x,(\mathbb{A}-x)+(x-\mathbb{B})\big) = \sum_{i=0}^{\infty} S_{1^n-1,m-n+1-i}(\mathbb{A}-x)S^i(x-\mathbb{B}) \ .$$

Now, this sum decomposes into 3 sums, according to the range of i, noting $k:=m-n+1$:

• $i \leq k-1$. The functions $S_{1^{n}-1,m-n+1-i}(\mathbb{A}-x)$ factorize into $S_{1^n}(\mathbb{A}-x)S_{m-n-i}(\mathbb{A})$, that is, are scalar multiple of $S_{1^n}(\mathbb{A}-x) = \pm S^n(x-\mathbb{A})$. The second factor $S_i(x-\mathbb{B})$ is of degree $\leq k-1$.

• $k < i < m$ Then the Schur functions $S_{1^{n-1}0}, \ldots, S_{1^{n-1}2-n}$ are null, having two identical columns.

• $i = m$. The corresponding term is $S_{1^{n-1},-n+1}(\mathbb{A}-x)\, S_m(x-\mathbb{B})$, that is $\pm S_{0\ldots0}(\mathbb{A}-x)\, S_m(x-\mathbb{B})$. This term also belongs to the space $\mathcal{P}ol_m(\mathbb{A},\mathbb{B})$, and this finishes the proof of the proposition. QED

We have given a detailed proof by checking Euler's conditions. In fact, there is a three-lines proof, by just writing $x-\mathbb{B} = (x\text{-}\mathbb{A}) + (\mathbb{A}\text{-}\mathbb{B})$. Then

$$S^m(x-\mathbb{B}) = \sum_{0\leq i\leq n-1} S^i(x-\mathbb{A})S^{m-i}(\mathbb{A}\text{-}\mathbb{B}) + S^n(x-\mathbb{A})\sum_{i\geq n} x^{i-n}S^{m-i}(\mathbb{A}\text{-}\mathbb{B}) \ ,$$

using the factorizations $S^{n+j}(x-\mathbb{A}) = x^j S^n(x-\mathbb{A})$. The remainder is equal to the first sum, which is nothing but the expansion of $(-1)^{n-1}$ times the determinant of $\mathbb{S}_{1^{n-1},m-n+1}(\mathbb{A}\text{-}x, \mathbb{A}\text{-}\mathbb{B})$. QED

Now, we are ready to give the general theorem. The Schur function given below is clearly of degree $\leq n-r$. To check that it belongs to the space $\mathcal{P}ol_{m+r-1}$ is straightforward for those persons fluent in manipulations of determinants. We shall rather use specializations to get hold of it.

Theorem 16 *The r-remainder in the division of $S^m(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$ is equal to*

$$S_{1^{n-r},\,(m-n+r)^r}(\mathbb{A}-x, \mathbb{A}\text{-}\mathbb{B}) \ . \qquad (Div7)$$

Proof. Put $x = a \in \mathbb{A}$, $\mathbb{A}' := \mathbb{A}\setminus a$. Subtract $\mathbb{A}'$ to the alphabets of the first row, then a in the last $r-1$ columns. The alphabets are now

$$\begin{bmatrix} 0 & \cdots & 0 & a-\mathbb{B} & -\mathbb{B} & \cdots & -\mathbb{B} \\ \mathbb{A} & \ldots & \mathbb{A} & \mathbb{A}-\mathbb{B} & \mathbb{A}'-\mathbb{B} & \cdots & \mathbb{A}'-\mathbb{B} \\ \vdots & & \vdots & \vdots & & \vdots & \\ \mathbb{A} & \ldots & \mathbb{A} & \mathbb{A}-\mathbb{B} & \mathbb{A}'-\mathbb{B} & \cdots & \mathbb{A}'-\mathbb{B} \end{bmatrix}$$

and the first row is null, but for the entry $R(a,\mathbb{B})$, whose cofactor is $S_{0\ldots0}(\mathbb{A}')\times S_{(r-1)^{m-n+r}}(\mathbb{A}-a-\mathbb{B})$. This specialization therefore coincides with the one of $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ and thus the Schur function is equal to the r-th remainder, being identical in sufficiently many points. QED

For example, for $r=3, n=6, m=7$, the third-remainder $S_{111;444}(\mathbb{A}\text{-}x; \mathbb{A}\text{-}\mathbb{B})$ is equal to the following determinant (writing the alphabets at the bottom of the column in which they appear, and writing a "·" for a zero entry; a "0" stands for the function S^0) :

$$\begin{array}{r|cccccc|}
 & 1 & 2 & 3 & 7 & 8 & 9 \\
 & 0 & 1 & 2 & 6 & 7 & 8 \\
 & \cdot & 0 & 1 & 5 & 6 & 7 \\
 & \cdot & \cdot & 0 & 4 & 5 & 6 \\
 & \cdot & \cdot & \cdot & 3 & 4 & 5 \\
 & \cdot & \cdot & \cdot & 2 & 3 & 4 \\
\scriptstyle alphabet & \mathbb{A}-x & \mathbb{A}-x & \mathbb{A}-x & \mathbb{A}-\mathbb{B} & \mathbb{A}-\mathbb{B} & \mathbb{A}-\mathbb{B}
\end{array}$$

This determinant specializes, for $x = a$ into

$$S^6(a-\mathbb{B})\, S_{000}(\mathbb{A})\, S_{44}(\mathbb{A}-a-\mathbb{B}) \ .$$

We have restricted to the "generic case" where $\mathbb{A}$ and $\mathbb{B}$ are independent alphabets. However, one can freely specialize variables in the algebraic identities that we have written, if no disaster occurs like a zero in denominator. For example, the determinantal expression $(Div7)$ of the remainder of degree d is an expression of the G.C.D. of $R(x,\mathbb{A})$ and $R(x,\mathbb{B})$ when $\mathbb{A}\cap\mathbb{B}$ is of cardinality d.

Companion Matrix

One can use linear algebra to express the first remainder. Let $\mathbb{A}$ be still of cardinality n. Let $\mathcal{P}ol_n$ be the space of polynomials of degree $\leq n-1$; it has basis $\{x^0, x^1, \dots, x^{n-1}\}$. Let u be the morphism: $\mathcal{P}ol_n \mapsto \mathcal{P}ol_n$ "multiplication by x modulo $Sn(x-\mathbb{A})$".

Let $\mathcal{C}(\mathbb{A})$ (resp. $C(\mathbb{A})$) be the matrix the $n\times\infty$ (resp. $n\times n$) matrix expressing the remainders of $x^0, x^1, \dots$ (resp. $x^1, \dots, x^n$) modulo $S^n(x-\mathbb{A})$. These two matrices are called *companion matrices of* $S^n(x-\mathbb{A})$. The matrix expressing u in the basis of powers of x is $C(\mathbb{A})$ is precisely $C(\mathbb{A})$ and its powers are sub-matrices of $\mathcal{C}(\mathbb{A})$. We know from (?) that the remainder of x^k is $S_{1^{n-1},k-n+1}(\mathbb{A}-x,\mathbb{A})$, and therefore the coefficients of the remainder of x^k are hook-Schur functions, up to sign. They can be written $S_{k,0^{n-1}}(\mathbb{A}), S_{k-1,0^{n-2}}(\mathbb{A}), \dots, S_{k-n+1}(\mathbb{A})$. Finally

$$\mathcal{C}(\mathbb{A}) = \left[S_{k-i,0^{n-i-1}}(\mathbb{A})\right]_{0\leq k,\, 0\leq i\leq n-1}$$

For example, for $n=3$, the infinite companion matrix is

$$\mathcal{C}(\mathbb{A}) = \begin{bmatrix} S_{000} & S_{100} & S_{200} & S_{300} & S_{400} & S_{500} & \cdots \\ S_{-10} & S_{00} & S_{10} & S_{20} & S_{30} & S_{40} & \cdots \\ S_{-2} & S_{-1} & S_{0} & S_{1} & S_{2} & S_{3} & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & S_{300} & S_{400} & S_{500} & \cdots \\ 0 & 1 & 0 & S_{20} & S_{30} & S_{40} & \cdots \\ 0 & 0 & 1 & S_1 & S_2 & S_3 & \cdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & S_{111} & S_{112} & S_{113} & \cdots \\ 0 & 1 & 0 & -S_{11} & -S_{12} & -S_{13} & \cdots \\ 0 & 0 & 1 & S_1 & S_2 & S_3 & \cdots \end{bmatrix}$$

and the finite one is

$$C(A) = \begin{bmatrix} 0 & 0 & S_{111} \\ 1 & 0 & -S_{11} \\ 0 & 1 & S_1 \end{bmatrix} .$$

Suppose that all the elements of $\mathbb{A}$ are distinct. Let f be any polynomial. Then the composite of "multiplication by $f(x)$, and taking the remainder" is diagonal in the basis $\{S^{n-1}(x+a-\mathbb{A})\}_{a\in\mathbb{A}}$, with diagonal composed of $f(a) : a \in \mathbb{A}$. Indeed, since $f(x) - f(a)$ is a multiple of $(x-a)$, then $f(x)\, S^{n-1}(x+a-\mathbb{A})$ is congruent to $f(a)\, S^{n-1}(x+a-\mathbb{A})$ modulo $S^n(x-\mathbb{A})$. The *trace* of this morphism is therefore $\sum_{a\in\mathbb{A}} f(a)$, and each power sum $\psi_k(\mathbb{A}) = \sum_{a\in\mathbb{A}} a^k$ appears as the trace of u^k.

The resultant of two polynomials $R(x,\mathbb{A})$ and $R(x,\mathbb{B})$ can, of course, be expressed with the help of a companion matrix. Indeed, the determinant of the matrix $R\,(C(\mathbb{A}),\mathbb{B})$ factorizes into the product

$$\prod_{b\in\mathbb{B}} \det\left(C(\mathbb{A}) - b\right) = \prod_{b\in\mathbb{B}} R(\mathbb{A}, b) = R(\mathbb{A},\mathbb{B}) \ ,$$

and the result applies to any pair of polynomials, not necessarily in factorized form.

Wronski Algorithm

Wronski's method is different: he uses only division by a single polynomial $S^n(x-\mathbb{A})$, instead of dividing by the successive remainders.

Let us call *normalized difference* of two polynomials of degree $m, n : m \geq n$, any linear combination of them which is of degree $\leq m-1$ (thus it is defined up to multiplication by a non-zero scalar).

Wronski's first step is to take the remainders of $S^m(x-\mathbb{B}), S^{m+1}(x-\mathbb{B}), \ldots$ modulo $S^n(x-\mathbb{A})$. Then he builds a triangular table with these remainders in the first row, the entries of the successive rows being the normalized differences of the two entries next to to it in the row just above (our definition of normalized difference is imprecise when two consecutive polynomials on row r are of degree $< n-r$, because any linear combination of them can be taken to fill the next row – we shall not be interested by possible degeneracies). The first entry of row r is a linear combination of the remainders of

$$S^m(x-\mathbb{B}), \ldots , S^{m+r-1}(x-\mathbb{B})$$

and it is not difficult to see that it is in fact a linear combination of

$$S^m(x-\mathbb{B}),\ldots,S^{m+r-1}(x-\mathbb{B});\ S^n(x-\mathbb{A}),\ldots,S^{m+r-1}(x-\mathbb{A})\ .$$

Therefore, it is the r-th remainder.

For example, for the pair $S_5(x-\mathbb{B}) = x^5+2x^4+3x^3+4x^2+3x+2$, $S_4(x-\mathbb{A}) = x^4+x^3+x^2+x+1$ seen above, Wronski algorithm produces the following table of normalized differences :

$$\begin{array}{ccccccc}
x^3+2x^2+x+1 & & x^3-1 & & -x^3-x^2-2x-1 & & -x^2+1 \\
& 2x^2+x+2 & & x^2+2x+2 & & x^2-1 & \\
& & 3x+2 & & 2x+3 & & \\
& & & 5 & & &
\end{array}$$

The first row contains the remainders of $S_5(x-\mathbb{B}),\ldots,S_8(x-\mathbb{B})$ modulo $S_4(x-\mathbb{A})$; the successive remainders are the first entries of each row.

Division and continued fractions

Let us take $m=n+1$, and perform the Euclidean division of $S^{n+1}(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$, normalizing the successive remainders, by taking them unitary: $\mathcal{R}_r = x^{n-r}+\cdots$.

The iterated division produces a sequence of unitary polynomials of degree 1, $\mathcal{Q}_r$, as well as constants c_r, such that $\mathcal{R}_{-1} = S^{n+1}(x-\mathbb{B})$, $\mathcal{R}_0 = S^n(x-\mathbb{A})$,

$$\mathcal{R}_{r-1} = cQ_r\mathcal{R}_r - c_{r+1}\mathcal{R}_{r+1}\ . \qquad (Div1)$$

From the expression of the remainders seen above, one deduce the quotients $Q_r = x+\zeta_r$, with

$$\zeta_r = \frac{S_{1,(r+1)^r}(\mathbb{A},\mathbb{A}-\mathbb{B})}{S_{(r+1)^r}(\mathbb{A}-\mathbb{B})} - \frac{S_{1,r^{r-1}}(\mathbb{A},\mathbb{A}-\mathbb{B})}{S_{r^{r-1}}(\mathbb{A}-\mathbb{B})}\ .$$

However, these coefficients can be written using only $\mathbb{A}-\mathbb{B}$,

$$\zeta_r = \frac{S_{1(r+1)^r}(\mathbb{A}-\mathbb{B})}{S_{(r+1)^r}(\mathbb{A}-\mathbb{B})} - \frac{S_{1r^{r-1}}(\mathbb{A}-\mathbb{B})}{S_{r^{r-1}}(\mathbb{A}-\mathbb{B})}\ . \qquad (Div2)$$

because

$$S_{1,(r+1)^r}(\mathbb{A},\mathbb{A}-\mathbb{B}) = S_{1,(r+1)^r}(\mathbb{A}-\mathbb{B}+\mathbb{B},\mathbb{A}-\mathbb{B}) = \\ S_{1,(r+1)^r}(\mathbb{A}-\mathbb{B}) + S^1(\mathbb{B})\,S_{(r+1)^r}(\mathbb{A}-\mathbb{B})\ .$$

The terms $S^1(\mathbb{B})$ compensate each other.

For example, writing J for $S_J(\mathbb{A}-\mathbb{B})$, one has

$$\zeta_0 = [1],\ \zeta_1 = \frac{[12]}{[2]} - [1],\ \zeta_2 = \frac{[133]}{[33]} - \frac{[12]}{[2]},\ \zeta_3 = \frac{[1444]}{[444]} - \frac{[133]}{[33]}, \ldots$$

Recursions *(Div1)* can be put in a continued fraction form :

Proposition 17 *Let n be an integer, $\mathbb{A}, \mathbb{B}$ be two alphabets, $|\mathbb{A}| = n$, $|\mathbb{B}| = n+1$. Let $y = 1/x$. Let*

$$\mu_r := S_{(r-1)^{r-2}}(\mathbb{A}-\mathbb{B})\, S_{(r+1)^r}(\mathbb{A}-\mathbb{B}) / S_{r^{r-1}}(\mathbb{A}-\mathbb{B})^2\ ,\ r \geq 1\ .$$

Let $\zeta_0, \zeta_1, \ldots$ be the coefficients appearing in the Euclidean division of $S^{n+1}(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$. Then the continued fraction expansion of the rational function

$$S^{n+1}(x-\mathbb{A}) \,/\, S^{n+1}(x-\mathbb{B}) = \prod(1-ay) \,/\, \prod(1-by) = \sum y^j\, S^j(\mathbb{B}-\mathbb{A})$$

is

$$\cfrac{1}{(1+\zeta_0 y) - \cfrac{y^2\mu_1}{(1+\zeta_1 y) - \cfrac{y^2\mu_2}{(1+\zeta_2 y) - \cfrac{y^2\mu_3}{\ddots}}}}$$

$$= \cfrac{1}{\big(1+[1]y\big) + \cfrac{y^2[2]}{1+\big(\frac{[12]}{[2]}-[1]\big)y + \cfrac{y^2[33]/[2]^2}{1+\big(\frac{[133]}{[33]}-\frac{[12]}{[2]}\big)y + \cfrac{y^2[2][444]/[33]^2}{\ddots}}}}$$

where J denotes $S_J(\mathbb{A}-\mathbb{B})$.

The above continued fraction expansion of $\sigma_y(\mathbb{B}-\mathbb{A})$ stops because $S_J(\mathbb{A}-\mathbb{B}) = 0$ for all $J \supseteq (n+2)^{n+1}$. However, letting n tends towards infinity, and putting $\mathbb{B} = 0$, one gets the continued fraction expansion of $\sigma_y(-\mathbb{A}) = \lambda_{-y}(\mathbb{A})$ for any $\mathbb{A}$ (but now the link with Euclidean division has become obscure !).

For example, for $\mathbb{A} := \{a\}$, one gets

$$1 - ay = \cfrac{1}{1 + ay + \cfrac{a^2y^2}{1-ay}}$$

For $\mathbb{A}$ of cardinality 2, it is still possible to check directly that

$$1 - y\Lambda^1(\mathbb{A}) + y^2\Lambda^2(\mathbb{A}) = \cfrac{1}{1 + S_1 y + \cfrac{y^2 S_2}{1 + (\frac{S_{12}}{S_2} - S_1)y + \cfrac{y^2 S_{33}/S_2^2}{1 - \frac{S_{12}}{S_2} y}}}$$

The Euclidean division of x^3 by $x^2 - x\Lambda^1 + \Lambda^2$ produces indeed the two successive quotients $x + \Lambda^1$, $x - \frac{(\Lambda^1)^3 - 2\Lambda^1\Lambda^2}{(\Lambda^1)^2 - \Lambda^2} = x - \frac{S^3}{S^2} = x + (\frac{S_{12}}{S_2} - S_1)$

Wronski's continued fraction can be obtained from Stieltjes'one by "condensation" of terms two by two. But to describe it, we need to approach continued fractions from a broader point of view than only Euclidean division, and this what is done in the next section.

Reciprocal Differences and continued fractions

Fractions $\frac{a}{b}$ were known to Egyptian and Chinese mathematicians some thousand years ago. It is natural to iterate them and consider expressions like $a_1 + \dfrac{b_2}{a_2}$, $a_1 + \dfrac{b_2}{a_2 + \dfrac{b_3}{a_3}}$ ... Putting the dots inside the fraction, one gets a *continued fraction*

$$a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{\ddots}}} \tag{$Cf1$}$$

that we already encountered in the process of Euclidean division.

Stopping the continued fraction at step n, and reducing it to a fraction, one gets the *n-th convergent* of the continued fraction. The French terminology *Les Réduites* is more appropriate, because convergence is not always *au rendez-vous.*

Euler wrote a fundamental treatise (De Fractionibus continuis, Comm. Acad. Sc. Imp. Petropol. **9** (1737)) and found recursions between consecutive convergents which can conveniently be written in matrix form :

Proposition 18 *The convergents of a continued fraction*

$$\frac{P_n}{Q_n} := a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{\ddots \; + \cfrac{b_n}{a_n}}}}$$

can be expressed through multiplication of 2×2 *matrices:*

$$\begin{bmatrix} P_n \\ Q_n \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 1 \\ b_2 & 0 \end{bmatrix} \begin{bmatrix} a_3 & 1 \\ b_3 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & 1 \\ b_{n-1} & 0 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

Proof. Write

$$\begin{bmatrix} a_3 & 1 \\ b_3 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & 1 \\ b_{n-1} & 0 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Suppose the theorem is true for

$$a_2 + \frac{b_3}{\cdots + \frac{b_n}{a_n}} = \frac{a_2\alpha + \beta}{\alpha}$$

On the other hand,

$$\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 1 \\ b_2 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a_1 a_2 \alpha + b_2 \alpha + a_1 \beta \\ a_2 \alpha + \beta \end{bmatrix}$$

The two entries of this last vector are exactly the numerator and denominator of

$$a_1 + \frac{b_2}{\dfrac{a_2\alpha + \beta}{\alpha}} = \frac{a_1(a_2\alpha + \beta) + b_2\alpha}{a_2\alpha + \beta}$$

QED

Continued fraction can be used in interpolation of one variable functions. There are two approaches : either one wants the interpolant to coincide with the function in a given point up to a certain order, or one wants the interpolant to coincide with the function in a set of distinct points (of course, one can mix the two approaches).

Let us first look at the approximation of a formal series that one can choose to be $\sigma_z(\mathbb{A})$.

Choosing all $a_i = 1$, and replacing the b_i's by zb_i, one gets a continued fraction of the type

$$1 + \cfrac{zb_2}{1 + \cfrac{zb_3}{1 + \cfrac{zb_4}{\ddots}}}$$

For generic b_i's, one sees by induction that the degrees in z of the successive convergents $P_n(z)/Q_n(z)$ are

$$d^o(P_{2n}) = n,\ d^o(Q_{2n}) = n-1,\ d^o(P_{2n+1}) = n,\ d^o(Q_{2n+1}) = n$$

starting with $P_2/Q_2 = (1 + zb_2)/1$. Choosing appropriate (and unique) $b_2, \ldots, b_n$ allows to approximate $\sigma_z(\mathbb{A})$ up to order $n-1$ included, and the rational fraction $P_n(z)/Q_n(z)$ is uniquely determined (up to multiplication of both numerator and denominator by a common scalar). The explicit expression of the b_i's for the expansion of a rational function of the type $\sigma_z(\mathbb{A} - \mathbb{B})$ was first found by Heilerman (Crelle J. **23**(1845) 174–188, **48**(1852)190-206, Muir II, p.361-365) and can be rewritten in the following theorem usually ascribed to Stieltjes :

Theorem 19 *The series $\sigma_z(\mathbb{A})$ admits a continued fraction expansion* $1+\cfrac{zb_2}{1+\cfrac{zb_3}{\ddots}}$ *with*

$$b_{2n} = \frac{S_{n^n} S_{(n-1)^{n-2}}}{S_{n^{n-1}} S_{(n-1)^{n-1}}} \quad \& \quad b_{2n+1} = \frac{S_{(n+1)^n} S_{(n-1)^{n-1}}}{S_{n^n} S_{n^{n-1}}}$$

2) The convergents are

$$\frac{P_{2n}}{Q_{2n}} = (-1)^{n-1} z \frac{S_{n^n}(\mathbb{A}+1/z)}{S_{(n-1)^{n-1}}(\mathbb{A}-1/z)} \quad \& \quad \frac{P_{2n+1}}{Q_{2n+1}} = (-1)^n z \frac{S_{n^{n+1}}(\mathbb{A}+1/z)}{S_{(n+1)^n}(\mathbb{A}-1/z)}$$

The first parameters are

$$b_2 = S_1,\ b_4 = S_{22}/S_2S_1,\ b_6 = S_{333}S_2/S_{33}S_{22},\ \ldots$$

$$b_3 = -S_2/S_1,\ b_5 = S_{33}S_1/S_{22}S_2,\ b_7 = S_{444}S_{22}/S_{333}S_{33},\ \ldots$$

while the first convergents are

$$\frac{P_2}{Q_2} = zS_1(\mathbb{A}+\frac{1}{z})\ ,\ \frac{P_4}{Q_4} = -z\frac{S_{22}(\mathbb{A}+\frac{1}{z})}{S_3(\mathbb{A}-\frac{1}{z})}\ ,\ \frac{P_6}{Q_6} = z\frac{S_{333}(\mathbb{A}+\frac{1}{z})}{S_{44}(\mathbb{A}-\frac{1}{z})}\ ,\ldots$$

$$\frac{P_1}{Q_1} = 1\ ,\ \frac{P_3}{Q_3} = -\frac{S_{11}(\mathbb{A}+\frac{1}{z})}{S_2(\mathbb{A}-\frac{1}{z})}\ ,\ \frac{P_5}{Q_5} = \frac{S_{222}(\mathbb{A}+\frac{1}{z})}{S_{33}(\mathbb{A}-\frac{1}{z})}\ ,\ldots$$

To prove the theorem, we need to rewrite some Schur functions :

Lemma 20 *Let $k, n \in \mathbb{N}$, $\mathbb{A}$ be arbitrary, and z be an indeterminate. Then*

$$(-\frac{1}{z})^n S_{k^n}(\mathbb{A}-\frac{1}{z})\left(S_0(\mathbb{A})+\cdots+z^{k+n-1}S_{k+n-1}(\mathbb{A})\right) \equiv S_{(k-1)^{n+1}}(\mathbb{A}+\frac{1}{z})$$

modulo terms of degree $\geq k(n+1)$ in $\mathbb{A}$.

