SYMMETRIC FUNCTIONS & COMBINATORIAL OPERATORS ON POLYNOMIALS

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ABSTRACT. Course about symmetric functions and combinatorial operators on polynomials.
KeyWords: Lambda-rings, Schur functions, Cauchy kernel, Euclidean algorithm, Continued fractions, Padé approximants, Divided differences, Symmetrizing Operators, Newton interpolation, Orthogonal polynomials, Schubert polynomials, Young tableaux, Plactic algebra, Non-commutative symmetric functions.

Contents

Preface

These notes present some applications of the theory of symmetric functions to algebraic identities related to the *Euclidean algorithm* : division of polynomials, continued fraction expansion of formal series, Padé approximants, orthogonal polynomials.

Most of the time, the statement will involve only *Schur functions*, specially those indexed by a rectangular partition which must be considered as generalizing the resultant of two polynomials in one variable.

Proofs relies on less than half a dozen formulas about symmetric functions, supplemented by two identities about minors (Plücker relations and Bazin relations). Some statements are proved several times in different manners, and we could have even restricted our toolkit by not using identities about minors, for example. Doing so, we would have missed the fact that many algebraic constructions involving Hankel or Toeplitz determinants are closely related to the theory of symmetric functions, and that such an interplay is a fruitful manner of avoiding unnecessary computations, and of providing connections between different fields of mathematics.

As an illustration, we have taken many examples in Muir's five volumes *the Theory of Determinants in the Historical Order of Development*, refering to MuirI, ..., MuirV. Much more examples could have been added. For example, all the sections of Muir about *alternants* are straightforward to reformulate in terms of symmetric functions.

Our last chapters show how to adapt the previous constructions to non-symmetric polynomials. Essentially, Schur functions are replaced by the more general family of *Schubert polynomials*. To introduce Schubert, we first need to forget about determinants, and keep only the underlying symmetric groups. Reducing to $\mathfrak{S}_2$, to 2×2 determinants and to polynomials in two variables is the key to compute in several variables.

This method was initiated by Newton, who applied it to obtain an interpolation formula which is a discrete version of Taylor's formula, replacing derivatives by *divided differences*. We show how to obtain a similar interpolation in several variables, Schubert polynomials occurring as universal coefficients. In the following chapter, we abandon interpolation methods, and rather show that Schubert polynomials can be obtained though a Cauchy-type kernel, exactly as Schur functions.

We have sometimes departed from the usual conventions about symmetric functions, as they can be found in the treatise of Macdonald. For example, Schur functions are indexed by increasing partitions, to make it easier to use minors, or also, to consider them as special Schubert polynomials.

We also use λ-rings as the proper set-up for symmetric functions. The reader will see that many identities boil down to $(\mathbb{A}+\mathbb{B})-\mathbb{C} = \mathbb{A}+(\mathbb{B}-\mathbb{C})$ in this formalism.

In fact, we have been preceded by a "Savant du Nord", who, according to Voltaire[1] used this symbolism to prove that sheep must be red. We shall not striclty follow his steps, but rather use λ-rings, and consider symmetric functions as *functors* acting on polynomials, the action being fixed by the single axiom describing how a power sum (an *Adams operation*) acts on a polynomial.

Summary

Chapter I defines symmetric functions through generating functions $\sigma_z(\mathbb{A})$, which become matrix generating functions when taking z to be a matrix. Schur functions first occur as minors of Toeplitz matrices. A second point of view is furnished by the Cauchy kernel $\prod_{a\in\mathbb{A},b\in\mathbb{B}}(1-ab)^{-1}$, which allows to define a scalar product, relative to which Schur functions are now the only $\mathbb{Z}$-orthogonal basis. We also present symmetric functions as operators on integral vectors (which can be the indices of symmetric functions, or the indices of minors). A special case of this action leads to the Pieri formulas which describe the product of a Schur function by an elementary symmetric one, and determine the multiplicative structure of the ring of symmetric polynomials in the basis of Schur functions.

Chapter II introduces the λ-ring structure of a ring of polynomials, symmetric functions being now considered as operators on it. One just need to require that power sums Ψ^i be ring homomorphisms, such that $\Psi^i(x) = x^i$, if x is a variable, and $\Psi^i(c) = c$ if c is a scalar. This one-line definition has far reaching implications. Even at the level of notations it is helpful[2] to denote coefficients of products of powers of formal series. For example $S^6\left(2\mathbb{A} - \frac{1}{3}\mathbb{B} + \sqrt{2}\mathbb{D}\right)$ denotes the coefficient of z^6 in the product $(\sigma_z(\mathbb{A}))^2\,(\sigma_z(\mathbb{B}))^{-1/3}\,(\sigma_z(\mathbb{D}))^{\sqrt{2}}$, and writing $2\mathbb{A} - \frac{1}{3}\mathbb{B} + \sqrt{2}\mathbb{D} = \left(\mathbb{A} - \frac{1}{3}\mathbb{B}\right) + \left(\mathbb{A} + \sqrt{2}\mathbb{D}\right)$ will induce an expansion of this coefficient.

In the set-up of $\lambda-$rings, the inversion of a formal series $z\sigma_z(\mathbb{A})$ (*Lagrange inversion*) will amount to computing the $S^n(k\mathbb{A})$, $k\in\mathbb{Z}, n\in\mathbb{N}$. This is not too far from what Lagrange did himself. However, because $k\mathbb{A}$ is a product, then $S^n(k\mathbb{A})$ can be expanded, thanks to the Cauchy formula, into any pair of adjoint bases of symmetric functions. Thus, for the price of one computation, one gets many different formulas (which indeed can be found in the classical literature, each time obtained through a new computation).

Chapter III attacks the main application of symmetric functions we present in these notes: the Euclidean algorithm for polynomials in one variable. This process generates symmetric functions of $\mathbb{A}$ and $\mathbb{B}$, if $\mathbb{A}$ and $\mathbb{B}$ are the alphabets of roots of the two starting polynomials. The problem is to identify these functions (of course Schur functions), and to give relations between them[3] A special case, first treated by Sturm, is to start from a polynomial and its derivative. The Euclidean algorithm can be written as a continued fraction, and Wroński was the first at the beginning

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[1]Voltaire: Candide ou l'optimisme. Candide ... *fut seulement très fâché de se séparer de son mouton qu'il laissa à l'Académie des Sciences de Bordeaux, laquelle proposa pour le sujet du prix de cette année de trouver pourquoi la laine de ce mouton était rouge; et le prix fut adjugé à un savant du Nord qui démontra, par A plus B, moins C, divisé par Z, que le mouton devait être rouge et mourir de la clavelée.* We must add that A, B, C are alphabets, and Z is a complex number.

[2]this aspect is much favoured in California under the brand mark of *plethystic notation.*

[3]The reader can ask himself, before looking at the notes, how he would write the coefficients of the remainder of degree 5 in the division of two polynomials of degree 10. Maple or Mathematica can be of some help, but will not clean the result if it accepts to send it on the screen.

of the 19^{th} century to write coefficients as determinants (Schur functions indexed by "almost rectangular" partitions).

Chapter IV obtains again the continued fraction expansion of a formal series $\sigma_z(\mathbb{A})$, and its convergents, by just writing out some multi-Schur function. Interpolating a general function by a continued fraction, following Thiele, leads to similar coefficients. One can use matrices to get continued fractions: 2×2 matrices are related to Euler's recursions, but one obtains more information from Jacobi's tridiagonal matrix. In fact, taking powers of this last matrix, one instantly gets different combinatorial interpretations in terms of Motzkin or Dyck paths.

In chapter V, we realize that one can further simplify the Euclidean algorithm by starting with a pair (f, g) of formal series instead of two polynomials. Indeed the pair $(f/g, 1)$, i.e. the case of division by 1, give the same successive quotients as the pair (f, g). Writing $g/f = \sigma_z(\mathbb{A})$, one has now to manipulate symmetric functions of a single alphabet $\mathbb{A}$. Writing similarly the successive remainders as $\sigma_z(\mathbb{A}^1), \sigma_z(\mathbb{A}^2), \sigma_z(\mathbb{A}^3), \ldots$, one obtains from $\mathbb{A}$ a sequence of "derived alphabets" which are closely related to the work of Stieltjes. As a matter of fact, one can simplify computations even further, at the cost of doubling the number of steps, and restrict to using only normalized differences:

$$(f = 1 + \alpha z + \cdots\ ,\ g = 1 + \beta z + \cdots) \to (f{-}g)/(\alpha z{-}\beta z)\ .$$

Stieltjes' construction was motivated by the problem of decomposing a Hankel form $\sum_{i,j} x^{i+j} S_{i+j}(\mathbb{A})$ into a sum of squares, and the answer is still in terms of derived alphabets.

Chapter VI treats of rational interpolation (Padé approximants) of a formal series $\sigma_z(\mathbb{A})$. Writing

$$\sigma_z(\mathbb{A}) \sim \frac{\prod_{b\in\mathbb{B}}(1 - zb)}{\prod_{d\in\mathbb{D}}(1 - zd)} = \pm x^{\star} \frac{R(x + \mathbb{D}, \mathbb{B})}{R(x + \mathbb{B}, \mathbb{D}}\ ,$$

with $x = 1/z$, one gets the answer, without computation, as a quotient of two resultants, and in final, as a quotient of Schur functions of $\mathbb{A} \pm x$.

From Chapter VII on, we adopt the point of view that symmetric functions should be considered as the output of some "symmetrizing operators". Starting with Newton's divided differences

$$f(a, b) \to \big(f(a, b) - f(b, a)\big)/(a - b)\ ,$$

we produce different types of divided differences. These operators are compatible with Schur functions, and constitute a powerful tool to generate identities on symmetric functions, passing though intermediary steps which are no more symmetrical.

For example, the Lagrange interpolation can be seen as an operator from $\mathfrak{Sym}(1) \otimes \mathfrak{Sym}(n{-}1)$ to $\mathfrak{Sym}(n)$ ($\mathfrak{Sym}(k)$ being the ring of symmetric functions in k variables). It can be expressed with divided differences. Its generalization gives a morphism from $\mathfrak{Sym}(n') \otimes \mathfrak{Sym}(n'')$ to $\mathfrak{Sym}(n'{+}n'')$ and reduces, on the basis of Schur functions, to concatenation of indices. Though a very simple operation, it has many implications in different fields, for example in geometry (cohomology of Grassmannians), or in the study of determinantal ideals.

All our symmetrizing operators are induced from a "maximal symmetrizer" due to Cauchy and Jacobi :

$$f(x_1,\ldots,x_n) \longrightarrow \sum\nolimits_{\sigma\in\mathfrak{S}_n} \pm f^\sigma \,/\, \prod\nolimits_{i<j}(x_i-x_j) \ .$$

This symmetrizer sends monomials onto Schur functions. Partial symmetrizations give generalizations of Schur functions, which satisfy many combinatorial properties in common with Schur functions, though they cannot be written in general as determinants.

We end the chapter by mentioning that interpolation methods give a discrete approximation of differential operators, and refer to Calogero's work for further developments.

Chapter VIII reinterprets many identities obtained in chapters IV,V,VI, in terms of orthogonal polynomials. Given "moments" $S^n(\mathbb{A})$, $n = 0, 1, 2, \ldots$, then the associated orthogonal polynomials can be written $S_{n^n}(\mathbb{A}-x)$, i.e. as can be written as Schur functions indexed by square partitions. More general functions occur: Darboux-Christoffel kernels $S_{n^{n-1}}(\mathbb{A}-x-y)$, Christoffel determinants $S_{(n+k)^n}(\mathbb{A} - x-x_1-\cdots-x_k)$. Orthogonal polynomials also occur as denominators of convergents of continued fractions, and the combinatorics of Jacobi's matrix and of Motzkin and Dyck paths can be used again without changes. However, new considerations arise, relative to the zeroes of $S_{n^n}(\mathbb{A}-x)$, or to quadrature problems for example.

The following two chapters introduce Schubert polynomials from two different points of view: interpolation, and non-symmetric Cauchy kernels. Newton's interpolation formula

$$f(t) = f(b_1) + f^{\partial}\,(t-b_1) + f^{\partial\partial}\,(t-b_1)(t-b_2) + \cdots$$

is a discrete version of Taylor's formula, where divided differences $f^\partial, f^{\partial\partial}, \ldots$ have replaced derivatives with respect to time. The discrete version of Taylor's formula in several variables can be explicited, Schubert polynomials $\mathbb{Y}_I(\mathbb{A},\mathbb{B})$ in two sets of variables $\mathbb{A},\mathbb{B}$ being the coefficients of the divided differences (the case of Newton is when $\mathbb{A} = \{t\}$). It is remarkable that the vanishing conditions $\mathbb{Y}_I(\mathbb{A},\mathbb{A}) = 0$, $I \neq 0$ (i.e. the specialization $a_n = b_n, n \in \mathbb{N}$) are sufficient to write down the (unique) expansion of any polynomial in terms of Schubert polynomials. In particular Schubert polynomials constitute a linear basis of the ring of polynomials in $a_1, a_2, \ldots$ with coefficients in $b_1, b_2, \ldots$. Schur functions are now seen as special Schubert polynomials, those such that their index is a partition followed by zeroes, with $\mathbb{B}$ specialized to 0.

In chapter X, we rather start from a non-symmetric Cauchy kernel $\mathbf{K}_n(\mathbb{A},\mathbb{B}) := \prod_{1\le i<j\le n}(b_i-a_j)$, fixing the number of variables. Expanding this kernel as a sum $\sum \pm f(\mathbb{A})g(\mathbb{B})$ has a unique solution, when imposing some constraints, and give back the Schubert polynomials. In fact the full theory is again deduced from trivial vanishing conditions : $\mathbf{K}_n(\mathbb{A},\mathbb{B})$ vanishes whenever $\mathbb{A}$ is specialized to a permutation of $\mathbb{B}$ different from the identity.

Schubert polynomials are now indexed by permutations; $\{\mathbb{X}_\sigma(\mathbb{A},\mathbb{B}) \,:\, \sigma \in \mathfrak{S}_n\}$ is a basis of the ring of polynomials in $a_1,\ldots,a_n$, as a module over symmetric polynomials (with coefficients in $\mathbb{B}$). Letting n tends towards infinity, one recovers the case considered in the previous chapter. However, the coefficients of the expansion of a polynomial in the Schubert basis are now to be interpreted as scalar products. On the other hand, expanding Schubert polynomials with respect to the first variable a_1 can be obtained by taking a copy of the algebra of divided differences, and make it act on the permutations indexing the Schubert polynomials and no more on the variables $a_1, \ldots, a_n$. This expansion can be formulated in terms of a "NilCauchy kernel". Using stricter relations than the braid relations, one can use a new kernel inside the nilplactic algebra, and obtain the expansion of Schubert polynomials in terms of other polynomials occurring in Demazure's character formula. Finally, we show that Schubert polynomials are also part of the combinatorics of Yang-Baxter relations.

The last chapter adopts a non-commutative point of view. Following Littlewood, one defines a non-commutative Schur function S_I as the sum of all tableaux of shape I (tableaux being special words, which can be planarly represented). The product of two tableaux is a word which is no more a tableau in general. To recover products of Schur functions, the easiest way is to use the Robinson-Schensted construction, which amounts using a quotient of the free algebra, the plactic algebra. Identifying Schur functions to their plactic version, one gets an embedding of the ring of symmetric polynomials inside the plactic algebra. This is the best way to explain the Littlewood-Richardson rule describing the product of two Schur functions. But it explains as well that the expansion of the determinant expressing a Schur function in terms of complete functions, which can be interpreted as a sum of $n-$tuples of lattice paths , with ± 1 coefficients, reduces to the subsum of non-intersecting paths. The Cauchy formula for the resultant $\prod(a_i - b_j)$ has a plactic generalization due to Bender and Knuth. This is also the case of the kernel $\prod_{i+j\leq n}(a_i + b_j)$, sitting inside the plactic algebra in $\mathbb{A}$, with coefficients in commutative variables $b_1, \ldots, b_n$. The coefficient of a Schubert polynomial $\mathbb{X}_\sigma(\mathbb{B}, \mathbf{0})$ is a sum of tableaux in $\mathbb{A}$, without multiplicities, which projects onto a Schubert polynomial $\mathbb{X}_{\sigma'}(\mathbb{A}, \mathbf{0})$ when the a_i's are allowed to commute.

Developing further the non-commutative theory would require many combinatorial tools that are not needed for computations of polynomials, the aim of these notes. We stop at this point.

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CHAPTER 1

Symmetric functions

1.1. Alphabets

We shall handle functions on different sets of indeterminates (called *alphabets*, though we shall mostly use commutative indeterminates for the moment).

A symmetric function of an alphabet $\mathbb{A}$ is a function of the letters which is invariant under permutation of the letters of $\mathbb{A}$.

The simpler symmetric functions are best defined through generating functions. We shall not use the classical notations for symmetric functions (as they can be found in Macdonald's book [**135**]), because it will become clear in the course of these lectures that we need to consider symmetric functions as *functors*, and connect them with operations on vector spaces and representations. It is a small burden imposed on the reader, but the compact notations that we propose greatly simplify manipulations of symmetric functions. Notice that exponents are used for products, and that S^J is different from S_J, except when J is of length one (i.e. is an integer).

$$J = [j_1, j_2, \ldots] \Rightarrow \ \Lambda^J = \Lambda^{j_1}\Lambda^{j_2}\cdots \ \ \& \quad S^J = S^{j_1}S^{j_2}\cdots \ \ \& \quad \Psi^J = \Psi^{j_1}\Psi^{j_2}\cdots$$

are different from S_J, ψ_J etc.

In case of length 1, we shall indifferently write indices or exponents for the same functions :

$$S^j = S_j \ , \ \Lambda^j = \Lambda_j \ , \ \Psi^j = \Psi_j \ .$$

We need operations on alphabets, the first one being the *addition*, that is the disjoint union that we shall denote by a '+'-sign :

$$\Big(\mathbb{A} = \{a\} \ , \ \mathbb{B} = \{b\}\Big) \quad \mapsto \quad \mathbb{A} + \mathbb{B} := \{a\} \cup \{b\}$$

More operations will be introduced in Chapter 2.

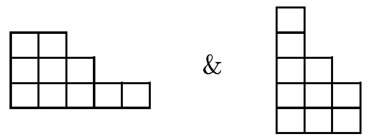
1.2. Partitions

A weakly increasing sequence of strictly positive numbers $I = [i_1, i_2, \ldots, i_\ell]$ is called a *partition of the number n of length $\ell(I) = \ell$*, where $n = |I| := i_1 + \cdots + i_\ell$. One also uses weakly decreasing sequences instead of increasing ones, and refer to them as *decreasing partitions*, but to handle minors of matrices, it is preferable to choose our convention.

A partition I has a graphical representation due to Ferrers, which is called its *diagram*: it is a diagram of square boxes left packed, $i_1, i_2, \ldots, i_\ell$ being the number of boxes in the successive rows. Reading the number of boxes in the successive columns, one obtains another partition $I^\sim$ which is called the *conjugate partition.*

Conjugating partitions is an involutive operation which can be interpreted as symmetry along the main diagonal, for what concerns diagrams.

For example, when $I = [2, 3, 5]$, then $I^{\sim} = [1, 1, 2, 3, 3]$ and their diagrams are



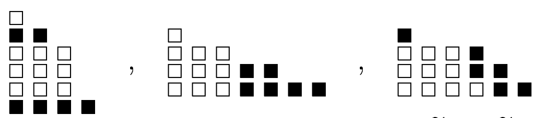
A partition I will be identified with any vector obtained by concatenating initial zeroes. This is coherent with identifying partitions and their diagrams, because one can start by reading empty rows!

Let $\mathfrak{Part}(n)$ be the set of partitions of n.

There are two operations on partitions, '$+$' and '$\cup$' :

$$\mathfrak{Part}(m) \times \mathfrak{Part}(n) \to \mathfrak{Part}(m+n)$$

which are exchanged under conjugation: $(I, J) \to I+J$ is the addition of partitions, considered as vectors, *normalizing them to belong to the same space* $\mathbb{N}^r$, and $I \cup J$ is the partition obtained by reordering the *concatenation* of I and J into a partition. With $I = [1, 3, 3, 3]$, $J = [2, 4]$, one has :



$I \cup J$, $I + J$, $I^{\sim} + J^{\sim}$

To order $\mathfrak{Part}(n)$, one uses, instead of partitions, their *cumulative sums* and one puts the *componentwise* order on these new vectors.

DEFINITION 1.2.1. Given a vector in $v \in N^n$, its *cumulative sum* $\overline{\overline{v}}$ is the vector

$$\overline{\overline{v}} = [v_1,\, v_1{+}v_2,\, \ldots, v_1{+}v_2{+}\cdots{+}v_n] \; .$$

The *n-th cumulative sum* (resp. *cumulative sum*) of a partition I of length $\ell(I) \le n$ is the cumulative sum of the vector obtained by concatenating $n - \ell(I)$ initial zeroes to I (resp. n being the weight of I).

Given two partitions I, J of the same integer n, I is smaller than J for the *dominance order* iff $\overline{\overline{I}}$ is smaller than $\overline{\overline{J}}$ componentwise, i.e.

$$\overline{\overline{I}}_1 \le \overline{\overline{J}}_1,\, \overline{\overline{I}}_2 \le \overline{\overline{J}}_2,\, \ldots,\, \overline{\overline{I}}_n \le \overline{\overline{J}}_n \; .$$

Cumulative sums of partitions satisfy convexity inequalities that characterize them. It is immediate to check :

LEMMA 1.2.2. *An element $u \in \mathbb{N}^n$ is the cumulative sum of a partition iff*

$$2u_i \le u_{i-1} + u_{i+1} \;,\; 1 < i < n \; . \tag{1.2.1}$$

But now the supremum (componentwise) of two vectors satisfying inequalities (1.2.1) also satisfies the same inequalities, and this allows to define the supremum of two partitions :

DEFINITION 1.2.3. The supremum $I \vee J$ of two partitions I, J of n is the only partition K such that $sup(\overline{\overline{I}}, \overline{\overline{J}})$ is the n-th cumulative sum of K.

One could have taken the r-th cumulative sum of I, J, for any $r \geq \ell(I), \ell(J)$.

One defines the infimum $I \wedge J$ of two partitions of the same weight by using conjugation :

$$I \wedge J := \left(I^{\sim} \vee J^{\sim}\right)^{\sim} \tag{1.2.2}$$

Beware that the minimum componentwise of two cumulative sums is not necessarily the cumulative sum of a partition.

For example, take $I = [1,1,1,5]$, $J = [0,2,2,4]$. Then $\overline{\overline{I}} = [1,2,3,8]$, $\overline{\overline{J}} = [0,2,4,8]$, $sup(\overline{\overline{I}},\overline{\overline{J}}) = [1,2,4,8] = \overline{\overline{K}}$, with $K = [1,1,2,4] = I \vee J$. On the other hand, $inf(\overline{\overline{I}},\overline{\overline{J}}) = [0,2,3,8]$ is the cumulative sum of $[0,2,1,5]$, which is not a partition.

Conjugating, one has $I^{\sim} = [1,1,1,1,4]$, $J^{\sim} = [0,1,1,3,3]$, with cumulative sums $[1,2,3,4,8]$,$[0,1,2,5,8]$ and supremum $[1,2,3,5,8]$. Therefore $I^{\sim} \vee J^{\sim} = [1,1,1,2,3]$ and

$$I \wedge J = \left(I^{\sim} \vee J^{\sim}\right)^{\sim} = [1,1,1,2,3]^{\sim} = [1,2,5]\ .$$

One could have used the cumulative sums starting from the right, or equivalently, the cumulative sums on decreasing partitions.

The two operations $\vee$, $\wedge$, define a lattice structure on $\mathfrak{Part}(n)$ (with minimum element $[n]$ and maximal element $[1^n]$). The poset of partitions (*partially ordered set* – bad terminology, because every order is partial, unless otherwise specified !) is not a rank poset, i.e. maximal chains between two comparable elements do not have the same length.