Proof. Up to a power of z and a sign, the left hand side is equal to

$$\frac{1}{z^n}S_{k^n,0}(\mathbb{A},\ \frac{1}{z})S_{k+n-1}(\mathbb{A}+\frac{1}{z}) = \begin{vmatrix} S_k(\mathbb{A}) & \cdots & S_{k+n-1}(\mathbb{A}) & \frac{1}{z^0}S_{k+n-1}(\mathbb{A}+\frac{1}{z}) \\ \vdots & & \vdots & \vdots \\ S_{k-n}(\mathbb{A}) & \cdots & S_{k-1}(\mathbb{A}) & \frac{1}{z^n}S_{k+n-1}(\mathbb{A}+\frac{1}{z}) \end{vmatrix}$$

Up to terms of degree $\geq k(n+1)$, this determinant coincide with $S_{k^n,k-1}(\mathbb{A},\ \mathbb{A}+\frac{1}{z}) = \pm S_{k-1+n,(k-1)^n}(\mathbb{A}+\frac{1}{z},\mathbb{A})$. The expansion of this determinant in terms of $1/z$ is a sum of determinants which, up to zero terms, is the same as the expansion of $z^{-n}S_{k-1,(k-1)^n}(\mathbb{A}+\frac{1}{z},\mathbb{A}) = z^{-n}S_{(k-1)^{n+1}}(\mathbb{A}+\frac{1}{z})$. Checking signs and powers of z, one gets the lemma. QED

Proof of theorem. The convergents are the only rational functions with Taylor's expansion coinciding with $\sigma_z(\mathbb{A})$ up to degree n. However, thanks to lemma, the expressions written for P_n/Q_n have also the same property.

As for the explicit expression of the b_i's, let us check by induction that they give the right convergents. Recall that the induction on convergents is $\begin{bmatrix} P_n \\ Q_n \end{bmatrix} = \begin{bmatrix} P_{n-1} \\ Q_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ b_n & 0 \end{bmatrix}$; the case $n=7$ will be convincing enough. Thus one has $P_7 = P_6 + zb_7\, P_5$, and this gives for b_7 the following expression (do not forget to normalize the P_i's !) :

$$b_7 = \frac{P_7 - P_6}{P_5} = \Big(\frac{z^4 S_{3333}(\mathbb{A}+\frac{1}{z})}{S_{333}(\mathbb{A})} - \frac{z^3 S_{333}(\mathbb{A}+\frac{1}{z})}{S_{33}(\mathbb{A})}\Big) \Big/ \frac{z^3 S_{222}(\mathbb{A}+\frac{1}{z})}{S_{22}(\mathbb{A})}$$

As usual, the looked for relation is afforded by a matrix

$$\begin{array}{r} \textit{indexing} \\ \\ \\ \\ \\ \textit{alphabets} \end{array} \left[\begin{array}{cccccc} \scriptstyle 0 & \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 \\ \cdot & S^0 & z^3S^3 & S^4 & S^5 & S^6 \\ \cdot & \cdot & z^2S^2 & S^3 & S^4 & S^5 \\ \cdot & \cdot & zS^1 & S^2 & S^3 & S^4 \\ S^0 & \cdot & S^0 & S^1 & S^2 & S^3 \\ 0 & 0 & \mathbb{A}+\frac{1}{z} & \mathbb{A} & \mathbb{A} & \mathbb{A} \end{array}\right]$$

The reader will recognize that the products of minors :

$$[2,4,5,6]\,[0,1,4,5] - [0,2,4,5]\,[1,4,5,6] = [0,4,5,6]\,[1,2,4,5]$$

indeed leads to $b_7 = -S_{444}S_{22}/S_{333}S_{33}$ QED

Given a function $f(x_1, x_2, \dots)$, one uses the following reciprocal of differences associated to a pair x_i, x_j

$$\frac{x_i - x_j}{f(\dots x_i \dots x_j \dots) - f(\dots x_j \dots x_i \dots)}\ .$$

One can start from a function of x_1 (considered as a function of the x_i's) and iterate these reciprocal of differences, at least while no zero comes in denominator. However, one prefers to modify slightly the preceding operations.

Given a function $f(x)$ and an alphabet $x_1, x_2, \dots$, define recursively functions $[i, j, \dots, h]$ of $x_i, x_j, \dots, x_h$, which are called *reciprocal differences of* f, by

$$[1] = f(x_1)\ ;\ [12] = \frac{x_1 - x_2}{f(x_1) - f(x_2)}$$

$$[123] = \frac{x_2 - x_3}{[12] - [13]} + [1]$$
$$\cdots$$
$$[12\cdots n] = \frac{x_{n-1} - x_n}{[1\cdots n-2, n-1] - [1\cdots n-2, n]} + [1\cdots n-2]$$

Associate to this sequence the continued fraction

$$g(x\,,\,1234\ldots) := [1] + \cfrac{x - x_1}{[12] + \cfrac{x - x_2}{[123] - [1] + \cfrac{x - x_3}{[1234] - [12] + \cdots}}}$$

Proposition 21 *(Thiele Interpolation) The continued fraction $g(x\,,\,1\ldots n)$ takes values $f(x_i)$ at points $x = x_1, \ldots, x_n$.*

Proof. Take $n = 4$ for simplicity of indices. One has a symmetry in the last two interpolation points: $g(x, 1234) = g(x, 1243)$. Indeed, $[123] - [1] + \frac{x-x_3}{[1234]-[12]} = [124] - [1] + \frac{x-x_4}{[1234]-[12]}$ because $[1234] = [1243]$ (a divided difference, or a reciprocal difference in x_3, x_4 furnishes a function symmetrical in x_3, x_4).

Now, $g(x, 1234)$ coincides with $g(x, 123)$ for $x = x_1, x_2, x_3$, and $g(x, 1243)$ coincides with $g(x, 124)$ for $x = x_1, x_2, x_4$. Supposing the proposition true for the previous order, one thus obtains it for the next one QED

Remark. One could have taken simpler reciprocal differences, recursively defined by

$$[1\ldots n]' := \frac{x_{n-1} - x_n}{[1\cdots n-2, n-1]' - [1\cdots n-2, n]'}$$

However these functions are not symmetrical in $x_1, \ldots, x_n$, contrary to the $[1\ldots n]$, as we shall see later, and are not easily expressed as Schur functions.

The continued fraction $g(x\,,\,1\ldots n)$ is of the type (Cf1), with $b_2, b_3, \ldots$ of degree 1 in x. As for convergents, for even n, $deg(P_n) = n/2$, $deg(Q_n) = n/2 - 1$; for odd n, $deg(P_n) = deg(Q_n) = (n-1)/2$. Now, since P_n/Q_n interpolates $f(x)$ in n points, counting the number of parameters, we see that $P_n(x)$ and $Q_n(x)$ are unique, up to multiplication of both by the same scalar. This uniqueness will allow us in the next theorem to express the convergents of Thiele's continued fraction as Schur functions.

Theorem 22 *Given an alphabet $\mathbb{Y}$ of cardinality k, let $f(x)$ be the polynomial $f(x) := S^k(x - \mathbb{Y})$. Then the convergents of Thiele continued fraction for f are expressible as Schur functions:*

$$\begin{cases} P_{2n} = S_{1^n,(k-n)^n}(\mathbb{X}_{2n} - x,\, X_{2n} - \mathbb{Y}) \\ Q_{2n} = \quad S_{(k-n)^{n-1}}(\mathbb{X}_{2n} - x - \mathbb{Y}) \end{cases}$$

$$\begin{cases} P_{2n+1} = S_{1^n,\,(k-n)^{n+1}}(\mathbb{X}_{2n+1}-x,\,\mathbb{X}_{2n+1}-\mathbb{Y}) \\ Q_{2n+1} = \qquad S_{(k-n)^n}(\mathbb{X}_{2n+1}-x-\mathbb{Y}) \end{cases}$$

Because of the uniqueness of a rational interpolant at n points with sum of degrees of the numerator and denominator $\le n-1$, we have only to check that P_n/Q_n coincides with $f(x)$ at points $x = x_1, \dots, x_n$.

Let us reason on an example, and take $n = 6$, $k = 7$. Then $P_6 = S_{111,\,444}(\mathbb{X}_6 - x,\, X_6 - \mathbb{Y}) =$

$$= \begin{vmatrix} 1 & 2 & 3 & 7 & 8 & 9 \\ 0 & 1 & 2 & 6 & 7 & 8 \\ \cdot & 0 & 1 & 5 & 6 & 7 \\ \cdot & \cdot & 0 & 4 & 5 & 6 \\ \cdot & \cdot & \cdot & 3 & 4 & 5 \\ \cdot & \cdot & \cdot & 2 & 3 & 4 \\ \mathbb{X}_6 - x & \mathbb{X}_6 - x & \mathbb{X}_6 - x & \mathbb{X}_6 - \mathbb{Y} & \mathbb{X}_6 - \mathbb{Y} & \mathbb{X}_6 - \mathbb{Y} \end{vmatrix}$$

Take any $i \le 6$ and subtract $(\mathbb{X} - x_i)$ to the alphabets in the top row. Since degrees are increasing by 1 when passing from column 4 to 5, 5 to 6, one can similarly subtract x_i to columns 5 and 6. Alphabets in the determinant are now, with $\mathbb{B} = \mathbb{X}_6 - x_i - \mathbb{Y}$,

$$\begin{vmatrix} 0 & 0 & 0 & x_i - \mathbb{Y} & -\mathbb{Y} & -\mathbb{Y} & -\mathbb{Y} \\ \mathbb{X}_6 - x_i & \mathbb{X}_6 - x_i & \mathbb{X}_6 - x_i & \mathbb{X}_6 - \mathbb{Y} & \mathbb{B} & \mathbb{B} & \mathbb{B} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mathbb{X}_6 - x_i & \mathbb{X}_6 - x_i & \mathbb{X}_6 - x_i & \mathbb{X}_6 - \mathbb{Y} & \mathbb{B} & \mathbb{B} & \mathbb{B} \end{vmatrix}$$

Because $S^j(-\mathbb{Y}) = 0$ for $j > 6$, the elements of the top row vanish all, except for $S^7(x_i - \mathbb{Y})$. The cofactor of this non-zero element factorizes into $S_{000}(\mathbb{X}_6 - x_i) = 1$, and $S_{44}(\mathbb{X}_6 - x_i - \mathbb{Y})$. On the other hand $Q_6(x) = S_{44}(\mathbb{X}_6 - x - \mathbb{Y})$, and therefore the specialization of $P_6(x)/Q_6(x)$ in x_i is equal to $S^7(x_i - \mathbb{Y})$ QED

One can notice that the numerators P_n of Thiele's convergents can be obtained in the Euclidean division of $f(x)$ by $S^n(x - x_1 - \cdots - x_n)$: they are the "middle remainders".

Reciprocal differences can themselves be expressed as Schur functions :

Proposition 23 *Given an alphabet $\mathbb{Y}$ and an integer k, let $f(x)$ be the polynomial $f(x) := S^k(x - \mathbb{Y})$. Then the reciprocal difference of f $[1, 2, \dots, n]$, $n = 1, 2, \dots$, of order n, is a quotient of two Schur functions of $x_1 + \cdots + x_n - \mathbb{Y}$:*

$$[1, \dots, 2m+1] = S_{(k-m)^{m+1}}/S_{(k-m-1)^m}\ ,$$

$$[1, \dots, 2m] = S_{(k-m-1)^{m-1}}/S_{(k-m)^m}$$

For example,

$$[1] = S_k,\ [123] = S_{(k-1)^2}/S_{k-2},\ [12345] = S_{(k-2)^3}/S_{(k-3)^2}\ \ \ldots$$

$$[12] = 1/S_{k-1},\ [1234] = S_{k-3}/S_{(k-2)^2},\ [123456] = S_{(k-4)^2}/S_{(k-3)^3}\ \ \ldots$$

Proof. Once more, we need only Plücker relations, but twice! Let us see how to obtain $[1,\ldots,7]$, knowing $[1,\ldots,6]$ and $[1,\ldots,5]$. Take $k=9$ for example, and write $S_J(\mathbb{A})$ for $S_J(\mathbb{A}+x_1+\cdots+x_5-\mathbb{Y})$. From the exercises, we know that

$$S_{55}(x_6)/S_{666}(x_6) - S_{55}(x_7)/S_{666}(x_7) = (x_7-x_6)S_{555}(x_6+x_7)S_{66}()/S_{666}(x_6)S_{666}(x_7)\ ,$$

and therefore

$$[1,\ldots,7] = -\frac{S_{666}(x_6)S_{666}(x_7)}{S_{555}(x_6+x_7)S_{66}()} + \frac{S_{777}()}{S_{66}()}\ .$$

However, one has also that

$$S_{666}(x_6)S_{666}(x_7) - S_{555}(x_6+x_7)S_{777}() = S_{6666}(x_6+x_7)S_{66}()\ ,$$

whence the required expression

$$[1,\ldots,7] = S_{6666}(x_1+\cdots+x_7-\mathbb{Y})/S_{555}(x_1+\cdots+x_7-\mathbb{Y})\ .$$

QED

Padé approximants

Given a formal series $f(z) := \sum z^n S_n$, with $S_0 = 1$, one wants a rational function $\prod_{b\in\mathbb{B}}(1-zb)/\prod_{a\in\mathbb{A}}(1-za)$ (with no factor common to the numerator and the denominator) whose Taylor expansion coincides with $f(z)$ up to the order $card(\mathbb{A})+card(\mathbb{B})$ (which is the maximum that one can expect, counting the number of parameters). In other words, one wants the first S_i to be the complete functions $S_i(\mathbb{A}-\mathbb{B})$.

Writing the rational function, up to a power of z, as follows

$$\frac{R(\frac{1}{z},\mathbb{B})}{R(\frac{1}{z},\mathbb{A})} = \pm\frac{R(\frac{1}{z}+\mathbb{A},\mathbb{B})}{R(\frac{1}{z}+\mathbb{B},\mathbb{A})}\ , \qquad (Pad1)$$

one could stop there, because the numerator and denominator of the last expression are functions of $\mathbb{A}-\mathbb{B}$, i.e. are expressible in terms of coefficients of the series. However, it would not be polite towards Padé, and we shall provide some comments about the different expressions of the rational function attached to his name.

Let us first remark that the extra factor that we have introduced, $R(\mathbb{A},\mathbb{B})$, is by hypothesis different from 0 (the rational function is reduced).

Now, our expression, with $card(\mathbb{A}) = \alpha$, $card(\mathbb{B}) = \beta$, is, thanks to the expression of the resultant

$$S_{(\alpha+1)^\beta}(\mathbb{B}-\mathbb{A}-\frac{1}{z})/S_{(\beta+1)^\alpha}(\mathbb{A}-\mathbb{B}-\frac{1}{z})$$

$$= \pm S_{\beta^{\alpha+1}}(\mathbb{A}-\mathbb{B}+\frac{1}{z})/S_{(\beta+1)^\alpha}(\mathbb{A}-\mathbb{B}-\frac{1}{z}) \qquad (Pad2)$$

Do not forget that, for any $\mathbb{A}$, any x of rank 1, on has the decompositions $S^k(\mathbb{A}+x) = \sum_i x^i S^{k-i}(\mathbb{A})$, $S^k(\mathbb{A}-x) = S^k(\mathbb{A}) - xS^{k-1}(\mathbb{A})$. This shows how to expand the preceding functions in terms of x.

In fact, using our favorite transformations on Schur functions, we can restrict $1/z$ to only one column and write the rational function :

$$S_{\beta,\beta^\alpha}(\mathbb{A}-\mathbb{B}+\frac{1}{z};\mathbb{A}-\mathbb{B})/S_{(\beta+1)^\alpha,0}(\mathbb{A}-\mathbb{B};\frac{1}{z}) \qquad (Pad3)$$

These expressions can be found in the literature: the last expression is due to Jacobi (?), while Sylvester (?) gave the right hand side of (Pad2).

The explicit expansion of the numerator and the denominator of the Padé approximant, in terms of the Schur functions of $\mathbb{A}-\mathbb{B}$, is

$$\frac{S_{\beta^\alpha} + zS_{1\beta^\alpha} + z^2 S_{2\beta^\alpha} + \cdots + z^\beta S_{\beta\beta^\alpha}}{S_{\beta^\alpha} - zS_{\beta^{\alpha-1},\beta+1} + z^2 S_{\beta^{\alpha-2},(\beta+1)^2} + \cdots + (-z)^\alpha S_{(\beta+1)^\alpha}} \,. \qquad (Pad4)$$

For example, for $\beta=1, \alpha=1$, one gets

$$\frac{S_1 + zS_{11}}{S_1 - zS_2} = 1 + zS^1 + z^2S^2 + z^3S^{22}/S^1 + z^4 S^{222}/S^{11} + \cdots \ ,$$

for $\beta=1, \alpha=2$,

$$\frac{S_{11} + zS_{111}}{S_{11} - zS_{12} + z^2 S_{22}} = 1 + zS^1 + z^2S^2 + z^3S^3 + z^4 \frac{S_{114} + S_{15} - S_{222}}{S_{11}} + \cdots \ ,$$

for $\beta=2, \alpha=1$,

$$\frac{S_2 + zS_{12} + z^2 S_{22}}{S_2 - zS_3} = 1 + zS^1 + z^2S^2 + z^3S^3 + z^4 \frac{S^{33}}{S^2} + \cdots$$

and for $\beta=2, \alpha=2$,

$$\frac{S_{22} + zS_{122} + z^2 S_{222}}{S_{22} - zS_{23} + z^2 S_{33}} = 1 + zS^1 + z^2S^2 + z^3S^3 + z^4S^4 + \frac{S^{144} - 2S^{234} + S^{333}}{S_{22}} + \cdots$$

We have supposed that $R(\mathbb{A},\mathbb{B})$ be different from 0, that is, we have supposed the non-vanishing of the Hankel determinant $S_{\beta^\alpha}(\mathbb{A}-\mathbb{B})$ of coefficients of the initial series. Specialists know what to do in the degenerate cases.

Write $[\beta/\alpha]$ for the approximant whose denominator is of degree α, and numerator of degree β (we only consider the generic case, where there is no indeterminacy, nor any drop of degree).

Usually, the $[\beta/\alpha]$ are displayed in a two-dimensional table (let us take $\beta = 0, 1, .. =$ horizontal axis), and one is interested in giving relations between neighbours. Now, the alphabets $\mathbb{A}$ and $\mathbb{B}$ are variable and we shall avoid using them. Say that the function to be interpolated is $\sigma_z(\mathbb{D})$.

Because $[\beta\text{-}1/\alpha]$ and $[\beta/\alpha]$ are approximations of consecutive orders of the same function, one sees that $[\beta/\alpha] - [\beta\text{-}1/\alpha]$ is a rational function with a numerator reduced to $cz^{\alpha+\beta}$, with $c \in \mathfrak{S}ym$. In fact, we have already encountered the relation

$$S_{k^n}(\mathbb{A}+x)S_{k^{n-1}}(\mathbb{A}-x) - xS_{(k-1)^n}(\mathbb{A}+x)S_{(k+1)^{n-1}}(\mathbb{A}-x) = S_{k^n}(\mathbb{A})S_{k^{n-1}}(\mathbb{A}) \ ,$$

and its image under $\mathbb{A} \mapsto -\mathbb{A}$.

This relation gives us

$$[\beta/\alpha] - [\beta\text{-}1/\alpha] = (-1)^\alpha z^{\beta-\alpha} \frac{S_{\beta^\alpha}(\mathbb{D}) S_{\beta^{\alpha+1}}(\mathbb{D})}{S_{\beta^\alpha}(\mathbb{D}-1/z) S_{(\beta+1)^\alpha}(\mathbb{D}-1/z)} , \qquad (Pad5)$$

as well as

$$[\beta/\alpha] - [\beta/\alpha\text{-}1] = (-1)^\alpha z^{\beta+1-\alpha} \frac{S_{\beta^\alpha}(\mathbb{D}) S_{(\beta+1)^\alpha}(\mathbb{D})}{S_{(\beta+1)^{\alpha-1}}(\mathbb{D}-1/z) S_{(\beta+1)^\alpha}(\mathbb{D}-1/z)} . \qquad (Pad6)$$

Now, one can relate the inverse of differences between $[\beta/\alpha]$ and its North, East, South, West neighbours, as shows the following lemma, due to Wynn.

Lemma 24 *The elements of a generic Padé table satisfy :*

$$\frac{1}{[\beta/\alpha\text{+}1] - [\beta/a]} + \frac{1}{[\beta/\alpha\text{-}1] - [\beta/a]} = \frac{1}{[\beta\text{-}1/\alpha] - [\beta/a]} + \frac{1}{[\beta\text{+}1/\alpha] - [\beta/a]} \qquad (Pad7)$$

Proof. Remember relation [Bazin ?]

$$\begin{vmatrix} [\mathcal{C}12] & [\mathcal{C}13] & [\mathcal{C}14] \\ [\mathcal{C}32] & [\mathcal{C}33] & [\mathcal{C}34] \\ [\mathcal{C}25] & [\mathcal{C}35] & [\mathcal{C}45] \end{vmatrix} = 0$$

for the minors of order n of a matrix $n \times \infty$, $\mathcal{C}$ being a set of integers of cardinality $n-2$.

Apply this relation to the matrix obtained from $\mathbb{S}_{(\beta+1)^\alpha}(\mathbb{D}-x)$ by repeating the first column, with the alphabet $\mathbb{D}$, and adding the two columns $[1,0,\dots,0]$ and $[0,\dots,0,1]$. Explicitely, with $\alpha = 3$, $\beta = 5$, I mean the matrix

$$\begin{array}{c} \\ \\ \\ \\ \textit{columns} \end{array} \begin{bmatrix} 1 & 0 & S^6(\mathbb{D}) & S^6(\mathbb{D}-x) & S^7(\mathbb{D}-x) & S^8(\mathbb{D}-x) & S^9(\mathbb{D}-x) \\ 0 & 0 & S^5(\mathbb{D}) & S^5(\mathbb{D}-x) & S^6(\mathbb{D}-x) & S^7(\mathbb{D}-x) & S^8(\mathbb{D}-x) \\ 0 & 0 & S^4(\mathbb{D}) & S^4(\mathbb{D}-x) & S^5(\mathbb{D}-x) & S^6(\mathbb{D}-x) & S^7(\mathbb{D}-x) \\ 0 & 1 & S^3(\mathbb{D}) & S^3(\mathbb{D}-x) & S^4(\mathbb{D}-x) & S^5(\mathbb{D}-x) & S^6(\mathbb{D}-x) \\ {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle \mathcal{C}} & {\scriptstyle \mathcal{C}} & {\scriptstyle 5} \end{bmatrix} .$$

Using that $[34\,\mathcal{C}] = S_{6,555}(\mathbb{D},\mathbb{D}\text{-}x) = S_{65,55}(\mathbb{D},\mathbb{D}\text{-}x) - xS_{64,55}(\mathbb{D},\mathbb{D}\text{-}x) = xS_{55,55}(\mathbb{D},\mathbb{D}\text{-}x) = xS_{5555}(\mathbb{D})$, one gets the relation

$$\begin{vmatrix} S_{66}(\mathbb{D}\text{-}x) & S_{555}(\mathbb{D}) & S_{555}(\mathbb{D}\text{-}x) \\ -S_{666}(\mathbb{D}) & 0 & xS_{5555}(\mathbb{D}) \\ S_{777}(\mathbb{D}\text{-}x) & S_{6666}(\mathbb{D}) & S_{6666}(\mathbb{D}\text{-}x) \end{vmatrix} = 0 ,$$

which expands into
$S_{777}(\mathbb{D}-x)S_{555}(\mathbb{D})xS_{5555}(\mathbb{D}) + S_{666}(\mathbb{D})S_{555}(\mathbb{D})S_{6666}(\mathbb{D}-x) -$
$S_{666}(\mathbb{D})S_{6666}(\mathbb{D})S_{555}(\mathbb{D}-x) - S_{66}(\mathbb{D}-x)S_{6666}(\mathbb{D})xS_{5555}(\mathbb{D}) = 0$,
which is the Wynn identity reduced to the same denominator (up to the common factor $S_{(\beta+1)^\alpha}(\mathbb{D}-x)$).

See in the exercises a step by step proof of Wynn's identity.

Euclidean division can be put into use to provide recursions between Padé entries. Let $[\beta/\alpha]$ denote a Padé approximant of the function $\lambda_{-z}(\mathbb{A})$. Then the following lemma, due to Kronecker, relates the approximants $[n-j/j]$, $j = 0, \ldots, n$ for any fixed n.

Lemma 25 *Let n be an integer. Define $P_{-1} = z^{n+1}$, $P_0 = S^n(1 - z\mathbb{A})$, $Q_{-1} = 0$, $Q_0 = 1$, and for $j = 0, \ldots, n-1$,*

$$\begin{aligned} P_{j+1} &= -P_{j-1} + \rho_j P_j \\ Q_{j+1} &= -Q_{j-1} + \rho_j Q_j \ , \end{aligned} \tag{Pad8}$$

that is, $-P_{j+1}$ is the remainder of the division of P_{j-1} by P_j, and ρ_j is the quotient.

Then P_j/Q_j is the Padé approximant of type $[n-j/j]$ of $\lambda_{-z}(\mathbb{A})$, and

$$P_j Q_{j+1} - P_{j+1} Q_j = z^{n+1} \ .$$

Proof. The successive remainders have described in the third section. Up to normalization, they are

$$z^{1-n} S_{2^{n-1}}(A-z), \ z^{2-n} S_{3^{n-2}}(A-z), \ z^{3-n} S_{4^{n-3}}(A-z), \ \ldots$$

On the other hand, equations (Pad8) implies that the determinant $\begin{vmatrix} P_j & P_{j+1} \\ Q_j & Q_{j+1} \end{vmatrix}$ is independent of j, and this determines the Q_j's knowing the P_j's. Recognizing in this determinant relation (?), one gets that $Q_1, Q_2, \ldots$ are, up to the same normalization as the P_j's,

$$S_1(\mathbb{A} + z), \ S_{22}(\mathbb{A} + z), \ \ldots, \ S_{j^j}(\mathbb{A} + z), \ldots \ .$$

Checking values in $z = 0$, one indeed recognizes that P_j/Q_j is the $[n-j/j]$-approximant. QED

Padé approximants are rational interpolants of $f(z)$. But in interpolation theory, one wants to write expressions depending upon the values of $f(z)$ at specific points $y_1, y_2, \ldots \in \mathcal{Y}$. The preceding case, that is, requiring the equality of the first terms in Taylor expansions around $z = 0$ must be considered as the case where all y_i's are concentrated in 0.

Already Cauchy had solved the rational interpolation problem in 1820, writing a summation generalizing Lagrange interpolation. We shall rather use the following determinantal expression, and give its relation with Padé expressions.

Lemma 26 *Let $f(z)$ be a function of one indeterminate, and let $\alpha, \beta \in \mathbb{N}$. Let $P(z)/Q(z)$, $deg(P) \le \beta$, $deg(Q) \le \alpha$, coïncide with $f(z)$ at points $y \in \mathcal{Y}$, $card(\mathcal{Y}) = \alpha + \beta + 1$, . Then P and Q are determined, up to a factor, by the following identity :*

$$\mathcal{Y}\left\{\begin{vmatrix} \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 1 & y & \cdots & y^\beta & y^\alpha f(y) & \cdots & yf(y) & f(y) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ Q(z) & zQ(z) & \cdots & z^\beta Q(z) & z^\alpha P(z) & \cdots & zP(z) & P(z) \end{vmatrix}\right. = 0 \qquad (Pad9)$$

Proof. Multiply row y by the corresponding factor $Q(y)$. The new determinant is an alternating function in $y_1, \ldots, y_{\alpha+\beta+1}, z$ of degree $\le \alpha+\beta$ in each letter. Being divisible by the Vandermonde in $\mathcal{Y} + z$, it must be null. QED

To get more compact expressions of P and Q, let us suppose that $f(z)$ is a polynomial that we shall write $f(z) = S^N(z - \mathbb{D})$.

The denominator $Q(z)$ is proportional to

$$\begin{vmatrix} \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 1 & y & \cdots & y^\beta & y^\alpha f(y) & \cdots & yf(y) & f(y) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & z^\alpha & \cdots & z & 1 \end{vmatrix} .$$

Extracting the Vandermonde $\Delta(\mathcal{Y})$ from the upper part, one transforms the preceding determinant into a determinant of complete functions :

$$\begin{vmatrix} S^0(\mathcal{Y}) & \cdots & S^\beta(\mathcal{Y}) & S^{N+\alpha}(\mathcal{Y} - \mathbb{D}) & \cdots & S^N((\mathcal{Y} - \mathbb{D}) \\ \vdots & & \vdots & \vdots & & \vdots \\ S^{-\alpha-\beta}(\mathcal{Y}) & \cdots & S^{-\alpha}(\mathcal{Y}) & S^{N-\beta}(\mathcal{Y} - \mathbb{D}) & \cdots & S^{N-\alpha-\beta}((\mathcal{Y} - \mathbb{D}) \\ 0 & \cdots & 0 & z^\alpha & \cdots & z^0 \end{vmatrix} .$$

The determinant factorizes into its North-West part (equal to $\mathbb{S}_{0^{\beta+1}}(\mathcal{Y}) = 1$) and its South-East part in which one recognizes the permuted of

$$S_{(N-\beta)^\alpha,0}(\mathcal{Y} - \mathbb{D}, z) = S_{(N-\beta)^\alpha}(\mathcal{Y} - \mathbb{D} - z) .$$

Similarly, $P(z)$ is proportional to

$$\begin{vmatrix} S^0(\mathcal{Y}) & \cdots & S^\beta(\mathcal{Y}) & S^{N+\alpha}(\mathcal{Y}-\mathbb{D}) & \cdots & S^N((\mathcal{Y}-\mathbb{D}) \\ \vdots & & \vdots & \vdots & & \vdots \\ S^{-\alpha-\beta}(\mathcal{Y}) & \cdots & S^{-\alpha}(\mathcal{Y}) & S^{N-\beta}(\mathcal{Y}-\mathbb{D}) & \cdots & S^{N-\alpha-\beta}((\mathcal{Y}-\mathbb{D}) \\ z^0 & \cdots & z^\beta & 0 & \cdots & 0 \end{vmatrix} .$$

Subtracting z to alphabets in columns $2, \ldots, \alpha+1$, the bottom row becomes $[1, 0, \ldots, 0]$ and what remains of the determinant after suppressing the first column and bottom row (and permuting the last columns) is the Schur function $S_{1^\beta,(N-\beta)^{\alpha+1}}(\mathcal{Y}-z, \mathcal{Y}-\mathbb{D})$. This function is in fact the remainder of degree β in the Euclidean division of $S^N(z-\mathbb{D})$ by $R(z, \mathcal{Y})$, as one could have realized beforehand without computation!

In summary, one has

Lemma 27 *The approximant of type $[\beta/\alpha]$ of the function $S^N(z-\mathbb{D})$ obtained from the interpolation set $\mathcal{Y}$ of cardinality $\alpha+\beta+1$ is*

$$S_{1^b,(N-\beta)^{\alpha+1}}(\mathcal{Y}-z, \mathcal{Y}-\mathbb{D}) \,/\, S_{(N-\beta)^\alpha}(\mathcal{Y}-\mathbb{D}-z) \ . \qquad (Pad10)$$

Questions of convergence are at the heart of the Padé approximation paradigm, and we have only touched upon the relevant algebraic identities. This algebraic part is a small part of the story, however for what concerns it proper, we disagree with most of the classics.

First, the theory is full of quadratic relations between minors, and no reference is usually given to Plücker relations, which are more general and more understandable that the 3 terms or 5 terms relations commonly met in the theory.

Secondly, many functions can be identified with Schur functions. When restricting to Schur functions indexed by rectangular partitions, one can dispense with symmetric function theory. However, inductions involve other partitions than the rectangular ones, and this becomes cumbersome when restricting to Hankel or Toeplitz determinants.

Thirdly, using partitions is the only way of easily seeing that a function appearing in a certain process is the same Schur function as the one pertaining to another theory. Even to identify some coefficients in the Euclidean division of two polynomials (when they are not the leading terms) would be cumbersome without reference to Schur functions.

In the fourth place, this is a book about the beauties of symmetric functions and their generalizations.

Symmetrizing operators

Divided differences ∂_σ have been introduced by Newton to solve the interpolation problem in one variable. We shall also need the *isobaric divided differences* π_σ.

In the case where σ is a simple transposition s_i exchanging a_i and a_{i+1}, writing then ∂_i, π_i, the operators (**acting on their left**) are :

$$\begin{cases} \partial_i &= \frac{1}{(a_i - a_{i+1})}(1+s_i) &= (1-s_i)\frac{1}{(a_i-a_{i+1})} \\ \pi_i &= \frac{1}{(1-a_{i+1}/a_i)}(1+s_i) &= a_i\partial_i \end{cases} \tag{3}$$

More explicitely, ∂_i and π_i are the operators on functions $f(a_1, \ldots, a_n)$ of several variables, acting only on the pair (a_i, a_{i+1}):

$$\begin{cases} f\,\partial_i &= \frac{f(\ldots a_i, a_{i+1}\ldots) - f(\ldots a_{i+1}, a_i \ldots)}{a_i - a_{i+1}} \\ f\,\pi_i &= \frac{a_i\, f(\ldots a_i, a_{i+1}\ldots) - a_{i+1}\, f(\ldots a_{i+1}, a_i \ldots)}{a_i - a_{i+1}} \end{cases} \tag{4}$$

We shall check below that each of the families of operators $\{s_i\}$, $\{\partial_i\}$, $\{\pi_i\}$ separately satisfies the Moore/Coxeter *braid relations* (writing $\{D_i\}$ in each of these three cases) :

$$D_iD_j = D_jD_i \text{ if } |i-j| \geq 2 \quad , \quad D_iD_{i+1}D_i = D_{i+1}D_iD_{i+1} \tag{5}$$

One can notice that, for what concerns squares, one has

$$\partial_i\partial_i = 0 \ , \ \pi_i\pi_i = \pi_i \ , \ s_is_i = 1 \ . \tag{6}$$

Both divided differences ∂_i and isobaric divided differences π_i are special cases of local rational operators of the type

$$D_i = P(a_i,\, a_{i+1}) + Q(a_i,\, a_{i+1})\, s_i \ , \tag{7}$$

where $P(x,y)$ and $Q(x,y)$ are two fixed rational functions of two variables. In [24] [26], one determines the conditions on P and Q which ensures that the D_i's satisfy braid relations and preserve polynomials. These operators happen to satisfy an *Hecke relation*, i.e. there exists constants q_1, q_2 such that

$$(D_i - q_1)(D_i - q_2) = 0 \ , \ \forall i \geq 1 \ . \tag{8}$$

To represent the Hecke algebra, one usually take

$$T_i = \pi_i(q_1+q_2) - q_2\, s_i\ ,\ i \geq 1. \tag{9}$$

It is convenient to represent the symmetric group by its *Cayley graph*, called *permutohedron*, which describes how permutations are successively generated by products of simple transpositions, starting from the identity permutation. Paths in this oriented graph are exactly the reduced decompositions.

Now the same graph can describe operators, the edges now being interpreted as simple divided differences, or isobaric divided differences instead of simple transpositions. One has in fact more general operator satisfying the braid relations, and more general relations involving parameters (*Yang-Baxter equations*).

Divided differences satisfy a Leibniz formula, as easily seen from the definition:

$$fg\partial_i = f\,(g\partial_i) + f\partial_i\, g^{s_i}\ . \tag{10}$$

In particular, symmetric functions in a_i, a_{i+1} are scalars for ∂_i and π_i:

$$g = g^{\sigma_i} \Rightarrow f\,g\,\partial_i = f\partial_i\, g \quad \text{and} \quad f\,g\,\pi_i = f\pi_i\, g\ . \tag{11}$$

When a_{i+1} tends towards a_i, the divided difference ∂_i becomes the usual derivative with respect to a_i. However, in that limit one loses the fact that symmetric functions in a_i, a_{i+1} commute with the operator, which is a property which greatly simplify computations. Indeed, to compute the image of a function under ∂_i, one will try to write it as a function in a_i, a_{i+1} as simple as possible, the complications being pushed to the coefficients (which are symmetrical in a_i, a_{i+1} and involve the other variables).

Complete functions behave "nicely" with respect to divided differences. Indeed, let $\mathbb{A}_i = \{a_1, \ldots, a_i\}$, and let $\mathbb{B}$ be a second alphabet symmetrical in a_i, a_{i+1} (i.e. invariant under s_i, which is the case for example if it contains p copies of a_i, and of a_{i+1}). Then

$$S^k(\mathbb{A}_i - \mathbb{B})\,\partial_i = S^{k-1}(\mathbb{A}_{i+1} - \mathbb{B})$$

and similarly

$$S^k(\mathbb{A}_i - \mathbb{B})\,\pi_i = S^k(\mathbb{A}_{i+1} - \mathbb{B})\ .$$

This property is easier to show on the generating function:

$$\sigma_z(\mathbb{A}_i - \mathbb{B}) = \frac{\prod_{b\in\mathbb{B}}(1-zb)}{\prod_{j=1}^{j=i}(1-a_j)} = \Big(\frac{\prod_{b\in\mathbb{B}}(1-zb)}{\prod_{j=1}^{j=i+1}(1-a_j)}\Big)(1-za_{i+1})$$

and, the factor inside parentheses being symmetrical, one is left to check that the image of $(1 - za_{i+1})$ under ∂_i is z.

Now, we can check the braid relations. Divided differences which involve two disjoint pairs of variables clearly commute, thus we have only to show $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$, and $\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}$. To simplify indices, let us check it for $i = 1$, and only for the divided differences, the computation being similar for the π_i's.

Instead of applying three times the definition of ∂_i to expand $\partial_2\partial_1\partial_2$, one makes it act on monomials that one writes as Schur functions:

$$a_1^\alpha a_2^\beta a_3^\gamma = S_{\gamma;\beta;\alpha}(\mathbb{A}_3\,;\mathbb{A}_2\,;\mathbb{A}_1\,)$$

Now, the first and third column of $S_{\gamma\beta\alpha}$ are symmetrical in a_2, a_3, and thus the image of the determinant is a similar determinant, because we have to operate only on the middle column:

$$S_{\gamma;\beta;\alpha}(\mathbb{A}_3;\ \mathbb{A}_2;\ \mathbb{A}_1) \xrightarrow{\partial_2} S_{\gamma;\beta-1;\alpha}(\mathbb{A}_3\,;\mathbb{A}_3;\ \mathbb{A}_1)\ .$$

Now, the new determinant has its first columns symmetrical in a_1, a_2 and becomes $S_{\gamma;\beta-1;\alpha-1}(\mathbb{A}_3;\ \mathbb{A}_3;\ \mathbb{A}_2)$ under ∂_1, and finally, under ∂_2 again :

$$S_{\gamma;\beta-1;\alpha-2}(\mathbb{A}_3;\ \mathbb{A}_3;\ \mathbb{A}_3) = S_{\gamma,\beta-1,\alpha-2}(\mathbb{A}_3)$$

which is a usual Schur function in three variables. The result being symmetrical in a_1, a_2, a_3, the image of the same monomial under $\partial_1\partial_2\partial_1$ produces the same Schur function (however, to apply $\partial_1\partial_2\partial_1$, one writes the starting monomial as a Schur function in $a_1 + a_2 + a_3$; $a_2 + a_3$; a_3).

Because monomials are a linear basis of polynomials, this computation implies the braid relation $\partial_1\partial_2\partial_1 = \partial_2\partial_1\partial_2$.

The preceding computation was not really necessary. Indeed, we did not use that ∂_1, d_2 commute with multiplication by symmetric functions in a_1, a_2, a_3. Knowing that the space $\mathbb{Z}[a_1, a_2, a_3]$ is a module (in fact free) over $\mathfrak{S}ym(a_1, a_2, a_3)$, with basis

$$1,\ a_1,\ a_2\ ,a_1^2,\ a_1a_2,\ a_1^2a_2$$

we see that both $\partial_1\partial_2\partial_1$ and $\partial_2\partial_1\partial_2$ decrease degree by 3, and thus send the basis to 0, apart from the last element which is sent to a constant (necessarily non-zero, because none of the operators is identically null). We shall see later a good reason why the constant is in both cases equal to 1.

The braids relations imply the following lemma (knowing that two reduced decompositions of a permutation differ by a sequence of transformations $s_is_j = s_js_i$ or $s_{i+1}s_is_{i+1} = s_is_{i+1}s_i$).

Lemma 28 *Given a permutation σ and a reduced decomposition $\sigma = s_i s_j \cdots s_k$, then the product $\partial_i \partial_j \cdots \partial_k$ (resp. $\pi_i \pi_j \cdots \pi_k$) is independent of the choice of the reduced decomposition of σ. One denotes it ∂_σ (resp. π_σ).*

Let now the cardinality of $\mathbb{A}$ be n, and let ω be the maximal element of $\mathfrak{S}_n$. Then writing any monomial in $\mathbb{A}$ as a Schur function, the same type of computation as in the case of three variables shows that the image of a monomial under ∂_ω or π_ω is a Schur function:

$$a_n^{i_1} \cdots a_1^{i_n}\, \pi_\omega = S_I(\mathbb{A}_n) \quad , \quad a_n^{i_1} \cdots a_1^{i_n}\, \partial_\omega = S_{I-\rho}(\mathbb{A}_n) \ ,$$

where $I = [i_1, \ldots, i_n]$, $\rho = [0, \ldots, n-1]$, $I - \rho = [i_1 - 0, \ldots, i_n - n + 1]$.

Instead of taking the group algebra of $\mathfrak{S}(\mathbb{A})$ with constant coefficients, one can use the algebra generated by rational functions in $\mathbb{A}$, and permutations. This is the proper way of combining multiplication by elements of $\mathcal{P}ol(\mathbb{A})$ and symmetrizing operators. Indeed, M.P. Schützenberger and I called this algebra *algebra of divided differences* to stress the fact that divided differences, or permutations act on the coefficients, that one can choose to write on the left or on the right. When writing P, it means now "multiplication by P". The algebra is nowadays called the *affine Hecke algebra* (Cherednik [4] defined a double affine Hecke algebra, by using the *affine symmetric group* instead of the symmetric group).

This algebra contains all our symmetrizing operators, for example :

$$\begin{aligned} \partial_i &= (1 - s_i)\frac{1}{x_i - x_{i+1}} = \frac{1}{x_i - x_{i+1}}(1 + s_i) \\ T_i &= (q_1 x_i + q_2 x_{i+1})\partial_i + q_2 = \frac{q_1 x_i + q_2 x_{i+1}}{x_i - x_{i+1}}(1 + s_i) + q_2 \end{aligned} \tag{12}$$

It has linear bases : $\{\sigma \in \mathfrak{S}(\mathbb{A})\}$, $\{\partial_\sigma,\, \sigma \in \mathfrak{S}(\mathbb{A})\}$, $\{\pi_\sigma,\, \sigma \in \mathfrak{S}(\mathbb{A})\}$, $\{T_\sigma,\, \sigma \in \mathfrak{S}(\mathbb{A})\}$, and many others. An interesting problem is to write explicit matrices of changes of bases.

For example, even to pass from the $T_\sigma^{q_1,q_2}$ to the $T_\sigma^{r_1,r_2}$, that is just to let the pair of eigenvalues be different, is not as straightforward as it would seem.

Indeed, by specializing the parameters q_1, q_2, r_1, r_2, one covers all the known change of bases. We shall encounter later Schubert polynomials, which appear for both changes $\sigma \mapsto \partial_\sigma$ and $\partial_\sigma \mapsto \sigma$. Let us just say a word about how to express any element of the affine Hecke algebra in the basis of permutations.

We first need a straightforward lemma, introducing a second alphabet $\mathbb{B}$ of cardinality n :

Lemma 29 *Let* $\mathbf{K} = \mathbf{K}(\mathbb{A}, \mathbb{B}) = \prod_{n \geq i > j \geq 1} (a_i - b_j)$. *Then* $\mathbf{K}(\mathbb{A}^\sigma, \mathbb{A}) = 0$ *if* σ *is not the identity permutation.*

Using these vanishing properties, one obtains expansions by just specializing the images of $\mathbf{K}$:

Proposition 30 *Let*

$$H = \sum_{\sigma \in \mathfrak{S}(\mathbb{A})} \sigma\, h_\sigma, \quad \textit{with} \quad h_\sigma \in Rat(\mathbb{A},\, \mathbb{B}) \ . \tag{13}$$

Then

$$\mathbf{K}H\big|_{\mathbb{B}=\mathbb{A}^\sigma} = (-1)^{\ell(\sigma)}\, h_\sigma \prod_{n \geq i > j \geq 1} (a_i - a_j) \tag{14}$$

We shall also need the explicit change of bases between the Hecke algebra with parameters $(1, 0)$ and the Hecke algebra with parameters $(-1, 0)$ (both are *0-Hecke algebras*. The π_i, $i = 1, \ldots, n{-}1$, are algebraic generators of the first one, the $\overline{\pi}_i := \pi_i - 1$, $i = 1, \ldots, n{-}1$, are generators of the second.

Proposition 31 *The* π_σ *and* $\overline{\pi}_\sigma$, $\sigma \in \mathfrak{S}(\mathbb{A})$ *are related by the relations*

$$\begin{aligned} \pi_\sigma &= \sum_{\nu \leq \sigma} \overline{\pi}_\nu \\ \overline{\pi}_\sigma &= \sum_{\nu \leq \sigma} (-1)^{\ell(\sigma) - \ell(\nu)}\, \pi_\nu \ , \end{aligned} \tag{15}$$

where the order is the Ehresmann-Bruhat order on the symmetric group.

Proof. Let s_i be such that $\ell(\sigma s_i) > \ell(\sigma)$. Then it is known that the interval $[1,\, \sigma]$ is composed of pairs $(\eta, \eta s_i)$, and of singletons ν such that $\ell(\nu s_i) > \ell(\nu)$. For the pairs, one has

$$(\overline{\pi}_\eta + \overline{\pi}_{\eta_i})\, \pi_i = \overline{\pi}_\eta + \overline{\pi}_{\eta_i}$$

and for the singletons, one has

$$\overline{\pi}_\nu\, \pi_i = \overline{\pi}_\nu\, (\overline{\pi}_i + 1) = \overline{\pi}_\nu + \overline{\pi}_{\nu_i} \ .$$

Knowing now that the complement of $[1,\, \sigma]$ in the interval $[1,\, \sigma\, s_i]$ is made of the permutations ν_i, this proves the first part by induction on the length of σ. The proof of the second part is similar. QED

Orthogonal Polynomials

To any "generic" linear functional L on $\mathcal{P}ol(x)$ is associated a (unique) family of orthogonal polynomials:

$$L(P_m(x)P_n(x)) = 0 \text{ if } m \neq n \ , \ L(P_n(x)P_n(x)) = 1 \ .$$

One can formally suppose that there exists an alphabet $\mathbb{A}$ such that the *moments* $L(x^n)$ are the complete functions of $\mathbb{A}$, i.e.

$$L(x^n) = S^n(\mathbb{A}) \ , \ n \geq 0 \ .$$

Now L is a linear functional, that we shall note $L_{\mathbb{A}}$, with values in symmetric functions:

$$L_{\mathbb{A}} : \mathcal{P}ol(x) \mapsto \mathfrak{S}ym(\mathbb{A}) \ .$$

The linear functional can be thought as a quadratic form on the space of polynomials in x, compatible with product :

$$(f(x)\,,\, g(x)) := L\mathbb{A}\big(f(x)g(x)\big) = (f(x)g(x)\,,\,1) \ .$$

As could be expected, the orthogonal polynomials $P_n(x)$ are Schur functions, as shown by the following theorem.

Theorem 32 *Given an alphabet $\mathbb{A}$ of infinite cardinality, then there exists non zero-constants $c_n \in \mathfrak{S}ym(\mathbb{A})$ such that the orthogonal polynomials associated to $L_{\mathbb{A}}$ are*

$$P_n(x) = c_n S_{n^n}(\mathbb{A} - x) \ , \ n = 0, 1, \ldots$$

Proof. On can write the orthogonality relations as $L_{\mathbb{A}}(P_n(x)x^m) = 0$, $m < n$. However, $S_{n^n}(\mathbb{A} - x)\, x^m = S_{n^n,m}(\mathbb{A}, x)$ and its image under $L_{\mathbb{A}}$ is obtained by replacing the last column of powers of x by complete functions of $\mathbb{A}$, i.e. one has

$$L_{\mathbb{A}}(S_{n^n,m}(\mathbb{A}, x)) = S_{n^n,m}(\mathbb{A}) \ .$$

This determinant vanishes for $m < n$, having two identical columns QED

Notice that the functional $L_{\mathbb{A}}$ can also be interpreted as a symmetrizing operator. Indeed, when $\mathbb{A}$ is of finite cardinality n, let ω be the maximal permutation in $\mathfrak{S}_n$. Then

$$a_1^k\, \pi_\omega = S_k(\mathbb{A}) \ , \ k = 0, 1, 2, \ldots \ ,$$

and thus, for any polynomial $f(x)$, one has

$$L_{\mathbb{A}}\big(f(x)\big) = f(a_1)\,\pi_\omega \ .$$

Since $a^J \pi_\omega = S_J(\mathbb{A})$, $J \in \mathbb{N}$, there is no difficulty in extending the definition of π_ω to an alphabet of infinite cardinality, as is needed in the theory of orthogonal polynomials.

Instead of handling orthogonality relations, one can introduce *reproducing kernels*, i.e. functions $K_n(x,y)$ of two variables having the property that $L_{\mathbb{A}}(K_n(x,y)\,f(x)) = f(y)$ for all $f(x)$ of degree $\le n$ in x.