For example, $\mathfrak{Part}(8)$ contains the following piece, which contains the four partitions that we have just considered, writing their cumulative sums on the right :

$$\begin{array}{ccccc} & 1124 & \\ \swarrow & & \searrow \\ & & 224 \\ 1115 & & \downarrow \\ & & 134 \\ \searrow & & \swarrow \\ & 125 & \end{array} \quad \longleftrightarrow \quad \begin{array}{ccccc} & 1248 & \\ \swarrow & & \searrow \\ & & 0248 \\ 1238 & & \downarrow \\ & & 0148 \\ \searrow & & \swarrow \\ & 0138 & \end{array}\ .$$

However, it is easy to characterize consecutive elements :

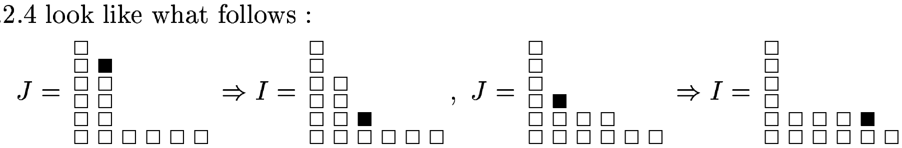
LEMMA 1.2.4. *Let I, J be two partitions of the same number. If*

$$I < J \quad \& \quad \text{there is no } K \text{ such that } I < K < J\ ,$$

then either

$$\exists\, p,k \in \mathbb{N} :\ J = J'\, p^{k+2}\, J'' \quad \& \quad I = J',\, p{-}1,\, p^k,\, p{+}1,\, J'' \quad or$$
$$\exists\, p,q \in \mathbb{N} :\ J = J'\, p\, q\, J'' \quad \& \quad I = J',\, p{-}1,\, q{+}1,\, J''\ .$$

Notice that the two cases are exchanged by conjugation of partition, so that the dominance order is reversed under conjugation. The possible pairs in Lemma 1.2.4 look like what follows :



Pushing a box down gives a smaller partition, but it is not true that it gives a pair of consecutive partitions : $\begin{array}{lll} \square & \blacksquare & \\ \square & \square & \square \\ \square & \square & \square \end{array}$ and $\begin{array}{llll} \square & & & \\ \square & \square & \square & \\ \square & \square & \square & \blacksquare \end{array}$ are not consecutive, because the move of the black box can be performed in two steps:

$$\begin{array}{lll} \square & \blacksquare & \\ \square & \square & \clubsuit \\ \square & \square & \square \\ \square & \square & \square \end{array} \Rightarrow \begin{array}{llll} \square & \blacksquare & & \\ \square & \square & & \\ \square & \square & \square & \\ \square & \square & \square & \clubsuit \end{array} \Rightarrow \begin{array}{llll} \square & & & \\ \square & \square & \blacksquare & \\ \square & \square & \square & \\ \square & \square & \square & \clubsuit \end{array}$$

Let J, I be a pair of partitions such that the diagram of J contains the diagram of I. Then the set difference of the two diagrams is called a *skew diagram* and denoted J/I (adding common boxes to I and J does not change J/I. In some problems, one has to consider pairs (J, I) rather than J/I).

If J/I contains no 2×2 sub-diagram and is connected (resp. J/I contains no two boxes in the same column, res. no two boxes in the same row), then J/I is called a *ribbon* (resp. *horizontal strip*, resp. *vertical strip*). There are strips which are both vertical and horizontal, for example a single box.

$$\begin{array}{lllll} \blacksquare & \blacksquare & & & \\ \square & \blacksquare & \blacksquare & \blacksquare & \\ \square & \square & \square & \blacksquare & \blacksquare \\ \square & \square & \square & \square & \square \end{array} \qquad \begin{array}{lllll} \blacksquare & \blacksquare & & & \\ \square & \square & \square & \blacksquare & \\ \square & \square & \square & \square & \blacksquare \\ \square & \square & \square & \square & \square \end{array} \qquad \begin{array}{lllll} \square & \blacksquare & & & \\ \square & \square & \square & \square & \\ \square & \square & \square & \square & \blacksquare \\ \square & \square & \square & \square & \blacksquare \end{array}$$

ribbon horizontal strip vertical strip

A partition of the type $[1^{\beta}, \alpha{+}1]$ is called a *hook* and is denoted $(\alpha \& \beta)$. The decomposition of the diagram of a partition I into its diagonal hooks (i.e. hooks having their head on the diagonal) is called the *Frobenius code* of I and denoted $\mathfrak{Frob}(I) = (\alpha_1, \alpha_2, \ldots, \alpha_r \,\&\, \beta_1, \beta_2, \ldots, b_r)$ (where r, the number of boxes in the main diagonal, is called the *rank* of the partition).

$$I = [2,4,5,6] = \begin{array}{llllll} \square & \blacksquare & & & & \\ \square & \blacksquare & \heartsuit & \heartsuit & & \\ \square & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \\ \square & \square & \square & \square & \square & \square \end{array} \text{ gives } \mathfrak{Frob}([2,4,5,6]) = (531 \,\&\, 320) \ .$$

Given a box in a diagram, its *content* is its distance to the main diagonal. The multiset of contents allows to recover the diagram, hence the partition. Replacing each box by its content, one has, for example, that $I = [2,4,5,6]$ has contents $\begin{array}{llllll} -3 & -2 & & & & \\ -2 & -1 & 0 & 1 & & \\ -1 & 0 & 1 & 2 & 3 & \\ 0 & 1 & 2 & 3 & 4 & 5 \end{array}$ and multiset $\{(-3)^1,\ (-2)^2,\ (-1)^2,\ 0^3,\ 1^3,\ 2^2,\ 3^2,\ 4^1,\ 5^1\}$.

Finally (for the moment!), one can code a partition by the word obtained by reading the *border* of its diagram : 0 for an horizontal step, 1 for a vertical step.

$$I = [2,4,5,6] \Rightarrow \ \text{border} = \begin{array}{lllllll} 0 & 0 & 1 & & & & \\ \square & \square & 0 & 0 & 1 & & \\ \square & \square & \square & \square & 0 & 1 & \\ \square & \square & \square & \square & \square & 0 & 1 \\ \square & \square & \square & \square & \square & \square & \end{array} = [0,0,1,0,0,1,0,1,0,1]$$

Taking reverse words and exchanging 0 and 1 corresponds to taking conjugate partitions. In fact, all operations on words on two letters induce operations on partitions that we leave to the reader the pleasure to discover.

It is appropriate to concatenate letters 1 on the left of a border word, and letters 0 on its right. This amounts to consider that the partition be contained in a rectangular partition m^n (where m is the total number of 0's and n the total number of 1's). Since a $0, 1$ word is specified by its length and the position of the 0's, one has the following lemma that one shall need in determinantal identities.

LEMMA 1.2.5. *Let $I = [i_1, \ldots, i_n] \in \mathbb{N}^n$ be a partition contained in m^n, and $J = [j_1, \ldots, j_m]$ be its conjugate. Then*

$\{i_1{+}1,\ i_2{+}2,\ \ldots, i_n{+}n\}$ *and* $\{m{+}n{+}1 - j_1{-}1,\ m{+}n{+}1 - j_2{-}2,\ \ldots, m{+}n{+}1 - j_m{-}m\}$

are complementary sets in $\{1, \ldots, m{+}n\}$.

For example, take $m = 5$, $n = 4$. Adding $[1, 2, 3, 4]$ to $I = [0, 0, 2, 4]$, one gets $[1, 2, 5, 8]$; the partition $I^{\sim} = [0, 1, 1, 2, 2]$ gives under the same operation $[1, 3, 4, 6, 7]$, and $[10-1, 10-3, 10-4, 10-6, 10-7] = [9, 7, 6, 4, 3]$ is the complementary set of $1, 2, 5, 8]$.

Given a box $\square$ in the diagram of a partition J, let its *leg* be the set of boxes in the same column above it, its *arm* be the set of boxes in the same row, on its right. The *hook* relative to $\square$ is the union of the leg, the arm and the box itself, the total number of boxes being the *hook length* of $\square$ in the diagram. It is usual to directly write the hook length of a box in the box itself.

The *content* of a box in a diagram is its distance to the main diagonal, counted negative in the North-West sector.

For example, for $J = [2, 3, 3, 6, 6]$, one has the following hook lengths and contents :

2	1				
4	3	1			
5	4	2			
9	8	6	3	2	1
10	9	7	4	3	2

Hook lengths

−4	−3				
−3	−2	−1			
−2	−1	0			
−1	0	1	2	3	4
0	1	2	3	4	5

Contents

These informations about a partition can also been read on the other codings of the partition, like its border, or its Frobenius decomposition in diagonal hooks.

One can relate the contents and hook lengths as follows. Let J be a partition in $\mathbb{N}^n$ (it can have initial zeros). Let $v := [j_1{+}0, j_1{+}1, \ldots, j_n{+}n{-}1]$. Taking row r of the diagram of J, one sees that the set of hook lengths in this row is such that

$$\{h_\square\} = \{1, 2, \ldots, v_r\} \setminus \{(v_r - v_i) : i < r\} ,$$

and therefore, in total, one has the following equality between multisets :

$$(1.2.3)\quad \{h_\square : \square \in Diagr(J)\} = \cup_{1\le r\le n}\{1, \ldots, v_r\} \setminus \cup\{(v_r{-}v_i) : 1 \le i < r \le n\} .$$

1.3. Generating Functions of Symmetric Functions

Taking an extra indeterminate z, one has three fundamental series

$$(1.3.1)\qquad \lambda_z(\mathbb{A}) := \prod_{a\in\mathbb{A}}(1+za) \ ,\ \sigma_z(\mathbb{A}) := \prod_{a\in\mathbb{A}} \frac{1}{1-za} \ ,\ \Psi_z(\mathbb{A}) := \sum_{i=1}^{\infty}\sum_{a\in\mathbb{A}} z^i a^i/i$$

the expansion of which gives the *elementary symmetric functions* $\Lambda^i(\mathbb{A})$, the *complete functions* $S^i(\mathbb{A})$, and the *power sums* $\Psi_i(\mathbb{A})$:

$$(1.3.2)\quad \lambda_z(\mathbb{A}) = \sum z^i\,\Lambda^i(\mathbb{A}) \ ,\ \sigma_z(\mathbb{A}) = \sum z^i\, S^i(\mathbb{A}) \ ,\ \mathbf{\Psi}_z(\mathbb{A}) = \sum_{i=1}^{\infty} z^i\Psi_i(\mathbb{A})/i \ .$$

Since $\log(1/(1-a)) = \sum_{i>0} a^i/i$, one has

$$(1.3.3)\qquad \sigma_z(\mathbb{A}) = \exp(\mathbf{\Psi}_z(\mathbb{A})) \quad , \quad \mathbf{\Psi}_z(\mathbb{A}) = \log(\sigma_z(\mathbb{A})) \ .$$

Addition of alphabets implies product of generating series

$$(1.3.4)\qquad \lambda_z(\mathbb{A}+\mathbb{B}) = \lambda_z(\mathbb{A})\,\lambda_z(\mathbb{B}) \quad , \quad \sigma_z(\mathbb{A}+\mathbb{B}) = \sigma_z(\mathbb{A})\,\sigma_z(\mathbb{B}) \ .$$

However, since one can invert formal series beginning by 1, or take any power of them, one can extend (1.3.1) by setting :

$$\sigma_z(\mathbb{A}-\mathbb{B}) := \frac{\prod_{b\in\mathbb{B}}(1-zb)}{\prod_{a\in\mathbb{A}}(1-za)} \quad , \quad \sigma_z(c\,\mathbb{A}) = (\sigma_z(\mathbb{A}))^c \ , \ c\in\mathbb{C}\ . \tag{1.3.5}$$

When $\mathbb{B}=0$, or $\mathbb{A}=0$, one recovers the two series $\sigma_z(\mathbb{A})$ and $\sigma_z(-\mathbb{B}) = \lambda_{-z}(\mathbb{B})$.

Notice that the addition of alphabets satisfies the usual properties of addition : $\sigma_z(-\mathbb{A})$ is the inverse of $\sigma_z(\mathbb{A})$ because $\mathbb{A}-\mathbb{A}=0$ and $\sigma_z(0)=1$. Similarly, the identity $(\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C}) = \mathbb{A}-\mathbb{B}$ translates, at the level of generating series, the fact that

$$\frac{\prod_b(1-zb)\prod_c(1-zc)}{\prod_a(1-za)\prod_c(1-zc)} = \frac{\prod_b(1-zb)}{\prod_a(1-za)}\ ,$$

and nobody will deny that one can simplify a factor common to the numerator and denominator of a rational function !

Formal series in z beginning by z^0 should be treated as generating series of complete or elementary symmetric functions of some formal alphabet, or of a difference of two alphabets (one alphabet is sufficient, but one has more flexibility with two alphabets), that one can manipulate without having access to the letters which compose them. This is indeed what one does with a polynomial, even when one is not able to factor it. One writes a monic polynomial $P(x)$ as $P(x)=\prod_{a\in\mathbb{A}}(x-a)$, $\mathbb{A}$ being the alphabet of zeros of $P(x)$. Now, $\prod_{a\in\mathbb{A}}(x-a)^2$, $\prod_{a\in\mathbb{A}}(x-a^2)$, to show a few examples, are perfectly defined polynomials whose coefficients can be written in terms of the coefficients of P, though they have been defined in terms of the roots of P (but we respected the symmetry between the roots of P!).

Having written $\mathbb{A}+\mathbb{B}$ for the disjoint union of alphabets forces us to consider a finite alphabet as the sum of its sub-alphabets of cardinality 1, i.e to identify $\mathbb{A}$ and $\sum_{a\in\mathbb{A}} a$, and write

$$S^k(a_1+a_2+\cdots+a_n-b_1-\cdots-b_m)$$

instead of $S^k(\mathbb{A}-\mathbb{B})$, when we shall need the letters composing the finite alphabets $\mathbb{A}$ and $\mathbb{B}$.

Given a finite alphabet $\mathbb{A}$, let $\mathfrak{Sym}(\mathbb{A})$ be the ring of symmetric polynomials in $\mathbb{A}$ over the rational numbers. As a vector space, it has (multiplicative) bases

$$\begin{cases} \Lambda^I(\mathbb{A}) & := & \Lambda^{i_1}(\mathbb{A})\Lambda^{i_2}(\mathbb{A})\cdots \\ S^I(\mathbb{A}) & := & S^{i_1}(\mathbb{A})S^{i_2}(\mathbb{A})\cdots \\ \Psi^I(\mathbb{A}) & := & \Psi^{i_1}(\mathbb{A})\Psi^{i_2}(\mathbb{A})\cdots \end{cases}\ , \tag{1.3.6}$$

where k runs over all integers in $\mathbb{N}$, and I runs over all partitions $I=[i_1,\ldots,i_k]$ such that $i_k \le card(\mathbb{A})$.

The fact that Λ^I is a linear basis is called "Newton's fundamental theorem". It is usually formulated as follows :

THEOREM 1.3.1. *(Newton). Let $\mathbb{A}$ be an alphabet of cardinality n. Then $\mathfrak{Sym}(\mathbb{A})$ is a polynomial ring with generators $\Lambda^1(\mathbb{A}),\ldots,\Lambda^n(\mathbb{A})$.*

Because of relations (1.3.1), it is easy to deduce from Newton's theorem the two other statements in (1.3.6), in other words, that $S^1(\mathbb{A}),\ldots,S^n(\mathbb{A})$ and $\Psi^1(\mathbb{A}),\ldots,\Psi^n(\mathbb{A})$ are also algebraic bases of $\mathfrak{Sym}(\mathbb{A})$. In other words, the ring $\mathfrak{Sym}(\mathbb{A})$, with coefficients in $\mathbb{Q}$, is isomorphic to each of the three polynomial rings

$$\mathbb{Q}[\Lambda^1(\mathbb{A}),\ldots,\Lambda^n(\mathbb{A})]\ ,\ \mathbb{Q}[S^1(\mathbb{A}),\ldots,S^n(\mathbb{A})]\ ,\ \mathbb{Q}[\Psi^1(\mathbb{A}),\ldots,\Psi^n(\mathbb{A})]\ .$$

In the case of the elementary symmetric functions, or complete functions, one does not need the condition that $Sym(\mathbb{A})$ contains the rationals. Newton's theorem is valid for symmetric polynomials with coefficients in $\mathbb{Z}$, and consequently, for any ring of coefficients.

When working with *rational* symmetric functions, one can use other generators. For example, Cauchy showed that any symmetric polynomial is a rational function in the odd power sums $\Psi^1(\mathbb{A})$, $\Psi^3(\mathbb{A})$, $\ldots$, $\Psi^{2n-1}(\mathbb{A})$. This property has been rediscovered and extended many times, and we shall comment it in exercises 1.22, 1.25, 1.26.

The sum of all elements in the orbits of a monomial a^J under the action of the symmetric group $\mathfrak{S}(\mathbb{A})$ is of course a symmetric function, called *monomial function*[1] and we shall denote it $\Psi_J(\mathbb{A})$, (J partition), rather than m_J. They must not be mistaken with the product of power sums denoted $\Psi^J(\mathbb{A})$ with exponents and not indices.

It has been since long realized that one should use alphabets of infinite cardinality, and thus consider a universal ring $\mathfrak{Sym}$ from which one gets by specialization the rings $\mathfrak{Sym}(\mathbb{A})$, for specific alphabets $\mathbb{A}$ of finite cardinality (more generally, we shall use specializations such that *letters* are no more algebraically independent).

1.4. Matrix Generating Functions

Let z stand now for the infinite matrix with diagonal $j-i=1$ filled with 1's, all other entries being 0's.

Since z^k, $k \in \mathbb{N}$, is the matrix with 1's in the k-th diagonal above the main diagonal, and 0 outside of it, we see that now $\sigma_z(\mathbb{A})$ is a Toeplitz matrix (i.e. a matrix with constant values in each diagonal) that we shall denote by $\mathbb{S}(\mathbb{A})$; similarly $\lambda_z(\mathbb{A})$ is a matrix denoted $\mathbb{L}(\mathbb{A})$:

$$\mathbb{S}(\mathbb{A}) = \left[S^{j-i}(\mathbb{A})\right]_{i,j\geq 0} \quad \& \quad \mathbb{L}(\mathbb{A}) = \left[\Lambda^{j-i}(\mathbb{A})\right]_{i,j\geq 0} . \tag{1.4.1}$$

$$\mathbb{S}(\mathbb{A}) = \begin{bmatrix} S^0(\mathbb{A}) & S^1(\mathbb{A}) & S^2(\mathbb{A}) & S^3(\mathbb{A}) & \cdots \\ S^{-1}(\mathbb{A}) & S^0(\mathbb{A}) & S^1(\mathbb{A}) & S^2(\mathbb{A}) & \cdots \\ S^{-2}(\mathbb{A}) & S^{-1}(\mathbb{A}) & S^0(\mathbb{A}) & S^1(\mathbb{A}) & \cdots \\ & \ddots & \ddots & \ddots & \end{bmatrix}$$

$$\mathbb{L}(\mathbb{A}) = \begin{bmatrix} \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) & \Lambda^3(\mathbb{A}) & \cdots \\ \Lambda^{-1}(\mathbb{A}) & \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) & \cdots \\ \Lambda^{-2}(\mathbb{A}) & \Lambda^{-1}(\mathbb{A}) & \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \cdots \\ & \ddots & \ddots & \ddots & \end{bmatrix} .$$

These matrices are upper triangular, but it is wiser to write entries S^{-k} rather than their value 0.

[1]For the classics, monomial functions were the *symmetric functions*. Alphabets being defined as sets of roots of polynomials, the problem was to express the symmetric functions in terms of the $\Lambda^I(\mathbb{A})$, which were the data. Vandermonde solved this problem, without explaining his method, and published tables in the *Mémoires de l'Académie* for degree up to 10 – with no mistake, as controlled by D.Knuth.

Addition or subtraction of alphabets still correspond to product of matrices, that z be an indeterminate or a matrix makes no difference :

$$\mathbb{S}(\mathbb{A} \pm \mathbb{B}) = \mathbb{S}(\mathbb{A})\,\mathbb{S}(\mathbb{B})^{\pm 1} \quad \& \quad \mathbb{L}(\mathbb{A} \pm \mathbb{B}) = \mathbb{L}(\mathbb{A})\,\mathbb{L}(\mathbb{B})^{\pm 1} \ . \tag{1.4.2}$$

The advantage of matrices, compared to formal series, is that they offer us their minors, that we shall index by (increasing) partitions, or more generally, by vectors with components in $\mathbb{Z}$. More precisely, given $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, $J = (j_1, \ldots, j_n) \in \mathbb{Z}^n$ one defines the *skew Schur function* $S_{J/I}(\mathbb{A})$ to be the minor of $\mathbb{S}(\mathbb{A})$ taken on rows $i_1+1, i_2+2, \ldots, i_n+n$ and columns $j_1+1, \ldots, j_n+n$ (we define the minor to be 0 if one of these numbers is < 0). When $I = 0^n$, the minor is called a *Schur function* and one writes $S_J(\mathbb{A})$ instead of $S_{J/0^n}(\mathbb{A})$.

In other words,

$$S_{J/I}(\mathbb{A}) = \left| S^{j_k - i_h + k - h}(\mathbb{A}) \right|_{1 \le h,k \le n} \ . \tag{1.4.3}$$

The expression of a Schur function as a determinant of complete functions is called the *Jacobi-Trudi* determinant (we shall see that there is also another expression in terms of elementary symmetric functions).

One can visualize the Schur function $S_{J/I}$ as being obtained from the initial minor of the same order, by shifting the columns by J, and the rows by I :

$$\begin{array}{ccc|c} 0 & 1 & 2 & -i_1 \\ -1 & 0 & 1 & -i_2 \\ -2 & -1 & 0 & -i_3 \\ \hline j_1 & j_2 & j_3 & \end{array} \Rightarrow \begin{vmatrix} 0+j_1-i_1 & 1+j_2-i_1 & 2+j_3-i_1 \\ -1+j_1-i_2 & 0+j_2-i_2 & 1+j_3-i_2 \\ -2+j_1-i_3 & -1+j_2-i_3 & 0+j_3-i_3 \end{vmatrix} .$$

It is convenient to also use determinants in elementary symmetric functions :

$$\Lambda_{J/I}(\mathbb{A}) = \left| \Lambda^{j_k - i_h + k - h}(\mathbb{A}) \right|_{1 \le h,k \le n} \ . \tag{1.4.4}$$

Of course, one must not forget that $\Lambda^i(\mathbb{A}) = (-1)^i S^i(-\mathbb{A})$, $i \in \mathbb{Z}$, and thus the $\Lambda_{J/I}(\mathbb{A})$ are also skew Schur functions in $-\mathbb{A}$ (we shall see that they also are Schur functions in $\mathbb{A}$, but indexed by "column lengths").

To write easily a Schur function $S_J(\mathbb{A})$, one first fill the diagonal, then complete the columns, increasing or decreasing indices by 1 when moving up or down :

$$J = [1,2,4] \Rightarrow \begin{vmatrix} S_1(\mathbb{A}) & & \\ & S_2(\mathbb{A}) & \\ & & S_4(\mathbb{A}) \end{vmatrix} \Rightarrow \begin{vmatrix} S_1(\mathbb{A}) & S_3(\mathbb{A}) & S_6(\mathbb{A}) \\ S_0(\mathbb{A}) & S_2(\mathbb{A}) & S_5(\mathbb{A}) \\ S_{-1}(\mathbb{A}) & S_1(\mathbb{A}) & S_4(\mathbb{A}) \end{vmatrix} = S_{124}(\mathbb{A})$$

We shall also need rectangular sub-matrices of $\mathbb{S}(\mathbb{A})$ that we shall continue to index the same way: $\mathbb{S}_{J/I}(\mathbb{A})$ is the sub-matrix of $\mathbb{S}(\mathbb{A})$ taken on rows $i_1+1, i_2+2, \ldots$, and columns $j_1+1, j_2+2, \ldots$.

Binet-Cauchy theorem for minors of the product of two matrices implies, in the case of $\mathbb{S}(\mathbb{A}+\mathbb{B})$, the following expansion of skew-Schur functions :

$$S_{J/I}(\mathbb{A}+\mathbb{B}) = \sum\nolimits_K S_{J/K}(\mathbb{A})\, S_{K/I}(\mathbb{B}) \ , \tag{1.4.5}$$

sum over all partitions (only those $K : I \subseteq K \subseteq J$ give a non-zero contribution).

Jacobi's theorem on minors of the inverse of a matrix gives, thanks to Lemma 1.2.5

$$\Lambda_{J/I}(\mathbb{A}) = S_{J^\sim/I^\sim}(\mathbb{A}) = (-1)^{|J/I|} S_{J/I}(-\mathbb{A}) \tag{1.4.6}$$

One needs to enlarge the definition of a Schur function, to be able to play with different alphabets at the same time.

Given n, given two sets of alphabets $\{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_n\}$, $\{\mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_n\}$, and $I, J \in \mathbb{N}^n$, we define the *multi-Schur function*

$$S_{J/I}(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_n - \mathbb{B}_n) := \left| S_{j_k - i_h + k - h}(\mathbb{A}_k - \mathbb{B}_k) \right|_{1 \le h, k \le n} . \tag{1.4.7}$$

In the case where the alphabets are repeated, we indicate by a semicolon the corresponding block separation : given $H \in \mathbb{Z}^p$, $K \in \mathbb{Z}^q$, then $S_{H;K}(\mathbb{A} - \mathbb{B}; \mathbb{C} - \mathbb{D})$ stands for the multi-Schur function with index the concatenation of H and K, and alphabets $\mathbb{A}_1 = \cdots = \mathbb{A}_p = \mathbb{A}$, $\mathbb{B}_1 = \cdots = \mathbb{B}_p = \mathbb{B}$, $\mathbb{A}_{p+1} = \cdots = \mathbb{A}_{p+q} = \mathbb{C}$, $\mathbb{B}_{p+1} = \cdots = \mathbb{B}_{p+q} = \mathbb{D}$.

To write a multi-Schur function easily, one first fill the diagonal, then complete columns by keeping the same alphabet in each column :

$$S_{12;\,4}(\mathbb{A};\, \mathbb{B}) \Rightarrow \begin{vmatrix} S_1(\mathbb{A}) & & \\ & S_2(\mathbb{A}) & \\ & & S_4(\mathbb{B}) \end{vmatrix} \Rightarrow \begin{vmatrix} S_1(\mathbb{A}) & S_3(\mathbb{A}) & S_6(\mathbb{B}) \\ S_0(\mathbb{A}) & S_2(\mathbb{A}) & S_5(\mathbb{B}) \\ S_{-1}(\mathbb{A}) & S_1(\mathbb{A}) & S_4(\mathbb{B}) \end{vmatrix}$$

These functions are now sufficiently general to allow easy inductions, thanks to the following transformation lemma.

LEMMA 1.4.1. *Let $S_J(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_n - \mathbb{B}_n)$ be a multi-Schur function, and $\mathbb{D}_0, \mathbb{D}_1, \ldots, \mathbb{D}_{n-1}$ be a family of finite alphabets such that card$(\mathbb{D}_i) \le i$, $0 \le i \le n-1$. Then $S_J(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_n - \mathbb{B}_n)$ is equal to the determinant*

$$\left| S_{j_k - i_h + k - h}(\mathbb{A}_k - \mathbb{B}_k - \mathbb{D}_{n-h}) \right|_{1 \le h, k \le n}$$

In other words, one does not change the value of a multi-Schur function S_J by replacing in row h the difference $\mathbb{A} - \mathbb{B}$ by $\mathbb{A} - \mathbb{B} - \mathbb{D}_{n-h}$. Indeed, thanks to the expansion (1.4.5) :

$$S_j(\mathbb{A} - \mathbb{B} - \mathbb{D}_h) = S_j(\mathbb{A} - \mathbb{B}) + S_1(-\mathbb{D}_h)\, S_{j-1}(\mathbb{A} - \mathbb{B}) + \cdots + S_h(-\mathbb{D}_h)\, S_{j-h}(\mathbb{A} - \mathbb{B}) \ ,$$

the sum terminating because the $S_k(-\mathbb{D}_h)$ are null for $k > h$, we see that the determinant has been transformed by multiplication by a triangular matrix with 1's in the diagonal, and therefore has kept its value. □

For example, taking $D_0 = \emptyset$, $\mathbb{D}_1 = \{x\}$, $\mathbb{D}_2 = \{y, z\}$, one has

$$\begin{bmatrix} S_i(\mathbb{A}_1 - y - z) & S_{j+1}(\mathbb{A}_2 - y - z) & S_{h+2}(\mathbb{A}_3 - y - z) \\ S_{i-1}(\mathbb{A}_1 - x) & S_j(\mathbb{A}_2 - x) & S_{h+1}(\mathbb{A}_3 - x) \\ S_{i-2}(\mathbb{A}_1) & S_{j-1}(\mathbb{A}_2) & S_h(\mathbb{A}_3) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & -y-z & yz \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} S_i(\mathbb{A}_1) & S_{j+1}(\mathbb{A}_2) & S_{h+2}(\mathbb{A}_3) \\ S_{i-1}(\mathbb{A}_1) & S_j(\mathbb{A}_2) & S_{h+1}(\mathbb{A}_3) \\ S_{i-2}(\mathbb{A}_1) & S_{j-1}(\mathbb{A}_2) & S_h(\mathbb{A}_3) \end{bmatrix}$$

and the determinant of the left matrix is equal to $S_{ijh}(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3)$.

To understand better the structure of such determinants, it is appropriate to use "umbral" notations and write alphabets on the border of the determinant: an entry k in position (i, j) will be interpreted as $S_k(\mathbb{A} \pm \mathbb{B})$ if $\mathbb{A}$ is written at the bottom of column j and $\pm\mathbb{B}$ on the right of row i.

$$\begin{array}{ccc|c} & \vdots & & \\ \cdots & k & \cdots & \pm\mathbb{B} \\ & \vdots & & \\ \hline & \mathbb{A} & & \end{array} \quad \text{means} \quad \begin{array}{ccc} & \vdots & \\ \cdots & S_k(\mathbb{A} \pm \mathbb{B}) & \cdots \\ & \vdots & \end{array} .$$

For example, during a computation, one would rather write the preceding determinant as :

$$\begin{array}{ccc|c} i & j+1 & h+2 & -y-z \\ i-1 & j & h+1 & -x \\ i-2 & j-1 & h & 0 \\ \hline \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{A}_3 & \end{array}$$

Taking $-\mathbb{A}_1, -\mathbb{A}_2, -\mathbb{A}_3$ instead of $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$, and getting rid of signs because of "isobarity", one gets by the same token

$$\Lambda_{i;j;h}(\mathbb{A}_1;\mathbb{A}_2;\mathbb{A}_3) = \begin{bmatrix} \Lambda_i(\mathbb{A}_1+y+z) & \Lambda_{j+1}(\mathbb{A}_2+y+z) & \Lambda_{h+2}(\mathbb{A}_3+y+z) \\ \Lambda_{i-1}(\mathbb{A}_1+x) & \Lambda_j(\mathbb{A}_2+x) & \Lambda_{h+1}(\mathbb{A}_3+x) \\ \Lambda_{i-2}(\mathbb{A}_1) & \Lambda_{j-1}(\mathbb{A}_2) & \Lambda_h(\mathbb{A}_3) \end{bmatrix}$$

In the preceding lemma, we needed only consecutive elements of a column to be complete functions of the same difference of alphabets, of consecutive degrees. A similar transformation can be performed in rows, when alphabets are repeated in some consecutive columns, for partitions having repeated parts.

LEMMA 1.4.2. *Let j, n be two integers, $\mathbb{D}_0, \ldots, \mathbb{D}_{n-1}$ be a family of alphabets such that $card(\mathbb{D}_i) \le i$, $0 \le i \le n-1$, and let $\mathbb{A}$, $\mathbb{B}$ be two arbitrary alphabets. Let $S_{\diamond;\, j^n;\, \circ}(\clubsuit;\ \mathbb{A}-\mathbb{B};\ \spadesuit)$ be a multi-Schur function of which we have specified only n columns.*

Then it is equal to the multi-Schur function

$$S_{\diamond;\, j,\ldots,j;\, \circ}(\clubsuit;\ \mathbb{A}-\mathbb{B}-\mathbb{D}_0,\ \mathbb{A}-\mathbb{B}-\mathbb{D}_1, \ldots,\ \mathbb{A}-\mathbb{B}-\mathbb{D}_{n-1};\ \spadesuit)\ .$$

For example,

$$S_{2;\,444}(\mathbb{A};\ \mathbb{B}) = S_{2;\,4;\,4;\,4}(\mathbb{A};\ \mathbb{B};\ \mathbb{B}-\mathbb{D}_1;\ \mathbb{B}-\mathbb{D}_2).$$

The above lemma implies many factorization properties, e.g. for $r \ge 0$,

$$S_J(\mathbb{A}-\mathbb{B}-x)\, x^r = S_{J,r}(\mathbb{A}-\mathbb{B}, x)\ , \tag{1.4.8}$$

since taking $\mathbb{D}_1 = \mathbb{D}_2 = \cdots = \{x\}$ factorizes the determinant $S_{J,r}(\mathbb{A}-\mathbb{B}, x)$.

More generally, for an alphabet $\mathbb{D}$ of cardinal $\le r$ and $J \in \mathbb{N}^r$, one has

$$S_I(\mathbb{A}-\mathbb{B}-\mathbb{D})\, S_J(\mathbb{D}) = S_{I,J}(\mathbb{A}-\mathbb{B},\ \mathbb{D})\ . \tag{1.4.9}$$

Monomials themselves can be written as multi-Schur functions. Given a totally ordered alphabet $\mathbb{A} = \{a_1, a_2, \ldots\}$, denote, for any n, $\mathbb{A}_n := \{a_1, \ldots, a_n\}$. Then, for any $J = [j_1, \ldots, j_n]$, denoting $J^\omega := [j_n, \ldots, j_1]$, one has

$$a^J := a_1^{j_1} \cdots a_n^{j_n} = S_{J^\omega}(\mathbb{A}_n, \ldots, \mathbb{A}_2, \mathbb{A}_1) \tag{1.4.10}$$

Indeed, subtract the *flag* $0, \mathbb{A}_1, \mathbb{A}_2, \ldots$ in the successive rows, starting from the bottom one. One sees the monomial appearing in the diagonal, the upper part of the matrix vanishing because it is constituted of $S_k\big(-(\mathbb{A}_j - \mathbb{A}_i)\big)$ for
$k > (j-i) \ge 0$.

$$a^{532} = a_1^5 a_2^3 a_3^2 = S_{2;3;5}(a_1+a_2+a_3;\ a_1+a_2;\ a_1) =$$

$$= \begin{array}{ccc|c} 2 & 4 & 7 & -\mathbb{A}_2 \\ 1 & 3 & 6 & -\mathbb{A}_1 \\ 0 & 2 & 5 & 0 \\ \hline \mathbb{A}_3 & \mathbb{A}_2 & \mathbb{A}_1 & \end{array} = \begin{vmatrix} S_2(a_3) & S_4(0) & S_7(-a_2) \\ S_1(a_2+a_3) & S_3(a_2) & S_6(0) \\ S_0(a_1+a_2+a_3) & S_2(a_1+a_2) & S_5(a_1) \end{vmatrix}.$$

One could have put any order on the letters in $\mathbb{A}$, and in general, a monomial on an alphabet of n letters can be written in $n!$ different manners as a multi-Schur functions. However, because it is appropriate to restrict to the flag of alphabets $\mathbb{A}_1 \subset \mathbb{A}_2 \subset \mathbb{A}_3 \subset \cdots$, we have preferred the convention which gives as the index of the multi-Schur function the reverse of the exponent of the monomial.

Given two finite alphabets, the following factorization and vanishing properties implicitly appear in many classical 19-th century texts about elimination theory (modern reference is Berele-Regev, [**14**]).

PROPOSITION 1.4.3. *Let $\mathbb{A}$, $\mathbb{B}$, be of cardinalities α, β, $p \in \mathbb{N}$, $I \in \mathbb{N}^p$, $J \in \mathbb{N}^\alpha$. Then*

$$S_{i_1,\ldots,i_p,\beta+j_1,\ldots,\beta+j_\alpha}(\mathbb{A}-\mathbb{B}) = S_I(-\mathbb{B})\, S_J(\mathbb{A}) \prod_{a\in\mathbb{A}, b\in\mathbb{B}} (a-b) \ .$$

Let J be a partition, $J \supseteq (\beta+1)^{\alpha+1}$. Then $S_J(\mathbb{A}-\mathbb{B}) = 0$.

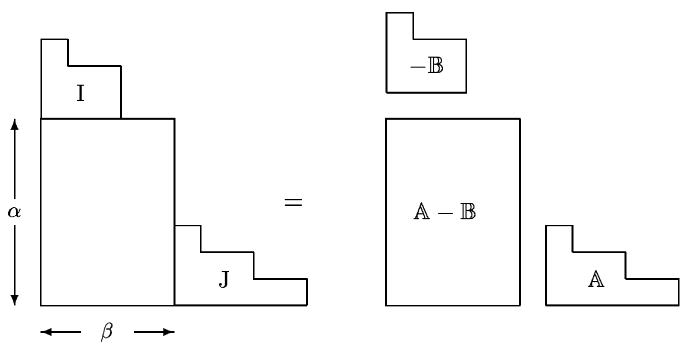
Proof. Subtract $\mathbb{A}$ in the first p rows. One gets the factorization

$$S_I(-\mathbb{B})\, S_{\beta+j_1,\ldots,\beta+j_\alpha}(\mathbb{A}-\mathbb{B}) \ .$$

Now, using the partition K conjugate to $[\beta+j_1,\ldots,\beta+j_\alpha]$, : one gets the factorization of $S_K(\mathbb{B}-\mathbb{A})$ into a Schur function of $\mathbb{A}$ and $S_{\alpha^\beta}(\mathbb{B}-\mathbb{A})$. This last function can be seen equal to the *resultant* $\prod_{a\in\mathbb{A}, b\in\mathbb{B}}(b-a)$ by subtracting the flag $0, \mathbb{B}_1, \mathbb{B}_2, \ldots$.

The case where $J \supseteq (\beta+1)^{\alpha+1}$ can be treated by adding the same letters to $\mathbb{A}$ and $\mathbb{B}$, so that one is reduced to the preceding case. But now the factor $\prod(a-b)$ vanishes because $\mathbb{A}$ and $\mathbb{B}$ have a letter in common. □

Pictorially, the relation is



Given a finite alphabet $\mathbb{A}$ (that one will totally order: $\mathbb{A} = \{a_1, \ldots, a_n\}$), Cauchy and Jacobi separately defined the Schur function $S_J(\mathbb{A})$ using the (infinite) *Vandermonde matrix*

$$V(\mathbb{A}) = \left[a_i^j\right]_{1\le i\le n\,;\, j\ge 0} = \begin{bmatrix} a_1^0 & a_1^1 & a_1^2 & \cdots \\ \vdots & \vdots & \vdots & \\ a_n^0 & a_n^1 & a_n^2 & \cdots \end{bmatrix}$$

and the *Vandermonde* (determinant)

$$\Delta(\mathbb{A}) := \prod_{i>j}(a_i - a_j) = \begin{vmatrix} a_1^0 & a_1^1 & \cdots & a_1^{n-1} \\ \vdots & \vdots & & \vdots \\ a_n^0 & a_n^1 & \cdots & a_n^{n-1} \end{vmatrix} \ .$$

PROPOSITION 1.4.4. *Let $J \in \mathbb{N}^n$. Then $S_J(\mathbb{A})\,\Delta(\mathbb{A})$ is equal to the minor of index $(0^n, J)$ of the Vandermonde matrix $V(\mathbb{A})$.*

Proof. Let $\mathbb{S}_J(\mathbb{A})$ denotes the sub-matrix of $\mathbb{S}(A)$ taken on columns $j_1{+}1$, $j_2{+}2, \ldots, j_n{+}n$. Consider the product $V(\mathbb{A})\,\mathbb{S}(-A)\,\mathbb{S}_J(\mathbb{A})$. It can be factorized in two manners, using (1.4.2):

$$V(\mathbb{A})\,\mathbb{S}(-A)\,\mathbb{S}_J(\mathbb{A}) = \left[S_j(a_i - \mathbb{A})\right]_{1\le i\le n; j\ge 0}\,\mathbb{S}_J(\mathbb{A}) = V(\mathbb{A})\,\mathbb{S}_J(0)\ .$$

However, the $S_j(a_i{-}\mathbb{A})$ are null for $j \ge n$, because they are the elementary functions (up to sign) of alphabets of cardinality $n-1$. On the other hand, $S_j(0) = 0$, if $j \ne 0$. In both cases, we have obtained matrices such that only one minor of order n is different from 0. □

For example, for $n = 3$, $J = [1, 3, 4]$, truncating the matrices, one has

$$\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^6 \\ 1 & a_2 & a_2^2 & \cdots & a_2^6 \\ 1 & a_3 & a_3^2 & \cdots & a_3^6 \end{bmatrix} \begin{bmatrix} S_0(-\mathbb{A}) & S_1(-\mathbb{A}) & S_2(-\mathbb{A}) & S_3(-\mathbb{A}) & S_4(-\mathbb{A}) & S_5(-\mathbb{A}) & S_6(-\mathbb{A}) \\ 0 & S_0(-\mathbb{A}) & S_1(-\mathbb{A}) & S_2(-\mathbb{A}) & S_3(-\mathbb{A}) & S_4(-\mathbb{A}) & S_5(-\mathbb{A}) \\ 0 & 0 & S_0(-\mathbb{A}) & S_1(-\mathbb{A}) & S_2(-\mathbb{A}) & S_3(-\mathbb{A}) & S_4(-\mathbb{A}) \\ 0 & 0 & 0 & S_0(-\mathbb{A}) & S_1(-\mathbb{A}) & S_2(-\mathbb{A}) & S_3(-\mathbb{A}) \\ 0 & 0 & 0 & 0 & S_0(-\mathbb{A}) & S_1(-\mathbb{A}) & S_2(-\mathbb{A}) \\ 0 & 0 & 0 & 0 & 0 & S_0(-\mathbb{A}) & S_1(-\mathbb{A}) \\ 0 & 0 & 0 & 0 & 0 & 0 & S_0(-\mathbb{A}) \end{bmatrix} \begin{bmatrix} S_1(\mathbb{A}) & S_4(\mathbb{A}) & S_6(\mathbb{A}) \\ S_0(\mathbb{A}) & S_3(\mathbb{A}) & S_5(\mathbb{A}) \\ 0 & S_2(\mathbb{A}) & S_4(\mathbb{A}) \\ 0 & S_1(\mathbb{A}) & S_3(\mathbb{A}) \\ 0 & S_0(\mathbb{A}) & S_2(\mathbb{A}) \\ 0 & 0 & S_1(\mathbb{A}) \\ 0 & 0 & S_0(\mathbb{A}) \end{bmatrix}$$

$$= \begin{bmatrix} S_0(a_1{-}\mathbb{A}) & S_1(a_1{-}\mathbb{A}) & S_2(a_1{-}\mathbb{A}) & 0 & 0 & 0 & 0 \\ S_0(a_2{-}\mathbb{A}) & S_1(a_2{-}\mathbb{A}) & S_2(a_2{-}\mathbb{A}) & 0 & 0 & 0 & 0 \\ S_0(a_3{-}\mathbb{A}) & S_1(a_1{-}\mathbb{A}) & S_2(a_1{-}\mathbb{A}) & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_1(\mathbb{A}) & S_4(\mathbb{A}) & S_6(\mathbb{A}) \\ S_0(\mathbb{A}) & S_3(\mathbb{A}) & S_5(\mathbb{A}) \\ 0 & S_2(\mathbb{A}) & S_4(\mathbb{A}) \\ 0 & S_1(\mathbb{A}) & S_3(\mathbb{A}) \\ 0 & S_0(\mathbb{A}) & S_2(\mathbb{A}) \\ 0 & 0 & S_1(\mathbb{A}) \\ 0 & 0 & S_0(\mathbb{A}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a_1 & a_1^2 & a_1^3 & a_1^4 & a_1^5 & a_1^6 \\ 1 & a_2 & a_2^2 & a_2^3 & a_2^4 & a_2^5 & a_2^6 \\ 1 & a_3 & a_3^2 & a_3^3 & a_3^4 & a_3^5 & a_3^6 \end{bmatrix} \begin{bmatrix} S_1(0) & S_4(0) & S_6(0) \\ S_0(0) & S_3(0) & S_5(0) \\ 0 & S_2(0) & S_4(0) \\ 0 & S_1(0) & S_3(0) \\ 0 & S_0(0) & S_2(0) \\ 0 & 0 & S_1(0) \\ 0 & 0 & S_0(0) \end{bmatrix}\ .$$

Still writing $\mathbb{A}_i = a_1 + \cdots + a_i$, one has an expression for Schur functions which "interpolates" between Jacobi-Trudi determinant and a minor of the Vandermonde matrix.

LEMMA 1.4.5. *Let $\mathbb{A}$ be of cardinality n, and $K = [k_1, \ldots, k_n] \in \mathbb{N}^n$. Then*

$$S_K(\mathbb{A}) = \det\left|S_{k_j+j-i}(\mathbb{A}_i)\right|_{1\le i,j\le n}\ . \tag{1.4.11}$$

Proof. Subtract 0, a_n, $a_n{+}a_{n-1}$, $\ldots$, $a_n{+}\cdots{+}a_2$ in the successive rows of $S_K(\mathbb{A})$. □

We shall later interpret such a determinant as a *discrete Wrońskian*. Notice that the top row is made of powers of a_1, and the bottom row is the same as in Jacobi-Trudi determinant.

Given an alphabet $\mathbb{A}$ of finite cardinality n, one will need the alphabet of inverses $\mathbb{A}^\vee = \{a^{-1} : a \in \mathbb{A}\}$. Noticing that

$$\Lambda^i(\mathbb{A}^\vee) = \Lambda^{n-i}(\mathbb{A})/\Lambda^n(\mathbb{A})\ ,\ 1 \le i \le n\ ,$$

and using the expression of a Schur function as a determinant of Λ^i, one has, for any r and any partition $I \subseteq \square = r^n$:

$$S_I(\mathbb{A}^\vee) = S_{\square/I}(\mathbb{A})/S_\square(\mathbb{A})\ . \tag{1.4.12}$$

1.5. Cauchy Formula

The most important formula in the theory of symmetric functions is the following expansion, due to Cauchy, relative to symmetric functions of a product of alphabets[2].

Let $\mathbb{A}, \mathbb{B}$ be two alphabets. Then :

$$\mathbf{K}(\mathbb{A}, \mathbb{B}) := \sigma_1(\mathbb{A}\mathbb{B}) = \prod_{a\in\mathbb{A}} \prod_{b\in\mathbb{B}} (1-ab)^{-1} = \sum\nolimits_J S_J(\mathbb{A}) S_J(\mathbb{B}) \ , \tag{1.5.1}$$

sum over all partitions J (the terms for which $\ell(J) > min\big(card(\mathbb{A})\, ,\, card(\mathbb{B})\big)$ vanish).

One will find later a proof of Cauchy's formula, using symmetrizing operators, starting from the straightforward identity

$$\frac{1}{1-a_1b_1} \frac{1}{1-a_1a_2b_1b_2} \frac{1}{1-a_1a_2a_3b_1b_2b_3} \cdots = \sum\nolimits_\lambda a^\lambda b^\lambda \ ,$$

sum over all weakly decreasing exponents λ.

For the moment, let us sketch a proof that one can find in the literature, supposing that the two alphabets have cardinality n.

Consider the Cauchy matrix $\big[1/(1-ab)\big]_{a\in\mathbb{A}, b\in\mathbb{B}}$. Each entry $1/(1-ab)$ can be considered as the scalar product of the two infinite vectors $[1, a, a^2, \ldots]$ and $[1, b, b^2, \ldots]$, and therefore the Cauchy matrix is equal to the product $V(\mathbb{A})\, V(\mathbb{B})^{tr}$ of two Vandermonde matrices.