Darboux-Christoffel theorem gives the explicit expression of these kernels:

Theorem 33 *Let* $K_n(x,y) = \frac{P_n(x)P_{n-1}(y)-P_n(y)P_{n-1}(x)}{x-y}$. *Then, up to a scalar,*

$$K_n(x,y) = S_{(n+1)^{n-1}}(\mathbb{A}-x-y)$$

and

$$L_{\mathbb{A}}(K_n(x,y)\,f(x)) = f(y) \ , \ \forall f \ : deg(f) \le n \ .$$

Proof. For simplicity of indices, let us take $n=3$. The two determinants expressing $P_3(x) = S_{333}(\mathbb{A}-x) = S_{3330}(\mathbb{A},x)$ and $P_2(x) = S_{22}(\mathbb{A}-x) = S_{220}(\mathbb{A},x)$ have many common entries. To take advantage of it, one can use Plücker relations between maximal minors of a rectangular matrix:

$$[a,b,\alpha,\beta,\ldots]\cdot[c,d,\alpha,\beta,\ldots]-[a,c,\alpha,\beta,\ldots]\cdot[b,d,\alpha,\beta,\ldots] = [a,d,\alpha,\beta,\ldots]\cdot[b,c,\alpha,\beta,\ldots]$$

Let us apply these relations to the 4×6 matrix of complete functions (denoted by their exponent; the alphabets are constant in each column and written at the bottom)

$$\begin{array}{r|cccccc}
 & 0 & 3 & 4 & 5 & 3 & 3 \\
 & \cdot & 2 & 3 & 4 & 2 & 2 \\
 & \cdot & 1 & 2 & 3 & 1 & 1 \\
 & \cdot & 0 & 1 & 2 & 0 & 0 \\
\hline
\textit{alphabets} & \mathbb{A} & \mathbb{A} & \mathbb{A} & \mathbb{A} & x & y \\
\textit{column index} & 0 & 3 & 4 & 5 & x & y
\end{array}$$

Taking for a,b,c,d the indices $0,3,x,y$ and corresponding minors (having in common columns 3,4), Plücker relation writes

$$\begin{matrix}[0,3,4,y]\\ S_{022,0}(\mathbb{A},y)\end{matrix}\ \begin{matrix}[3,4,5,x]\\ S_{333,0}(\mathbb{A},x)\end{matrix} - \begin{matrix}[0,3,4,x]\\ S_{022,0}(\mathbb{A},y)\end{matrix}\ \begin{matrix}[3,4,5,x]\\ S_{333,0}(\mathbb{A},x)\end{matrix} = \begin{matrix}[0,3,4,5]\\ S_{0222}(\mathbb{A})\end{matrix}\ \begin{matrix}[3,4,x,y]\\ S_{3,3,1,0}(\mathbb{A},x,y)\end{matrix}$$

Now, subtracting $x+y$ to all rows but the last two ones of S_{3310}, one gets

$$S_{33,1,0}(\mathbb{A},x,y) = S_{33}(\mathbb{A}-x-y)\,S_{1,0}(x,y)$$

and finally

$$\big(P_2(y)P_3(x) - P_2(x)P_3(y)\big)\frac{1}{x-y} = S_{222}(\mathbb{A})\, S_{33}(\mathbb{A} - x - y)$$

QED

In [L-Sh] I have shown with Shi He how to generalize Darboux-Christoffel formula to a set $\{x_1, \dots, x_k\}$, $k \geq 2$, instead of $\{x, y\}$.

The functions to consider are now Schur functions

$$S_{n^{n-k+1}}(\mathbb{A} - x_1 - \cdots - x_k)\ ,$$

and they still have some reproducing properties which are explained in the note [L-Sh].

The reader will have noticed that the functions $S_{n^n}(\mathbb{A} - x)$ already appeared many times. Indeed, one has

Theorem 34 *Let $\mathbb{A}$ be an alphabet and let $y = 1/x$. Let*

$$\sigma_y(\mathbb{A}) = \cfrac{1}{(1-\zeta_0 y) - \cfrac{y^2\beta_1}{(1-\zeta_1 y) - \cfrac{y^2\beta_2}{(1-\zeta_2 y) - \cfrac{y^2\beta_3}{\ddots}}}}$$

be the Wronski continued fraction for the series $\sigma_y(\mathbb{A})$. Then the orthogonal polynomials $S_{n^n}(\mathbb{A} - x)$ are the denominators of the convergents. The numerators are the "adjoint polynomials" $Q_n := xS_{(n-1)^{n+1}}(x + \mathbb{A})$, $n \geq 2$.

Moreover, $Q_n(1/y)/yP_n(1/y)$ is the Padé approximant of degree $[n-1, n]$ in x of $\sigma_y(A)$.

For example the first terms are

$$\frac{1}{1 - yS_1} = \frac{Q_1}{P_1}\ ;\ \cfrac{1}{1 - S_1 y + \cfrac{y^2 S_{11}}{1 - (\frac{S_{12}}{S_{11}} - S_1)y}} = \frac{Q_2}{P_2} = \frac{xS_{111}(x+\mathbb{A})}{S_{22}(\mathbb{A}-x)}$$

$$\cfrac{1}{1 - S_1 y + \cfrac{y^2 S_{11}}{1 - (\frac{S_{12}}{S_{11}} - S_1)y + \cfrac{y^2 S_{222}}{(S_{11})^2(1 - (\frac{S_{223}}{S_{222}} - \frac{S_{12}}{S_{11}})y)}}} = \frac{Q_3}{P_3} = \frac{xS_{2222}(x+\mathbb{A})}{S_{333}(\mathbb{A}-x)}$$

In presenting the work of Wronski, we had interpolated the series $\lambda_y(\mathbb{A})$ instead of $\sigma_y(\mathbb{A})$. The two expansions are exchanged by $y \mapsto y$, and conjugating partitions indexing Schur functions.

Orthogonal polynomials satisfy three-terms recurrence relations, which are equivalent to the continued fraction expansion given above (with $\beta_0 = 1$, $P_0 = 1$) :

$$P_{n+1} - (x - \zeta_n) + \beta_n P_{n-1} = 0$$

The adjoint polynomials Q_n satisfy the same recursion, but with initial conditions $Q_0 = 0, Q_1 = 1$.

We have preferred to give similar expressions to P_n and Q_n. However, one can also write $P_n = S_{n^n,0}(\mathbb{A}, x)$ and $Q_n = \pm S_{n^n,-1}(\mathbb{A}, \mathbb{A} + x)$ to break the symmetry in P and Q.

All formulas satisfied by the polynomials P_n can be adapted to the adjoint polynomials Q_n. For example, one has a Darboux-Christoffel kernel :

$$\frac{Q_n(x)Q_{n-1}(y) - Q_n(y)Q_{n-1}(x)}{x - y} = \Lambda_{n^{n-1}}(\mathbb{A})\,\Lambda_{(n+1)^{n-2}}(\mathbb{A} + x + y)\ .$$

For example,

$$\big(Q_2(y)Q_3(x) - Q_2(x)Q_3(y)\big)\frac{1}{x - y} = \Lambda_{33}(\mathbb{A})\,\Lambda_4(\mathbb{A} + x + y)\ .$$

One can "shift" the linear functional $L_{\mathbb{A}}$ by a finite alphabet $\mathbb{B}$, defining

$$\mathcal{P}ol(x) \ni f\ ,\ L_{\mathbb{A},\mathbb{B}}(f) = L_{\mathbb{A}}(f(x)\,R(x, \mathbb{B}))\ .$$

Christoffel obtained the associated orthogonal polynomials :

Proposition 35 *Let $\mathbb{B} = \{b_1, \ldots, b_k\}$. Then the orthogonal polynomials relative to $L_{\mathbb{A},\mathbb{B}}$ are*

$$P_{n,k}(x) = S_{(n+k)^n}(\mathbb{A} - \mathbb{B} - x)\ ,$$

and $P_{n,k}(x)\,R(x, B)$ is proportional to the determinant

$$\big|P_{n-1+j}(b_i)\big|_{1\le i,j\le k+1}\ ,$$

with $b_{k+1} := x$.

Proof. The verification that $P_{n,k}(x)$ is orthogonal to $x^0, \ldots, x^{n-1}$ is the same as for $L_{\mathbb{A}}$, apart from changing $\mathbb{A}$ into $\mathbb{A} - \mathbb{B}$, and shifting indices.

The determinant is divisible by the Vandermonde $\Delta(\mathbb{B} + x)$. Evaluating the image of the quotient multiplied by a function of x under $L_{\mathbb{A},\mathbb{B}}$ is the same as computing the image of the last row, multiplied by the same function, under $L_{\mathbb{A}}$. Therefore $P_{n,k}(x)$ is orthogonal (with respect to $L_{\mathbb{A},\mathbb{B}}$) to $x^0, \ldots, x^{n-1}$, while being of degree n in x. It must be proportional to $S_{(n+k)^n}(\mathbb{A} - \mathbb{B} - x)$. The explicit factor comes from the polarized Bazin formula given in the Appendix. QED

One can notice that Christoffel polynomials generalize the Darboux-Christoffel kernels, and indeed, they have reproducing properties already alluded to.

Schubert Polynomials

We want to be able to perform algebraic computations in the ring of polynomials in $a_1, a_2, \dots$, in a manner similar to what we did in the case of $\mathfrak{S}ym(\mathbb{A})$, that is independently of the number of variables; in particular, we want to avoid expanding a polynomial in the basis of monomials, but rather use operations on combinatorial objects to replace algebraic computations on polynomials. Moreover, we want this combinatorics to extend the combinatorics of symmetric polynomials.

Of course, a generic polynomial of degree 5 in 6 variables depends on $\binom{10}{5}$ coefficients, whatever way one chooses to represent it. Any specific tool is unlikely to be interesting in the generic case, but will apply only to the description of spaces of polynomials satisfying some requirements.

Let us show how Newton point of view can be used.

Given a function of one variable (the time t), let $\mathbb{B} = \{b_1, b_2, b_3, \dots\}$, consider the divided differences with respect to $\mathbb{B}$ and let

$$f^{\partial}\,,\ f^{\partial\partial}\,,\ f^{\partial\partial\partial}\,,\ \dots$$

denote

$$f(b_1)\partial_1,\ f(b_1)\,\partial_1\partial_2,\ f(b_1)\,\partial_1, \partial_2\partial_3\,,\ \dots$$

Then the formula of Newton to interpolate $f(t)$ from its values at time b_1, b_2, $\dots$ is

$$f(t) = f(b_1) + f^{\partial}\,(t-b_1) + \\ f^{\partial\partial}\,(t-b_1)(t-b_2) + f^{\partial\partial\partial}\,(t-b_1)(t-b_2)(t-b_3) + \cdots \quad (16)$$

the expansion being exact if $n+1$ points b_i are used and f is a polynomial of degree $\le n$.

When b_1, b_2, $\dots$ all collapse to 0, then equation (16) becomes *Taylor's formula*

$$f(t) = f(0) + f'(0)\,t + f''(0)/2!\,t^2 + \cdots\ ,$$

the factorials in the denominator being clear by putting $b_i = i\epsilon$ and letting ϵ tend to 0.

Since divided differences are the discrete analogues of derivatives, the question is :

What is the discrete analogue of Taylor's formula in several variables ?

In other words, what are the coefficients of the images of $f(a_1, a_2, \ldots)$ under the possible different divided differences (evaluated in $b_1, b_2, b_3, \ldots$) in the expansion of $f(a_1, a_2, \ldots)$?

The answer, that I obtained with Marcel-Paul Schützenberger some years ago (cf. [23]), is as simple as in the case of one variable. The universal coefficients, called *Schubert polynomials* due to their relevance to geometry, can be defined recursively as follows:

1) They are polynomials $Y_v(\mathbb{A}, \mathbb{B})$ in two infinite totally ordered sets of variables $\mathbb{A} = \{a_1, a_2, \ldots\}$, $\mathbb{B} = \{b_1, b_2, \ldots\}$, indexed by vectors $v \in \mathbb{N}^\infty$ (from now on, we shall suppose that they have only a finite number of non-zero components).

2) They are globally stable under divided differences in the a_i's, that is, the image of a Schubert polynomial is either 0 or a Schubert polynomial.

3) When v is weakly decreasing (one says v is *dominant*, or is a *decreasing partition*, deleting the terminal zeros), then

$$Y_v = \prod_{(i,j) \in Diagr(v)} (a_i - b_j) \ , \tag{17}$$

product on all boxes in the diagram of v: one stacks $v_1, v_2, v_3, \ldots$ boxes in successive rows, packing them to the left. A box in position (i, j) gives a factor $(a_i - b_j)$.

4) Given $v \in \mathbb{N}^\infty$ and $i \in \mathbb{N}$ such that $v_i > v_{i+1}$, writing vs_i for $[v_1, \ldots, v_{i-1}, v_{i+1}, v_i - 1, v_{i+2}, \ldots]$, one has

$$\partial_i(Y_v) = Y_{vs_i} \ . \tag{18}$$

One does not write what happens when $v_i \leq v_{i+1}$, because in that case Y_v is the image of some $Y_{v'}$ under ∂_i, (precisely $v' = [v_1, \ldots, v_{i-1}, v_{i+1} + 1, v_i, v_{i+2}, \ldots]$). The vanishing $\partial_i^2 = 0$ implies that $\partial_i(Y_v) = 0$ in that case.

On the indices of Schubert polynomials, the action of divided differences amounts to sorting (and decreasing), in other words, amounts to use the symmetric group. Thus, because divided differences satisfy the braid relations, it just remains, for ensuring that the above definition is consistent, to check that if a dominant polynomial can be obtained from another one by a sequence of divided differences, then it coincides with the product given by the diagram indexing it.

Because the set of partitions is a lattice (with respect to intersection and union of diagrams), compatibility of dominant polynomials boils down to the following lemma.

Lemma 36 *Given a dominant $v \in \mathbb{N}^\infty$, and $n \in \mathbb{N}$ such that $v \leq \rho := [n-1, n-2, \ldots, 1, 0, 0, \ldots]$ (componentwise), then there exists at least one* *chain of divided differences such at each step a ∂_i is applied to a dominant Schubert polynomial of the type $(a_i - b_j)\, g$, with g symmetrical in a_i, a_{i+1} (and therefore the image is a dominant Schubert polynomial corresponding to a diagram with the box of coordinate (i,j) erased, the factor $(a_i - b_j)$ becoming 1 under ∂_i).*

The proof of the lemma consists in peeling off boxes of a staircase diagram, in such a way as to have, at a step where ∂_i will be used, columns i and $i+1$ of lengths differing by 1. This can be realized by erasing boxes from top to bottom, in successive diagonals, as indicated by the following example for $v = [6,6,6,2,2,0,0,\ldots]$, $n = 9$ (boxes are numbered in the order they are peeled off).

1							
8	2						
13	9	3					
		10	4				
		14	11	5			
						6	
						12	7

Now, if Y_ρ was the product $\prod_{i,j,i+j\leq n}(a_i - b_j)$, then Y_v, as obtained by the above process, will be the product of factors $(a_i - b_j)$ corresponding to the boxes which are left. Therefore, the definition of Schubert polynomials is consistent. QED

Newton's polynomials could be characterized by their vanishing properties, because they are written in terms of their roots. Similarly, when v is dominant, and different from $[0,0,\ldots]$, one can also easily write the vanishing properties of Y_v.

When v is dominant and $\neq 0^n$, then Y_v vanishes under the specialization $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3, \ldots$ (we write $Y_v(\mathbb{B},\mathbb{B}) = 0$). Indeed all the factors in the main diagonal of the diagram of v vanish.

Using that $\partial_\sigma = \sum_{\nu\leq\sigma} h_\nu(\mathbb{A})\,\nu$, with some rational coefficients $h_\nu(\mathbb{A})$ depending only on $\mathbb{A}$, the order on $\mathfrak{S}(\mathbb{A})$ being the Ehresmann-Bruhat order, one can check the vanishing property $Y_v(\mathbb{B},\mathbb{B})$ for all indices $v \in \mathbb{N}^n$, $v \neq 0^n$.

Let us for a moment index products of divided differences by vectors, instead of permutations.

Given $K = [k_1, k_2, \ldots] \in \mathbb{N}^n$, let

$$\partial^k := (\partial_1 \cdots \partial_{k_1})\,(\partial_2 \cdots \partial_{k_2+1})\,(\partial_3 \cdots \partial_{k_3+2})\ \ldots\ ,$$

that is, ∂^k consists in products of blocks of consecutive divided differences, of respective lengths $k_1, k_2, \ldots$, starting with ∂_1, ∂_2, $\ldots$ respectively.

We can now state the multivariate Newton interpolation formula. For any polynomial in $a_1, a_2, \ldots$, one has :

$$f(\mathbb{A}) = \sum_{K \in \mathbb{N}^\infty} f^{\partial^K}(\mathbb{B})\, Y_K(\mathbb{A}, \mathbb{B}) \ . \tag{19}$$

Proof. One has to test (19) on a linear basis of polynomials in $\mathbb{A}$ (with coefficients in anything, for example functions of $\mathbb{B}$). One takes the Schubert polynomials $\{Y_J\}$ as such a basis. Now, one has to check whether the following equations are true :

$$Y_J(\mathbb{A}, \mathbb{B}) = \sum_{K \in \mathbb{N}^\infty} (Y_J)^{\partial^K}(\mathbb{B})\, Y_K(\mathbb{A}, \mathbb{B}) \ . \tag{20}$$

But the $Y_J^{\partial^K}$ are either zero, or Schubert polynomials, and they all vanish under the specialization $\mathbb{A} = \mathbb{B}$, except for $Y_{[0,0,\ldots]} = 1$. Now, $Y_J^{\partial^K}$ can be $Y_{[0,0,\ldots]}$ only when $k_1 + k_2 + \cdots = j_1 + j_2 + \cdots$, for degree reason. By recursion on the rightmost non-zero component of J, say j_ℓ, one sees that $k_i = 0$, $i > \ell$ and $k_\ell = j_\ell$. Finally $K = J$ and equation (20) becomes the identity

$$Y_J(\mathbb{A}, \mathbb{B}) = Y_{[0,0,\ldots]}(\mathbb{B}, \mathbb{B})\, Y_J(\mathbb{A}, \mathbb{B}) \ . \tag{21}$$

This proves equation (19) in full generality. QED

The original Newton formula has not been lost. When f depends only on a_1, then f^{∂^K} is 0 if K is not of the type $K = [n, 0, 0, \ldots]$ for some integer n. In that case, Y_K is dominant, corresponding to a row diagram with n boxes, and indeed the Newton polynomial is

$$Y_{[n,0,0,\ldots]} = (a_1 - b_1)\,(a_2 - b_2) \cdots (a_n - b_n) \ .$$

The ring of polynomials as a module over $\mathfrak{S}ym$

We have obtained many properties of symmetric functions from the existence of a Cauchy kernel giving rise to a scalar product. This approach can be extended to the ring $\mathbb{Z}[\mathbb{A}]$ of polynomials in n indeterminates : $\mathbb{A} = \{a_1, a_2, \ldots, a_n\}$. This time, we shall have a $\mathfrak{S}ym(\mathbb{A})$-quadratic form. In fact, we shall consider $\mathbb{Z}[\mathbb{A}]$ as a module over $\mathfrak{S}ym(\mathbb{A})$, and to have more flexibility, we introduce a second alphabet $\mathbb{B} = \{b_1, b_2, \ldots, b_n\}$, and denote by $\mathcal{P}ol(\mathbb{A})$ the ring

$$\mathcal{P}ol(\mathbb{A}) = \mathbb{Q}[\mathbb{A}, \mathbb{B}] / \big(\mathfrak{S}ym(\mathbb{A}) = \mathfrak{S}ym(\mathbb{B})\big)$$

as a module over $\mathbb{Q}[\mathbb{B}] \otimes \mathfrak{S}ym(\mathbb{A})$ (that we shall still denote $\mathfrak{S}ym(\mathbb{A})$, having identified any symmetric function of $\mathbb{A}$ with the same symmetric function in $\mathbb{B}$.

We shall see that $\mathcal{P}ol(\mathbb{A})$ is a free $\mathfrak{S}ym(\mathbb{A})$-module. Indeed, we have already at our disposal the tools to describe it.

First, the appropriate quadratic form is defined by using the maximal divided difference of $\mathfrak{S}(\mathbb{A})$:

$$\mathcal{P}ol(\mathbb{A}) \ni f, g \Rightarrow (f\,,\, g) = (fg\,,\, 1) := \partial_\omega(fg) \in \mathfrak{S}ym(\mathbb{A}) \tag{22}$$

Lemma 37 *The divided differences ∂_i, $i = 1, \ldots, n-1$, are self-adjoint with respect to $(\ ,\ )$. Therefore, for $\sigma \in \mathfrak{S}(\mathbb{A})$, ∂_σ is adjoint to $\partial_{\sigma^{-1}}$.*

Moreover σ is adjoint to $(-1)^{\ell(\sigma)}\sigma^{-1}$.

Proof. Given any $i < n$, factorize $\omega = \sigma_i\,\sigma$. Then $(f\,\partial_i, g) = \big(f\partial_i\, g\,\partial_i\big)\partial_\sigma$. Because $f\,\partial_i$ is a scalar for ∂_i, the preceeding expression is equal to

$$\big((f\partial_i)\,(g\partial_i)\big)\partial_\sigma$$

and the symmetry between f and g implies that it is also equal to $(f\,,\, g\partial_i)$. By product, one gets the assertion for every ∂_σ.

Similarly

$$(f\sigma\,,\, g) = f^\sigma\, g\, \partial_\omega = f\, g^{\sigma^{-1}}\, \sigma\partial_\omega = (-1)^{\ell(\sigma)} f\, g^{\sigma^{-1}}\, \partial_\omega = (-1)^{\ell(\sigma)}(f\,,\, g\sigma^{-1})\ .$$

QED

Let

$$\mathbf{K}(\mathbb{A},\mathbb{B}) := \prod_{n\geq i>j\geq 1} (a_i - b_j) \ . \tag{23}$$

The following proposition shows that it is a reproducing kernel.

Proposition 38 *For any $f \in \mathcal{P}ol(\mathbb{A})$, one has*

$$f\big(\mathbb{A},\ \mathbf{K}(\mathbb{A},\mathbb{B})\big) \equiv f(\mathbb{B}) \tag{24}$$

(we have written $\equiv$ instead of an equality to point out that we are identifying symmetric functions in $\mathbb{A}$ with those of $\mathbb{B}$).

Proof. Any polynomial in $\mathbb{A}$ can be written as a linear combination of a^I, $I \leq \rho = [n-1,\dots,1,0]$, with coefficients in $\mathfrak{S}ym(\mathbb{A})$. However, $a^I\,\mathbf{K}(\mathbb{A},\mathbb{B})$ is a linear combination of monomials a^{I+J}, $I \leq \rho$, $J \leq [0,1,\dots n{-}1]$, that is, of monomials having an exponent $I+J \leq (n-1)^n$. The image of such a monomial under ∂_ω is 0, except if all its exponents are different, and this is precisely the case where $I+J$ is a permutation of ρ. In that case $\partial_\omega(a^{I+J}) = \pm 1$.

Therefore, the image of each monomial a^{I+J} is independent of $\mathbb{A}$, and one can specialize $\mathbb{A}$ to $\mathbb{B}$ in the expression

$$\partial_\omega\big(f(\mathbb{A})\mathbf{K}(\mathbb{A},\mathbb{B})\big) = \sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)} f(\mathbb{A}^\sigma)\mathbf{K}(\mathbb{A}^\sigma,\mathbb{B})\frac{1}{\Delta(\mathbb{A})}$$

However $\mathbf{K}(\mathbb{B}^\sigma,\mathbb{B})$ is 0 if σ is not the identity permutation, and the summation reduces to a single term

$$f(\mathbb{B})\mathbf{K}(\mathbb{B},\mathbb{B})/\Delta(\mathbb{B}) = f(\mathbb{B}) \ .$$

QED

Now, we shall see that Schubert polynomials are compatible with the quadratic form (are "almost" an orthonormal basis, up to a twist).

We first need to write them differently, using permutations rather than vectors. Indeed, given $\sigma = [\sigma_1,\dots,\sigma_n] \in \mathfrak{S}(n)$, its *code* is the vector $c[\sigma] = [c_1,\dots,c_n]$ such that $c_i := card(j > i, \sigma_j < \sigma_i)$, $1 \leq i \leq n$.

It is immediate that $\sigma \mapsto c[\sigma]$ is a bijection between $\mathfrak{S}(n)$ and the vectors $c \in \mathbb{N}^n$ such that $c \leq \rho$ (i.e. $c_i \leq n-i$, $i = 1,\dots,n$).

We shall write $\mathbb{X}_\sigma$ or $\mathbb{Y}_{c(\sigma)}$ indifferently for the same Schubert polynomial.

Equivalently, Schubert polynomials for $\mathfrak{S}(n)$ are defined by

$$\begin{cases} \mathbb{X}_\omega(\mathbb{A},\mathbb{B}) = \displaystyle\prod_{1\leq i,j\leq n,\, i+j\leq n} (a_i - b_j) \\ \mathbb{X}_\sigma(\mathbb{A},\mathbb{B}) = \mathbb{X}_\omega(\mathbb{A},\mathbb{B})\,\partial_{\omega\sigma} \end{cases} \tag{25}$$

Notice that $\mathbf{K}(\mathbb{A}, \mathbb{B}) = \mathbb{X}_\omega(A^\omega, \mathbb{B})$.