Since minors of each matrix are Schur functions multiplied by a Vandermonde, Binet-Cauchy expansion gives :

$$\big|1/(1-ab)\big|_{a\in\mathbb{A}, b\in\mathbb{B}} = \Delta(\mathbb{A})\Delta(\mathbb{B}) \ \sum\nolimits_J S_J(\mathbb{A}) S_J(\mathbb{B}) \ .$$

Now, the determinant itself is equal to the sum ($\ell(\sigma)$ denoting the *length* of a permutation σ in the symmetric group $\mathfrak{S}(\mathbb{A})$) :

$$\sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)} \frac{1}{\Big((1-a_1b_1)\cdots(1-a_nb_n)\Big)^\sigma} \ .$$

Extracting the full denominator $\prod_{a\in\mathbb{A},\, b\in\mathbb{B}} 1/(1-ab)$, one has to compute the sum

$$\sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)} \Big(S^{n-1}(1+b_1a_1-b_1\mathbb{A}) \cdots S^{n-1}(1+b_na_n-b_n\mathbb{A})\Big)^\sigma$$

This sum is divisible by $\Delta(\mathbb{A})\,\Delta(\mathbb{B})$, because it vanishes when two of the a's, or two of the b's coincide. The degree in each a or b is at most $n-1$, and therefore the quotient is of degree zero in each variable. One has to check that it is equal to 1. $\square$

The last step in the above demonstration misses the crucial fact that what is really involved is *Jacobi symmetrizer*

$$\mathbb{C}[a_1, \ldots, a_n] \ni f \mapsto \left(\sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)} f^\sigma\right) \frac{1}{\Delta(\mathbb{A})} \ ,$$

[2] Given $\mathbb{A}, \mathbb{B}$, then $\mathbb{A}\mathbb{B}$ is the alphabet with letters ab, $a\in\mathbb{A}$, $b\in\mathbb{B}$.

sum over all permutations σ of the letters of $\mathbb{A}$. Jacobi's symmetrizer provides a connection with the theory of characters (and extends to Weyl's character formula for the classical groups). We postpone this point of view to another chapter, where we shall express Jacobi's symmetrizer as a product of divided differences.

There are other forms of Cauchy's formula, for two alphabets of finite cardinalities :

$$(1.5.2)\qquad \prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (1-ab) = \sigma_1(-\mathbb{A}\mathbb{B}) = \sum_I (-1)^{|I|} S_I(\mathbb{A}) S_{I^\sim}(\mathbb{B})$$

$$(1.5.3)\qquad \mathbf{R}(\mathbb{A},\mathbb{B}) := \prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (a-b) = \sum_I S_I(\mathbb{A}) S_{\square/I}(-\mathbb{B}) = \sum_I (-1)^{|\square/I|} S_I(\mathbb{A}) S_{\square^\sim/I^\sim}(\mathbb{B}) \ ,$$

where $\square = \beta^\alpha$, $\alpha = card(\mathbb{A})$, $\beta = card(\mathbb{B})$.

Formula (1.5.2) is equivalent to (1.5.1), changing $\mathbb{B}$ into $-\mathbb{B}$, and using (1.4.6). Changing $\mathbb{A}$ into $\mathbb{A}^\vee = \{a^{-1}\}$, one gets (1.5.3), thanks to the relations (1.4.12) between the Schur functions of $\mathbb{A}$ and those of $\mathbb{A}^\vee$.

Thus, the equivalence of the three forms of Cauchy formulas is purely formal, once one proves them for any cardinality. However, we already have stated a property implying (1.5.3) in theorem 1.4.3 and using only the transformation lemma 1.4.1. Let us repeat the proof in detail.

The right hand side of (1.5.3) is the expansion of $S_\square(\mathbb{A}-\mathbb{B})$, according to (1.4.3). One subtracts the alphabet $\mathbb{A}-a_1$ in the first row of $S_\square(\mathbb{A}-\mathbb{B})$, and a_1 in all the columns, except the first one. Now, the first row of the determinant has become

$$S_\beta(a_1-\mathbb{B}),\ S_{\beta+1}(-\mathbb{B}),\ \ldots,\ S_{\beta+\alpha-1}(-\mathbb{B}) \ .$$

Since $S_{\beta+1}(-\mathbb{B}),\ldots$ are null, the new determinant factorizes into

$$S_\beta(a_1-\mathbb{B})\, S_{\beta^{\alpha-1}}((\mathbb{A}-a_1)-\mathbb{B}) \ ,$$

and this gives (1.5.3) by induction on α.

1.6. Scalar Product

There are other decompositions of $\mathbf{K}(\mathbb{A},\mathbb{B})$ as a sum of products of symmetric functions in $\mathbb{A}$ and in $\mathbb{B}$. However, there is only one of the type $\sum P(\mathbb{A})P(\mathbb{B})$ over $\mathbb{Z}$: up to signs, the P's are all the Schur functions indexed by partitions in $\mathbb{N}$. Thus $\mathbf{K}(\mathbb{A},\mathbb{B})$, that we shall call *Cauchy kernel*, determines the Schur functions, because it is the only $\mathbb{Z}$-basis in which $\mathbf{K}(\mathbb{A},\mathbb{B})$ is diagonal.

One can interpret differently the kernel, as defining a scalar product on the space of symmetric functions, the Schur functions constituting the only orthogonal basis. Now, any expansion of the type

$$(1.6.1)\qquad \mathbf{K}(\mathbb{A},\mathbb{B}) = \sum P_J(\mathbb{A})\, Q_J(\mathbb{B})$$

define a pair of adjoint bases $\{P_J\}$, $\{Q_J\}$, with respect to the *canonical scalar product* (,) induced by $\mathbf{K}(\mathbb{A},\mathbb{B})$, i.e. the scalar product such that $(S_J, S_J) = 1$, for all partition J. In other words (1.6.1) is equivalent to

$$(1.6.2)\qquad (P_J(\mathbb{A})\,,\, Q_J(\mathbb{A})) = 1 \qquad \& \qquad (P_I(\mathbb{A})\,,\, Q_J(\mathbb{A})) = 0 \text{ for } I \neq J \ .$$

There are some difficulties for what concerns scalar products when taking finite alphabets, and in the rest of the section, we shall consider only infinite alphabets.

The expansion

$$\mathbf{K}(\mathbb{A},\mathbb{B}) = \prod_b \left(\prod_a \frac{1}{1-ab}\right) = \prod_b \Big(\sum b^i S^i(\mathbb{A})\Big) = \sum_I \Psi_I(\mathbb{B})\, S^I(\mathbb{A}) \tag{1.6.3}$$

shows that the basis adjoint to S^I, I partition, is the monomial basis Ψ_I.

From the identity (1.3.2), one gets

$$\begin{aligned} \sigma_1(\mathbb{A}) &= \exp\big(\sum_{i=1}^{\infty} \Psi^i(\mathbb{A})/i\big) = \prod_{i=1}^{\infty} \exp(\Psi^i(\mathbb{A})/i) \\ &= \sum\nolimits_I \Psi^I(\mathbb{A})/z_I \ , \end{aligned} \tag{1.6.4}$$

sum over all partitions $I = [1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots]$, defining

$$z_I = m_1!\, 1^{m_1}\, m_2!\, 2^{m_2}\, m_3!\, 3^{m_3} \cdots \tag{1.6.5}$$

Since $\Psi^i(\mathbb{A}\mathbb{B}) = \Psi^i(\mathbb{A})\, \Psi^i(\mathbb{B})$, it implies the expansion

$$\mathbf{K}(\mathbb{A},\mathbb{B}) = \sigma_1(\mathbb{A}\mathbb{B}) = \sum_I \Psi^I(\mathbb{A})\Psi^I(\mathbb{B})/z_I \ , \tag{1.6.6}$$

which shows that the basis of products of power sums is orthogonal, with scalar product $(\Psi_I, \Psi_I) = z_I$.

The name kernel is justified by the following property, which is just another way of stating that $\mathbf{K}(\mathbb{A},\mathbb{B})$ defines a scalar product:

LEMMA 1.6.1. *Let f be a symmetric function and $(\ ,\ )$ be the canonical scalar product on symmetric functions in $\mathbb{A}$. Then*

$$(\mathbf{K}(\mathbb{A},\mathbb{B})\ ,\ f(\mathbb{A})) = f(\mathbb{B})$$

Proof. The identity is linear in $f \in \mathfrak{Sym}(\mathbb{A})$. We check it on the basis of Schur functions :

$$\Big(\sum\nolimits_I S_I(\mathbb{A})S_I(\mathbb{B})\ ,\ S_J(\mathbb{A})\Big) = S_J(\mathbb{B}) \ .$$

$\square$

One can use simultaneoulsy several alphabets $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}, \ldots$, as well as the scalar products corresponding to them. We shall specify in which symmetric ring we are evaluating the scalar product by indexing it by the alphabet : $(\ ,\ )_{\mathbb{A}}$.

Let us give a first example. Let $f, g, h \in \mathfrak{Sym}$. Then one has

$$\Big(\sigma_1\big((\mathbb{A}{+}\mathbb{B})\mathbb{C}\big)\ ,\ f(\mathbb{A})\, g(\mathbb{B})\, h(\mathbb{C})\Big)_{\mathbb{A}} = \sigma_1(\mathbb{B}\mathbb{C})\, f(\mathbb{C})\, g(\mathbb{B})\, h(\mathbb{C}) \ . \tag{1.6.7}$$

Proof. One factors out $\sigma_1(\mathbb{B}\mathbb{C})$ and $g(\mathbb{B})h(\mathbb{C})$, which are scalars in $\mathfrak{Sym}(\mathbb{A})$. One is left with $(\sigma_1(\mathbb{A}\mathbb{C})\ ,\ f(\mathbb{A}))_{\mathbb{A}} = f(\mathbb{C})$. $\square$

1.7. Differential Calculus

Having a scalar product, one can now obtain operators *adjoint* to some simple ones. We did not use the multiplicative structure of $\mathfrak{Sym}$ up to now, but only the vector space structure of $\mathfrak{Sym}$. Now we shall use that any symmetric function f may be thought of as the operator

“ multiplication by f ” .

DEFINITION 1.7.1. For $f \in \mathfrak{Sym}$, D_f is the operator adjoint to the multiplication by f, i.e. for every $S', S'' \in \mathfrak{Sym}$ one has

$$(D_f(S')\,,\, S'') = (S'\,,\, f\, S'') \ .$$

The following lemma shows that Schur functions play a special rôle; the best proof of the following lemma is interpreting $\mathfrak{Sym}$ as a ring of shifting operators on isobaric determinants (as explained in the next section).

LEMMA 1.7.2. *For every $I, J \in \mathbb{N}^n$, one has*

$$D_{S_I}(S_J) = S_{J/I} \ , \tag{1.7.1}$$

$$D_{S^I}(\Psi_J) = \begin{cases} \Psi_K & \text{if } J = K \cup I \\ 0 & \text{otherwise} \end{cases} \ . \tag{1.7.2}$$

Proof. Instead of a single $S_{J/I}$, let us introduce two other alphabets $\mathbb{B}$, $\mathbb{C}$ and take the generating function

$$\sigma_1((\mathbb{A}+\mathbb{B})\,\mathbb{C}) = \sum\nolimits_{J,H} S_{J/H}(\mathbb{A})\, S_H(\mathbb{B})\, S_J(\mathbb{C}) \ .$$

The scalar product (in $\mathfrak{Sym}(\mathbb{B})$) of this function with $S_I(\mathbb{B})$ is equal to $\sum_J S_{J/I}(\mathbb{A})\, S_J(\mathbb{C})$, which is, according to (1.6.7), equal to $\sigma_1(\mathbb{A}\mathbb{C})\, S_I(\mathbb{C})$. Therefore

$$\begin{aligned} S_{J/I}(\mathbb{A}) &= \big(\sigma_1(\mathbb{A}\mathbb{C}) S_I(\mathbb{C})\,,\, S_J(\mathbb{C})\big)_{\mathbb{C}} \\ &= \big(\sigma_1(\mathbb{A}\mathbb{C})\,,\, D_{S_I} S_J(\mathbb{C})\big)_{\mathbb{C}} \end{aligned}$$

and, because $\sigma_1(\mathbb{A}\mathbb{C})$ is a reproducing kernel, one gets the wanted identity $S_{J/I}(\mathbb{A}) = D_{S_I} S_J(\mathbb{A})$.

As for the second identity, it is just another way of stating that monomial functions Ψ_J and product of complete functions S^J constitute two adjoint bases. Indeed,

$$(D_{S^I}(\Psi_J)\,,\, S^K) = (\Psi_J\,,\, S^{I\cup K})$$

allows to conclude. □

The operators D_{Ψ_i} are differential operators. Indeed, for every integer $i > 0$, one has

$$D_{\Psi_i} = i\partial_{\Psi_i} \tag{1.7.3}$$

as can be checked by operating on the basis Ψ^J: the scalar product is compatible with the tensor decomposition

$$\mathfrak{Sym} \simeq \mathbb{C}[\Psi_1] \otimes \mathbb{C}[\Psi_2] \otimes \mathbb{C}[\Psi_3] \otimes \cdots \ ,$$

and the equality

$$\big(\Psi_i(\Psi_i)^k\,,\, (\Psi_i)^m\big) = \delta_{k,m-1}\, i^m\, m! = \delta_{k,m-1}\, z_{i^m}$$

proves the assertion.

In the usual differential calculus on polynomials, one can define derivatives without having recourse to vanishing epsilon's, but just as the successive coefficients in Taylors's formula, in other words, just using $f(y) \to f(y+x)$.

The same property is true in the ring $\mathfrak{Sym}$.

PROPOSITION 1.7.3. *For any pair of adjoint bases $\{P_I\}$, $\{Q_I\}$, and any element $f \in \mathfrak{Sym}$, one has the decomposition*

$$f(\mathbb{A}+\mathbb{B}) = \sum_I D_{P_I}\big(f(\mathbb{A})\big)\, Q_I(\mathbb{B}) \ . \tag{1.7.4}$$

Proof. This is a linear statement that it is sufficient to prove for one pair of adjoint bases, for example for the Schur basis, and for a f a generic Schur function.

But in that case, the identity to be proven is the expansion of $S_J(\mathbb{A}+\mathbb{B})$ given in (1.4.5). $\square$

We shall use very often the identity

$$\forall f \in \mathfrak{Sym}\,,\ f(\mathbb{B}+\mathbb{A}) = \sum\nolimits_I D_{S_I}\big(f(\mathbb{B})\big)\, S_I(\mathbb{A}) \tag{1.7.5}$$

when $\mathbb{A}$ is a single letter a or when it is replaced by $-a$. In that case the identity becomes

$$f(\mathbb{B}+a) \;=\; \sum\nolimits_i D_{S^i}\big(f(\mathbb{B})\big)\, a^i \tag{1.7.6}$$

$$f(\mathbb{B}-a) \;=\; \sum\nolimits_i D_{\Lambda^i}\big(f(\mathbb{B})\big)\, (-a)^i\ . \tag{1.7.7}$$

Another corollary of (1.7.4) is :

$$\forall f \in \mathfrak{Sym}\,,\ f(\mathbb{A}+\mathbb{B}) \;=\; \sum_I \Psi_I(\mathbb{A})\, D_{S^I}\big(f(\mathbb{B})\big)\ , \tag{1.7.8}$$

$$= \sum_J \frac{1}{(\Psi^I\,,\,\Psi^I)} \Psi^I(\mathbb{A})\, D_{\Psi^I}\big(f(\mathbb{B})\big)\ . \tag{1.7.9}$$

The operators D_{S_I} are usually called *Hammond operators.* They satisfy a kind of Leibnitz[3] formula :

LEMMA 1.7.4. *For every $f, g \in \mathfrak{Sym}$, and every partition J, one has*

$$D_{S_J}(fg) = \sum\nolimits_I D_{S_{J/I}}(f)\, D_{S_I}(g)\ . \tag{1.7.10}$$

Proof. $D_{S_J}(fg)(\mathbb{A})$ is the coefficient of $S_J(\mathbb{B})$ in

$$f(\mathbb{A}+\mathbb{B})\, g(\mathbb{A}+\mathbb{B}) = \sum\nolimits_{H,K} (S_H(\mathbb{B}) S_K(\mathbb{B})\,,\ S_J(\mathbb{B}))_{\mathbb{B}}\, D_{S_{H(\mathbb{A})}} f(\mathbb{A})\, D_{S_K}(g)(\mathbb{A})\ .$$

The lemma follows from the fact that $\sum_H (S_H\, S_K\,,\, S_J)\, S_H = S_{J/K}$. $\square$

1.8. Operators on Isobaric Determinants

We have not yet used an evident property of the different determinants that we have written up to now, *that all terms in their expansion have the same total degree.* We shall say that such a determinant is *isobaric* (in fact, more generally, we are only dealing with *homogeneous polynomials*).

But now, given an isobaric determinant, how to operate on it obtaining only isobaric determinants ? A simpler question even :

how to increase degrees by 1 ?

It is appropriate to consider more general "weights" than degree, and we shall begin by vectors before considering determinants.

Let V be a vector space of functions of a variable $x \in \mathbb{C}$, with values in a commutative ring. Given two integers $1 \le k \le n$, define T_k^+ (resp. T_k^-) to be the operator on $V^{\otimes n}$ sending $f_1(x_1) \otimes \cdots \otimes f_n(x_n)$ onto

$$\sum_{\epsilon\in\{0,1\}^n,\, \epsilon_1+\cdots+\epsilon_n=k} f_1(x_1 \pm \epsilon_1) \otimes \cdots \otimes f_n(x_n \pm \epsilon_n)\ .$$

[3] Leibnitz, in his communications to the Académie des Sciences, was spelling his name with a 't' which is ignored by most anglo-saxons authors.

For example,

$$T_2^+\big(f_1(x_1)\otimes f_2(x_2)\otimes f_3(x_3)\big) = f_1(x_1+1)\otimes f_2(x_2+1)\otimes f_3(x_3)$$
$$+ f_1(x_1+1)\otimes f_2(x_2)\otimes f_3(x_3+1) + f_1(x_1)\otimes f_2(x_2+1)\otimes f_3(x_3+1)\ .$$

The following theorem is not difficult to check. It amounts to the fact that the T_i's, $i=1,\ldots,n$ commute between themselves, and are algebraically independent (they have been my first encounter with the theory of symmetric functions, through manipulations of isobaric determinants ([**93**]).

THEOREM 1.8.1. *Let n be a positive integer. Then $T_0^+=1, T_1^+,\ldots,T_n^+$ (resp. $T_0^-=1, T_1^-,\ldots,T_n^-$) generate a commutative algebra isomorphic to $\mathfrak{Sym}(n)$, the algebra of symmetric polynomials in n variables, the image of $T_k^\pm$ being Λ_k.*

Any symmetric polynomial S gives rise to two operators T_S^+ and T_S^-. Decomposing S into the basis Λ^J: $S=\sum_{J\in\mathbb{N}^n} c_J\,\Lambda^J$, one has

$$T_S^+ = \sum\nolimits_J c_J\, T_{j_1}^+\cdots T_{j_n}^+ \quad \& \quad T_S^- = \sum\nolimits_J c_J\, T_{j_1}^-\cdots T_{j_n}^- \tag{1.8.1}$$

The operators T_k^+, T_k^- can be made to act on $V^{\otimes n^2}$, considered as a space of matrices $\big[f_{ij}(x_{ij})\big]_{1\le i,j\le n}$, but now there are two natural ways to define their action, either by columns :

$${}^cT_k^\pm\big([f_{ij}(x_{ij})]\big) = \sum_{\epsilon\in\{0,1\}^n,\,|\epsilon|=k} \big[f_{ij}(x_{ij}\pm\epsilon_j)\big]\ ,$$

or by rows:

$${}^rT_k^\pm\big([f_{ij}(x_{ij})]\big) = \sum_{\epsilon\in\{0,1\}^n,\,|\epsilon|=k} \big[f_{ij}(x_{ij}\pm\epsilon_i)\big]\ .$$

Symmetry between rows and columns of a matrix entails the following lemma, when evaluating the determinants appearing in ${}^cT$ or ${}^rT$:

LEMMA 1.8.2. *Given any matrix $\big[f_{ij}(x_{ij})\big]_{1\le i,j\le n}$, fixing a sign $\pm$, then one has*

$$\sum_{\epsilon\in\{0,1\}^n,\,|\epsilon|=k} \big|f_{ij}(x_{ij}\pm\epsilon_j)\big| = \sum_{\epsilon\in\{0,1\}^n,\,|\epsilon|=k} \big|f_{ij}(x_{ij}\pm\epsilon_i)\big|\ . \tag{1.8.2}$$

Indeed, any term in the expansion of the determinant of the original matrix will be transformed according to $T_k^\pm$ and will be found in the expansion of one of the determinants of each side of equation (1.8.2).

By abuse of language, we shall write ${}^cT_k^\pm\big(|f_{ij}(x_{ij})|\big)$ and ${}^rT_k^\pm\big(|f_{ij}(x_{ij})|\big)$ for the above two sums of determinants.

For example, writing only the shifts in the x_{ij}'s, the equality ${}^cT_1^+={}^rT_1^+$ reads, for a determinant of order 3 :

$$\begin{vmatrix} +1 & \cdot & \cdot \\ +1 & \cdot & \cdot \\ +1 & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & +1 & \cdot \\ \cdot & +1 & \cdot \\ \cdot & +1 & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & +1 \\ \cdot & \cdot & +1 \\ \cdot & \cdot & +1 \end{vmatrix} = \begin{vmatrix} +1 & +1 & +1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & \cdot \\ +1 & +1 & +1 \\ \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ +1 & +1 & +1 \end{vmatrix}\ .$$

Notice that the determinants appearing on the right and on the left are different, though their sums are equal.

We could have written the sum of all T_k, $k=0,\ldots,n$, in one stroke :

$$(T_0+zT_1+\cdots+z^nT_n)\Big(\Big|f_{ij}(x_{ij})\Big|\Big) = \Big|f_{ij}(x_{ij})+z\,f_{ij}(x_{ij}+1)\Big|\,, \tag{1.8.3}$$

and this renders symmetry between rows and columns even more evident.

Let $\mathbb{A}$ be of cardinality n. Then the two algebras generated by the $T_k^{\pm}$ can be made act on $Sym(\mathbb{A})$, by making them operate on the Jacobi-Trudi determinants of complete functions expressing Schur functions, and extending the action by linearity.

THEOREM 1.8.3. *Let $\mathbb{A}$ be of cardinality n and S belong to $\mathfrak{Sym}(n)$. Then T_S^+ is the operator "multiplication by $S(\mathbb{A})$" and T_S^- acts by $f(\mathbb{A}) \mapsto D_S(f)(\mathbb{A})$.*

Proof. We have to test the action of the $T_k^{\pm} = T_{\Lambda^k}^{\pm}$ on the linear basis of Schur functions $S_J(\mathbb{A})$, but we shall rather take the Vandermonde matrices $\mathbb{V}_J$. Shifting the exponents uniformly by r in row i of $\mathbb{V}_J$ has the effect of multiplying its determinant by a_i^r, and more generally, multiplying $\mathbb{V}_J$ by a monomial a^K can be realized by increasing exponents by $k_1, \ldots, k_n$ in successive rows. Therefore

$$\Lambda^k(\mathbb{A}) V_J(\mathbb{A}) = {}^rT_k^+(V_J(\mathbb{A})) \ . \tag{1.8.4}$$

Notice that
(1.8.5)
$${}^cT_k^+(V_J(\mathbb{A})) = \sum_{\epsilon \in \{0,1\}^n,\, |\epsilon|=k} V_{J+\epsilon}(\mathbb{A}) = \Delta(\mathbb{A}) \sum_{\epsilon} S_{J+\epsilon}(\mathbb{A}) = \Delta(\mathbb{A})\, {}^cT_k^+(S_J(\mathbb{A}))$$

gives a description of the product of a Schur function by an elementary symmetric function that is called *Pieri formula*, and that we shall comment with more details in the next section.

Some care is needed to identify T_k^- by using its action on $\mathbb{V}_J(\mathbb{A})$ rather than $S_J(\mathbb{A})$. The operator is not "multiplication by $\Lambda^k(\mathbb{A}^\vee)$", because the operation of decreasing degrees by 1 sends a_i^0 onto 0, and not onto $1/a_i$, if we want this action to be coherent with the convention $S_j = 0$ for $j < 0$. However, if $J \supseteq 1^n$, then

$$\Lambda^k(\mathbb{A}^\vee)\, S_J(\mathbb{A}) = \Lambda^{n-k}(\mathbb{A})\, S_{J/1^n}(\mathbb{A}) = T_k^-(S_J(\mathbb{A})) \ .$$

To identify T_k^-, we use ${}^rT_k^-$ operating on Schur functions. The image of $S_j(\mathbb{A})$ is a sum of $\binom{n}{k}$ determinants, each of which is null except for the last one

$$S_{J/0^{n-k}1^k}(\mathbb{A}) = D_{\Lambda^k} S_J(\mathbb{A}) \ .$$

The operator ${}^cT_k^-$ allows us to recover the dual Pieri formula :

$$S_{J/0^{n-k}1^k}(\mathbb{A}) = {}^cT_k^-(S_J(\mathbb{A})) = {}^rT_k^-(S_J(\mathbb{A})) = \sum_{\epsilon \in \{0,1\}^n,\, |\epsilon|=k} S_{J-\epsilon}(A) \ ,$$

because one can restrict the preceding sum to terms such that $J - \epsilon$ is a partition, the other terms are 0 (having two identical columns). $\square$

For example, the action of T_2^- on Schur functions of order 3 reads :

$$\begin{vmatrix} \cdot & \cdot & \cdot \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -1 & \cdot \\ -1 & -1 & \cdot \\ -1 & -1 & \cdot \end{vmatrix} + \begin{vmatrix} -1 & \cdot & -1 \\ -1 & \cdot & -1 \\ -1 & \cdot & -1 \end{vmatrix} + \begin{vmatrix} \cdot & -1 & -1 \\ \cdot & -1 & -1 \\ \cdot & -1 & -1 \end{vmatrix} \ .$$

The operators $T_{\Psi_K}^+$ have been in particular considered by Muir, who obtained the following corollary, that one can check directly on minors of the Vandermonde matrix (and that we also give for $T_{\Psi_K}^-$ at the same time) :

COROLLARY 1.8.4. *Let K, J be two partitions in $\mathbb{N}^n$ and $\mathbb{A}$ be of cardinality n. Then*

$$\Psi_K(\mathbb{A})\, S_J(\mathbb{A}) \;=\; \sum_{H=perm(K)} S_{J+H}(\mathbb{A}) \ , \tag{1.8.6}$$

$$D_{\Psi_K}\big(S_J(\mathbb{A})\big) \;=\; \sum_{H=perm(K)} S_{J-H}(\mathbb{A}) \ , \tag{1.8.7}$$

sum over all different permutations H of K.

For example, for $\mathbb{A} = \{a, b, c\}$, the product of $\Delta(\mathbb{A})S_{125}(\mathbb{A})$ by $\Psi_{22}(\mathbb{A})$ is :

$$\begin{array}{c} a^2 \\ b^2 \\ \end{array} \begin{vmatrix} a^1 & a^3 & a^7 \\ b^1 & b^3 & b^7 \\ c^1 & c^3 & c^7 \end{vmatrix} + \begin{array}{c} a^2 \\ \\ c^2 \end{array} \begin{vmatrix} a^1 & a^3 & a^7 \\ b^1 & b^3 & b^7 \\ c^1 & c^3 & c^7 \end{vmatrix} + \begin{array}{c} \\ b^2 \\ c^2 \end{array} \begin{vmatrix} a^1 & a^3 & a^7 \\ b^1 & b^3 & b^7 \\ c^1 & c^3 & c^7 \end{vmatrix}$$

$$= \begin{vmatrix} a^{1+2} & a^{3+2} & a^7 \\ b^{1+2} & b^{3+2} & b^7 \\ c^{1+2} & c^{3+2} & c^7 \end{vmatrix} + \begin{vmatrix} a^{1+2} & a^3 & a^{7+2} \\ b^{1+2} & b^3 & b^{7+2} \\ c^{1+2} & c^3 & c^{7+2} \end{vmatrix} + \begin{vmatrix} a^1 & a^{3+2} & a^{7+2} \\ b^1 & b^{3+2} & b^{7+2} \\ c^1 & c^{3+2} & c^{7+2} \end{vmatrix} .$$

As in the case of elementary functions seen above, if there exists ℓ such that $K \subseteq \square := \ell^n \subseteq J$, then

$$T^-_{\Psi_K}(\mathbb{V}_J(\mathbb{A})) = T^-_{\Psi_{\square-K}}(\mathbb{V}_{J-\square}(\mathbb{A})) = \frac{\Psi_{\square-K}(\mathbb{A})}{\Psi_\square(\mathbb{A})}\, V_J(\mathbb{A}) = \Psi_K(\mathbb{A}^\vee)\, V_J(\mathbb{A}) \ ,$$

and in that case $T^-_{\Psi_K}$ is also realized by a multiplication.

The special case of Muir's rule for $K = [k, 0^{n-1}]$ is called *Murnaghan-Nakayama rule.* The action of D_{Ψ_k} on a Schur function S_J consists in subtracting in all possible manners k to a part of J. To get Schur functions indexed by partitions, one must reorder the columns of the determinant $S_{J+[0\cdots 0\, k\, 0\cdots 0]}$. This correspond to subtracting in all possible manners a connected ribbon of length k to the diagram of J, taking as a sign $(-1)^{h-1}$, h being the height of the ribbon (counting 1 for an horizontal ribbon). This rule is iterated to compute values of irreducible characters of symmetric groups.

Conversely, multiplication by $\Psi_k(\mathbb{A})$ is realized by adding connected ribbons of length k to the diagram of J.

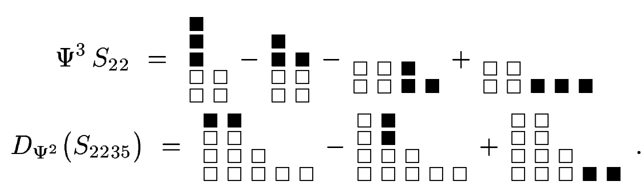
COROLLARY 1.8.5. *(Murnaghan-Nakayama). Let k be an integer, J be a partition. Then*

$$\Psi^k(\mathbb{A})\, S_J(\mathbb{A}) \;=\; \sum\nolimits_H (-1)^{h-1} S_H(\mathbb{A}) \tag{1.8.8}$$

$$D_{\Psi^k}\big(S_J(\mathbb{A})\big) \;=\; \sum\nolimits_H (-1)^{h-1} S_H(\mathbb{A}) \tag{1.8.9}$$

sum over all partitions H such that H/J (resp. J/H) is a connected ribbon of length k, h being its height.

For example, the black boxes being added in the first case, and erased in the second,



Formula (1.8.6) can be used to express a monomial function into the basis of Schur functions :

$$\Psi_K = \Psi_K \, S_{0^{|K|}} = \sum_{H=perm(K)} S_H \ . \tag{1.8.10}$$

For example, writing only the non-zero terms,

$$\Psi_{22} = S_{000022} + S_{000202} + S_{002020} = S_{22} - S_{112} + S_{1111} \ .$$

We have seen that ${}^rT_k^{\pm}(S_J(\mathbb{A}))$ is restricted to a single non-zero determinant. This property is in fact true for any Schur function :

LEMMA 1.8.6. *Let $\mathbb{A}$ be arbitrary, and K, L be two partitions in $\mathbb{N}^n$. Then*

$$ {}^rT_{S_L}^{+}\Big(S_K(\mathbb{A})\Big) \;=\; det\Big(S_{k_j+j-i+\ell_{n+1-i}}(\mathbb{A})\Big) \ , \tag{1.8.11}$$

$$ {}^rT_{S_L}^{-}\Big(S_K(\mathbb{A})\Big) \;=\; det\Big(S_{k_j+j-i-\ell_i}(\mathbb{A})\Big) = S_{K/L}(\mathbb{A}) \ . \tag{1.8.12}$$

Proof. The terms in the action of ${}^rT_{S_L}^{+}$ on S_K are in bijection with the terms in the action of ${}^cT_{S_L}^{+}$ on the Vandermonde matrix $\mathbb{V}_{0^n}(\mathbb{B})$, where $\mathbb{B}$ is of cardinality n. Two terms equal, up to a sign, in the second sum, are also equal, up to a sign, in the first sum. Since the second sum reduces to a single term, the first one also does. One can pass from the first statement to the second one, by taking complementary partitions. $\square$

For example, because $S_{12} = \Psi_{12} + 2\Psi_{111}$, the action of ${}^rT_{S_{12}}^{+}$ on a determinant of order 4 will be, writing the shifts in successive rows as a vector, and eliminating the null determinants :

$$[2100] + [1020] + [0210] + 2[1110] \ .$$

The terms [1020] and [0210] are each obtained from [1110] by a transposition of rows, and therefore the sum reduces to the single term [2100].

Notice that if $\mathbb{A}$ is of cardinality equal to n, then the action of $T_{S_L}^{+}$ is "multiplication by $S_L(\mathbb{A})$". If, on the contrary, $\mathbb{A}$ is of cardinality bigger than n, one still has, for $K, L \in \mathbb{N}^n$

$$ {}^rT_{S_L}^{+}\Big(S_K(\mathbb{A})\Big) = {}^cT_{S_L}^{+}\Big(S_K(\mathbb{A})\Big) = \sum_{H\in\mathbb{N}^n} (S_K S_L \, , \, S_H)\, S_H(\mathbb{A}) \ ,$$

but $T_{S_L}^{+}(S_K(\mathbb{A}))$ is not equal to $S_L(\mathbb{A})\, S_K(\mathbb{A})$.

For example,

$$ {}^rT_1^{+}(S_{25}(\mathbb{A})) = \begin{bmatrix} S_3 & S_7 \\ S_1 & S_5 \end{bmatrix} = S_{36/01}(\mathbb{A}) = S_{35}(\mathbb{A}) + S_{26}(\mathbb{A}) = S_1(\mathbb{A})S_{25}(\mathbb{A}) - S_{125}(\mathbb{A}) \ .$$

The T_k operators can be applied to other determinants than Jacobi-Trudi determinants, or Vandermondes. For example, let $f(x)$ be the function

$$f(x) = 1/x! \text{ if } x \in \mathbb{N} \text{ and } 0 \text{ otherwise}$$

To a vector $x = [x_1, \ldots, x_n]$, associate the matrix

$$D(x) := [f(x_j + j - i)] \ .$$

Then

$$
{}^{r}T_2^{+}\big(D([2,4,4,6])\big) = \begin{vmatrix} \frac{1}{3!} & \frac{1}{6!} & \frac{1}{7!} & \frac{1}{10!} \\ \frac{1}{2!} & \frac{1}{5!} & \frac{1}{6!} & \frac{1}{9!} \\ \frac{1}{0!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{8!} \\ 0 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{7!} \end{vmatrix} = {}^{c}T_2^{+}\big(D([2,4,4,6])\big) =
$$
$$
= D([3,4,5,6]) + D([3,4,4,7]) + D([2,5,5,6]) + D([2,4,5,7]) \ ,
$$

the zero determinants like $D([3,5,4,6])$ having not been written.

In that case, the operator T_2^+ does not correspond to a multiplication. In fact $D(x)$ is the dimension of the irreducible representation of index x of the symmetric group, and the equality of dimension can be obtained by specialization of the Schur function

$$
S_{3557/0011} = S_{3456} + S_{3447} + S_{2556} + S_{2457} \ .
$$

We leave it to the reader to use the T operators to show that, for $\mathbb{A}$ of cardinality n, and two partitions $K = [k_1, \ldots, k_n]$, $L = [\ell_1, \ldots, \ell_n]$, one has

$$
(1.8.13) \qquad \det\Big(\Psi_{k_i+\ell_j+i+j-2}(\mathbb{A}) \Big)_{1\le i,j\le n} = S_K(\mathbb{A})\, S_L(\mathbb{A}) \ \det\Big(\Psi_{i+j}(\mathbb{A}) \Big) \ .
$$

For example, for $n = 3$, one has

$$
\begin{vmatrix} \Psi_1 & \Psi_2 & \Psi_5 \\ \Psi_4 & \Psi_5 & \Psi_8 \\ \Psi_7 & \Psi_8 & \Psi_{11} \end{vmatrix} = S_{023}(\mathbb{A})\, S_{113}(\mathbb{A}) \begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix} \ .
$$

1.9. Pieri Formulas

We are mostly using the linear basis of Schur functions. Still, $\mathfrak{Sym}$ is a ring, and to recover its multiplicative structure, one needs to describe the product of two Schur functions. This is given by the so-called *Littlewood-Richardson Rule*, which is better stated, and proved, in terms of non-commutative symmetric functions (cf. the chapter about the Plactic Algebra). Since products Λ^I or S^I also constitute linear bases, we shall content ourselves, for the moment, to describe the product of general Schur functions by a complete or elementary symmetric function to determine the multiplicative structure. The products $S^r S_J$ and $\Lambda^r S_J$, $r \in \mathbb{N}$, J partition, were in fact determined by the Italian geometer Pieri in 1873. They have the remarkable property of being multiplicity free.

In fact, since an elementary symmetric function is a monomial function, we already know, from (1.8.5) how it acts by multiplication on Schur functions. We shall however reinterpret this multiplication.

First, let us introduce two notations for family of partitions deduced from a given one $I = [i_1, \ldots, i_n]$. The symbols $\{I \otimes k\}$ and $\{I \otimes 1^k\}$ respectively denote all the partitions J of weight $|J| = |I| + k$ such that

$$
(1.9.1) \quad \{I \otimes k\} \ := \ \{J = [j_0, \ldots, j_n], 0 \le j_0 \le i_1 \le j_1 \le \cdots \le i_n \le j_n\},
$$
$$
(1.9.2) \quad \{I \otimes 1^k\} \ := \ \{J : J^\sim \in \{I^\sim \otimes k\}\} \ ,
$$

that is, the families of partitions obtained from I by adding horizontal (resp. vertical) strips of k boxes.

The two above sets of partitions satisfy the property of being distributive sublattices of the lattice of partitions of a given weight. To show it, we first must code differently the partitions $J \in \{I \otimes k\}$. Let us write $J^\diamond := J - I$ (as vectors). Instead of J, we use the cumulative sum $\overline{\overline{J^\diamond}}$.

For example, for $I = [0,1,3,4]$, $k = 2$, we shall have the following lattice (with three labellings) :

$$
\begin{tikzcd}[column sep=tiny, row sep=tiny]
& & 1234 \arrow[dl] \arrow[dr] & \\
& 1144 \arrow[dl] \arrow[dr] & & 0334 \arrow[dl] \\
1135 \arrow[dr] & & 0244 \arrow[dl] & \\
& 0235 \arrow[d] & & \\
& 0145 \arrow[d] & J & \\
& 0136 & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny, row sep=tiny]
& & 1100 \arrow[dl] \arrow[dr] & \\
& 1010 \arrow[dl] \arrow[dr] & & 0200 \arrow[dl] \\
1001 \arrow[dr] & & 0110 \arrow[dl] & \\
& 0101 \arrow[d] & & \\
& 0011 \arrow[d] & J^\diamond & \\
& 0002 & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny, row sep=tiny]
& & 1222 \arrow[dl] \arrow[dr] & \\
& 1122 \arrow[dl] \arrow[dr] & & 0222 \arrow[dl] \\
1112 \arrow[dr] & & 0122 \arrow[dl] & \\
& 0112 \arrow[d] & & \\
& 0012 \arrow[d] & \overline{\overline{J^\diamond}} & \\
& 0002 & &
\end{tikzcd}
$$

LEMMA 1.9.1. *Given a partition I and $k \in \mathbb{N}$, then $\{I \otimes k\}$ is a distributive sublattice of the lattice of partitions, with minimal element $I + [0 \cdots 0k]$ and maximal element $I \cup k$.*

Proof. The lattice structure is seen on the vectors $\overline{\overline{J^\diamond}}$ (notice that $\overline{\overline{J^\diamond}} = \overline{\overline{J}} - \overline{\overline{I}}$). The condition that J belongs to $\{I \otimes k\}$ is equivalent to the fact that $J^\diamond$ is integral vector, with $|J^\diamond| = k$, componentwise majorized by $[i_1 - 0,\, i_2 - i_1,\, i_3 - i_2,\, \ldots,\, i_\ell - i_{\ell-1},\, \infty]$. □

More properties of the Pieri lattice can be found in the work of Kim [**80**].

LEMMA 1.9.2. *Let k be a positive integer, I be a partition. Then one has the following decompositions :*

$$S^k\, S_I \;=\; \sum\nolimits_{J \in \{I \otimes k\}} S_J \;, \tag{1.9.3}$$

$$\Lambda^k\, S_I \;=\; \sum\nolimits_{J \in \{I \otimes 1^k\}} S_J \;. \tag{1.9.4}$$

Replacing I by $[0^k, I]$, the last sum can be written $\sum_H S_H$, sum over all $H \in \mathbb{N}^{n+k}$ such that $h_1 - i_1, \ldots, h_{n+k} - i_{n+k}$ are 0 or 1, and $|H| = |I| + k$, because the extra terms such that H is not a partition index null determinants. Now (1.8.5) states that this last sum is equal to the product $\Lambda^k\, S_I$.

The involution $\mathbb{A} \mapsto -\mathbb{A}$ exchanges the two Pieri formulas (and graphically, conjugation exchanges horizontal strips with vertical ones). □

Pieri rules also occur in the non commutative world, as shows the following proposition.

PROPOSITION 1.9.3. *Let $\mathcal{R}$ be a ring, not necessarily commutative, containing a family of elements s_J, J = all partitions, satisfying (1.9.3) for all r, J, putting $S^r = s_r$. Then the $\mathbb{Z}$-module generated by the s_J is a subring, which is a quotient of the ring of symmetric polynomials with coefficient in $\mathbb{Z}$.*

The proof follows from the fact that the leading term (for the dominance order) in a product $S_r S_J$ is $S_{(r,J)}$, with (r, J) := increasing reordering of $[r, j_1, j_2, \ldots]$. By recursion, it allows to obtain all Schur functions, starting from the S_r's; it moreover proves that they commute and that they can be expressed as determinants in the S_r's .

In the commutative case, this proposition is due to Giambelli [**59**], who used it to characterize the cohomology ring of a Grassmann manifold as a quotient of $\mathfrak{Sym}$.

Schensted algorithm satisfy the non commutative product identity $S_2 S_J = \sum S_K$, and this imply the non commutative Pieri rules[4]. Thus, starting from the geometrical problem of intersecting Schubert cycles, one is led to the non commutative world as the simplest way to describe intersections in the cohomology ring of a Grassmann manifold.

Exercises

Ex. 1.1. Let $\mathbb{A}$ be arbitrary, k be a positive integer. Expand the product $S^k(1 - z\mathbb{A})\, \sigma_z(\mathbb{A})$ in the basis of Schur functions.

Ex. 1.2. Let $I \in \mathbb{N}^m$, $J \in \mathbb{N}^n$. Given $H \in \mathbb{N}^n$, denote $H^\omega + J := (j_1 + h_n, \ldots, j_n + h_1)$. Check that

$$S_{I;J}(\mathbb{A}; \mathbb{B}) = \sum_H (-1)^{|H|} S_{I/H^\sim}(\mathbb{A})\, S_{J+H^\omega}(\mathbb{B}) \ ,$$

sum over all partitions $H \in \mathbb{N}^n$, $H \subseteq m^n$.

In particular, for $n = 1$, one has

$$S_{I,j}(\mathbb{A}, \mathbb{B}) = \sum_{0 \le h \le m} (-1)^h S_{I/1^h}(\mathbb{A})\, S_{j+h}(\mathbb{B}) \ .$$

Ex. 1.3. Let $I = [i_1, \ldots, i_r]$ be a partition, n an integer : $n \ge i_r$. Check that

$$S_{I;0^n}(\mathbb{A}, \mathbb{B}) = S_I(\mathbb{A} - \mathbb{B}) \ .$$

Ex. 1.4. Let $\mathbb{A}$ be arbitrary. Define $\mathbb{B}$ by the equations $\Lambda^j(\mathbb{B}) = j\Lambda^j(\mathbb{A})$, $j \ge 1$. Compute $S^k(\mathbb{A} - \mathbb{B})$, any k, in the bases of Schur functions, power sums, monomial functions, of $\mathbb{A}$.

Ex. 1.5. Let $F(z) = \sigma_z(\mathbb{A})$, $\Phi(z) = F'(z)/F(z)$. Zhi Guo Liu[**128**] gives $S^1(\mathbb{A}), \ldots, S^6(\mathbb{A})$ in terms of the value in $z = 0$ of the derivatives of $\Phi(z)$ $\big($he interprets $S^n(\mathbb{A})$ as the residue in $z = 0$ of $z^{-n-1}\sigma_z(\mathbb{A})\big)$. Go further and give the expression of $S^n(\mathbb{A})$ for any n.

Ex. 1.6. Let n be an integer. Build a $n \times n$ matrix with a first column of indeterminates y_i, and entries $[i, i] = x$, entries $[i - 1, i] = -1$, for $i = 2, \ldots, n$, all other entries being 0, and compute its determinant without expanding it. For example, for $n = 4$, the matrix is $\begin{vmatrix} y_1 & -1 & 0 & 0 \\ y_2 & x & -1 & 0 \\ y_3 & 0 & x & -1 \\ y_4 & 0 & 0 & x \end{vmatrix}$.

Ex. 1.7. Let m, n be two integers. Compute, after Composto (1916; MuirV, p.349), the value of the determinant

$$\left| \binom{m + i + j - 2}{j} \right|_{1 \le i,j \le n} \ .$$

Ex. 1.8. Express $S_{J;r}(\mathbb{A}; x)$ as a determinant of smaller order when $r < 0$ (when $r = 0$, we have expressed it as $S_J(\mathbb{A} - x)$ in(1.4.8)).

[4] In the non commutative case, because multiplication is an associative operation, one needs only to describe products $xy\,t$, x, y = letters, t = tableau, to get all products of tableaux by increasing or decreasing words.

Ex. 1.9. Let $\mathbb{A}$, $\mathbb{B}$ be arbitrary, and let k be a positive integer. Show that

$$(-1)^{k+2} S_{1^k;2}(\mathbb{A};\mathbb{A}-\mathbb{B}) = S_{1;k+1}(\mathbb{B}-\mathbb{A};-\mathbb{A}) + S_{k+2}(-\mathbb{B}) \ .$$

Ex. 1.10. Given two finite alphabets $\mathbb{A},\mathbb{B}$, and a partition I, write $S_I(\mathbb{A}+\mathbb{B})\prod_{a\in\mathbb{A},b\in\mathbb{B}}(a-b)$ as a multi-Schur function.

Ex. 1.11. Let $\mathbb{A}=\{a_1,a_2,\ldots\}$ be infinite, $\mathbb{A}_n=\{a_1,\ldots,a_n\}$, $n\in\mathbb{N}$ be its initial subalphabets. Denote by T the shift operator $a_i\to a_{i+1}$. Given any $n,r\geq 1$, $K=[k_1,\ldots,k_n]\in\mathbb{N}^n$, denote by $D(K,r)$ the determinant $\left|S^{k_j}(T^{i-1}\mathbb{A}_r)\right|_{1\leq i,j\leq n}$.

After A. Regev [**151**], evaluate $D(K,r)$.

Ex. 1.12. Using the factorization property (1.4.3), compute the Schur functions $S_J(\mathbb{A}-\mathbb{B})$ when $\mathbb{A}$ and $\mathbb{B}$ are of cardinality 1 or 2.

Ex. 1.13. Let I,J be partitions such that the diagram of J/I decomposes into disjoint blocks K_1/H_1, K_2/H_2, $\ldots$, K_ℓ/H_ℓ. Show that, for any $\mathbb{A}$, $S_{J/I}(\mathbb{A})$ factorizes into

$$S_{J/I}(\mathbb{A}) = S_{K_1/H_1}(\mathbb{A})\cdots S_{K_\ell/H_\ell}(\mathbb{A}) \ .$$

Ex. 1.14. Let $\mathbb{A}$ be of cardinality n, and $I,J\in\mathbb{N}^n$. Show that the determinant

$$\left|S_{j_k+k-h+i_{n-h+1}}(\mathbb{A})\right|$$

factorizes into $S_J(\mathbb{A})\,S_I(\mathbb{A})$. What can be said if n is not the cardinality of $\mathbb{A}$?