Proposition (38) can be generalized.

Theorem 39 *For any $f \in \mathcal{P}ol(\mathbb{A})$, any permutation $\sigma \in \mathfrak{S}(n)$, one has*

$$\big(f(\mathbb{A}),\, \mathbb{X}_{\sigma\omega}(\mathbb{A}^\omega, \mathbb{B})\big) \equiv f\,\partial_{\sigma^{-1}}(\mathbb{B})$$

Proof. One has

$$\mathbb{X}_{\sigma\omega}(\mathbb{A}^\omega, \mathbb{B}) = \mathbb{X}_\omega(\mathbb{A}, \mathbb{B})\,\partial_{\omega\sigma\omega}\,\omega = \mathbb{X}_\omega(\mathbb{A}, \mathbb{B})\omega\,\big(\omega\partial_{\omega\sigma\omega}\,\omega\big) = \mathbb{X}_\omega(\mathbb{A}, \mathbb{B})\,\omega\,\partial_\sigma\ ,$$

and one can shift the divided difference ∂_σ to the left of (,) thanks to lemma (37). Now the formula results from proposition 38 for the function $f\,\partial_{\sigma^{-1}}$ QED

Corollary 40 *The quadratic form (,) is non-degenerate. The space $\mathcal{P}ol(\mathbb{A})$ is a free module over $\mathfrak{S}ym$, with the two adjoint bases $\{\mathbb{X}_\sigma(\mathbb{A}, \mathbb{B})\}$ and $\{\mathbb{X}_{\sigma\omega}(\mathbb{A}^\omega, \mathbb{B})\}$, $\sigma \in \mathfrak{S}(\mathbb{A})$.*

More precisely

$$\big(\mathbb{X}_\sigma(\mathbb{A}, \mathbb{B}),\, X_{\nu\omega}(\mathbb{A}^\omega, \mathbb{B})\big) = \delta_{\sigma,\nu}\ .$$

Any element of $\mathcal{P}ol(\mathbb{A})$ expands into the basis of Schubert polynomials $\mathbb{X}_\sigma$, $\sigma \in \mathfrak{S}(\mathbb{A})$ as follows :

$$\begin{aligned} f(\mathbb{A}) &\equiv \sum_\sigma \big(f,\, \mathbb{X}_{\sigma\omega}(\mathbb{A}^\omega, \mathbb{B})\big)\mathbb{X}_\sigma(\mathbb{A}, \mathbb{B}) \\ &\equiv f\partial_{\sigma^{-1}}(\mathbb{B})\,\mathbb{X}_\sigma(\mathbb{A}, \mathbb{B}) \end{aligned} \tag{26}$$

Newton interpolation has given another expansion in terms of Schubert polynomials, this time with coefficients independent of $\mathbb{A}$; in expansion26, the coefficients, on the contrary, belong to $\mathfrak{S}ym(\mathbb{A})$. For example, for $n = 2$, this expansion is

$$a_1^2 \equiv b_1^2 + (a_1 + a_2)(a_1 - b_1)$$

while Newton states

$$a_1^2 = b_1^2 + (b_1 + b_2)(a_1 - b_1) + (a_1 - b_1)(a_1 - b_2)\ .$$

Both equations coincide modulo $Sym(\mathbb{A}) = \mathfrak{S}ym(\mathbb{B})$.

More generally, the two expansions are identical if $f(\mathbb{A})$ is a linear combination of the a^I, $I \leq \rho$, with coefficients in $\mathbb{B}$.

One can view Schubert polynomials as a basis of $\mathcal{P}ol(\mathbb{A})$ depending on parameters $b_1, \dots, b_n$, and let the parameters vary. Indeed, let $\mathbb{C} = \{c_1, \dots, c_n\}$. Then the following theorem shows that the matrix of change of bases is given by specializations of Schubert polynomials :

Theorem 41 *Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ be three alphabets of cardinal n, and let $\zeta \in \mathfrak{S}(n)$. Then*

$$\mathbb{X}_\zeta(\mathbb{A},\mathbb{C}) = \sum_{\sigma,\eta} \mathbb{X}_\sigma(\mathbb{A},\mathbb{B})\,\mathbb{X}_\eta(\mathbb{B},\mathbb{C}) \ ,$$

sum over all reduced factorizations $\zeta = \sigma\eta$ (i.e. factorizations such that $\ell(\sigma) + \ell(\eta) = \ell(\zeta)$).

To have the non nullity of $\mathbb{X}_\zeta \partial_{\eta^{-1}}$ is the same as to require that the product $(\zeta\eta^{-1})(\eta)$ be reduced. Taking into account that $\mathbb{X}_\zeta(\mathbb{A},\mathbb{C})$ belong to the span of monomials in $\mathbb{A}$ of degree $\leq \rho$, the theorem follows from corollary(40) QED

Differential calculus on symmetric functions.

To be able to perform computations independently of the number of variables motivated the creation of algebraic and combinatorial tools already at the start of the theory of symmetric functions.

We have mostly used a scalar product on $\mathfrak{S}ym$ to get expansions in different bases, and symmetrizing operators like divided differences to provide recursions.

However, Cayley, Sylvester, MacMahon were taking another approach, by systematizing the fact that $\mathfrak{S}ym$ can be considered in different ways as a ring of polynomials (in the elementary symmetric functions, the complete functions, or the powers sums for example).

They used differential calculus on these variables, and characterized for example the invariants they were looking for, like the discriminant, as solutions of differential equations.

This point of view is now much relevant to the modern trends of the theory of symmetric functions. One can even mix differentials and divided differences, as in the case of Dunkl operators.

We shall adopt an even simpler strategy, by operating on the first variable a_1 only, inside an element of $\mathfrak{S}ym(a_1, \ldots , a_n)$, and symmetrizing afterwards to go back to $\mathfrak{S}ym(\mathbb{A})$. This is how one can write the *Debiard-Sekiguchi-Macdonald operator*

$$\mathfrak{S}ym(\mathbb{A}) \ni f(a_1, \ldots , a_n) \mapsto f(qa_1, \ldots , a_n) \mapsto \\ f(qa_1, \ldots , a_n) \prod_{i \geq 2} (ta_1 - a_i) \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{n-1}} DSM(f)$$

Let us first use derivatives in a_1, with $D := \frac{d}{d_{a_1}}$, and let $\mathbb{A} = \{a_1, \ldots , a_n\}$.

Lemma 42 *Let K be any function of a_1, with coefficients in $\mathfrak{S}ym(\mathbb{A})$. Let $\nabla = \nabla_K$ be the operator*

$$\mathfrak{S}ym(\mathbb{A}) \ni f \mapsto \nabla(f) := \pi_\omega(KD(P)) \ .$$

Then ∇ is a derivation, i.e.

$$\nabla(fg) = \nabla(f) \cdot g + f \cdot \nabla(g) \ .$$

Indeed $D(fg) = D(f)\,g + f\,D(g)$, and f, g being symmetrical, commutes with π_ω.

By varying $K = K(a_1)$, one can obtain as many differential operators as is needed to fill a grant application :

Proposition 43 *Let $j \in \mathbb{N}$, $1 \le j \le n$.*
1) Let $K = (-a_1)^{1-j}$. Then $\nabla = \frac{d}{d\Lambda_j}$
2) Let $K = (-1)^{n-1} a_1^{1-j-n}\,(a_1 \cdots a_n)$. Then $\nabla = j\,\frac{d}{d\Psi_j}$

Proof. Because the ∇'s are derivations, we have only to check their action on algebraic bases of $\mathfrak{S}ym(\mathbb{A})$. In the first case,

$$D(\Lambda^i(\mathbb{A})) = D(\Lambda^i(\mathbb{A} - a_1) + a_1 \Lambda^{i-1}(\mathbb{A} - a_1) = \Lambda^{i-1}(\mathbb{A} - a_1)$$

and $\Lambda^{i-1}(\mathbb{A} - a_1)\, a_1^{1-j} = S_{1^{i-1};1-j}(\mathbb{A}; a_1)$ is sent by π_ω on $S_{1^{i-1},1-j}(\mathbb{A})$ which is 0, except for $S_{1^{i-1};1-j}(\mathbb{A}) = (-1)^{j-1} S_{0\ldots 0}(\mathbb{A})$.

In the second, $D(\Psi_i(\mathbb{A}) = i a_1^{i-1}$, and $i a_1^{i-1} a_1^{2-j-n}\, a_2 \cdots a_n$ is sent by π_ω onto $i S_{1^{n-1},i-j-(n-1)}(\mathbb{A}) = (-1)^{n-1}\, i$ if $i = j$, and sent to 0 otherwise. QED

Having at our disposal the derivatives with respect to elementary symmetric functions or power sums, one can combine them to obtain more sophisticated differential operators. One can for example find in this way representations of the Witt or Virasoro algebras ([15]). Another application is to describe the product by a conjugacy class, in the group algebra of the symmetric group, as a differential operator.

We shall detail in a more complete version vertex operators which are needed in the theory of symmetric functions.

The plactic algebra

One way to better understand symmetric functions is to define *non-commutative symmetric functions.* There are several ways to do it. For example, since symmetric polynomials are polynomials in the elementary symmetric functions, then one can take formal non commutative variables Λ^i and claim that the free algebra generated by them is the ring of non-commutative symmetric polynomials. There is some grounds to this claim, see [8].

We shall take here a simpler point of view, defining an action on the free algebra, and picking out some appropriate invariants as generalizations of Schur functions. Indeed, considering Schur functions are sums of "tableaux" is the point of view introduced by Littlewood [34], which could be better understood after the work of Schensted [45].

Let us take a totally ordered alphabet $\mathbb{A} = \{a_1 < a_2 < \dots\}$. In the examples, we shall usually take $\mathbb{A} = \{1, 2, \dots, n\}$.

A nondecreasing word $v \in \mathbb{A}^*$ is called a *row.* Let $u = x_1 \cdots x_r$ and $v = y_1 \cdots y_s$ be two rows ($x_i, y_j \in A$). We say that u *dominates* v ($u \rhd v$) if $r \leq s$ and for $i = 1, \dots, r$, $x_i > y_i$. Clearly, every word w has a unique factorization $w = u_1 \cdots u_k$ as a product of rows of maximal length. A *tableau* is a word w such that $u_1 \rhd u_2 \rhd \dots \rhd u_k$. It is customary to think of tableaux as planar objects and to represent w as the left justified superposition of its rows. For instance, taking $\mathbb{A} = \{1 < 2 < \dots\}$,

$$t = 68\ 4556\ 223357\ 1112444$$

is a tableau whose planar representation is

$$\begin{array}{|c|c|c|c|c|c|c|}
\cline{1-2}
6 & 8 \\
\cline{1-4}
4 & 5 & 5 & 6 \\
\cline{1-6}
2 & 2 & 3 & 3 & 5 & 7 \\
\hline
1 & 1 & 1 & 2 & 4 & 4 & 4 \\
\hline
\end{array}$$

Similarly, a strictly decreasing word is called a *column.* Reading from top to bottom the lengths of the rows of a tableau t, one gets an (increasing) partition $sh(t)$ which is called the *shape* of t.

A word (in particular a tableau) is *standard* if it is a permutation of $1, 2, 3, \dots$. Any word can be *standardized* by indexing, from left to right, the different occurrences of the same letter (and renormalizing after that, the alphabet to be $1, 2, \ldots$). For example

$$2\,1\,22\,111 \mapsto 2_1\,1_1\,2_2 2_3\,1_2 1_3 1_4 \mapsto stand(2\,122111) = 5\,1\,67\,234\ .$$

Schensted described an algorithm to associate to any word w a tableau $P(w)$. This algorithm can in fact be traced back to Robinson. One can give an algebraic formulation to this algorithm by defining, after Knuth [16] the *plactic relations*

$$xzy \equiv zxy \quad (x \leq y < z)\,, \tag{27}$$

$$yxz \equiv yzx \quad (x < y \leq z)\,. \tag{28}$$

Two words are *congruent* iff they differ by a sequence of plactic relations. The *plactic algebra* $\mathcal{P}lac(\mathbb{A})$ is the quotient of the free algebra under the plactic relations. The plactic algebra is an intermediate quotient between the free algebra $\mathbb{Z}[\mathbb{A}^*]$ and the algebra of polynomial :

$$\mathbb{Z}[\mathbb{A}^*] \ \twoheadrightarrow\ \mathcal{P}lac(\mathbb{A}) = \mathbb{Z}[\mathbb{A}^*/\equiv] \ \twoheadrightarrow\ \mathcal{P}ol(\mathbb{A}) = \mathbb{Z}[\mathbb{A}]\ .$$

Let us note ev the projection map (*evaluation*) onto $\mathcal{P}ol(\mathbb{A})$.

Given a word $w = [x_1, \ldots, x_\ell]$, not only can one find the tableau $P(w)$, but one can also record the sequence of shapes of

$$P(x_1),\ P(x_1x_2),\ \ldots\ ,\ P(x_1x_2\cdots x_\ell)\ .$$

This sequence can be encoded into a standard tableau $Q(w)$ by labeling the boxes of the successive shapes in the order of their creation.

We can now give Schensted's result :

Theorem 44 *Each word w is congruent to a tableau and only one tableau, which is no other than $P(w)$.*

The morphism $w \mapsto (P(w), Q(w))$ is a bijection, and thus the elements of the plactic class of w are in bijection with standard tableaux of the same shape as $P(w)$.

The class of w is mapped, under standardization, to the class of $stand(w)$. Moreover $Q(w) = Q(stand(w))$.

Let u be a word. Define the *free Schur function $\mathcal{S}_u$ of index u* to be the sum of all words w such that $Q(w) = stand(P(u))$.

We shall see that the commutative image of $\mathcal{S}_u$ is the Schur function $S_J(\mathbb{A})$, where J is the shape of $P(u)$. But already, we notice that the image of $\mathcal{S}_u$ in $\mathcal{P}lac(\mathbb{A})$ is equal to the sum of all tableaux of shape J, sum that we shall still denote $S_J(\mathbb{A})$.

Now, given t, and a row $[x_1, x_2, \ldots]$, then an analysis of Schensted's construction [16] shows that the shape of $P(t\, x_1 x_2 \cdots)$ is obtained by adding to the shape of t an horizontal strip of boxes (the boxes being added from left to right). This implies, in fact, that Pieri's formulas are satisfied by Schur functions in the plactic algebra. Thanks to Giambelli's characterization of the ring of symmetric function, it gives the following theorem which allow us to recover symmetric functions at the level of the plactic algebra [20] :

Theorem 45 *The ring $\mathfrak{S}ym(\mathbb{A})$ is canonically embedded into the plactic algebra, by sending a Schur function $S_J(\mathbb{A})$ to the sum of all tableaux of shape J.*

Pieri formula (for any partition J, any positive integer k)

$$S_J(\mathbb{A})\, S_k(\mathbb{A}) = \sum_{I \in \{J \otimes k\}} S_I(\mathbb{A})$$

induces a bijection

$$\{(t,\, u) : sh(t) = J, shape(u) = k\} \longleftrightarrow \{t : sh(t) = I\} \ .$$

Commutativity of the product in $\mathfrak{S}ym(\mathbb{A})$ has interesting consequences at the level of $\mathcal{P}lac(\mathbb{A})$.

For example, using also Pieri for left multiplication, one gets a bijection

$$\{(t,\, u) : sh(t) = J, sh(u) = k\} \longleftrightarrow \{(v,\, t) : sh(v) = k,\, sh(t) = J\}$$

which is used to provide a statistics on tableaux related to Hall-Littlewood polynomials ($J = 1$ suffices for that purpose).

The fact the module generated by the non commutative Schur functions is an algebra gives an interesting description of the decomposition coefficients of products

$$S_I(\mathbb{A})\, S_J(\mathbb{A}) = \sum_K g_{I,J}^K\, S_K(\mathbb{A}) \ .$$

Say that a word w is a *Yamanouchi word* iff, for every factorization $w = w'\, w''$, then $|w''|_1 \geq |w''|_2 \geq |w''|_3 \geq \cdots$. In particular, a *Yamanouchi tableau* is of the type

$$\mathbf{y}_\lambda := \cdots 3^{\lambda_3} 2^{\lambda_2} 1^{\lambda_1} \ , \text{ with } \cdots \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 \ .$$

Notice that a Yamanouchi tableau is determined by its shape, as well as by its commutative evaluation.

It is easy to check :

Lemma 46 *A word is Yamanouchi iff it belongs to the class of a Yamanouchi tableau.*

The second part of the following proposition is the celebrated *Littlewood-Richardson rule* [35], the first part is due to M.P. Schützenberger [46] :

Proposition 47 *Given three partitions I, J, K, and a tableau $t_3 : sh(t_3) = K$, the number of solutions of the equation*

$$t_1 t_2 \equiv t_3 \ , \ sh(t_1) = I, \, sh(t_2) = J, \, sh(t_3) = K$$

is independent of the choice of the tableau t_3, and equal to $g^K_{I,J}$.

In particular, the number of solutions is equal to the number of tableaux t_1 such that $t_1 \mathbf{y}_J$ is a Yamanouchi word in the class of $\mathbf{y}_K$.

We have just seen that Littlewood-Richardson rule is a consequence of the embedding of $\mathfrak{S}ym(\mathbb{A})$ in $\mathcal{P}lac(\mathbb{A})$, which in turn follows from Pieri formula at the level of Schensted construction.

There is a stronger statement in the free algebra, which is even easier to prove, and that I have used in different constructions with M.P. Schützenberger [22].

Theorem 48 *Let $\mathbb{A}'$ and $\mathbb{A}''$ be two sub-alphabets such that $a' < a''$, for all $a \in \mathbb{A}'$, $a'' \in \mathbb{A}''$. Given two tableaux $t' \in \mathbb{A}'^*$ and $t'' \in \mathbb{A}''^*$ one has*

$$\left(\sum_{P(w')=t'} w' \right) ⧢ \left(\sum_{P(w'')=t''} w'' \right) = \sum_{t \in SH(t',t'')} \sum_{P(w)=t} w$$

where $SH(t', t'')$ is the set of all tableaux t such that $t|_{\mathbb{A}'} = t'$ and $P(t|_{\mathbb{A}''}) = t''$, that is, of all tableaux t occurring in the shuffle product of t' and a word in the plactic class of t''.

Thus the shuffle of a plactic class of A' and a plactic class of A'' is a union of plactic classes of A (identifying a class and the sum of its elements). It is in fact a direct consequence of the following :

Lemma 49 *Let $\mathbb{B}$ be an interval in the alphabet $\mathbb{A}$. Then*

$$w \equiv w' \Rightarrow w|_{\mathbb{B}} \equiv w'|_{\mathbb{B}}$$

Proof. of lemma. It is enough to check the lemma in the case when w' differs from w by a single plactic transformation, and this amounts to the observation that erasing x or z in 27 or 28, we are left with $xy = xy$ or $yz = yz$. QED

Proof. of Theorem 48. The words occurring in the shuffle are exactly those w such that $w|_{\mathbb{A}'} \equiv t'$ and $w|_{\mathbb{A}''} \equiv t''$. By the above Lemma, this set of words is saturated with respect to the plactic congruence, hence is a union of plactic classes. QED

We refer to [30] for a stronger statement involving free Schur functions S_u.

Let us now reintroduce the missing entity, $\mathfrak{S}(\mathbb{A})$, by defining an action on words compatible with the plactic relations.

In the case of an alphabet of cardinality 2, the action of $\mathfrak{S}_2$ is in fact commonly used in computer science. Let the letters be ')' and '(', with the order $) \leq ($. Given a word w, pair all successive parentheses $(\ldots)$. The sub-word of un-paired letters is of the type $)^m\,(^n$. Change it, inside w, into the subword $)^n\,(^m$.

More generally, given an integer i, and a word w, treat a_i as ')' and a_{i+1} as '(' to exchange the number of occurrences of a_i and a_{i+1} in the word w, and denote by $s_i(w)$ the resulting word (this is an action on values, we denote it on the left). It is clear that $w \mapsto s_i(w)$ is an involution on words, but in fact it is even a representation of the symmetric group, as show the following theorem (cf. [22]).

Theorem 50 *The operator s_i on words preserves the Q-symbol and preserves tableaux :*

$$P(s_i(w)) = s_i\big(P(w)\big) \quad \& \quad Q(s_i(w)) = Q(w)\ .$$

The operators σ_i satisfy the Moore-Coxeter relations

$$
\begin{aligned}
s_i^2 &= 1\,, && (29)\\
s_i s_j &= s_j s_i \quad (|i-j|>1)\,, && (30)\\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}\,. && (31)
\end{aligned}
$$

Proof. The first part is checked directly, the exchange of unpaired letters being compatible with Schensted's construction.

Equations (29,30) are immediate. The first part of the theorem restricts checking of (31) to tableaux in $1, 2, 3$, with no more than two rows because complete columns 321 can be ignored. We refer to ([30]) to finish the proof. QED

Because the action of the symmetric group preserves the Q-symbol, all the free Schur functions S_u are invariants under this action, but they are no more a linear basis of the invariants.

The easiest way to generate words in a plactic class is to use the celebrated *jeu de taquin* due to M.P. Schützenberger. To define it, one needs the notion of *skew Young tableaux*, and their deformations through vertical and horizontal moves of boxes in such a way that weakly increasing conditions in rows, and strictly decreasing conditions in columns are satisfied at any step.

Notice that the plactic class of a tableau with two rows contains products of two rows which are determined by their lengths :

Lemma 51 *Let t be a tableau of shape $[\alpha, \beta]$. Then for any pair of integers (γ, δ), $\alpha \leq \gamma, \delta \leq \beta$, $\gamma + \delta = \alpha + \beta$, there exists a unique pair of rows (u, v): $|u| = \gamma$, $|v| = \delta$, such that $uv \equiv t$.*

The jeu de taquin is a way to get successively all these words; in fact they come as a *string*, the two ends being the tableau and the *contre-tableau.*

Exchanging the rôle of the P and Q-symbols, one gets a *right action* of the symmetric group which is the counterpart of theorem (50), and such that elementary transpositions are described by (51).

Theorem 52 *Given a tableau t of shape $J \in \mathbb{N}^\ell$, then for any permutation $\sigma \in \mathfrak{S}_\ell$, there exist in the plactic class of t a unique word which is a product of rows, whose sequence of lengths is J^σ.*

We have lifted the permutation action of the symmetric group at the level of the free algebra. More generally, one can lift the action of elementary isobaric differences, or of the generators T_i of the Hecke algebra, to the free algebra, but these elements do not satisfy Moore-Coxeter relations any more, though they behave as if they indeed were satisfying these relations when one starts from a *dominant weight* (cf. [27]). We refer to Littlemann [32], [33] for more general constructions in the framework of Kac-Moody algebras.

Operators on isobaric determinants

One of the main properties of the different determinants expressing a Schur function is that all terms in their expansion have the same degree. A natural question is then :

how to increase degrees by 1 ?

It is appropriate to consider more general "weights" than degree. For our purposes, the following case will be general enough.

Let V be a vector space of functions of a variable $x \in \mathbb{C}$, with values in a commutative ring. Given two integers $1 \le k \le n$, define T_k^+ (resp. T_k^-) to be the operator on $V^{\otimes n}$ sending $f_1(x_1) \otimes \cdots \otimes f_n(x_n)$ on

$$\sum_{\epsilon \in \{0,1\}^n,\, \epsilon_1+\cdots+\epsilon_n=k} f_1(x_1 \pm \epsilon_1) \otimes \cdots \otimes f_n(x_n \pm \epsilon_n) \ .$$

The following theorem is not difficult to check (this has been my first encounter with the theory of symmetric functions through manipulations of isobaric determinants ([18]).

Theorem 53 *Let n be a positive integer. Then $T_0^+ = 1, T_1^+, \ldots, T_n^+$ (resp. $T_0^- = 1, T_1^-, \ldots, T_n^-$ generate a commutative algebra isomorphic to $\mathfrak{S}ym(n)$, the algebra of symmetric polynomials in n variables, the image of $T_k^\pm$ being Λ_k.*

Thus any symmetric polynomial S gives rise to two operators $T_S^\pm$, obtained by expressing S in the basis Λ^J.

The operators T_k^+, T_k^- can be made act on $V^{\otimes n^2}$, considered as a space of matrices $\big[f_{ij}(x_{ij})\big]_{1 \le i,j \le n}$, but now there are two natural ways to define their action, either by columns :

$${}^cT_k^\pm\big([f_{ij}(x_{ij})]\big) = \sum_{\epsilon \in \{0,1\}^n,\, |\epsilon|=k} \big[f_{ij}(x_{ij\pm\epsilon_j})\big] \ ,$$

or by rows:

$${}^rT_k^\pm\big([f_{ij}(x_{ij})]\big) = \sum_{\epsilon \in \{0,1\}^n,\, |\epsilon|=k} \big[f_{ij}(x_{ij\pm\epsilon_i})\big] \ .$$

Symmetry between rows and columns of a matrix entails the following lemma, when evaluating the determinants appearing in ${}^cT$ or ${}^rT$:

Lemma 54 *Given any matrix $\big[f_{ij}(x_{ij})\big]_{1 \le i,j \le n}$, then*

$$\sum_{\epsilon \in \{0,1\}^n,\, |\epsilon|=k} \big|f_{ij}(x_{ij\pm\epsilon_j})\big| = \sum_{\epsilon \in \{0,1\}^n,\, |\epsilon|=k} \big|f_{ij}(x_{ij\pm\epsilon_i})\big| \ . \tag{32}$$

Indeed, any term in the expansion of the determinant of the original matrix will be transformed according to $T_k^{\pm}$ and will be found in the expansion of one of the determinants of each side of equation (32).

By abuse of language, we shall write ${}^cT_k^{\pm}\big(|f_{ij}(x_{ij})|\big)$ and ${}^rT_k^{\pm}\big(|f_{ij}(x_{ij})|\big)$ for the above two sums of determinants.

Let $\mathbb{A}$ be of cardinality n. Then the two algebras generated by the $T_k^{\pm}$ can be made act on $Sym(\mathbb{A})$, by making them operate on the Jacobi-Trudi determinants of complete functions expressing Schur functions, and extending the action by linearity.

Theorem 55 *Let $\mathbb{A}$ be of cardinality n and S belong to $\mathfrak{S}ym(n)$. Then T_S^+ is the operator "multiplication by $S(\mathbb{A})$" and T_S^- acts by $f(\mathbb{A}) \mapsto D_S(f)(\mathbb{A})$.*

Proof. We have to test the action of the $T_k^{\pm} = T_{\Lambda^k}^{\pm}$ on the linear basis of Schur functions $S_J(\mathbb{A})$, but we shall rather take the Vandermonde minors $\mathbb{V}_J$. Multiplying $\mathbb{V}_J$ by a monomial a^K can be realized by increasing exponents by $k_1, \ldots, k_n$ in successive rows. Therefore

$$\Lambda^k(\mathbb{A}) V_J(\mathbb{A}) = {}^rT_k^+(V_J(\mathbb{A})) \ .$$

Notice that

$${}^cT_k^r(V_J(\mathbb{A})) = \sum_{\epsilon\in\{0,1\}^n,\, |\epsilon|=k} V_{J+\epsilon}(\mathbb{A}) = \Delta(\mathbb{A}) \sum_{\epsilon} S_{J+\epsilon}(\mathbb{A}) = \Delta(\mathbb{A}){}^cT_k^+(S_J(\mathbb{A})) \ .$$

Therefore one has $T^k(S_J(\mathbb{A})) = \Lambda^k(\mathbb{A})\, S_J(\mathbb{A})$ as wanted, but one has also proved Pieri formula by the same token.

Some care is needed to identify T_k^- by using its action on $\mathbb{V}_J(\mathbb{A})$ rather than $S_J(\mathbb{A})$. The operator is not "multiplication by $\Lambda^k(\mathbb{A}^\vee)$", because the operation of decreasing degrees by 1 sends a_i^0 onto 0, and not onto $1/a_i$. However, if $J \supseteq 1^n$, then

$$\Lambda^k(\mathbb{A}^\vee)\, S_J(\mathbb{A}) = \Lambda^{n-k}(\mathbb{A})\, S_{J/1^n}(\mathbb{A}) = T_k^-(S_J(\mathbb{A}) \ .$$

To identify T_k^-, we use ${}^rT_k^-$ operating on Schur functions. The image of $S_j(\mathbb{A})$ is a sum of $\binom{n}{k}$ determinants, each of which is null except for the last one

$$S_{J/0^{n-k}1^k}(\mathbb{A}) = D_{\Lambda^k} S_J(\mathbb{A}) \ .$$

The operator ${}^cT_k^-$ allows us to recover the dual Pieri formula :

$$S_{J/0^{n-k}1^k}(\mathbb{A}) = {}^cT_k^-(S_J(\mathbb{A})) = {}^rT_k^-(S_J(\mathbb{A})) = \sum_{\epsilon\in\{0,1\}^n,\, |\epsilon|=k} S_{J-\epsilon}(A) \ ,$$

because one can restrict the preceding sum to terms such that $J-\epsilon$ is a partition, the other terms are 0 (having two identical columns). QED

The operators $T^{\pm}_{\Psi_K}$ have been considered by Muir, who obtained the following corollary :

Corollary 56 *Let K, J be two partitions in $\mathbb{N}^n$. Then*

$$\Psi_K(\mathbb{A})\, S_J(\mathbb{A}) = \sum_{H=perm(K)} S_{J+H}(\mathbb{A}) \ ,$$

$$D_{\Psi_K}\big(S_J(\mathbb{A})\big) = \sum_{H=perm(K)} S_{J-H}(\mathbb{A}) \ ,$$

sum over all different permutations H of K.

As in the case of elementary functions seen above, if there exists ℓ such that $K \subseteq \square := \ell^n \subseteq J$, then

$$T^{-}_{\Psi_K}(\mathbb{V}_J(\mathbb{A})) = T^{-}_{\Psi_{\square-K}}(\mathbb{V}_{J-\square}(\mathbb{A})) = \frac{\Psi_{\square-K}(\mathbb{A})}{\Psi_{\square}(\mathbb{A})}\, V_J(\mathbb{A}) = \Psi_K(\mathbb{A}^{\vee})\, V_J(\mathbb{A}) \ ,$$

and in that case $T^{-}_{\Psi_K}$ is also realized by a multiplication.

The special case of Muir's rule for $K=[k,0^{n-1}]$ is called *Murnaghan-Nakayama rule*. The action of D_{Ψ_k} on a Schur function S_J consists in subtracting in all possible manners k to a part of J. To get Schur functions, up to sign, indexed by partitions, one must reorder the columns of the determinant $S_{J+[0\cdots 0\, k\, 0\cdots 0]}$. This correspond to subtracting in all possible manners a connected ribbon of length k to the diagram of J, taking as a sign $(-1)^{h-1}$, h being the height of the ribbon. This rule is iterated to compute values of irreducible characters of symmetric groups.

Conversely, multiplication by $\Psi_k(\mathbb{A})$ is realized by adding connected ribbons of length k to the diagram of J.

We have seen that ${}^cT^{\pm}(S_J(\mathbb{A}))$ is restricted to a single non-zero determinant. This property is in fact true for any Schur function :

Lemma 57 *Let $\mathbb{A}$ be of cardinality n, and K, L be two partitions in $\mathbb{N}^n$. Then*

$${}^rT^{+}_{S_L}\Big(S_K(\mathbb{A})\Big) = det\Big(S_{k_j+j-i+\ell_{n+1-i}}(\mathbb{A})\Big) \ ,$$

$${}^rT^{-}_{S_L}\Big(S_K(\mathbb{A})\Big) = det\Big(S_{k_j+j-i-\ell_i}(\mathbb{A})\Big) = S_{K/L}(\mathbb{A}) \ .$$

If A is of cardinality bigger than n, and $K, L \in \mathbb{N}^n$, one still has

$$ {}^rT^+_{S_L}\Big(S_K(\mathbb{A})\Big) = {}^cT^+_{S_L}\Big(S_K(\mathbb{A})\Big) = \sum_{H\in\mathbb{N}^n} (S_K S_L\,,\, S_H)\, S_H(\mathbb{A}) \ , $$

but it is not equal to $S_L(\mathbb{A})\,S_K(\mathbb{A})$.

For example,

$$ {}^rT_1(S_{25}(\mathbb{A})) = \left[\begin{smallmatrix} S_3 & S_7 \\ S_1 & S_5 \end{smallmatrix}\right] = S_{35}(\mathbb{A}) + S_{26}(\mathbb{A}) = S_1(\mathbb{A}) - S_{125}(\mathbb{A}) \ . $$

The T_k operators can be applied to other determinants than Jacobi-Trudi determinants, or Vandermondes. For example, let $f(x)$ be the function

$$ f(x) = 1/x \text{ if } x \in \mathbb{N}^n \text{ and 0 otherwise} $$

To a vector $x = [x_1, \ldots, x_n]$, associate the matrix

$$ D(x) := [f(x_j + j - i)] \ . $$

Then

$$ {}^rT_2^+\big(D([2,4,4,6])\big) = \begin{vmatrix} \frac{1}{3!} & \frac{1}{6!} & \frac{1}{7!} & \frac{1}{10!} \\ \frac{1}{2!} & \frac{1}{5!} & \frac{1}{6!} & \frac{1}{9!} \\ \frac{1}{0!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{8!} \\ 0 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{7!} \end{vmatrix} = {}^cT_2^+\big(D([2,4,4,6])\big) = $$

$$ = D([3,4,5,6]) + D([3,4,4,7]) + D([2,5,5,6]) + D([2,4,5,7]) \ , $$

the zero determinants like $D([3,5,4,6])$ not having been written.

In that case, the operator T_2^+ does not correspond to a multiplication. In fact $D(x)$ is the dimension of the irreducible representation of index x of the symmetric group, and the equality of dimension can be obtained by specialization of

$$ S_{3557/0011} = S_{3456} + S_{3447} + S_{2556} + S_{2457} \ . $$

We leave it to the reader to use the T operators to show that, for $\mathbb{A}$ of cardinality n, and two partitions $K = [k_1, \ldots, k_n]$, $L = [\ell_1, \ldots, \ell_n]$, one has

$$ \det\Big(\Psi_{k_i+\ell_j+i+j}(\mathbb{A})\Big)_{1\le i,j\le n} = S_K(\mathbb{A})\, S_L(\mathbb{A}) \ \det\Big(\Psi_{i+j}(\mathbb{A})\Big) \ . $$

Exact sequences

The notion of exact sequence is widely used in algebraic geometry. It applies to categories of objects we do not wish to use here. Let us just use the notion of *exact sequence of integers.*

Definition 58 $(0 = m_0, m_1, \ldots, m_k, m_{k+1} = 0)$ *is an* exact sequence *iff the partial alternating sums*

$$m_j - m_{j-1} + m_{j-2} - \cdots \pm m_0 \ , \ j = 0, \ldots, k+1$$

are non-negative.

This notion also applies to representations of the symmetric group (in char. 0) : $(0 = V_0, V_1, \ldots, V_k, V_{k+1})$ is an *exact sequence* iff all sequences $m_0(\lambda), \ldots, m_{k+1}(\lambda)$ are exact, for all partitions λ, where $m_i(\lambda)$ is the multiplicity of the irreducible representation of index λ in V_i.

It no less applies to elements of the free or plactic algebra : $(0 = f_0, f_1, \ldots)$ is exact iff for any word, the sequence of its coefficients in $f_0, f_1, \ldots$ is exact.

The reader will have understood alone what is an exact sequence of symmetric functions, because he will favor the Schur functions as the basis on which to test coefficients.

I have used in my thesis [?] that determinantal expressions like Jacobi-Trudi determinant can be replaced by exact sequences.

Given a square matrix M, denote by M^j the minor obained by suppressing its last row and its j-th column.

Lemma 59 *Let $K = [k_1, \ldots, k_n]$ be a partition, let M be the Jacobi-Trudi matrix expressing S_K. Then*

$$0, M^1 S_{k_1-n+1},\ M^2 S_{k_2-n+2},\ \ldots,\ M^n S_{k_n},\ S_K,\ 0$$

is an exact sequence in $\mathfrak{S}ym$.

Proof. Each minor M^i is a Schur function, and its product by a complete function is given explicitely by Pieri formula. One notices that if a Schur function appears in the sequence, then it apears also at the preceding or following step, and only there. Thus, the restriction of the sequence to any S_J appearing in it is of the type

$$0,\ 0, \ldots,\ 0,\ S_J,\ S_J,\ 0, \ldots,\ 0\ ,$$

and this insures exactness.

Since $\mathfrak{S}ym(\mathbb{A})$ is a sub-algebra of the plactic algebra, the preceding lemma entails a similar statement in the plactic algebra, but now Schur functions are sums of words :

Lemma 60 *Let $K = [k_1, \dots, k_n]$ be a partition. Then*

$$0,\ S_{k_2+1,\dots,k_n+1}\, S_{k_1-n+1},\ S_{k_1,k_3+1,\dots,k_n+1}\, S_{k_2-n+2},\ \dots\ ,\ S_{k_1,\dots,k_{n-1}}\, S_{k_n},\ S_K,\ 0$$

is an exact sequence in the plactic algebra.

The restriction of the sequence to any word w which appears in it, is of the type

$$0, \dots, 0,\ w,\ w,\ 0, \dots, 0\ .$$

Iterating Laplace expansion of Jacobi-Trudi determinant, one obtains a sequence of products of complete functions which is no less exact :

Proposition 61 *Let $K \in \mathbb{N}$ be a partition, $\mathbb{A}$ and alphabet. Let*

$$M_\ell := \sum_{\sigma,\, \ell(\sigma)=\ell} S^{(K+\rho)^\sigma-\rho}(\mathbb{A})\ .$$

Then

$$0,\ S_K,\ M_0,\ M_1,\ \dots\ ,\ M_{\binom{n}{2}},\ 0$$

is an exact sequence in the plactic algebra.

For example, in the case of $K = [2, 2, 2]$, one has the exact sequence

$$0,\ S_{222},\ S^{222},\ S^{312} + S^{213},\ S^{330} + S^{411},\ S^{420},\ 0\ .$$

Restricting it to the class of the tableau $\begin{smallmatrix} 3 & & \\ 2 & 2 & \\ 1 & 1 & 1 \end{smallmatrix}$, one finds

$$0,\ 0,\ 23\,12\,11 + 13\,22\,11,\ 123\,2\,11 + 23\,112\,1,\ 0,\ 0,\ 0\ .$$

Instead of tableaux and products S^J, one could use paths, whether intersecting or not, and relate the preceding proposition to Gessel and Viennot [9] constructions on Jacobi-Trudi type determinants.

Multiplication by a Schur function, and Littlewood-Richardson rule can be formulated in terms of exact sequences.

Instead of Schur functions, one can also handle vexillary Schubert polynomials, or flag Schur functions, and their specializations such as factorial Schur functions.

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EXERCISES

Chapter 1

Ex. 1 Let $\mathbb{A}$ be arbitrary, k be a positive integer. Expand the product $S^k(1 - z\mathbb{A})$ in the basis of Schur functions.

Ex. 2 Let $I \in \mathbb{N}^m$, $J \in \mathbb{N}^n$. Given $H \in \mathbb{N}^n$, denote $H^\omega + J := (j_1 + h_n, \ldots, j_n + h_1)$. Check that

$$S_{I;J}(\mathbb{A};\mathbb{B}) = \sum_H (-1)^{|H|} S_{I/H^\sim}(\mathbb{A})\, S_{J+H^\omega}(\mathbb{B}) \ ,$$

sum over all partitions $H \in \mathbb{N}^n$, $H \subseteq m^n$.
In particular, for $n = 1$, one has

$$S_{I,j}(\mathbb{A},\mathbb{B}) = \sum_{0 \le h \le m} (-1)^h S_{I/1^h}(\mathbb{A})\, S_{j+h}(\mathbb{B}) \ .$$

Ex. 3 Define a vertex operator on $\mathfrak{S}ym$ as a formal series in z by

$$\nabla := \exp\Big(\sum_{n\ge 1} z^n\, \Psi_n/n\Big) \exp\Big(\sum_{n \ge 1} z^{-n}\, D_{\Psi_n}/n\Big)$$

and expand it according to powers of z:

$$\nabla = \sum_{-\infty}^{+\infty} \nabla_n \ .$$

For any $\ell \in \mathbb{N}$, any $v = [v_1, \ldots, v_\ell] \in \mathbb{Z}^\ell$, show that $\nabla_{v_\ell} \cdots \nabla_{v_1}(1)$ is equal to the Schur function of index v.
See **??** for more general operators, due to Jing, "creating" Hall-Littlewood polynomials.

Ex. 4 Give the relations between the Schur functions in $\mathbb{A}$, and the Schur functions in $\mathbb{A}^\vee := \{1/a : a \in \mathbb{A}\}$, $\mathbb{A}$ a finite alphabet

Ex. 5 Let $I = [i_1, \ldots, i_r]$ be a partition, n an integer : $n \ge i_r$. Check that

$$\Lambda_{I;0^n}(\mathbb{A},\mathbb{B}) = \Lambda_I(\mathbb{A} - \mathbb{B}) \ .$$

Ex. 6 Express $S_{J;r}(\mathbb{A};x)$ as a determinant of smaller order when $r<0$ (this is not the case (1.13)).

Ex. 7 Let $\mathbb{A}$, $\mathbb{B}$ be arbitrary, and let k be a positive integer. Show that

$$(-1)^{k+2} S_{1^k;2}(\mathbb{A};\mathbb{A}-\mathbb{B}) = S_{1;k+1}(\mathbb{B}-\mathbb{A};-\mathbb{A}) + S_{k+2}(\mathbb{B}) \ .$$

Ex. 8 Compute the Schur functions $S_J(\mathbb{A}-\mathbb{B})$ when $\mathbb{A}$ and $\mathbb{B}$ are of cardinality 1 or 2.

Ex. 9 Let $\mathbb{A}$ be of cardinality n, and $I,J\in\mathbb{N}^n$. Show that the determinant

$$\left|S_{j_k+k-h+i_{n-h+1}}(\mathbb{A})\right|$$

factorizes into $S_J(\mathbb{A})\,S_I(\mathbb{A})$. What can be said if n is not the cardinality of $\mathbb{A}$?

Ex. 10 Let $\mathbb{A}$ be an alphabet of cardinality n. Describe an algorithm to express the $S^k(\mathbb{A})$ and $\Psi_k(\mathbb{A})$, $k>n$, in terms of respectively $S^1(\mathbb{A}),\ldots,S^n(\mathbb{A})$, $\Psi_1(\mathbb{A}),\ldots,\Psi_n(\mathbb{A})$.

Ex. 11 Let $\mathbb{A}$ be an alphabet of cardinality n. Show, after Cauchy and Laguerre, that elements of $\mathfrak{S}ym(\mathbb{A})$ are rational functions of

$$\Psi_1(\mathbb{A}),\Psi_3(\mathbb{A}),\ldots,\Psi_{2n-1}(\mathbb{A})$$

(resp. $S^1(\mathbb{A}),S^3(\mathbb{A}),\ldots,S^{2n-1}(\mathbb{A})$).

Ex. 12 Let $\mathbb{A}$ be any alphabet, m,n be two integers. Check that the adjoint matrix of $\mathbb{S}_{m^n}(\mathbb{A})$ is the matrix of the quadratic form $Q(x,y) := S_{(m+1)^{n-1}}(\mathbb{A}-x-y)$.

Chapter 2

Ex. 13 Let $x_1,\ldots,x_n$, y be rank 1-elements. Compute $\sum_i 1/(y-x_i)$.

Ex. 14 Let $\mathbb{A}$ be arbitrary, x of rank 1, $n,p\in\mathbb{N}$. Show that

$$S^n(\mathbb{A})-\binom{k}{1}S^n(\mathbb{A}-x)+\binom{k}{2}S^n(\mathbb{A}-2x)+\cdots+(-1)^k\binom{k}{k}S^n(\mathbb{A}-kx) = x^k S^{n-k}(\mathbb{A}) \ .$$

Ex. 15 Given two positive integers m,n, border the matrix $\mathbb{S}_{m^n}(\mathbb{A})$ by a first column $[x^n,\ldots,x^0,0]$ and a bottom row $[0,y^0,\ldots,y^n]$. Show that the determinant of this new matrix is equal to $S_{(m+1)^{n-1}}(\mathbb{A}-x-y)$, taking x,y to be rank 1-elements.

Ex. 16 Let $\mathbb{A}$ be an alphabet. Grothendieck defined functions $\gamma^i(\mathbb{A})$ by the generating function

$$\gamma_z(\mathbb{A}) = \sum z^i \gamma^i(\mathbb{A}) := \lambda_{z/(1-z)}(\mathbb{A}) \ .$$

Define accordingly the alphabet $\mathbb{A}^\diamond$ by $\Lambda^i(\mathbb{A}^\diamond) = \gamma^i(\mathbb{A})$, $i \geq 0$.
Show that

$$\begin{aligned} \Lambda^i(\mathbb{A}^\diamond) &= \Lambda^i(\mathbb{A} + i - 1) \\ rS^i(\mathbb{A}^\diamond) &= S^i(\mathbb{A} + i - 1) \\ r\Psi^i(\mathbb{A}^\diamond) &= \sum_{j=0}^{i-1} (-1)^j \binom{i}{j} \Psi^{i-j}(\mathbb{A}) \ . \end{aligned}$$

Show that the transformation $\mathbb{A} \mapsto \mathbb{A}^\diamond$ satisfies $(-\mathbb{A})^\diamond = -\mathbb{A}^\diamond$, $(\mathbb{A}+\mathbb{B})^\diamond = \mathbb{A}^\diamond + \mathbb{B}^\diamond$, but that in general, $(\mathbb{A}\mathbb{B})^\diamond \neq \mathbb{A}^\diamond \mathbb{B}^\diamond$.

Ex. 17 Let the Gauss polynomial $\begin{bmatrix} n \\ i \end{bmatrix}$ be $S^i((1-q^{n-i+1})/(1-q))$, with q of rank 1. Show that

$$\begin{bmatrix} m+n+1 \\ n \end{bmatrix} = \sum_0^n q^j \begin{bmatrix} m+j \\ j \end{bmatrix}$$

$$\begin{bmatrix} m+n \\ k \end{bmatrix} = \sum_j q^{(n-j)(k-j)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m \\ k-j \end{bmatrix} \ .$$

Ex. 18 Let p, r be positive integers. Show that

$$\sum_{k \geq 0} q^{k(p+r)} \frac{(1-q^{-p}) \cdots (1-q^{k-1-p})}{(1-q^{1+r}) \cdots (1-q^{k+r})(1-q^{-1}) \cdots (1-q^{-k})} = \frac{1}{(1-q^{1+r}) \cdots (1-q^{p+r})}$$

Ex. 19 Let $\mathbb{A}$ be arbitrary, p binomial, $n \in \mathbb{N}$. Show that

$$nS^n(p\mathbb{A}) - (p{+}n{-}1)S^{n-1}(p\mathbb{A})\,\Lambda^1(\mathbb{A}) + (2p{+}n{-}2)S^{n-2}(p\mathbb{A})\,\Lambda^2(\mathbb{A}) - \cdots \\ \cdots \pm ((n{-}1)p + 1)S^0(p\mathbb{A})\,\Lambda^n(\mathbb{A}) = 0 \ .$$

Ex. 20 Let $\mathbb{A}$ be arbitrary. Show that the product of $\Big[(-1)^{i+j}\Lambda^j(i)\Big]_{0 \leq i,j \leq n}$ by $\Big[S_j(i\mathbb{A})\Big]_{0 \leq i,j \leq n}$ is a triangular matrix. Express its entries in the $S^I(\mathbb{A})$ basis.

Ex. 21 Let $\mathbb{A}$ be of cardinality n, and $\mathbb{A}^{der}$ be the alphabet of roots of $S^{n-1}(2x - \mathbb{A})$ (we shall call it the *derived alphabet*). Show that

$$S^k(\mathbb{A} - \mathbb{A}^{der}) = \psi_k(\mathbb{A})/n \ , \ k = 1, 2, \dots \ .$$

Show that, for any $a \in \mathbb{A}$,

$$S_{n^{n-1}}(\mathbb{A} - \mathbb{A}^{der} - 2a) = (-1)^{n-1}\Delta(\mathbb{A})^2 \ ,$$

i.e. is equal to the discriminant $\prod_{i<j}(a_i - a_j)^2$, up to a sign.

Ex. 22 (Sylvester, Collect.W.1, p.502). Let $\mathbb{A}$ be of cardinality n, and let $\mathbb{A}^{der}$ its derived alphabet. For any positive integer k, let $P_k(x) := S_{k^k}(\mathbb{A} - \mathbb{A}^{der} - x)$, with x of rank 1. Show that for any $k > 0$, any $0 \le j < k$, one has

$$\sum\nolimits_{a\in\mathbb{A}} a^j \, P_k(a) = 0 \ ,$$

and that

$$\Big| P_0(a)^2 \, , \, P_1(a)^2 \, , \, \dots \, , \, P_{n-1}(a)^2 \Big|_{a\in\mathbb{A}} = 0 \ .$$

Ex. 23 Let x be such that $\forall n \in \mathbb{N}$, $S^n(1+x) = 1 + nx$. Compute the $S_J(1+x)$, J partition, without expansion of determinants.