For example, for $n=3$, $I=[1,3,3]$, $J=[2,4,6]$, one has

$$\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 3 \\ -2 & -1 & 0 & 1 \\ \hline 2 & 4 & 6 & \end{array} = \begin{vmatrix} S_5 & S_8 & S_{11} \\ S_4 & S_7 & S_{10} \\ S_1 & S_4 & S_7 \end{vmatrix} = S_{133}(\mathbb{A})\,S_{246}(\mathbb{A}) \ .$$

hint: Use the expression in terms of Λ^i.

Ex. 1.15. Compute the adjoint matrix of a Jacobi-Trudi matrix.

For example, for the partition 145, one has the following matrix and its adjoint :

$$\begin{bmatrix} S_1 & S_5 & S_7 \\ 1 & S_4 & S_6 \\ 0 & S_3 & S_5 \end{bmatrix} \ , \ \begin{bmatrix} S_{45} & -S_{55}-S_{46} & S_{56} \\ -S_5 & S_{15}+S_6 & -S_{16} \\ S_3 & -S_{13}-S_4 & S_{14} \end{bmatrix} \ .$$

Ex. 1.16. Recall that $\mathbb{S}_{123\ldots}$ is the submatrix of the Toeplitz matrix $\left[S^{j-i}\right]$ taken on columns $1,3,5,\ldots$ and rows $0,1,2,\ldots$. Find the lower triangular matrix N with unit diagonal such that $M=N\,\mathbb{S}_{12\ldots n}$ be upper triangular.

Ex. 1.17. Let $\mathbb{A}$ be infinite, $\mathbb{A}_n$, $n\geq 0$, be its initial sub-alphabets: $\mathbb{A}_n := \{a_1,\ldots,a_n\}$.

Compute the inverse of the matrix $\left[S_{j-i}(\mathbb{A}_i)\right]_{1\leq i,j\leq n}$.

Ex. 1.18. Let $f_1(x),\ldots,f_n(x)$ be n functions of x, $W=\left|\frac{d^i}{dx^i}f_j(x)\right|_{0\leq i\leq n-1,\,1\leq j\leq n}$ be the Wrońskian in these functions. Compute the derivatives of any order of W.

The reader may have advantage to denote a "shifted Wrońskian" $\left|\frac{d^a}{dx^a}f_j(x)\right|_{a\in\mathbb{A}}$ by the vector $[a_1-0,\ a_2-1,\ldots,a_n-n+1]$, so that W is the origin $[0,\ldots,0]$.

Ex. 1.19. Let $\mathbb{A}$ be an alphabet of cardinality n. Describe an algorithm to express the $S^k(\mathbb{A})$ and $\Psi_k(\mathbb{A})$, $k > n$, in terms of respectively $S^1(\mathbb{A}), \ldots, S^n(\mathbb{A})$, $\Psi_1(\mathbb{A}), \ldots, \Psi_n(\mathbb{A})$.

Ex. 1.20. Let r be a positive integer, $S_n^{(r)} = \sum_{I:|I|=n,\ell(I)=r} \Psi_I$. Show, after MacMahon [**136**] (vol.1, formula 57 p.42), that these functions satisfy a Newton-type recursion:

$$S_n^{(r)} - \Lambda^1 S_{n-1}^{(r)} + \cdots + (-1)^{n-1}\Lambda^{n-1} S_1^{(r)} = (-1)^n \binom{n}{r} \Lambda^n$$

(Newton's formula is for $r = 1$)

Ex. 1.21. (Ribbon functions). Check that the Jacobi-Trudi matrix of a ribbon of height r is built from its diagonal $S_{c_1}, \ldots, S_{c_r}$ as follows: the diagonal below the main diagonal is filled with S_0's. The entries below are 0. Each entry $[i,j]$, $j \geq i$, is equal to $S_{c_i+\cdots+c_{j-1}}$.

Show that the product of two ribbon Schur functions is equal to the sum of two ribbon Schur functions.

Writing $\mathcal{R}_k$ for the ribbon $S_{12\ldots k/12\ldots k-2}$, show that

$$S_{123456} = \begin{vmatrix} \mathcal{R}_2 & \mathcal{R}_3 & \mathcal{R}_4 \\ \mathcal{R}_3 & \mathcal{R}_4 & \mathcal{R}_5 \\ \mathcal{R}_4 & \mathcal{R}_5 & \mathcal{R}_6 \end{vmatrix} ,$$

using the preceding property ($\mathcal{R}_0 = S_0$, $\mathcal{R}_1 = S_1$, $\mathcal{R}_2 = S_{12}$, $\mathcal{R}_3 = S_{123/1}, \ldots$).

When $\mathbb{A} = \{a_1, \ldots, a_n\}$ is finite, show that if a ribbon $\mathcal{R}$ has a column of height $> n$, then $S_{\mathcal{R}}(\mathbb{A})$ vanishes; if it has a column of height n, then $S_{\mathcal{R}}(\mathbb{A})$ factorizes into the product of two ribbons multiplied by $a_1 \cdots a_n$.

Ex. 1.22. Let $\mathcal{Z} = z\Psi^1 - z^3\Psi^3/3 + z^5\Psi^5/5 - \cdots$. Show that

$$\tan(\mathcal{Z}) = \frac{z\Lambda^1 - z^3\Lambda^3 + z^5\Lambda^5 - \cdots}{1 - z^2\Lambda^2 + z^4\Lambda^4 + \cdots} = zS_1 + z^3 S_{12} + z^5 S_{123/1} + \cdots + z^{2n-1} S_{1\ldots n/1\ldots n-2} + \cdots$$

Deduce from the above, after Cauchy [**29**], Laguerre [**92**] and many other mathematicians, that the elements of $\mathfrak{Sym}(\mathbb{A})$, when $\mathbb{A}$ is of cardinality n, are rational functions of

$$\Psi^1(\mathbb{A}), \Psi^3(\mathbb{A}), \ldots, \Psi^{2n-1}(\mathbb{A}) .$$

Beware that it is not true that elements of $\mathfrak{Sym}(\mathbb{A})$ can be expressed as rational functions of $S^1(\mathbb{A}), S^3(\mathbb{A}), \ldots, S^{2n-1}(\mathbb{A})$, for $n \geq 3$. The rationality property is specific to the basis of power sums.

Ex. 1.23. Knowing that the cotangent function satisfies the addition formula

$$\cot(a+b) = \frac{\cot(a)\cot(b) - 1}{\cot(a) + \cot(b)} ,$$

give the expansion of $\cot(a_1 + \cdots + a_n)$, $n \in \mathbb{N}$.

Ex. 1.24. Defining the $\mathbb{A}$-cotangent $\cot(z;\mathbb{A})$ to be $\cot\bigl(z\Psi^1 - z^3\Psi^3/3 + z^5\Psi^5/5 - \cdots\bigr)$, give the Taylor expansion of $z\cot(z;\mathbb{A})$ around $z = 0$.

Ex. 1.25. Let $\mathbb{A}$ be of cardinality n, $\overline{\mathbb{A}} := \{(-a) : a \in \mathbb{A}\}$ (i.e. $\overline{\mathbb{A}}$ is the alphabet obtained from $\mathbb{A}$ by changing each letter into its negative). In other words, $\overline{\mathbb{A}}$ is defined by the equations $S^i(\overline{\mathbb{A}}) = (-1)^i S^i(\mathbb{A})$, $i \geq 1$.

Compute the Schur functions $S_{j,n^n}\left(\mathbb{A}-\overline{\mathbb{A}}\right)$, $0 \le j \le n$, and, after Laguerre [**92**], deduce from it that the elementary symmetric functions of $\mathbb{A}$ are rational functions of $\Psi_1(\mathbb{A}), \Psi_3(\mathbb{A}), \ldots, \Psi_{2n-1}(\mathbb{A})$, with denominator $S_{1,\ldots,n-1}(\mathbb{A})\, S_{1,\ldots,n}(\mathbb{A})$.

To show that the denominator can be reduced to $S_{1,\ldots,n-1}(\mathbb{A})$, see next exercise, or Polya [**147**].

Ex. 1.26. Let $\mathbb{A}$ be of cardinality n. Show, after Foulkes [**46**], that the explicit expression of $\Lambda^{n-k}(\mathbb{A})$, $0 \le k \le n$ as a rational function in the odd power sums is

$$\Lambda^{n-k}(\mathbb{A}) = S_{1\ldots n/1^k}(\mathbb{A})/S_{1\ldots n-1}(\mathbb{A}) \ ,$$

the two Schur functions being expressed in terms of power sums.

Ex. 1.27. Let $\mathbb{A}$ be any alphabet, m, n be two integers. Check that the adjoint matrix of $\mathbb{S}_{m^n}(\mathbb{A})$ is the matrix of the quadratic form $Q(x,y) := S_{(m+1)^{n-1}}(\mathbb{A}-x-y)$.

Ex. 1.28. Define a vertex operator on $\mathfrak{Sym}$ as a formal series in z by

$$\nabla := \exp\Big(\sum_{n\ge 1} z^n\, \Psi_n/n\Big)\exp\Big(\sum_{n\ge 1} z^{-n}\, D_{\Psi_n}/n\Big)$$

and expand it according to powers of z:

$$\nabla = \sum_{-\infty}^{+\infty} z^n\, \nabla_n \ .$$

For any $\ell \in \mathbb{N}$, any $v = [v_1, \ldots, v_\ell] \in \mathbb{Z}^\ell$, show that $\nabla_{v_\ell} \cdots \nabla_{v_1}(1)$ is equal to the Schur function of index v.

See Jing [**75**] for more general operators "creating" Hall-Littlewood polynomials.

Ex. 1.29. Let $I \in \mathbb{N}^n$, $\mathbb{A}$ be finite. Show that $\sum_0^\infty S_{I,j}(\mathbb{A})x^j$ is a rational function with denominator $\lambda_{-x}(\mathbb{A})$. Compute it.

Ex. 1.30. Let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be three alphabets, k a positive integer. Express the following determinant of order 10 : $\begin{array}{|c|c|}\hline M & N \\ \hline P & Q \\ \hline\end{array}$ as a Schur function, where $Q = 0$,

$$M = \begin{bmatrix} S_k(\mathbb{C}) & \cdots & S_{k+7}(\mathbb{C}) \\ \vdots & & \vdots \\ S_{k-6}(\mathbb{C}) & \cdots & S_{k+1}(\mathbb{C}) \end{bmatrix},\ N = \begin{bmatrix} S_5(\mathbb{A}) & S_6(\mathbb{A}) \\ \vdots & \vdots \\ S_{-1}(\mathbb{A}) & S_0(\mathbb{A}) \end{bmatrix},\ P = \begin{bmatrix} S_0(\mathbb{B}) & \cdots & S_7(\mathbb{B}) \\ S_{-1}(\mathbb{B}) & \cdots & S_6(\mathbb{B}) \\ S_{-2}(\mathbb{B}) & \cdots & S_5(\mathbb{B}) \end{bmatrix}$$

Generalize to any order.

Ex. 1.31. Let $\mathbb{A}$ be finite and $\mathbb{B}$ arbitrary. Following Mattia (MuirV, p.204), show that one can replace, in the Vandermonde $\Delta(\mathbb{A})$ of $\mathbb{A}$, each entry a^j by $S^j(\mathbb{A}+\mathbb{B})$ without changing the value of $\Delta(\mathbb{A})$. Show that it implies that one can replace each a^j by the complete function $\Lambda^j(\mathbb{A}-a)$.

Ex. 1.32. Let $\mathbb{A}$ be of cardinality n, and let $x_1, \ldots, x_n$ be n indeterminates. Compute the determinant

$$\det\Big(\sum_k \Lambda^{n+j-2-2k}(\mathbb{A})x_i^{2k}\Big)_{1\le i,j\le n} \ .$$

Ex. 1.33. Let n be a positive integer and $\mathbb{A} = \{1, \dots, n\}$. Show, after Theisinger (1915; MuirV p. 205) that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \frac{1}{1! \cdots n!} \det(a^0, a^2, \dots, a^n)_{a \in \mathbb{A}} .$$

Ex. 1.34. Let n be an integer. Evaluate

$$\frac{1}{S_1} + \frac{S_2}{S_1 S_{11}} + \frac{S_{22}}{S_{11} S_{111}} + \cdots + \frac{S_{2^n}}{S_{1^n} S_{1^{n+1}}} .$$

Ex. 1.35. Use Muir's formula (1.8.6) to expand the monomial functions $\Psi_{1^\alpha 2^\beta}$, $\alpha, \beta \in \mathbb{N}$, in the basis of Schur functions.

For example,

$$\Psi_{1222} = S_{1222} - 2S_{11122} + 3S_{1^5 2} - 4S_{1^7} .$$

Ex. 1.36. Given two positive integers k, n, show that Ψ_{k^n} expands as a sum of Schur functions with coefficients ± 1.

Ex. 1.37. Using shift operators on Jacobi-Trudi determinants, reprove Pieri formulas for multiplication by Λ^k, by S^2 and by S^3.

Ex. 1.38. Show that the set of partitions $\{[0038] \otimes 5\}$ is equal to the disjoint union

$$\{[0035] \otimes 8\} \cup \{[0068] \otimes 2\} .$$

Relate this property to the vanishing of the determinant $\begin{vmatrix} S_3 & S_6 & S_9 \\ S_2 & S_5 & S_8 \\ S_2 & S_5 & S_8 \end{vmatrix}$. Give an analogous statement, when taking $\begin{vmatrix} S_3 & S_6 & S_9 \\ S_2 & S_5 & S_8 \\ S_1 & S_4 & S_7 \end{vmatrix}$ instead.

Ex. 1.39. For any $k \in \mathbb{N}$, let

$$\begin{aligned} \Phi_k &= \Psi_{1^k 2} + \Psi_{1^{k-2} 22} + \Psi_{1^{k-4} 222} + \cdots \\ P_k &= \Phi_0 S^{k-2} - \Phi_1 S^{k-3} + \Phi_2 S^{k-4} - \cdots \\ Q_k &= \sum \{\Psi_I : |I| = k,\, 1 \notin I\} \end{aligned}$$

Show that $P_k = Q_k$ for all k by using the Hammond's operators D_{S^i}.

Ex. 1.40. Let $f, g \in \mathfrak{Sym}$, $K \in \mathbb{N}^n$. Show that the following identity on scalar products :

$$(fg\,,\, \Psi^K) = \sum\nolimits_{I,J:\, I+J=K} (f\,,\, \Psi^I)\ (g,\,,\, \Psi^J)\ \binom{k_1}{i_1}\binom{k_2}{i_2} \cdots ,$$

sum over all $I = [i_1, \dots, i_n]$, $J = [j_1, \dots, j_n]$ in $\mathbb{N}^n$ such that $I + J = K$.

Ex. 1.41. For any $i \in \mathbb{N}$, let ξ_i be the operator on $\mathfrak{Sym}$ defined by $\Psi_J \to \Psi_{i,J}$, for all partitions J (reordering i, J). Decompose the identity operator as

$$1 = \xi_1(\Omega_1) + \xi_2(\Omega_2) + \xi_3(\Omega_3) + \cdots ,$$

with some operators $\Omega_1, \Omega_2, \dots$ to determine.

Ex. 1.42. Compute the inverse of a Hankel matrix $[c_{i+j}]_{0 \le i,j \le n}$.

EX. 1.43. Let M be a matrix of order n, $\mathbb{A}$ be the (multi)-set of its eigenvalues. Check that

$$\lambda_z(\mathbb{A}) = \det(1+zM) \quad \& \quad \sigma_z(\mathbb{A}) = \det(1-zM)^{-1} \ ,$$
$$\Psi_k(\mathbb{A}) = trace(M^k) \ , \ k \geq 0 \ .$$

EX. 1.44. (*Faddeev's algorithm for computing the adjoint*) Let M be a matrix of order n, $\mathbb{A}$ its eigenvalues. Let $B_0 = \mathbf{1}$ be the identity matrix of order n. Define recursively

$$c_i := \frac{1}{i} tr(B_{i-1}M) \quad \& \quad B_i = c_i\,\mathbf{1} - B_{i-1}M \ , \ i \geq 1$$

Show that $c_i = \Lambda^i(\mathbb{A})$, that $B_n = 0$ and that B_{n-1} is the adjoint of M.

EX. 1.45. Let M be an $n \times n$ matrix having (multi)-set of eigenvalues $\mathbb{A}$. Show that

$$\sum_{i=1}^{n} (-1)^{2i} S_{2,\ldots,i,\,i+3,\ldots,n+2}(\mathbb{A}) = 0 \ .$$

EX. 1.46. (*Regularized determinant*). Let M be a square matrix. The formal power series in y, $\det(1+yM)$, involves the trace of M. To get rid of it, one defines

$$det_2(1+yM) := \det\left((1+M)\exp(-M)\right) = \det(1+M)\ \exp(-trace(M)) =$$
$$\det\left(\sum_0^{\infty} (-1)^{j+1} \frac{j-1}{j!} M^j\right) = \det\left(1 - \frac{1}{2}M^2 + \frac{2}{3}M^3 - \cdots\right) \ .$$

Writing $t_k = trace(M^k)$, show that

$$(*) \qquad det_2(1+yM) = \sum \frac{y^k}{k!} \begin{vmatrix} 0 & k-1 & 0 & \cdots & \cdots \\ t_2 & 0 & k-2 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \\ t_{k-1} & t_{k-2} & \cdots & 0 & 1 \\ t_k & t_{k-1} & \cdots & t_2 & 0 \end{vmatrix} \ .$$

CHAPTER 2

Symmetric Functions as Operators and λ-Rings

2.1. Algebraic Operations on Alphabets

We have used algebraic operations on alphabets, like addition, subtraction and multiplication of two alphabets, without expliciting the underlying algebraic structure. We shall show in this section that we have in fact used λ-rings, a structure due to Alexandre Grothendieck.

Already we met an interpretation of symmetric functions, as *operators* on isobaric determinants, but λ-rings are a more powerful way of considering symmetric functions as "functors". The first exposition of this point of view is due to Knutson [**83**].

Let us first consider the product of an alphabet by a positive integer. This, we defined by taking powers of generating functions :

$$\sigma_z(k\,\mathbb{A}) := (\sigma_z(\mathbb{A}))^k \ .$$

However, it can be given a more concrete interpretation. In the simplest case, passing from $\sigma_z(\mathbb{A})$ to $\sigma_z(2\mathbb{A}) := (\sigma_z(\mathbb{A}))^2$ can be interpreted as doubling the alphabet $\mathbb{A} = \{a\} \mapsto 2\mathbb{A} := \{a'\} \cup \{a''\}$, computing the symmetric functions in $2\mathbb{A}$, and eventually erasing the diacritics on the letters a's. Similarly, one introduces, for each letter $a \in \mathbb{A}$ other letters a', a'' a''', ..., $a^{(k)}$, compute the symmetric functions in this new alphabet, and erase the diacritics, obtaining functions of $\mathbb{A}$ again.

However, the inverse operations, say $\mathbb{A} = \{a\} \mapsto \frac{1}{2}\mathbb{A}$, cannot be described in the same way, which means that one must not stick to considering alphabets as composed of letters.

More explicitly, at the level of power sums, multiplication by 2 is just

$$\Psi_i(2\mathbb{A}) = \sum (a')^i + \sum (a'')^i = 2\,\Psi_i(\mathbb{A}) \ .$$

Therefore the "alphabet" $\mathbb{B} = k\,\mathbb{A}$, $k \in \mathbb{C}$ will be defined by

$$\Psi_i(\mathbb{B}) := k\,\Psi_i(\mathbb{A}) \ , \ i \geq 1 \ , \tag{2.1.1}$$

and now, k needs no more be an integer, we can take $k = 1/2$ as we desired.

Instead of only multiplying alphabets by constants, one can go one step further and realize that every polynomial can play the role of an alphabet, i.e. that the Ψ_i are operators on polynomials ($\alpha \in \mathbb{C}$, u monom):

$$P = \sum_{\alpha,u} \alpha\, u \Rightarrow \Psi_i(P) = \sum_{\alpha,u} \alpha\, u^i \ . \tag{2.1.2}$$

The ring $\mathfrak{Sym}$ being generated by the Ψ_i, $i = 1, 2, \ldots$, formulas (2.1.2) extend to an action of $\mathfrak{Sym}$ on the ring of polynomials.

Thus, the generating functions of elementary and symmetric functions become operators, their explicit action (determined by $\sigma_z = 1/\lambda_{-z} = \exp\left(\sum_{i\geq 1} z^i \Psi^i/i\right)$) being

$$P = \sum_{\alpha,u} \alpha\, u \mapsto \lambda_z(P) = \prod (1+zu)^{\alpha} \ , \tag{2.1.3}$$

$$P = \sum_{\alpha,u} \alpha\, u \mapsto \sigma_z(P) = \prod (1-zu)^{-\alpha} \ . \tag{2.1.4}$$

Each of formulas (2.1.2), (2.1.3) or (2.1.4) can be chosen at will to define the action of symmetric polynomials and implies the two others. The ring of polynomials, with $\mathfrak{Sym}$ operating on it, is called a *λ-ring*.

We shall refer to Thibon[**169**] for what concerns the operation $\mathbb{A} \to \mathbb{A}_1 \oplus \mathbb{A}_2$, which requires the notion of *Hopf algebra* to be fully understood.

2.2. Lambda Operations

Grothendieck choose *lambda operations*, that is, the *exterior powers* of vector spaces, or more generally, fiber bundles, to introduce λ-rings. In the same interpretation, the S^i are the symmetric powers. Algebraic topologists prefers the *Adams operations* Ψ_i.

Having taken polynomials as our building blocks, rather than more sophisticated mathematical entities, we needed only the single axiom (2.1.2). The general theory of λ-rings require three axioms: the compatibility of the λ-operations with addition, product and composition.

Let us remark the different role played by constants α and monomials u :

$$\begin{cases} \Psi_i(\alpha) = \alpha\ , \quad S^i(\alpha) = \binom{\alpha+i-1}{i}\ , \qquad \Lambda^i(\alpha) = \binom{\alpha}{i} \\ \Psi_i(u) = u^i\ , \qquad S^i(u) = u^i\ , \qquad \Lambda^i(u) = 0,\ i>1, \quad \Lambda^1(u) = u \end{cases} \tag{2.2.1}$$

When implementing λ-rings, one must distinguish between indeterminate coefficients α and variables u. The preceding terminology is not satisfactory; it is preferable, instead of using the term "monomial" to say *element of rank 1* (i.e. non zero element x such that $\Lambda^i(x) = 0\ \forall i > 1$), and avoid the term "constant" for the elements invariant under the Ψ_i, but rather say *binomial element* as a tribute to Rota[1], because these elements are such that their images $\Lambda^i(\alpha) = \binom{\alpha}{i}$ are binomial coefficients.

Because $\Psi_i(1) = 1 = S^i(1)$, one can specialize rank 1-elements to 1 (the only other specialization compatible with λ-rings is specialization to 0). It has the consequence that, for a positive integer n, $\Psi_I(n)$, $S^i(n)$, $\Lambda^i(n)$ are the number of terms in the expansion, in terms of monomials, of $\Psi_I(\mathbb{A})$, $S^i(\mathbb{A})$, $\Lambda^i(\mathbb{A})$ when $card(\mathbb{A}) = n$. The identity

$$\Psi_I(n) = \frac{n(n-1)\cdots(n-\ell(I)+1)}{m_1!\,m_2!\,m_3!\cdots} \quad \text{with } I = 1^{m_1}2^{m_2}3^{m_3}\cdots\ , \tag{2.2.2}$$

being true for any positive integer n, is true for any complex number $n \in \mathbb{C}$.

[1]The *umbral calculus* and *operator calculus* [**155**] are full of sequences of polynomials of binomial type.

2.3. Interpreting Polynomials and q-series

Polynomials in one indeterminate are conveniently coded in a λ-ring. Let $\mathbb{A}$ be an alphabet of cardinality n. Then

$$\prod_{a\in\mathbb{A}}(x-a) = S^n(x-\mathbb{A}) = \sum x^{n-i}S^i(-\mathbb{A})$$

Now, expanding $S^{n+k}(x-\mathbb{A})$, $k\in\mathbb{N}$, one sees that

$$S^{n+k}(x-\mathbb{A}) = x^k\, S^n(x-\mathbb{A})\ . \tag{2.3.1}$$

On the other hand, $S^{n-k}(x-\mathbb{A})$ is the component of positive degree of the Laurent polynomial

$$x^{-k}\, S^n(x-\mathbb{A}) = x^{-k}S^n(-\mathbb{A})+\cdots+x^{-1}S^{n-k+1}(-\mathbb{A})+x^0\, S^{n-k}(-\mathbb{A})+\cdots+x^{n-k}S^0(-\mathbb{A})\ .$$

Derivating with respect to x the generating function $\sigma_z(x\mathbb{A})$ of the $S^k(x-\mathbb{A})$, one sees that for any $k\in\mathbb{N}$, the successive derivatives of $S^k(x-\mathbb{A})$ are

$$S^{k-1}(2x-\mathbb{A})\,,\ 2!\,S^{k-2}(3x-\mathbb{A})\,,\ 3!\,S^{k-3}(4x-\mathbb{A})\,,\ldots,j!\,S^{k-j}((j+1)x-\mathbb{A}),\ldots$$

There are other possible codings. For example, given any $\mathbb{A}$, any $n\in\mathbb{N}$, and an element x of rank 1,

$$\Lambda^n(nx+\mathbb{A}) = \sum\binom{n}{i}x^{n-i}\,\Lambda^i(\mathbb{A}) \tag{2.3.2}$$

is a polynomial of degree n.

Derivating with respect to x the generating function $\lambda_z(nx+\mathbb{A}) = (1+zx)^n\lambda_z(\mathbb{A})$, one sees that the successive derivatives are now

$$n\,\Lambda^{n-1}((n-1)x+\mathbb{A})\,,\ n(n-1)\,\Lambda^{n-2}((n-2)x+\mathbb{A})\,,\ \ldots$$

Therefore, the integral of $\Lambda^n(nx+\mathbb{A})$ is $\frac{1}{n+1}\Lambda^{n+1}((n+1)x+\mathbb{A})$, up to the addition of a constant, and more generally, the k-th integral will be

$$\frac{1}{(n+1)\cdots(n+k)}\Lambda^{n+k}((n+k)x+\mathbb{A})\ .$$

We shall see in another section that orthogonal polynomials are also conveniently expressed in λ-rings.

One can extends the action of symmetric polynomials to rational functions or formal power series :

$$\Psi_i\left(\frac{\sum\alpha u}{\sum\beta v}\right) = \frac{\sum\alpha u^i}{\sum\beta v^i}\ , \tag{2.3.3}$$

If q is of rank 1, then one has $\frac{1}{1-q} = 1+q+q^2+\cdots$, i.e. $\frac{1}{1-q}$ is the infinite alphabet $\{1,q,q^2,\ldots\}$ and $\Psi_i(\frac{1}{1-q}) = \frac{1}{1-q^i}$, $i\geq 1$. From

$$\sigma_x\left(\frac{1}{1-q}\right) = \exp\left(\sum_{i=1}^{\infty}\frac{x^i}{i\,(1-q^i)}\right)\ ,$$

Cauchy obtained that :

$$S_i\left(\frac{1}{1-q}\right) = \frac{1}{(1-q)\cdots(1-q^i)}\ . \tag{2.3.4}$$

Therefore

$$\sigma_x\left(\frac{1}{1-q}\right) = \sum \frac{x^i}{(1-q)\cdots(1-q^i)} \tag{2.3.5}$$

is the *q-exponential*, equality $\frac{1}{1-q} = 1 + q + q^2 + \cdots$ implying the factorization

$$\sigma_x\left(\frac{1}{1-q}\right) = \prod_{i=0}^{\infty} \frac{1}{1-xq^i} \tag{2.3.6}$$

Such a factorization renders the q-exponential sometimes easier to use than the classical exponential.

Some classical identities on q-series come from addition in λ-rings. For example, take the *q-binomial identity*

$$\sum_{i\geq 0} z^i \frac{(1-a)(1-aq)\cdots(1-aq^{i-1})}{(1-q)(1-q^2)\cdots(1-q^i)} = \prod_{i\geq 0} \frac{1-zaq^i}{1-zq^i}\ . \tag{2.3.7}$$

Considering z, a, q to be of rank 1, one recognizes the left hand side to be $\sigma_1(z\frac{1-a}{1-q})$ and the right one to be $\sigma_1(\frac{z}{1-q})\,\sigma_1(\frac{-za}{1-q})$. Therefore, the identity does not need a proof, because it is just

$$z\,\frac{1-a}{1-q} = \frac{z}{1-q} + \frac{-za}{1-q}\ . \tag{2.3.8}$$

Using addition and product in a λ-ring, it is easy to get q-identities. For example, the equality $(1-q^2)^{-1}+q(1-q^2)^{-1} = (1-q)^{-1}$, writing $(a:q)_n := (1-aq^0)\cdots(1-aq^{n-1})$, implies :

$$\Lambda^n\left(\frac{1+q}{1-q^2}\right) = \sum_{i+j=n} q^{i^2+j^2-i} \frac{1}{(1:q^2)_i\,(1:q^2)_j}$$
$$= \Lambda^n\left(\frac{1}{1-q}\right) = q^{n(n-1)/2}\frac{1}{(q:q)_n}\ .$$

The reader will find in the exercises some other examples of this type.

Let us stress that if we identify a formal series with $\sigma_z(\mathbb{A})$, then determining all the $\Lambda^i(\mathbb{A})$, $i \in \mathbb{N}$, means writing explicitely the inverse $1/\sigma_z(\mathbb{A})$, determining the $\Psi^i(\mathbb{A})$ means writing the logarithm of $\sigma_z(\mathbb{A})$. For example, the *q-binomial identity* (2.3.7) implies that the logarithm of $\prod_{i\geq 0}(1-zaq^i)/(1-zq^i)$ is equal to $\sum_{i\geq 1} z^i/i\,(1-a^i)/(1-q^i)$.

We shall also see later that computing the Schur functions $S_{n^n}(\mathbb{A})$ and $S_{(n-1)^n}(\mathbb{A})$ for all $n \in \mathbb{N}$ is equivalent to writing $\sigma_z(\mathbb{A})$ as a continued fraction.

2.4. Lagrange Inversion

Let us illustrate the advantages of λ-rings on another very classical exemple, the Lagrange inversion of formal series in one variable. Given $f(z) = z + f_1z^2 + f_2z^3 + \cdots$, one has to find $g(z) = z + g_1z^2 + g_2z^3 + \cdots$ such that

$$f(g(z)) = z \quad \& \quad g(f(z)) = z\ .$$

Lagrange inversion has many applications in classical analysis and combinatorics. Many extensions (q-generalizations, or multivariate extensions, &c.) have been proposed. We shall refer in particular to Gessel [**57**].

Lagrange and Bürman solved the original problem by expressing the coefficients of g in terms of coefficients of derivatives of powers of f.

Another powerful approach is due to Jabotinsky [**66**] who associates to f a matrix $\mathfrak{Jab}(f)$, whose entries are all the coefficients of all the integral powers of f. Now, composition or inversion of series becomes multiplication or inversion of the associated Jabotinsky's matrices.

Lagrange's solution uses only the fact that the logarithmic derivative of a series has no residue. More generally, one has the following lemma :

LEMMA 2.4.1. *Given an alphabet* $\mathbb{A}$, *let* $f(z) := z\sigma_{-z}(\mathbb{A})$. *Then for any* $m, n \in \mathbb{Z}$, *the residue of* $f^{-n}\frac{d}{dz}(f^m)$ *is null if* $m \neq n$, *and equal to* n *if* $m = n$.

Equivalently, one has the relations

$$\sum\nolimits_{k\in\mathbb{Z}} k\, S^{n-k}(-n\,\mathbb{A})\, S^{k-m}(m\,\mathbb{A}) = n\delta_{m,n} \ . \tag{2.4.1}$$

Proof. Because $f^m = z^m\sigma_{-z}(m\mathbb{A})$ and $f^{-n} = z^{-n}\sigma_{-z}(-n\mathbb{A})$, the two statements are equivalent.

The residue to determine is equal to the coefficient of z^{n-1} in $\sigma_{-z}(-n\mathbb{A})\,\frac{d}{dz}(z^m\sigma_{-z}(m\mathbb{A}))$. If $m \neq n$, it is the coefficient of z^{n-1} in

$$\frac{m}{m-n}\frac{d}{dz}\left(z^m\sigma_{-z}((m-n)\mathbb{A})\right) + \left(m - \frac{m^2}{m-n}\right) z^{m-1}\sigma_{-z}((m-n)\mathbb{A}) \ ,$$

which is

$$(-1)^{n-m}\frac{m}{m-n} n S^{n-m}((m-n)\mathbb{A}) + \left(m - \frac{m^2}{m-n}\right)(-1)^{n-m}S^{n-1-m+1}((m-n)\mathbb{A}) \ .$$

It is indeed null; the case $m = n$ comes from a similar computation. □

Equations (2.4.1) determine the coefficients of the powers of the series inverse to f :

THEOREM 2.4.2 (Lagrange-Bürman). *Let* $f(z) = z\sigma_{-z}(\mathbb{A})$, *and* $g(z)$ *its inverse series. Then for any* $k \neq 0$, *one has*

$$g^k(z) \ = \ k\, z^k \sum \frac{z^i}{i+k}\Lambda^i((i+k)\mathbb{A}) \ , \tag{2.4.2}$$

$$\log\big(g(z)/z\big) \ = \ \sum \frac{z^i}{i}\Lambda^i(i\,\mathbb{A}) \ . \tag{2.4.3}$$

Proof. Writing $\big(g(f(z))\big)^k = z^k$, one sees that the expressions given by the theorem satisfy the right recursions thanks to relations (2.4.1). One gets the logarithm as the limit $k \to 0$ of g^k/k. □

Since Lagrange's expression involves taking elementary symmetric functions in the product of two alphabets ($(i{+}k)$ and $\mathbb{A}$), then the Cauchy formula gives an expansion of it in any pair of adjoint bases of symmetric functions, without further computations. In particular, one has

(2.4.4)

$$\begin{aligned}
\Lambda^i((i+k)\mathbb{A}) \ &= \ (-1)^i \sum\nolimits_{|J|=i} \Psi_J(-(i+k))S^J(\mathbb{A}) && (a)\\
&= \ \sum\nolimits_{|J|=i} \Psi_J(i+k)\Lambda^J(\mathbb{A}) && (b)\\
&= \ \sum\nolimits_{|J|=i} S_J(i+k)S_{J^\sim}(\mathbb{A}) && (c)\\
&= \ \sum\nolimits_{|J|=i} \Lambda^J(i+k)\Psi_J(\mathbb{A}) && (d)\\
&= \ \sum\nolimits_{|J|=i} S^J(i+k)F_J(\mathbb{A}) && (e)\\
&= \ (-1)^i \sum\nolimits_{|J|=i} \tfrac{(-i-k)^{\ell(J)}}{m_1!m_2!\cdots}\left(\tfrac{\Psi_1}{1}\right)^{m_1}\left(\tfrac{\Psi_2}{2}\right)^{m_2}\left(\tfrac{\Psi_3}{3}\right)^{m_3}\cdots && (f)
\end{aligned}$$

using forgotten symmetric functions F_J in equation (e) and using exponential notations for partitions in the last one. It is interesting to note that all these equations (except the one involving forgotten functions) can be found in the mathematical literature, each time obtained from a new computation.

For example, for $i = 3$, the different expressions of Lagrange coefficient $\Lambda^3((3+k)\mathbb{A})$ are :

$$\frac{1}{6}(k+3)(k+2)(k+1)\Lambda^{111} + (3+k)(2+k)\Lambda^{12} + (k+3)\Lambda^3$$

$$(k+3)S^3 - (k+4)(k+3)S^{12} + \frac{1}{6}(k+5)(k+4)(k+3)S^{111}$$

$$\frac{1}{6}(k+5)(k+4)(k+3)S_{111} + \frac{1}{3}(k+4)(k+3)(k+2)S_{12} + \frac{1}{6}(k+3)(k+2)(k+1)S_3$$

$$\frac{1}{6}(k+3)(k+2)(k+1)\Psi_3 + \frac{1}{2}(k+2)(k+3)^2\Psi_{12} + (3+k)^3\Psi_{111}$$

$$\frac{1}{6}(k+5)(k+4)(k+3)F_3 + \frac{1}{2}(k+4)(k+3)^2F_{12} + (k+3)^3F_{111}$$

$$\frac{1}{3}(k+3)\Psi^3 - \frac{1}{2}(3+k)^2\Psi^{12} + \frac{1}{6}(3+k)^3\Psi^{111}$$

Writing $g(z) = z\sigma_z(\mathbb{B})$, then (2.4.2) and (2.4.3) imply that, for any $i \geq 1$,

$$S^i(\mathbb{B}) = \frac{1}{i+1}\,\Lambda^i\big((i{+}1)\mathbb{A}\big) \qquad \& \qquad \Psi^i(\mathbb{B}) = \Lambda^i(i\mathbb{A}) \ . \tag{2.4.5}$$

Riordan [**153**], p. 150, gives tables for $S^i(\mathbb{B})$ expressed in the basis $S^J(\mathbb{A})$ (with a change of signs).

One can also write Lagrange coefficients as determinants, as shows the next lemma.

LEMMA 2.4.3. *Let $\mathbb{A}$ be an alphabet, $k \in \mathbb{C}$, $k \neq 0$, $n \in \mathbb{N}$. Then*

$$\begin{aligned} n!\,S_n(k\,\mathbb{A}) &= \det\Big|\big((j{-}i{+}1)k + 1 - i\big)\, S_{j-i+1}(\mathbb{A})\Big|_{1\leq i,j\leq n} \\ &= \det\Big|\big((j{-}i{+}1)k - j + n\delta_{j,n}\big)\, S_{j-i+1}(\mathbb{A})\Big|_{1\leq i,j\leq n} \end{aligned} \tag{2.4.6}$$

Proof. Each of the two matrices is the product, on the right or on the left respectively, of the matrix $\big[S_{j-i}(\mathbb{A})\big]$ (of determinant 1) by the Newton matrix[2] $\big[\Psi^{j-i+1}(k\mathbb{A})\big]_{1\leq i,j\leq n}$, which is of determinant $n!\,S_n(k\mathbb{A})$. □

Replacing k by $-i-k$, and n by i, one gets two determinantal expressions of Lagrange coefficient $\Lambda^i((i+k)\mathbb{A})$.

[2]with subdiagonal $-1, -2, \ldots, -n{+}1$, and not Ψ^0. Recall also that $\Psi^j(k\mathbb{A}) = k\Psi^j(\mathbb{A})$.

For example, $4!\,S_4(k\mathbb{A})$ is equal to the determinant of each of the following matrices :

$$\begin{bmatrix} kS_1(\mathbb{A}) & 2kS_2(\mathbb{A}) & 3kS_3(\mathbb{A}) & 4kS_4(\mathbb{A}) \\ -1 & (k{-}1)S_1(\mathbb{A}) & (2k{-}1)S_2(\mathbb{A}) & (3k{-}1)S_3(\mathbb{A}) \\ 0 & -2 & (k{-}2)S_1(\mathbb{A}) & (2k{-}2)S_2(\mathbb{A}) \\ 0 & 0 & -3 & (k{-}3)S_1(\mathbb{A}) \end{bmatrix} =$$

$$\begin{bmatrix} k\Psi^1(\mathbb{A}) & k\Psi^2(\mathbb{A}) & k\Psi^3(\mathbb{A}) & k\Psi^4(\mathbb{A}) \\ -1 & k\Psi^1(\mathbb{A}) & k\Psi^2(\mathbb{A}) & k\Psi^3(\mathbb{A}) \\ 0 & -2 & k\Psi^1(\mathbb{A}) & k\Psi^2(\mathbb{A}) \\ 0 & 0 & -3 & k\Psi^1(\mathbb{A}) \end{bmatrix} \begin{bmatrix} S_0(\mathbb{A}) & S_1(\mathbb{A}) & S_2(\mathbb{A}) & S_3(\mathbb{A}) \\ 0 & S_0(\mathbb{A}) & S_1(\mathbb{A}) & S_2(\mathbb{A}) \\ 0 & 0 & S_0(\mathbb{A}) & S_1(\mathbb{A}) \\ 0 & 0 & 0 & S_0(\mathbb{A}) \end{bmatrix},$$

$$\begin{bmatrix} (k{-}1)S_1(\mathbb{A}) & (2k{-}2)S_2(\mathbb{A}) & (3k{-}3)S_3(\mathbb{A}) & 4kS_4(\mathbb{A}) \\ -1 & (k{-}2)S_1(\mathbb{A}) & (2k{-}3)S_2(\mathbb{A}) & 3kS_3(\mathbb{A}) \\ 0 & -2 & (k{-}3)S_1(\mathbb{A}) & 2kS_2(\mathbb{A}) \\ 0 & 0 & -3 & kS_1(\mathbb{A}) \end{bmatrix} =$$

$$\begin{bmatrix} S_0(\mathbb{A}) & S_1(\mathbb{A}) & S_2(\mathbb{A}) & S_3(\mathbb{A}) \\ 0 & S_0(\mathbb{A}) & S_1(\mathbb{A}) & S_2(\mathbb{A}) \\ 0 & 0 & S_0(\mathbb{A}) & S_1(\mathbb{A}) \\ 0 & 0 & 0 & S_0(\mathbb{A}) \end{bmatrix} \begin{bmatrix} k\Psi^1(\mathbb{A}) & k\Psi^2(\mathbb{A}) & k\Psi^3(\mathbb{A}) & k\Psi^4(\mathbb{A}) \\ -1 & k\Psi^1(\mathbb{A}) & k\Psi^2(\mathbb{A}) & k\Psi^3(\mathbb{A}) \\ 0 & -2 & k\Psi^1(\mathbb{A}) & k\Psi^2(\mathbb{A}) \\ 0 & 0 & -3 & k\Psi^1(\mathbb{A}) \end{bmatrix}$$

Exercises

EX. 2.1. Let $x_1, \ldots, x_n, y$ be rank 1-elements. Compute $\sum_i 1/(y - x_i)$.

EX. 2.2. Let $\mathbb{A}$ be arbitrary, x of rank 1, $n, p \in \mathbb{N}$. Show that

$$S^n(\mathbb{A}) - \binom{k}{1} S^n(\mathbb{A}{-}x) + \binom{k}{2} S^n(\mathbb{A}{-}2x) + \cdots + (-1)^k \binom{k}{k} S^n(\mathbb{A}{-}kx) = x^k S^{n-k}(\mathbb{A}) .$$

EX. 2.3. Let k, n be two positive integers, $\mathbb{A}$ be of cardinality $n{+}1$, x be of rank 1. Show that the multi-Schur function

$$S_{n;2n;\ldots;(k-1)n}(kx - \mathbb{A}; kx - 2\mathbb{A}; \ldots ; kx - (k-1)\mathbb{A})$$

is a non trivial power.

EX. 2.4. Let r be an element of binomial type, $\mathbb{A}$ be arbitrary. Express the weighted sum of monomial functions $\sum_I S^{|I|}\big(r + 1 - \ell[I]\big)\, \Psi_I(\mathbb{A})$, sum over all partitions, in the basis of Schur functions.

EX. 2.5. Let α, q be rank-1 elements, and $\mathbb{A}$ be such that $S_n(\mathbb{A}) = (1{-}\alpha q^n)^{-1}$, $n \geq 1$. Let I be a partition in $\mathbb{N}^n$ such that $I \supseteq n^n$. Evaluate $S_I(\mathbb{A})$.

EX. 2.6. Let α, β be elements of binomial type. Define $\mathbb{A}$ by $S_n(\mathbb{A}) = \alpha(\alpha + n\beta)^{-1}\, S^n(\alpha + n\beta)$, $n > 0$. Compute $\Lambda^n(\mathbb{A})$ and $\Psi^n(\mathbb{A})$.

EX. 2.7. Given two positive integers m, n, border the matrix $\mathbb{S}_{m^n}(\mathbb{A})$ by a first column $[x^n, \ldots, x^0, 0]$ and a bottom row $[0, y^0, \ldots, y^n]$. Show that the determinant of this new matrix is equal to $S_{(m+1)^{n-1}}(\mathbb{A}{-}x{-}y)$, taking x, y to be rank 1-elements.

Ex. 2.8. Let $\mathbb{A}$ be an alphabet of cardinality n, $\{m_a \in \mathbb{N}\}_{a\in\mathbb{A}}$ be a family of "multiplicities". Let $\mathbb{X} = \sum_a m_a\, a$, the letters a being rank-1 elements.

Show that minors of order $> n$ of the matrix $\left[\Psi^{i+j}(\mathbb{X})\right]_{i,j\geq 0}$ are null, and compute the minors of order n.

Ex. 2.9. Compute the exponential of $f(x)$, with

$$f(x) = -\sum_1^\infty \frac{(-x)^n}{n} \begin{vmatrix} 1^2 & 2^2 & 3^2 & \cdots & n^2 \\ 1 & 1 & 2 & \cdots & n-1 \\ \cdot & 1 & 1 & \cdots & n-2 \\ \vdots & & \ddots & & \vdots \\ \cdot & \cdot & & & 1 \end{vmatrix}$$

or

$$f(x) = -\sum_1^\infty \frac{(-x)^n}{n} \begin{vmatrix} 1^3 & 2^3 & 3^3 & \cdots & n^3 \\ 1^2 & 2^2 & 3^2 & \cdots & n^2 \\ 0 & 1^2 & 2^2 & \cdots & (n-1)^2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & & 2^2 \end{vmatrix}$$

Ex. 2.10. To each sequence of alphabets $\mathcal{A} = (\mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_5, \ldots, \mathbb{A}_p, \ldots)$, all primes p, associate a *Moëbius function* on the positive integers :

$$\mu_{\mathcal{A}}(n) := \prod_p (S_k(\mathbb{A}_p)\ ,$$

where k is the order of p in n (p^k is the greatest power dividing n).

Given two sequences $\mathcal{A}$, $\mathcal{B}$, show that $\mu_{\mathcal{A}+\mathcal{B}}$ is the Dirichlet convolution product of $\mu_{\mathcal{A}}$ and $\mu_{\mathcal{B}}$, i.e. that

$$\mu_{\mathcal{A}+\mathcal{B}}(n) = \sum_{d|n} \mu_{\mathcal{A}}(d)\mu_{\mathcal{B}}(n/d)\ ,$$

summation over all divisors of n, where the sum $\mathcal{A}+\mathcal{B} = (\mathbb{A}_2+\mathbb{B}_2, \mathbb{A}_3+\mathbb{B}_3, \ldots)$ is componentwise.

Show that the Moebius function μ, and the Euler functions φ, τ, σ correspond respectively to

$$(-1,\ldots,-1,\ldots)\,,\ (\mathbb{A}_2-1,\ldots,\mathbb{A}_p-1,\ldots)\,,\ (2,\ldots,2,\ldots)\,,\ (\mathbb{A}_2+1,\ldots,\mathbb{A}_p+1,\ldots)\,,$$

where $\mathbb{A}_p$ is such that $S_k(\mathbb{A}_p) = p^k$ (Recall that $\mu(n)$ is 0 if n is not square-free, and ± 1 otherwise; the *totient function* $\varphi(n)$ is the number of positive integers less than n and prime to n, $\tau(n)$ is the number of divisors of n and $\sigma(n)$ is their sum).

Ex. 2.11. Let $f(n)$ be a function on $\mathbb{N}$ having values in $\mathbb{Q}$-vector space. After Désiré André [**3**], check the following identity on formal series :

$$\sum_0^\infty x^n/n! = \exp(x) \sum_0^\infty f\partial_1\partial_2\cdots\partial_n x^n\ ,$$

where the divided differences are evaluated on $\mathbb{B} = \{0, 1, 2, \ldots\}$.