Ex. 24 Let the *Bernoulli alphabet* $\mathbb{B}$ be defined by $\sigma_z(\mathbb{B}) = z/(1-\exp(-z))$, i.e. $S^k(\mathbb{B}) = B_k/k!$, with B_k= Bernoulli number (normalization $B_1 = 1/2$). Show that this alphabet can be defined by the equations

$$\forall k > 0, \ S_k((k+1)\mathbb{B}) = 1 \ ,$$

or by the equations

$$\forall k > 1, \ \Psi_k(\mathbb{B}) = -S^k(\mathbb{B}) \quad \& \quad \Psi_1(\mathbb{B}) = 1/2 \ .$$

Ex. 25 *Séries de Facultés* (Nördlund, Leçons sur les séries d'interpolation, Gauthiers-Villars, Paris(1926)). A "série de facultés" is a series of the type

$$f(z,\mathbb{A}) := 1 + \frac{S^1(\mathbb{A})}{z} + \frac{S^2(\mathbb{A})}{z(z+1)} + \frac{2!\, S^3(\mathbb{A})}{z(z+1)(z+2)} + \cdots = \sum + \frac{S^n(\mathbb{A})}{(n+1)S^{n+1}(z)} \ ,$$

where z is of binomial type and $\mathbb{A}$ arbitrary.
Show that for every binomial element y, one has $f(z,\mathbb{A}) = f(z+y, \mathbb{A}+y)$.
Deduce that

$$\frac{d}{dz} f(z,\mathbb{A}) = -\sum \Big(S^{n-1}(\mathbb{A})/1 + \cdots + S^0(\mathbb{A})/n\Big) / (n+1)S^{n+1}(z) \ .$$

Express the product $f(z,\mathbb{A})\, f(z,\mathbb{B})$ as a série de facultés.

Ex. 26 *Vertex operators* [Vertex2] Let $\mathbb{A}$ be of infinite cardinality, x, y be rank-1 elements.
Define T_x, T_{-x} to be the ring homomorphisms $\mathfrak{S}ym(\mathbb{A}) \mapsto Sym(\mathbb{A})[[x]]$:

$$T_x(S^k(\mathbb{A})) = \sum_i x^i S^{k-i}(\mathbb{A}) \ , \tag{$\star$}$$

$$T_{-x}(\Lambda^k(\mathbb{A})) = \sum_i (-x)^i \Lambda^{k-i}(\mathbb{A}) \ . \tag{$\star\star$}$$

Check the twisted commutations, for any $f \in \mathfrak{S}ym(\mathbb{A})$:

$$\begin{aligned} T_x(\sigma_y(\mathbb{A})\, f) &= \frac{1}{1-xy}\sigma_y(\mathbb{A}) T_x(f) \\ T_{-x}(\lambda_{-y}(\mathbb{A})\, f) &= \frac{1}{1-xy}\lambda_{-y}(\mathbb{A}) T_{-x}(f) \\ T_x(\lambda_{-y}(\mathbb{A})\, f) &= (1-xy)\, \lambda_{-y}(\mathbb{A}) T_x(f) \\ T_{-x}(\sigma_y(\mathbb{A})\, f) &= (1-xy)\, \sigma_y(\mathbb{A}) T_{-x}(f) \end{aligned}$$

Ex. 27 Let $\mathbb{A}$ be arbitrary, k, n be two integers, x be a rank-1 element. Show that

$$S_{k^n}(\mathbb{A}-x)\, S_{k^{n+1}}(\mathbb{A}) - S_{k^n}(\mathbb{A})\, S_{k^{n+1}}(\mathbb{A}-x) = x\, S_{(k+1)^n}(\mathbb{A}-x)\, S_{(k-1)^{n+1}}(\mathbb{A}) \ .$$

For example,

$$S_{55}(\mathbb{A}-x)\, S_{555}(\mathbb{A}) - S_{55}(\mathbb{A})\, S_{555}(\mathbb{A}-x) = x\, S_{66}(\mathbb{A}-x)\, S_{444}(\mathbb{A}) \ .$$

Ex. 28 Let $\mathbb{A}, \mathbb{B}$ be alphabets; take on $\mathfrak{S}ym(\mathbb{A}) \otimes \mathfrak{S}ym(\mathbb{B})$ the tensor product of the usual scalar products.
Show that $\Psi^I(\mathbb{A}-\mathbb{B})\, \Psi^J(A+\mathbb{B})$, (I, J) running over all pairs of partitions, is an orthogonal basis of $\mathfrak{S}ym(\mathbb{A}) \otimes \mathfrak{S}ym(\mathbb{B})$.

Ex. 29 Let $k, n \in \mathbb{N}$, $\mathbb{A}_0, \dots, \mathbb{A}_{n-1}$ be n alphabets. Take on $\mathfrak{S}ym(\mathbb{A}_0) \otimes \cdots \otimes \mathfrak{S}ym(\mathbb{A}_{n-1})$ the tensor product of the usual scalar product on each $\mathfrak{S}ym(\mathbb{A}_i)$.
Show that

$$\prod_i S_{\nu^i}\big(\mathbb{A}_i - \Lambda^1([k])\mathbb{A}_{i-1} + \Lambda^2([k])\mathbb{A}_{i-2} - \cdots \pm \Lambda^k([k])\mathbb{A}_{i-k}\big)$$

is the adjoint basis of

$$\prod_i S_{\nu^i}\big(\mathbb{A}_i + S^1([k])\mathbb{A}_{i+1} + \cdots + S^{n-1-i}([k])\mathbb{A}_{n-1}\big)$$

the indexing set being all n-tuple of partitions $(\nu^0, \dots, \nu^{n-1})$; $[k] := 1 + q + \cdots + q^k$, with q of rank 1; $\mathbb{A}_i := 0$ for $i < 0$.
The case $k = 1$ is due to B.Leclerc and H.Miyachi (*Some closed formulas for canonical bases of Fock spaces* (2001)). For example, for $n = 4$, $k = 1$, the adjoint bases are the following elements, indexed by all possible quadruples of partitions :

$$S_{\nu^0}(\mathbb{A}_0) S_{\nu^1}(\mathbb{A}_1 - q\mathbb{A}_0) S_{\nu^2}(\mathbb{A}_2 - q\mathbb{A}_1) S_{\nu^3}(\mathbb{A}_3 - q\mathbb{A}_2) \ ,$$

$$S_{\nu^0}(\mathbb{A}_0 + q\mathbb{A}_1 + q^2\mathbb{A}_2 + q^3\mathbb{A}_3) S_{\nu^1}(\mathbb{A}_1 + q\mathbb{A}_2 + q^2\mathbb{A}_3) S_{\nu^2}(\mathbb{A}_2 + q\mathbb{A}_3) S_{\nu^3}(\mathbb{A}_3) \ .$$

Chapter 3

Ex. 30 Let $f(x)$ be a polynomial and $\mathbb{A}$ be an alphabet of cardinality n. Show that $f(x)$ is congruent, modulo $S^n(x - \mathbb{A})$, to the quotient

$$\begin{vmatrix} f(a) & 1 & a & a^2 & \cdots & a^{n-1} \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 1 & x & x^2 & \cdots & x^{n-1} \end{vmatrix} \frac{1}{\Delta(\mathbb{A})} \ .$$

Ex. 31 Find back the first remainder of the division of $S^m(x - \mathbb{B})$ by $S^n(x - \mathbb{A})$, applying Plücker relations to the following matrix of complete functions (denoted only by their exponents) :

$$\begin{array}{r} \\ \\ \\ \\ \textit{alphabets} \end{array} \begin{bmatrix} 0 & \cdots & n & m & n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -n & \cdots & 0 & m - n & 0 \\ \mathbb{A} & \cdots & \mathbb{A} & \mathbb{A} - \mathbb{B} + x & x \end{bmatrix}$$

Ex. 32 Use Plücker relations to recover, following Wronski, that the $r+1$-th remainder in the division of $S^m(x - \mathbb{B})$ by $S^n(x - \mathbb{A})$ is the normalized difference of the r-th remainders of $S^m(x - \mathbb{B})$ and $S^{m+1}(x - \mathbb{B})$.

Ex. 33 Let $\mathbb{A}$ be of cardinality n, x of rank 1. Give the successive remainders in the Euclidean division of x^{n+1} by $S^n(1 - x\mathbb{A})$.

Ex. 34 Let $\mathbb{A}$ be an alphabet, with $|\mathbb{A}| = n$. Extract from the companion matrix of $\mathbb{A}$ the (infinite) Vandermonde matrix. Deduce from it that any Schur function can be expressed as determinant of hook functions $S_{1^i j}(\mathbb{A})$.

Ex. 35 Let $\mathbb{A}$ be an alphabet, with $|\mathbb{A}| = n$. Let $\mathbb{X} := \{x_1, \dots, x_k\}$ be an alphabet, $k \leq n$. Show that the remainder of any minor of order k of the infinite Vandermonde matrix $\mathbb{V}(\mathbb{X})$ modulo the polynomials $S^n(x_i - \mathbb{A})$, is a sum of products of minors of the finite Vandermonde matrix of width n, by Schur functions in $\mathbb{A}$.

Ex. 36 Let $\mathbb{A}, \mathbb{B}$ be two alphabets, $|\mathbb{A}| = |\mathbb{B}| = n$. Express the matrix of Bézout's morphism

"Multiplication by $S^n(x-\mathbb{B})$ modulo $S^n(x-\mathbb{A})$"

in the basis $\{S^0(x-\mathbb{A}), \dots, S^{n-1}(x-\mathbb{A})\}$, and find back the expression of Bézout's matrix in terms of $\mathbb{S}_{n^n}(\mathbb{A}-\mathbb{B})$, using that $S^i(x-\mathbb{A})S^n(x-\mathbb{B}) = (-1)^i S_{1^i,n}(\mathbb{A}, x-\mathbb{B}) = (-1)^i S_{1^i,n}(\mathbb{A}, (x-\mathbb{A})+(\mathbb{A}-\mathbb{B}))$.

Ex. 37 The coefficients of the successive remainders in Sturm sequence (for the division of a polynomial $R(x,\mathbb{A})$ by its derivative) are symmetric functions of $\mathbb{A}$. Find them.

Ex. 38 (*Hypergeometric alphabet*). Let a, b be elements of binomial type, $a, b \notin \mathbb{N}$, and x be rank-1 element. Define the alphabet $\mathbb{A}$ by $\Lambda^i(\mathbb{A}) := \Lambda^i(a)/\Lambda^i(b)$, $i = 1, 2, \dots$. Let, for any $n \in \mathbb{N}$, $\phi(n, a, b) = \Lambda^n(nx - \mathbb{A})$.
Show that the derivative of $\phi(n, a, b)$ with respect to x is equal to $n\phi(n-1, a, b)$, and that the respective remainders of $\phi(n, a, b)$ with respect to $\phi(n-1, a+k, c+\ell)$, with $[k, l] = [1, 0], [0, 0], [0, -1], [-1, 0], [-1, -1]$ or $[-1, -2]$, are proportional to some $\phi(n-2, a', b')$.
In particular, show after Stieltjes (*Sur les polynômes de Jacobi*, Comptes Rendus, **100** (1885) 620), that the Sturm sequence associated to $\phi(n, a, b)$ (i.e. the Euclidean division of $\phi(n, a, b)$ by $\phi(n-1, a, b)$) is the sequence

$$\phi(n-i-1, a-i, b-2i),\ i = 1, 2, \dots ,$$

up to factors to be determined.
Show in particular that the discriminant of $\phi(n, a, b)$ factorizes into factors which are linear in a and b.
For another method to prove this last point, see next exercise.

Ex. 39 Let a, b be elements of binomial type, $a, b \notin \mathbb{N}$, and let $n \in \mathbb{N}$. Define $\mathbb{A}$ and B by the equations

$$\Lambda^i(\mathbb{A}) := \binom{n}{i}\Lambda^i(a)/\Lambda^i(b) \quad \& \quad \Lambda^i(\mathbb{B}) := \binom{n-1}{i}\Lambda^i(a)/\Lambda^i(b),\ i = 1, 2, \dots$$

Compute the $S_{k^n}(\mathbb{A}-\mathbb{B})$, for $k \geq n-1$.

Ex. 40 Let $\mathbb{A}$, $\mathbb{B}$ be two alphabets: $|\mathbb{A}| = n$, $|\mathbb{B}| = m$. Let $\mathbb{S}$ be the $n \times \infty$ matrix $\mathbb{S} := \left[S^{m+j-i}(\mathbb{A}-\mathbb{B})\right]_{0 \leq i \leq n-1, 0 \leq j}$. Let $\mathcal{C}(\mathbb{A})$ be the infinite companion matrix of $S^n(x-\mathbb{A})$. Show that $\mathbb{S}$ factorizes into

$$\mathbb{S} = \mathbb{S}_{m^n}(\mathbb{A}-\mathbb{B})\, \mathcal{C}(\mathbb{A}) \ .$$

Deduce, for every $J \in \mathbb{N}^n$ the following factorization :

$$S_{m^n+J}(\mathbb{A}-\mathbb{B}) = R(\mathbb{A},\mathbb{B})\, S_J(\mathbb{A}) \ .$$

Deduce also the expression of a Schur function $S_J(\mathbb{A})$ as a determinant of hook-functions $S_{1^i j}(\mathbb{A})$.
For example, for $n = 3$, $m = 5$, writing the alphabets below the matrices, and truncating them to width 7 as a first approximation to ∞, one has

$$\underset{\mathbb{A}-\mathbb{B}}{\begin{bmatrix} S^5 & S^6 & S^7 & S^8 & S^9 & S^{10} & S^{11} \\ S^4 & S^5 & S^6 & S^7 & S^8 & S^9 & S^{10} \\ S^3 & S^4 & S^5 & S^6 & S^7 & S^8 & S^9 \end{bmatrix}} = \underset{\mathbb{A}-\mathbb{B}}{\begin{bmatrix} S^5 & S^6 & S^7 \\ S^4 & S^5 & S^6 \\ S^3 & S^4 & S^5 \end{bmatrix}} \underset{\mathbb{A}}{\begin{bmatrix} 1 & \cdot & \cdot & S_{300} & S_{400} & S_{500} & S_{600} \\ \cdot & 1 & \cdot & S_{20} & S_{30} & S_{40} & S_{50} \\ \cdot & \cdot & 1 & S_1 & S_2 & S_3 & S_4 \end{bmatrix}}$$

Ex. 41 Let $\mathbb{A}$: $|\mathbb{A}| = n$ be such that all $a \in \mathbb{A}$ are distinct, and let $Q_a := S^{n-1}(x - \mathbb{A} + a)$. Show that the morphism u defined in section(Bézout) is diagonal in basis $\{Q_a\}_{a\in\mathbb{A}}$, with diagonal $\{S^n(a-\mathbb{B})\}$.

Ex. 42 Relate the expressions given by Sturm for the successive remainders of a polynomial by its derivative to the expression of the remainders as Schur functions.

Chapter 4

Ex. 43 Let $a_1 + \frac{b_2}{a_2+b_3/\cdots}$ be a continued fraction. Check from the matrix expression that the parameters can be recovered from the convergents :

$$a_n = \frac{P_{n-2}Q_n - P_nQ_{n-2}}{P_{n-2}Q_{n-1} - P_{n-1}Q_{n-2}} \quad \& \quad b_n = \frac{P_nQ_{n-1} - P_{n-1}Q_n}{P_{n-2}Q_{n-1} - P_{n-1}Q_{n-2}}$$

Ex. 44 (R.Rogers, On the representation of certain series as continued fractions, Proc. London M.S. **4** (1907)72-89) Give the Taylor expansion of a continued fraction $1\Big/\Big(1 - \big(b_1 z/(1 - b_3 z/(\cdots))\big)\Big)$

Ex. 45 Given $\alpha \in \mathbb{C}$, show that $(1+z)^\alpha$ has a continued fraction expansion $a_1 + \frac{b_2}{a_2+b_3/\cdots}$ with $a_1 = 1, a_{2n} = 2n-1$, $a_{2n+2} = 2$, $b_{2n} = (n-1+\alpha)z$, $b_{2n+1} = (n-\alpha)z$. Obtain, as a limit, the continued fraction for the exponential: $a_1 = 1, a_{2n} = 2n-1$, $a_{2n+2} = 2$, $b_{2n} = z$, $b_{2n+1} = -z$. Do not forget the logarithm: $\log(1+z)$ has a continued fraction expansion with $a_n = n$, $n = 0, 1, \ldots$, and $b_2 = z$, $b_{2n+1} = b_{2n+2} = z\, n^2$, $n = 1, 2, \ldots$

Ex. 46 Continued fraction expansions can also be found for divergent series. Check, after Euler (De transformatione seriei divergentis.., Acta Acad. Sc.Petropoli (1784)), that the series $\sum z^n\, n!$ has a continued fraction expansion $1+\frac{b_2 z}{1+b_3 z/\cdots}$ with $[b_2, b_3, \dots] = [1, -2, -1, -3, -2, \dots, -n-1, -n, \dots]$. Introduce an extra parameter $\beta \in \mathbb{C}$ and show that $\sum z^n\, n! S^n(\beta)$ admits a continued fraction expansion with $[b_2, b_3, \dots] = [\beta, -\beta-1, -1, -\beta-2, -2, \dots, -\beta-n, -n, \dots]$ (recall that $n! S^n(\beta) = \beta(\beta+1)\cdots(\beta+n-1)$).

Ex. 47 Given two infinite sequences $[a_1, a_2, \dots]$, $[b_2, \dots]$, let

$$K_n([a_1, \dots, a_n], [b_2, \dots, b_n])$$

be the $n \times n$ following determinant (with only three non zero diagonals):

$$K_n([a_1, \dots, a_n], [b_2, \dots, b_n] := \begin{vmatrix} a_1 & b_2 & 0 & \cdots & & \\ -1 & a_2 & b_3 & 0 & \cdots & \\ 0 & \ddots & \ddots & \ddots & & \\ \vdots & & & & a_{n-1} & b_n \\ & & & & -1 & a_n \end{vmatrix}$$

Show, after Sylvester (Collected Work I, p.609-629 and 641-644) that the convergents of the fraction $a_1 + \frac{b_2}{a_2+b_3/\cdots}$ are

$$P_n/Q_n = K([a_1, \dots, a_n], [b_2, \dots, b_n])/K([a_2, \dots, a_n], [b_3, \dots, b_n])$$

Ex. 48 Let $[a_1, \dots, a_n]$, $[b_2, \dots, b_n]$ be two sequences. The *Jacobi matrix* $J([a_1, \dots, a_n], [b_2, \dots, b_n])$ is the tridiagonal symmetric matrix, with main diagonal equal to $[a_1, \dots, a_n]$ and its neighbours to $[b_2, \dots, b_n]$. Jacobi's continued fraction is $\dfrac{1}{u - a_1 - \dfrac{{b_2}^2}{u - a_2 - \dfrac{{b_3}^2}{u - a_3 - \cdots}}}$. Show that its n-th convergent P_n/Q_n is such that P_n is the characteristic polynomial of $J([a_2, \dots, a_n], [b_3, \dots, b_n])$ and Q_n, of $J([a_1, \dots, a_n], [b_2, \dots, b_n])$.

Ex. 49 Let $\mathbb{A}$ be an alphabet of cardinality 2, and let $\Lambda^1 = \Lambda^1(\mathbb{A})$, $\Lambda^2 = \Lambda^2(\mathbb{A})$. Show that the n-th convergent of $\Lambda^1 - \frac{\Lambda^2}{\Lambda^1 - \Lambda^2/\cdots}$ is $S^n(\mathbb{A})/S^{n-1}(\mathbb{A})$.

Ex. 50 (Contraction of a continued fraction). Since

$$1 + \frac{az}{1+bz/X} = 1 + az - \frac{abz^2}{bz + X}$$

one can contract Stieltjes' continuous fraction for $\sigma_z(\mathbb{A})$, or its inverse $\lambda_{-z}(\mathbb{A})$ by grouping terms two by two. There are two manners to do it:

$$\frac{1}{1+\cfrac{za_1}{1+\cfrac{za_2}{1+\cfrac{za_3}{\ddots}}}} = \frac{1}{1+za_1+\cfrac{z^2a_1a_2}{1+z(a_2+a_3)+\cfrac{z^2a_3a_4}{1+z(a_4+a_5)+\cfrac{z^2a_5a_6}{\ddots}}}}$$

$$1+\frac{za_1}{1+\cfrac{za_2}{1+\cfrac{za_3}{\ddots}}} = 1+\frac{za_1}{1+za_2-\cfrac{z^2a_2a_3}{1+z(a_3+a_4)-\cfrac{z^2a_4a_5}{1+z(a_5+a_6)+\cfrac{z^2a_6a_7}{\ddots}}}}$$

Show that the first contraction gives back Wronski's continuous fraction for $\lambda_{-z}(\mathbb{A})$.
For the second contraction, show that one can rewrite the coefficients

$$a_3+a_4=\frac{S_{222}-S_{33}}{S_1S_{22}}\ ,\ a_5+a_6=\frac{S_2S_{3333}-S_{11}S_{444}}{S_{22}S_{33}}\ ,$$

$$a_{2n-1}+a_{2n}=\frac{S_{(n-1)^{n-2}}S_{n^{n+1}}-S_{(n-2)^{n-1}}S_{(n+1)^n}}{S_{(n-1)^{n-1}}S_{n^n}}\ .$$

Ex. 51 Let $f(x)$ be a function. Let $T_r(f)$ be the limit of Thiele's reciprocal difference $[x_1,\dots,x_{r+1}]$, for $x_1=\cdots=x_{r+1}=x$. Show that $T_0(f)=f$, $T_1(f)=1/D(f)$, where D is the derivative with respect to x, and that

$$T_r(f)=T_{r-2}(f)+\frac{r}{D(T_{r-1}(f))}$$

Deduce Thiele's expansion of f in the neighbourhood of x :

$$f(x+\epsilon)=f(x)+\frac{\epsilon}{T_1(f)+\cfrac{\epsilon}{2T_1(T_1(f))+\cfrac{\epsilon}{3T_1(T_2(f))+\cfrac{\epsilon}{4T_1(T_2(f))+\cfrac{\epsilon}{\ddots}}}}}$$

Ex. 52 (O.Perron, Die Lehre von den Kettenbrüchen, Leipzig 1929) Put $f(x) = e^x$ in the preceding exercise. Show that $T_{2n}(e^x) = (-1)^n\, e^x$ and $T_{2n+1}(e^x) = (-1)^n(n+1)\, e^{-x}$. Transform Thiele's expression into

$$e^{\epsilon} = 1 + \cfrac{\epsilon}{1 - \cfrac{\epsilon}{2 + \cfrac{\epsilon}{3 - \cfrac{\epsilon}{2 + \cfrac{\epsilon}{5 - \cfrac{\epsilon}{\ddots}}}}}}$$

Chapter 5

Ex. 53 Let $f(z) := \lambda_z(\mathbb{B})/\lambda_z(\mathbb{A})$ be a rational fraction. Given two integers m, n, write the numerator and the denominator of the Padé approximant P/Q of $f(z)$ of degree $[\beta/\alpha]$ as determinants with entries the elementary symmetric functions in $\mathbb{A}$ or $\mathbb{B}$.

Ex. 54 Let $\mathbb{A}$ be an alphabet of infinite cardinality, m, n be two positive integers, $K \subset [1, \dots, m{+}n]$ be a subset of cardinality n. Find a polynomial $Q(z) = S^n(1 - z\mathbb{B})$ of degree n such that

$$S^k(\mathbb{A} - \mathbb{B}) = 0 \;\forall k \in K \ .$$

Thus $P(z) := S^{m+n}(1 + z(\mathbb{A}{-}\mathbb{B}))$ is a polynomial of degree $\leq m + n$ such that

$$P/Q \equiv \sigma_z(A)\ mod(x^{m+n+1}) \ ,$$

but, contrary to the case of a Padé approximant, the numerator is of degree $\leq m + n$, having n coefficients null.

Ex. 55 Let α, β be two integers, and $[\beta/\alpha]$ the Padé approximant of $\sigma_z(\mathbb{D})$. Give the Taylor expansion of $[\beta/\alpha]$.
In particular describe explicitely the coefficient of $z^{\alpha+\beta+1}$.

Ex. 56 (Baker and Grave-Morris, p.339) Let $f(z)$ be a function of one variable, α, β two positive integers, $\mathcal{Y} = \{y_0, \dots, y_{\alpha+\beta}\} \subset \mathbb{C}$. Write cY_n for $y_0, \dots, y_n$, and $\mathcal{Y}_{-1}$ for the void set. Write $[i \dots j]$ for the image of $f(y_i)$ under the product of divided differences $\partial_{y_i, y_{i+1}} \cdots \partial_{y_{j-1}, y_j}$, and $[i \dots j, z]$ for the image of $[i \dots j]$ under $\partial_{y_j, z}$. When $i > j$ put $[i \dots j] = 0$.

Let moreover $P(z)/Q(z)$ be the rational interpolant of $f(z)$ of type $[\beta/\alpha]$ at points $\mathcal{Y}$.
Show that $Q(z)$ is proportional to

$$\begin{vmatrix} [\alpha\ldots(\beta{+}1)] & \cdots & [\alpha\ldots(\alpha{+}\beta)] & R(z,\mathcal{Y}_{\alpha-1}) \\ \vdots & & \vdots & \vdots \\ [1\ldots(\beta{+}1)] & \cdots & [1\ldots(\alpha{+}\beta)] & R(z,\mathcal{Y}_0) \\ [0\ldots(\beta{+}1)] & \cdots & [0\ldots(\alpha{+}\beta)] & R(z,\mathcal{Y}_{-1}) \end{vmatrix} .$$

Show that $\big(f(z)Q(z) - P(z)\big)/R(z,\mathcal{Y})$ is equal to the determinant

$$\begin{vmatrix} [\alpha\ldots(\beta{+}1)] & \cdots & [\alpha\ldots(\alpha{+}\beta)] & [\alpha\ldots(\alpha{+}\beta),z] \\ \vdots & & \vdots & \vdots \\ [1\ldots(\beta{+}1)] & \cdots & [1\ldots(\alpha{+}\beta)] & [1\ldots(\alpha{+}\beta),z] \\ [0\ldots(\beta{+}1)] & \cdots & [0\ldots(\alpha{+}\beta)] & [0\ldots(\alpha{+}\beta),z] \end{vmatrix} . \qquad (\star)$$

Ex. 57 Compute the vertical and horizontal distance 2 differences in a Padé table (for $\sigma_z(\mathbb{A})$) : $[\beta/\alpha] - [\beta{-}2/\alpha]$ and $[\beta/\alpha] - [\beta/\alpha{-}2]$.

Ex. 58 Let $\mathbb{A}$ be arbitrary, k, n be integers, x be a rank-1 element. Show that

$$S_{k^n}(\mathbb{A}{-}x)\, S_{(k+1)^n}(\mathbb{A}) = S_{k^n}(\mathbb{A})\, S_{(k+1)^n}(\mathbb{A}{-}x) - xS_{k^{n+1}}(\mathbb{A})S_{(k+1)^{n-1}}(\mathbb{A}{-}x) .$$

Use this formula, in conjunction with ex(Wynn1) to reprove Wynn's formula.

Ex. 59 (Montsessus) Let $f(z)$ be a meromorphic function in the unit disk. Poles are of total multiplicity n, and different from 0.
Show that the sequence of Padé approximants $[m/n]$, $m \to \infty$, converges towards $f(z)$ in any point in the disk, different from a pole.
The reader, having only algebraic tools at his disposal, can suppose that $f(z) = \sigma_z(\mathbb{A} - \mathbb{B})$, with $card(\mathbb{A}) = n$, $card(\mathbb{B}) < \infty$, and $|a| > 1, \forall a \in \mathbb{A}$.

Chapter 6

Ex. 60 Show how to obtain the first remainder in the Euclidean division of two polynomials, using divided differences.

Ex. 61 Given two alphabets $\mathbb{A}, \mathbb{B}$, of respective cardinalities n, m, $n \leq m$ and let $0 < r \leq m, n$. Find the remainder of $S_{(r-1)^{m-n+r}}(\mathbb{A}{-}\mathbb{B}{-}x)\, S^m(x{-}\mathbb{B})$ modulo $S^n(x - \mathbb{A})$.

Ex. 62 : Factorization of the Vandermonde matrix.
Let $\mathbb{A}$ be of cardinality n, and $V := \left[a_i^{n-j}\right]_{1\le i,j\le n}$. Let $\mathbb{A}_k := \{a_1, \ldots, a_k\}$. Using Newton to interpolate each a_i^k in $\mathbb{A}_{i-1}$ show that V factorizes into

$$V = \left[R(a_{i-1}, \mathbb{A}_{j-1})\right]\left[S^{j-i}(\mathbb{A}_i)\right]$$

(with $R(x, \emptyset) = 1$). Decompose the two matrices into products of matrices with entries in the set $\{0, 1, a_1, \ldots, a_n, (a_j - a_i) : 1 \le i < j \le n\}$.

Ex. 63 Let $\mathbb{A}$ be infinite, and, for any k, let $\mathbb{A}_k$ denote $\{a_1, \ldots, a_k\}$. Expand the characteristic polynomial of the (right-justified) $n \times n$ Pascal matrix

$$M := \left[\Lambda^{j-i}(\mathbb{A}_{i-1})\right]_{1\le i,j\le n} .$$

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Ex. 64 Let $\mathbb{A}$ be an alphabet and let $\int$ be the linear functional such that $\int t^k = S_k(\mathbb{A})$, $k \ge 0$. Show that $\int \partial_{t,x}(P(t))$ is equal to the adjoint polynomial $Q_n(x) = S_{(n-1)^{n+1}}(\mathbb{A} + x)$.

Ex. 65 *Gauss quadrature.* Let $L_{\mathbb{A}}$ be the linear functional attached to a generic alphabet $\mathbb{A}$:

$$L_{\mathbb{A}}(x^k) = S_k(\mathbb{A}) \ , \ k \ge 0 .$$

Let $\mathbb{B}$ be the set of zeroes of $P_n(x) := S_{n^n}(\mathbb{A} - x)$. Show that

$$L_{\mathbb{A}}(f(x)) = (P_{n-1}, P_{n-1}) \sum_{b\in\mathbb{B}} f(b)/P_{n-1}(b)R(b, \mathbb{B} - b)$$

for any polynomial of degree $\le 2n - 1$. The reader can use the basis

$$\{x^k P_n(x) : k = 0, \ldots, n{-}1\} \cup \{x^k P_{n-1}(x) : k = 0, \ldots, n{-}2\} \cup \{P_{n-1}^2(x)\}$$

to test the statement.
When the linear functional $L_{\mathbb{A}}$ is defined by an integral, the formula gives a way of computing the integral of a polynomial by a finite sum on the zeroes of P_n.

Ex. 66 Given an alphabet $\mathbb{A} = \{a_1, \ldots, a_n\}$, write $\sum_{1<i\le n} 1/(a_1 - a_i)$ as a function of a_1 and $\mathbb{A}$.
Knowing that the Hermite polynomial $H_n(x)$ satisfy the differential equation

$$D^2 H_n - 2x\, DH_n + 2nH_n = 0 ,$$

with $D := \frac{d}{dx}$, use the preceding result to prove that

$$a_1 = \sum\nolimits_{1<i\leq n} 1/(a_1 - a_i)$$

when $\mathbb{A}$ is specialized to the set of roots of H_n, a_1 being any root.

Ex. 67 Let $\mathbb{A}$ be an infinite alphabet, x an indeterminate, and Ψ_0 a parameter. Define a linear functional L on polynomials in x by

$$L(x^0) = \psi_0 \,, \qquad L(x^i) = \Psi_i(\mathbb{A}) \,, i > 0 \,.$$

Find the orthogonal polynomials associated to L.

Ex. 68 Let $\mathbb{A}$ be an infinite alphabet, $x_1, \ldots, x_n$ be indeterminates of sum $\mathbb{X}$. Let L_j, $j = 1, \ldots, n$, be the linear functional on $\mathbb{C}[x_j]$ defined by $L_j(x_j^k) = S^k(\mathbb{A})$, $k \geq 0$, and let L be the product of the L_j's.
Show that, for every $J \in \mathbb{N}^n$,

$$L\big(S_J(\mathbb{X})\,\Delta(\mathbb{X})^2\big) = (-1)^{n(n-1)/2} n!\, S_{J+(n-1)^n}(\mathbb{A}) \ .$$

Notice the corollary corresponding to $J = 1^n$, with z of rank 1 :

$$L\big(R(\mathbb{X}, z)\Delta(\mathbb{X})^2\big) = \pm S_{n^n}(\mathbb{A} - z) \ .$$

Ex. 69 Let x be a rank-1 element, L be the linear functional $x^j \mapsto S^j(\mathbb{A})$, k be a positive integer. Show that

$$L\big(S_{(k+1)^k}(\mathbb{A} - 2x)\big) = (-1)^k (k+1)\, S_{k^{k+1}}(\mathbb{A}) \ .$$

Deduce that

$$L\big(S_{k^k}(\mathbb{A} - x)^2\big) = (-1)^k S_{k^{k-1}}(\mathbb{A}) S_{k^{k+1}}(\mathbb{A}) \ .$$

Ex. 70 (Stieltjes). Let $g(t)$ be a positive continuous function of $t \in [0, 1]$, satisfying $\int_0^1 g(t)dt = 1$. Define $\mathbb{A}$ by $S^k(\mathbb{A}) := \int_0^1 g(t)t^k dt$, $k = 1, 2, \ldots$.
Given any positive integer n, and indeterminates $x_1, \ldots, x_n$, remark that the quadratic form

$$Q(x_1, \ldots, x_n) := \int_0^1 g(t)(\sum x_i t^{i-1})^2 dt$$

is positive, and deduce from it that

$$(-1)^{n(n-1)/2} S_{(n-1)^n}(\mathbb{A}) \geq 0 \ .$$

More generally, show that any minor of the Hankel matrix $\left[S_{i+j}(\mathbb{A})\right]_{i,j\geq 0}$ is positive.
Consider the case $g(t) = \sigma_t(\mathbb{B})$ (which is 'generic'), and express the Schur functions of $\mathbb{A}$ in terms of $\mathbb{B}$.
Given any k, evaluate the multiple integral

$$\int_0^1 x_1^k g(x_1) \int_0^1 \cdots \int_0^1 x_n^k g(x_n) \prod_{1\leq i<j\leq n} ((x_i - x_j)^2 dx_1 \cdots dx_n$$

and conclude, once more, that the Hankel determinant $|S_{k+j-i}(\mathbb{A})|_{1\leq i,j\leq n}$ is positive (the preceding exercise could have been used).

Mixed Exercises, to reach 100

Ex. 71 (Newton's recursions) Show that

$$k\Lambda^k = \Lambda^{k-1}\Psi_1 - \Lambda^{k-2}\Psi_2 + \Lambda^{k-3}\Psi_3 - \cdots \pm \Lambda^0\Psi_k \ .$$

Deduce Brioschi' recursions

$$kS^k = S^{k-1}\Psi_1 + S^{k-2}\Psi_2 + S^{k-3}\Psi_3 + \cdots + S^0\Psi_k \ .$$

Ex. 72 Show that

$$n!\, S^n = \begin{vmatrix} 2\Lambda^1 & \Lambda^0 & 0 & \cdots & 0 \\ 4\Lambda^2 & 3\Lambda^1 & 2\Lambda^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (2n-2)\Lambda^{n-1} & (2n-3)\Lambda^{n-2} & (2n-4)\Lambda^{n-3} & \cdots & (n-1)\Lambda^0 \\ n\Lambda^n & (n-1)\Lambda^{n-1} & (n-2)\Lambda^{n-2} & \cdots & \Lambda^1 \end{vmatrix}$$

(only the last row is irregular).

Ex. 73 (Waring's Formula) Show that

$$\frac{1}{k}\Psi_k = \sum (-1)^{\ell(J)+k} \frac{(\ell(J)-1)!}{m_1!m_2!\cdots} \Lambda^J \ ,$$

sum over all partitions $J = 1^{m_1}2^{m_2}\cdots$ of weight $|J| = k$.
For example, $\psi_6/6 = -\Lambda_6 + \Lambda_5\Lambda_1 - \Lambda_4\Lambda_1^2 + \Lambda_3\Lambda_1^3 - \Lambda_2\Lambda_1^4 + 1/6\Lambda_1^6 + 3/2\Lambda_1^2\Lambda_2^2 - 2\Lambda_1\Lambda_2\Lambda_3 + \Lambda_2\Lambda_4 - 1/3\Lambda_2^3 + 1/2\Lambda_3^2$
Identify the scalar products (Ψ_k, Ψ_J).

Ex. 74 Compute the scalar products (Λ^k, Ψ_J).

Ex. 75 Let $\mathbb{A}$ be an alphabet of cardinality n. Show that the $S_k(\mathbb{A})$ satisfy a linear recurrence of order n. What are the terms of negative index of this recurrent sequence ?

Ex. 76 Let $\mathbb{A}, \mathbb{B}$ be two alphabets, $card(\mathbb{A}) = n > card(\mathbb{B})$. Define $\$^k$, $k \in \mathbb{Z}$ as the Taylor coefficient of degree k of $\sigma_z(\mathbb{A} - \mathbb{B})$ in $z = 0$ when $k \geq 0$, and as the Taylor coefficient of $-\sigma_z(\mathbb{A} - \mathbb{B})$ in $z = \infty$ when $k < 0$.
Show that $\$^k$ is a linear recurrence of order n, and that the preceding exercise is the case $\mathbb{B} = 0$.
For $I \in \mathbb{N}^n$, $J \in \mathbb{Z}^n$, define $\$_{J/I}$ to be the minor of the Toeplitz matrix $\left[\$^{j-i}\right]_{i \geq 0, j \in \mathbb{Z}}$ taken on rows I and columns J. Show that

$$\$_{(J \pm 1^n)/I} = (\Lambda^n(\mathbb{A}))^{\pm 1} \$_{J/I} \ .$$

Ex. 77 Let $\mathbb{A}$ be an alphabet of cardinality n, and let $\mathbb{B}$ be arbitrary. Replace each entry a^k of the Vandermonde matrix $V(\mathbb{A})$ by $S^k(a + \mathbb{B})$. Compute the minors of order n of this new matrix.

Ex. 78 Show that

$$\Psi_k = S_k - S_{1,k-1} + S_{1,1,k-2} - \cdots + \pm S_{1^k} \ .$$

Ex. 79 Murnaghan-Nakayama formula) Let $J \in N^n$. Show that

$$D_{\Psi^k}(S_J) = \sum_i S_{J - [0^i, k, 0^{n-1-i}]} \ .$$

Ex. 80 For $n \in \mathbb{N}$, express $\sin(r\theta)/\sin(\theta)$ and $\cos(n\theta)$ in terms of $\cos(\theta)$, by identifying them to symmetric functions in $\pm \exp(\sqrt{-1}\theta)$.

Ex. 81 Let $\mathbb{A}$ be an alphabet of cardinality n, and $k \in \mathbb{N}$. Show that

$$\prod_{1 \leq i < j \leq n} S^k(a_i + a_j)$$

is equal to a Schur function.

Ex. 82 Let $\mathbb{A}$ be an alphabet of cardinality n, and S a symmetric function. Show that the determinant obtained from the Vandermonde $\Delta(\mathbb{A})$ by replacing its last column $[a_1^{n-1}, a_2^{n-1}, \ldots]$ by the partial derivatives $[\frac{d}{d_{a_1}}(S), \frac{d}{\partial_{a_2}}(S), \ldots]$ is equal to

$$(S, \Psi_n) \, \Delta(\mathbb{A}) \ .$$

Ex. 83 Let $\mathbb{A}$ be an alphabet, $\mathbb{A}^{der}$ the alphabet of roots of the derivative of $R(x,\mathbb{A})$. For any integer k, express $S^k(x-\mathbb{A}^{der})$ in the basis of $\{S^i(x-\mathbb{A})\}$.

Ex. 84 Let $\mathbb{A}$ be an alphabet of cardinality n, and let S be an homogeneous symmetric polynomial in $\mathbb{A}$ of degree $d \le n$. Show that

$$(S, \Lambda^d) = S(\mathbb{B}) \ ,$$

where B is the set of roots of

$$x^n - x^{n-1} + x^{n-2} - \cdots + (-1)^n \ .$$

Ex. 85 Show that, for any $r \in \mathbb{N}$,

$$x(x+1)\cdots(x+r-1) = \sum_{J:|J|=r} x^{\ell(J)} \frac{r!}{(\Psi^J, \Psi^J)} \ .$$

What becomes the left hand side when the summation is restricted to partitions with all parts odd ? For example, one gets, for $n = 2, \ldots, 6$ the following values x^2, x^3+2x, $8x^2+x^4$, x^5+20x^3+24x, $x^6+40x^4+184x^2$.

Ex. 86 Let $k \in \mathbb{N}$. Show that

$$\sum_J m_k \frac{\Psi^J}{(\Psi^J, \Psi^J)}$$

is a sum of Schur functions with coefficients ± 1, where $m_k :=$ multiplicity of k in $J = 1^{m_1} 2^{m_2} \cdots$.

Ex. 87 $\mathbb{A}$ be an alphabet of cardinality n, and let $v \in \mathbb{N}^n$. Write $\Delta(\mathbb{A}+v)$ for $\prod_{i<j}(a_i+v_i-a_j-v_j)$. Let K be partition. Evaluate

$$\sum_I (S_K \, , \, S^I) \, \Delta(\mathbb{A}+I) \ ,$$

sum over all vectors $I \in \mathbb{N}^n$, $|I| = |K|$.

Ex. 88 Compute

$$\det(\, 1/(a_i+a_j)\,)_{1 \le i,j \le n} \ .$$

Ex. 89 Let $\mathbb{A}, \mathbb{B}$ be alphabets of cardinality n, $\mathbb{D}$ another alphabet. Compute

$$\det \left| \frac{R(a,\mathbb{D}) - R(b,\mathbb{D})}{a-b} \right|_{a\in\mathbb{A}, b\in\mathbb{B}} \ .$$

Ex. 90 Let $\mathbb{A}, \mathbb{B}$ be alphabets of cardinality n, $p \in \mathbb{N}$. Compute

$$\det\Big((a_i + b_j)^p \Big)_{1 \le i,j \le n} .$$

Ex. 91 Let $\mathbb{A}$ be of cardinality n, $\mathbb{B}$ of cardinality $2n$. Compute

$$\det\Big(\frac{1}{b_i - a_1}, \frac{1}{(b_i - a_1)^2}, \ldots, \frac{1}{b_i - a_n}, \frac{1}{(b_i - a_n)^2} \Big)_{1 \le i \le n}$$

Ex. 92 Let $\mathbb{A}, \mathbb{B}$ be alphabets of cardinality n. Compute

$$\det\Big(\frac{a_i + b_j}{a_i - b_j} \Big)_{1 \le i,j \le n} .$$

Ex. 93 Let $\mathbb{A}$ be of cardinality n, $\mathbb{B}$ be a set of n complex numbers. Express

$$\det((1 - a_i)^{b_j})_{1 \le i,j \le n} \,/\, \Delta(\mathbb{A})$$

in the basis of Schur functions of $\mathbb{A}$.

Ex. 94 Let $\mathbb{A}, \mathbb{B}$ be arbitrary. Expand

$$\begin{vmatrix} \sum_{i\ge 0} S^i(\mathbb{A}) S^i(\mathbb{B}) & \sum_{i \ge 0} S^{i+1}(\mathbb{A}) S^i(\mathbb{B}) \\ \sum_{i \ge 0} S^i(\mathbb{A}) S^{i+1}(\mathbb{B}) & \sum_{i \ge 0} S^i(\mathbb{A}) S^i(\mathbb{B}) \end{vmatrix}$$

Ex. 95 Border the determinant $S_{m^n}(\mathbb{A})$ by a first column $[x^{n-1}, \ldots, x^0, 0]$ and a bottom row $[0, y^0, \ldots, y^{n-1}]$. Use it to show that

$$S_\Box(\mathbb{A}) S_\Box(\mathbb{A} - x - y) = S_\Box(\mathbb{A} - x) S_\Box(\mathbb{A} - y) - xy\, S_{m^n}(\mathbb{A}) S_{(m+2)^{n-2}}(\mathbb{A} - x - y) ,$$

with $\Box := (m+1)^{n-1}$. For example, the determinant $\begin{vmatrix} x^2 & S_2(\mathbb{A}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) \\ x & S_1(\mathbb{A}) & S_2(\mathbb{A}) & S_3(\mathbb{A}) \\ x^0 & S_0(\mathbb{A}) & S_1(\mathbb{A}) & S_2(\mathbb{A}) \\ 0 & 1 & y & y^2 \end{vmatrix}$ allows to check that

$$S_{33}(\mathbb{A}) S_{33}(\mathbb{A} - x - y) = S_{33}(\mathbb{A} - x) S_{33}(\mathbb{A} - y) - xy\, S_{222}(\mathbb{A}) S_4(\mathbb{A} - x - y) .$$

Ex. 96 Let $\mathbb{A}, \mathbb{B}$ be alphabets of respective cardinalities α,β. Show that the resultant $R(\mathbb{A}, \mathbb{B})$ is a divisor of

$$\begin{vmatrix} \Psi_0(\mathbb{A}) & \Psi_1(\mathbb{A}) & \cdots & \Psi_{\alpha+\beta}(\mathbb{A}) \\ \vdots & \vdots & & \vdots \\ \Psi_\alpha(\mathbb{A}) & \Psi_{\alpha+1}(\mathbb{A}) & \cdots & \Psi_{\alpha+\beta}(\mathbb{A}) \\ 1 & b_1 & \cdots & b_1^{\alpha+\beta} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & b_\beta & \cdots & b_\beta^{\alpha+\beta} \end{vmatrix}$$

and find the quotient.
In particular, if $\beta = 1$, $\mathbb{B} = x$, then the preceding determinant is a multiple of the polynomial in x with $\mathbb{A}$ as roots (Brioschi 1854, Muir II, p.342).

Ex. 97 Let $\mathbb{A}, \mathbb{B}$ be alphabets of respective cardinalities α,β. For any $k \in \mathbb{N}$, let

$$P_k := \sum_{a \in \mathbb{A}} a^{r-1} R(a, \mathbb{B}) / S^{\alpha}(2a - A)$$

Show (after Brioschi, 1854, Muir II p.349) that the resultant is equal, up to sign, to

$$det(P_{i+j})_{0 \leq i,j \leq \alpha-1} \ .$$

Ex. 98 Let $\mathbb{A}, \mathbb{B}$ be alphabets of respective cardinalities α,β. Let p be an integer. Show, after Pomey, that the greatest common divisor of $R(x, \mathbb{A})$ and $R(x, \mathbb{B})$ is of degree p if $\Lambda_{(\beta-p+1)^{\alpha-p};n}(\mathbb{B} - \mathbb{A}; \mathbb{B} - \mathbb{A} - x)$ is identically null, and $\Lambda_{(\beta-p)^{\alpha-p}}(\mathbb{B} - \mathbb{A})$ is different from 0.

Ex. 99 Let $\mathbb{A}, \mathbb{B}$ be alphabets of respective cardinalities α,β. Write $\mathbf{P} = R(\mathbb{A}, x)$, $\mathbf{Q} = R(\mathbb{B}, x)$. Show that

$$0 = \begin{vmatrix} \Lambda^{\alpha}(\mathbb{A}) - \mathbf{P} & 0 & 0 & \cdots & \Lambda^{\beta}(\mathbb{B}) - \mathbf{Q} & 0 & \cdots \\ \Lambda^{\alpha-1}(\mathbb{A}) & \Lambda^{\alpha}(\mathbb{A}) - \mathbf{P} & 0 & \cdots & \Lambda^{\beta-1}(\mathbb{B}) & \Lambda^{\beta}(\mathbb{B}) - \mathbf{Q} & \cdots \\ \Lambda^{\alpha-2}(\mathbb{A}) & \Lambda^{\alpha-1}(\mathbb{A}) & \Lambda^{\alpha}(\mathbb{A}) - \mathbf{P} & \cdots & \Lambda^{\beta-2}(\mathbb{B}) & \Lambda^{\beta-1}(\mathbb{A}) & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \end{vmatrix}$$

i.e. that the determinant that one obtain from the usual expression of the resultant by replacing $\Lambda^{\alpha}(\mathbb{A})$ by $\Lambda^{\alpha}(\mathbb{A}) - P$ and $\Lambda^{\beta}(\mathbb{B})$ by $\Lambda^{\beta}(\mathbb{B}) - Q$ is null.

Ex. 100 Let $\mathbb{A}$ be an alphabet, and $\mathcal{B}$ be the ring $\mathcal{B} := \mathbb{C}[x]/R(x, \mathbb{A})$. Let $g(x)$ be any polynomial in x. Show that the norm of $g(x)$ in $\mathcal{B}$ (i.e. the determinant of the morphism "multiplication by $g(x)$") is equal to the resultant of $g(x)$ and $R(x, \mathbb{A})$.

Ex. 101 Let $\mathbb{A}$ be an alphabet of cardinality n. Show that the determinant of order $2n$

$$\begin{vmatrix} \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \cdots & \Lambda^{2n-1}(\mathbb{A}) \\ \vdots & \vdots & & \vdots \\ \Lambda^{-n+1}(\mathbb{A}) & \Lambda^{-n+2}(\mathbb{A}) & \cdots & \Lambda^{n}(\mathbb{A}) \\ S^0(-\mathbb{A}) & S^1(-\mathbb{A}) & \cdots & S^{2n-1}(-\mathbb{A}) \\ \vdots & \vdots & & \vdots \\ S^{-n+1}(-\mathbb{A}) & S^{-n+2}(-\mathbb{A}) & \cdots & S^{n}(-\mathbb{A}) \end{vmatrix}$$

is equal to $\prod_{i \leq j} (a_i + a_j)^2$.