Deduce that, for any $\mathbb{A}$, one has the identities

$$\sum x^n S^n(\mathbb{A})/n! \quad = \quad \exp(x) \sum x^n S^n(\mathbb{A}-n)/n! \ , \tag{2.4.7}$$

$$\sum x^n \Lambda^n(\mathbb{A})/n! \quad = \quad \exp(-x) \sum x^n \Lambda^n(\mathbb{A}+n)/n! \ . \tag{2.4.8}$$

In particular, when z is a rank-1 element, and $\mathbb{A}$ is such that $S^n(\mathbb{A}) = (z+n)^{-1}$, $\forall n \geq 1$, one obtains a formula of Ramanujan :

$$\sum_0^\infty \frac{x^n}{n!\,(z+n)} = \exp(x) \sum_0^\infty \frac{(-x)^n}{z(z+1)\cdots(z+n)} \ .$$

Ex. 2.12. Let f, y be two functions of one variable, having derivatives of any order, y vanishing at 0.

Show, after Arbogast[**32**], art. 533, Faa de Bruno [**25**], and many other people, that

$$\frac{1}{n!} D^n\big(f(y(x))\big) = \sum_{p,I} \frac{\partial^p f}{\partial y^p} \frac{1}{m_1!\, m_2! \cdots} \left(\frac{D^1 y}{1!}\right)^{m_1} \left(\frac{D^2 y}{2!}\right)^{m_2} \cdots \ ,$$

sum over all $p \geq 1$, all partitions $I = 1^{m_1} 2_{m_2} \cdots : |I| = p$, $\ell(I) = p$, noting by D the derivative with respect to x.

The reader may suppose that $f(y) = \sigma_y(\mathbb{A})$, $y(x) = \sum_{n=1}^\infty x^n S^n(\mathbb{B})$.

Ex. 2.13. Let $\mathbb{A}$ be an alphabet. Grothendieck defined functions $\gamma^i(\mathbb{A})$ by the generating function

$$\gamma_z(\mathbb{A}) = \sum z^i \gamma^i(\mathbb{A}) := \lambda_{z/(1-z)}(\mathbb{A}) \ .$$

Define accordingly the alphabet $\mathbb{A}^\diamond$ by $\Lambda^i(\mathbb{A}^\diamond) = \gamma^i(\mathbb{A})$, $i \geq 0$.

Show that

$$\Lambda^i(\mathbb{A}^\diamond) = \Lambda^i(\mathbb{A} + i - 1)$$
$$S^i(\mathbb{A}^\diamond) = S^i(\mathbb{A} + 1 - i)$$
$$\Psi^i(\mathbb{A}^\diamond) = \sum_{j=0}^{i-1} (-1)^j \binom{i}{j} \Psi^{i-j}(\mathbb{A}) \ .$$

Show that the transformation $\mathbb{A} \mapsto \mathbb{A}^\diamond$ satisfies $(-\mathbb{A})^\diamond = -\mathbb{A}^\diamond$, $(\mathbb{A}+\mathbb{B})^\diamond = \mathbb{A}^\diamond + \mathbb{B}^\diamond$, but that in general, $(\mathbb{A}\mathbb{B})^\diamond \neq \mathbb{A}^\diamond \mathbb{B}^\diamond$.

Ex. 2.14. Given an arbitrary $\mathbb{A}$, and a rank-1 element q, define the alphabet $\mathbb{A}^\diamond$ by $\Lambda^i(\mathbb{A}^\diamond) = \Lambda^i(\mathbb{A} + q(i-1))$, $i \geq 0$.

Show that for any $n \in \mathbb{N}$, $S_{n^n}(\mathbb{A}^\diamond) = S_{n^n}(\mathbb{A})$, give the expression of $S_{(n-1)^n}(\mathbb{A}^\diamond)$ and a continued fraction expansion of $\sigma_z(\mathbb{A}^\diamond)$.

Ex. 2.15. Let ξ be such $S^n(\xi) = 1/n!$, $n \in \mathbb{N}$, and let x, y be rank-1 elements. The polynomials $S^n(x\xi - (n+1)y)$, $n \in \mathbb{N}$ are a basis a polynomials in x.

Express the powers of x in this basis.

Ex. 2.16. Following A.Kirillov [**81**], define the q-analog of Euler's dilogarithm to be

$$Li_2(z,q) := \sum_{n \geq 1} z^n \big(n(1-q^n)\big)^{-1} \ .$$

Check that the logarithm of the q-exponential of z is equal to $Li_2(z,q)$.

Ex. 2.17. Let x be of binomial type, $i, j \in \mathbb{N}$. Expand the product $\Lambda^i(x)\Lambda^j(x)$ in the basis $\Lambda^n(x)$.

Ex. 2.18. Let x be of binomial type, q be of rank 1, y be such that $y(1-q)$ be of rank 1.

Define the *q-Charlier polynomials* $h_n(x, y, q)$ by

$$h_n(x, y, q)/n! = \Lambda^n(x-y),\ n \in \mathbb{N}.$$

Show that they have exponential generating function equal to

$$\sum_n h_n(x, y, q)\frac{z^n}{n!} = (1+z)^x \sum_n \frac{\big(-zy(1-q)\big)^n}{(1-q)\cdots(1-q^n)} .$$

Ex. 2.19. Let the Gauss polynomial $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]$ be $S^i((1-q^{n-i+1})/(1-q))$, with q of rank 1. Show that

$$\begin{bmatrix} m+n+1 \\ n \end{bmatrix} = \sum_0^n q^j \begin{bmatrix} m+j \\ j \end{bmatrix}$$

$$\begin{bmatrix} m+n \\ k \end{bmatrix} = \sum_j q^{(n-j)(k-j)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m \\ k-j \end{bmatrix} .$$

Ex. 2.20. Let $m, n \in \mathbb{N}$. Show the identity

$$\sum_{j=1}^m (-1)^{m-j} \begin{bmatrix} m+n \\ j \end{bmatrix} \begin{bmatrix} m-j+n-1 \\ m-j \end{bmatrix} q^{(m-j)(m-j+1)/2} = 1+(-1)^{m+1} \begin{bmatrix} m+n-1 \\ m \end{bmatrix} q^{m(m+1)/2} .$$

Ex. 2.21. Let a, x, q be of rank 1.

Define the *Rogers-Szegö polynomial* of order n to be

$$r_n(x, a) = S^n\big((a+x)(1-q)^{-1}\big)/S^n\big((1-q)^{-1}\big) .$$

Find its expansion in terms of powers of x. Show the recursion

$$r_{n+1}(x, a) = (x+a)\, r_n(x, a) - ax(1-q)\, r_{n-1}(x, a) .$$

Show, after Gauss, that

$$r_{2n}(1, -1) = (1-q)(1-q^3)\cdots(1-q^{2n-1}) .$$

Ex. 2.22. Let $n \in \mathbb{N}$. Compute the following sum of Gauss polynomials:

$$\sum_{i=0}^n q^i \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} .$$

Ex. 2.23. Let k be an integer. Write $(a : q)_n$ for $(1-a)(1-aq)\cdots(1-aq^{n-1})$. Prove that

$$\sum_{i\geq 0} \frac{q^{i(2k+1)}}{(q^2 : q^2)_i} = (q : q^2)_k \sum_{i\geq 0} \frac{q^i}{(q^2 : q^2)_i} .$$

Ex. 2.24. Let $n \in \mathbb{N}$, $n \geq 1$. Show that

$$\sum_k (-1)^k q^{\binom{k}{2}} \frac{\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]}{(aq^k : q)_k\, (aq^{2k+1} : q)_{n-k}} = 0 .$$

Ex. 2.25. Let $\theta \in \mathbb{C}$, α, q, z be rank-1 elements. Define the *Rogers polynomials* $C_n^\alpha(\cos(\theta))$ by the generating function

$$\frac{(\alpha z \exp(\theta\sqrt{-1}) : q)_\infty \, (\alpha z \exp(-\theta\sqrt{-1}) : q)_\infty}{(z \exp(\theta\sqrt{-1}) : q)_\infty \, (z \exp(-\theta\sqrt{-1}) : q)_\infty} = \sum z^n \, C_n^\alpha(\cos(\theta)) \ .$$

Show that

$$C_n^\alpha(\cos(\theta) = \sum_{k=0}^{n} \frac{(\alpha : q)_n (\alpha : q)_{n-k}}{(q : q)_n (q : q)_{n-k}} \cos\bigl((n-2k)\theta\bigr) \ .$$

Ex. 2.26. Given $n \in \mathbb{N}$, q a rank-1 element, let $\int$ be the linear functional on functions of one variable defined by

$$f \to \int f = \sum_0^n (a : q)_k (aq^{n+1+k} : q)_{n-k} S^k \left(\frac{q^n - 1}{1 - q}\right) f(k) \ .$$

Give a compact expression of this functional. Show in particular that it vanishes on $f : f(k) = 1 - aq^{2k}$, for $n \geq 1$, and on $f : f(k) = q^{-k} - a^2 q^{3k}$ for $n \geq 2$. Give more general vanishing properties.

Ex. 2.27. Let $\mathbb{A}$ be arbitrary, p binomial, $n \in \mathbb{N}$. Show that

$$nS^n(p\mathbb{A}) - (p{+}n{-}1)S^{n-1}(p\mathbb{A})\,\Lambda^1(\mathbb{A}) + (2p{+}n{-}2)S^{n-2}(p\mathbb{A})\,\Lambda^2(\mathbb{A}) - \cdots$$
$$\cdots \pm ((n{-}1)p + 1)S^0(p\mathbb{A})\,\Lambda^n(\mathbb{A}) = 0 \ .$$

Ex. 2.28. Let $\mathbb{A}$ be arbitrary. Show that the product of $\Bigl[(-1)^{i+j}\Lambda^j(i)\Bigr]_{0\leq i,j\leq n}$ by $\Bigl[S_j(i\mathbb{A})\Bigr]_{0\leq i,j\leq n}$ is a triangular matrix. Express its entries in the $S^I(\mathbb{A})$ basis.

Ex. 2.29. Let $\mathbb{A}$ be of finite cardinality n, x be a rank 1 element. Define $\mathbb{B} = \mathbb{A} + x$, and $\mathbb{E} = \{(a{+}x) : a \in \mathbb{A}\}$, i.e. the alphabet obtained from $\mathbb{A}$ by changing each letter a into a letter $(a{+}x)$.

Express $\Lambda^i(\mathbb{E})$, any i, in terms of $\Lambda^j(\mathbb{B})$.

Ex. 2.30. Let $\mathbb{A}$ be of cardinality n, and $\mathbb{A}^{der}$ be the alphabet of roots of $S^{n-1}(2x - \mathbb{A})$ (we shall call it the *derived alphabet*). Show that

$$S^k(\mathbb{A} - \mathbb{A}^{der}) = \Psi_k(\mathbb{A})/n \ , \ k = 1, 2, \ldots \ .$$

Show that, for any $a \in \mathbb{A}$,

$$S_{n^{n-1}}(\mathbb{A} - \mathbb{A}^{der} - 2a) = (-1)^{n-1}\Delta(\mathbb{A})^2 \ ,$$

i.e. is equal to the discriminant $\prod_{i<j}(a_i - a_j)^2$, up to a sign.

Ex. 2.31. Let $\mathbb{A}$ be of cardinality n. Show that

$$\sum_0^\infty \Psi_i(\mathbb{A}) = \frac{n - (n{-}1)\Lambda^1(\mathbb{A}) + (n{-}2)\Lambda^2(\mathbb{A}) - (n{-}3)\Lambda^3(\mathbb{A}) + \cdots}{\sum(-1)^i\Lambda^i(\mathbb{A})} \ .$$

Ex. 2.32. Let $\mathbb{A}$ be an alphabet, $\mathbb{A}^{der}$ the alphabet of roots of the derivative of $R(x, \mathbb{A})$. For any integer k, express $S^k(x - \mathbb{A}^{der})$ in the basis $\{S^i(x - \mathbb{A}),\ i \geq 0\}$.

Ex. 2.33. (Sylvester, [**166**] **1**, p.502). Let $\mathbb{A}$ be of cardinality n, and let $\mathbb{A}^{der}$ its derived alphabet. For any positive integer k, and x of rank 1, let $P_k(x) := S_{k^k}(\mathbb{A} - \mathbb{A}^{der} - x)$. Show that for any $k > 0$, any $0 \le j < k$, one has

$$\sum\nolimits_{a\in\mathbb{A}} a^j \, P_k(a) = 0 \ ,$$

and that

$$\Big| P_0(a)^2 \, , \, P_1(a)^2 \, , \, \ldots , \, P_{n-1}(a)^2 \Big|_{a\in\mathbb{A}} = 0 \ .$$

Ex. 2.34. Let $\mathbb{A}$ be of cardinality n. One defines the *cumulants* K_j by the generating function $\sum z^i \Psi_i(\mathbb{A}) = n \exp\left(\sum_1^\infty z^j K_j / j!\right)$. Express them in the basis $\Lambda^I(\mathbb{A})$.

Ex. 2.35. Let x be such that $\forall n \in \mathbb{N}$, $S^n(1+x) = 1 + nx$. Compute the Schur function $S_J(1+x)$, J partition, without expansion of determinants.

Ex. 2.36. Let $\mathbb{A}$ be arbitrary, β be of binomial type. Show, after Han Guo Niu, that

$$\frac{n}{\beta+1} S_n\big((\beta{+}1)\mathbb{A}\big) = \sum_{j=0}^{n} j \, S_j(\mathbb{A}) S_{n-j}(\beta\mathbb{A}) \ .$$

Ex. 2.37. Show that

$$\log\left(\frac{1+2zx+z^2}{1-2zx+z^2}\right) = 4 \sum \frac{z^{2k+1}}{2k+1} T_{2k+1}(x) \ ,$$

where the $T_n(x)$ are the *Tchebychef polynomials* (cf. Ex. 8.9).

Ex. 2.38. Show that, for any $r \in \mathbb{N}$,

$$x(x+1)\cdots(x+r-1) = \sum_{J:|J|=r} x^{\ell(J)} \frac{r!}{(\Psi^J, \Psi^J)} \ .$$

What becomes the left hand side when the summation is restricted to partitions with all parts odd ? For example, one gets, for $n = 2, \ldots, 6$ the following values x^2, $x^3 + 2x$, $x^4 + 8x^2$, $x^5 + 20x^3 + 24x$, $x^6 + 40x^4 + 184x^2$.

Ex. 2.39. Let x be of binomial type, and *sf* be a symmetric function. Express *sf*(x) in the basis of Newton's polynomials $N_i := x(x-1)\cdots(x-i+1)$.

Ex. 2.40. Let x be of binomial type, I be a partition. Compute the derivative of $S_I(x)$ with respect to x. Express it in terms of the $S_{I/H}(x)$, where H runs over hook-partitions.

Ex. 2.41. Let x be of binomial type, ζ be a rank-1 element. Compute the specializations $\zeta = -1$ of the monomial functions and Schur functions in $\mathbb{A} = x(1+\zeta)$.

Give as a corollary Gillis's formula :

$$S^{2n}(x(1+\zeta))\Big|_{\zeta=-1} = S^n(x) \ .$$

Ex. 2.42. A "série de facultés", according to Nördlund [**142**], is a series of the type

$$f(z,\mathbb{A}) := 1 + \frac{S^1(\mathbb{A})}{z} + \frac{S^2(\mathbb{A})}{z(z+1)} + \frac{2!\, S^3(\mathbb{A})}{z(z+1)(z+2)} + \cdots = \sum \frac{S^n(\mathbb{A})}{(n+1)S^{n+1}(z)} \ ,$$

where z is of binomial type and $\mathbb{A}$ arbitrary.

Show that for every binomial element y, one has $f(z,\mathbb{A}) = f(z+y, \mathbb{A}+y)$. Deduce that

$$\frac{d}{dz} f(z,\mathbb{A}) = -\sum \Big(S^{n-1}(\mathbb{A})/1 + \cdots + S^0(\mathbb{A})/n \Big) / (n+1) S^{n+1}(z) \ .$$

Express the product $f(z,\mathbb{A})\, f(z,\mathbb{B})$ as a série de facultés.

Ex. 2.43. Let x be of binomial type, r be an integer, $J \in \mathbb{N}^r$ be a partition. Compute the generating series $\sum_n z^n S_{Jn}(x)$, where Jn means the concatenation of J and n.

Ex. 2.44. Let $k \in \mathbb{N}$. Writing partitions exponentially, $J = 1^{m_1} 2^{m_2} \cdots$, show that

$$\sum_J m_k \frac{\Psi^J}{(\Psi^J, \Psi^J)}$$

is a sum of Schur functions with coefficients ± 1.

Ex. 2.45. Let $f(x)$ be a formal series $f(x) = f_0 + x f_1 + x^2 f_2 + \cdots$. Let y be another indeterminate. For any positive integer m, define, after Catalan,

$$\Omega = (f_0 + x f_1 + x^2 f_2 + \cdots) + \binom{m+1}{1} y (x f_1 + x^2 f_2 + \cdots) \\ + \binom{m+2}{2} y (x^2 f_2 + x^3 f_3 + \cdots) + \cdots$$

Let $\phi(x,y) := y^m \Big(f(x) - y f(xy) \Big) / (1-y)$. Show that

$$\Omega = \frac{1}{m!} \frac{d}{dy^m} \phi(x,y) \ .$$

Ex. 2.46. Define the *Hermite polynomials* $H_n(x)$ and the alphabet $\mathbb{A}$ by

$$S_n(\mathbb{A}) = H_n/n! := \sum_{r=0}^{n} (-1)^r (2x)^{n-2r} / r!(n-2r)! \ .$$

Compute the forgotten symmetric functions[3] in $\mathbb{A}$.

Ex. 2.47. Define the *Gegenbauer polynomials* $G(n,\alpha,x)$ by the generating function

$$\sum_{n \geq 0} z^n G(n,\alpha,x) = (1 - 2xz + z^2)^{-\alpha} \ .$$

Let $k \in \mathbb{C}$. Evaluate the determinant of order n with first row $[jkG(j,\alpha,x),\ j = 1 \ldots n]$, entries $[i,j] = ((j-i+1)k + i - 1)\, G(j-i+1, \alpha, x)$, $2 \leq i \leq j \leq n$, subdiagonal equal to $[1, 2, \ldots, n-1]$, and other entries 0.

[3]They are the images of the monomial symmetric functions under the involution of $\mathfrak{Sym}$ which exchanges complete and elementary symmetric functions. They have been defined by Kostka and forgotten afterwards.

For example, for $n = 4$, evaluate the determinant

$$\begin{vmatrix} kG(1,\alpha,x) & 2\,kG(2,\alpha,x) & 3\,kG(3,\alpha,x) & 4\,kG(4,\alpha,x) \\ 1 & (k+1)\,G(1,\alpha,x) & (2\,k+1)\,G(2,\alpha,x) & (3\,k+1)\,G(3,\alpha,x) \\ 0 & 2 & (k+2)\,G(1,\alpha,x) & (2+2\,k)\,G(2,\alpha,x) \\ 0 & 0 & 3 & (k+3)\,G(1,\alpha,x) \end{vmatrix} .$$

Ex. 2.48. Let $G(n,\alpha,x)$, $n \in \mathbb{N}$, be the Gegenbauer polynomials. For any $n \in \mathbb{N}$, evaluate the sum

$$\sum_J \Psi_J(-1/\alpha) \prod_i G(i,\alpha,x)^{m_i} \ ,$$

sum over all partitions $J = 1^{m_1}2^{m_2}\cdots$ of n.

Ex. 2.49. Let x be such that $\Lambda^2(2x) = 1$, $\Lambda^k(2x) = 0$, $k > 2$. Compute $\sigma_z(x)$.

Ex. 2.50. Let $x, y, \zeta, \alpha, \beta, \gamma$ be rank-1 elements. Show that the n-th Legendre polynomial $P_n(x)$ can be written

$$\begin{aligned} P_n &= S^n\left(\frac{1}{2}(y + y^{-1})\right) \\ &= S^n\big((n{+}1)x - n(\zeta{+}1)\big)\, 2^{-n} \\ &= S^n\big((n{+}1)x - n\zeta\big) \\ &= \Lambda^n\big(n - (n{+}1)c\big) \\ &= \Lambda^n(na + nb) \ , \end{aligned}$$

where the rank-1 elements are specialized respectively to

$$y = x + \sqrt{x^2{-}1} \ , \ \zeta = -1 \ , \ a = \frac{x{+}1}{2} \ , \ b = \frac{x{-}1}{2} \ , \ c = \frac{1{-}x}{2} \ .$$

Ex. 2.51. Let $P_1, P_2, \ldots$ be the Legendre polynomials. Compute the series inverse to $z\exp\big(\sum_1^\infty z^n P_n/n\big)$.

Ex. 2.52. Let $\mathbb{A}$ be an alphabet, and n a positive integer. Show that there exists a unique polynomial (the *Faber polynomial*) $F(x)$ of degree n such that

$$F\big(z^{-1}\lambda_z(\mathbb{A})\big) = z^{-n} + z(\cdots)$$

i.e. such that its evaluation in $x = z^{-1}\lambda_z(\mathbb{A})$ has no term of degree $-n{+}1, \ldots, 0$.

Ex. 2.53. (*Vertex operators*) Let $\mathbb{A}$ be of infinite cardinality, x, y be rank-1 elements.

Define T_x, T_{-x} to be the ring homomorphisms $\mathfrak{Sym}(\mathbb{A}) \mapsto \mathfrak{Sym}(\mathbb{A})[[x]]$:

$$T_x(S^k(\mathbb{A})) = \sum_i x^i S^{k-i}(\mathbb{A}) \ , \tag{$\star$}$$

$$T_{-x}(\Lambda^k(\mathbb{A})) = \sum_i (-x)^i \Lambda^{k-i}(\mathbb{A}) \ . \tag{$\star\star$}$$

Check the twisted commutations, for any $f \in \mathfrak{Sym}(\mathbb{A})$:

$$\begin{aligned} T_x(\sigma_y(\mathbb{A})\, f) &= \frac{1}{1-xy}\sigma_y(\mathbb{A})T_x(f) \\ T_{-x}(\lambda_{-y}(\mathbb{A})\, f) &= \frac{1}{1-xy}\lambda_{-y}(\mathbb{A})T_{-x}(f) \\ T_x(\lambda_{-y}(\mathbb{A})\, f) &= (1-xy)\,\lambda_{-y}(\mathbb{A})T_x(f) \\ T_{-x}(\sigma_y(\mathbb{A})\, f) &= (1-xy)\,\sigma_y(\mathbb{A})T_{-x}(f) \end{aligned}$$

Ex. 2.54. (*Vertex operators*) Let $\mathbb{A}$ be arbitrary, t, z be rank-1 elements. After Jing [**75**] define ∇_z to be the following vertex operator on $\mathfrak{Sym}(\mathbb{A})$ (acting on its right) :

$$\nabla_z = \exp\left((1-t^i)z^i\Psi^i/i\right)\ \exp\left((-z_i)^i D_{\Psi^i}/i\right)\ .$$

Let $z_1, \ldots, z_n$ be rank-1 elements, $\mathbb{Z} = z_1 + \cdots + z_n$, $\mathbb{Z}^\vee = z_1^{-1} + \cdots + z_n^{-1}$. Show that $\nabla_{z_1} \cdots \nabla_{z_n}$ is the operator

$$\mathfrak{Sym}(\mathbb{A}) \ni f \longrightarrow \prod_{i<j} \frac{1 - z_j z_i^{-1}}{1 - t z_j z_i^{-1}}\, \sigma_1\left((1-t)\mathbb{A}\mathbb{Z}\right)\, f(\mathbb{A} - \mathbb{Z}^\vee)\ .$$

Ex. 2.55. Let $\mathbb{A}$ be arbitrary, k, n be two integers, x be a rank-1 element. Show that

$$S_{k^n}(\mathbb{A}-x)\, S_{k^{n+1}}(\mathbb{A}) - S_{k^n}(\mathbb{A})\, S_{k^{n+1}}(\mathbb{A}-x) = x\, S_{(k+1)^n}(\mathbb{A}-x)\, S_{(k-1)^{n+1}}(\mathbb{A})\ .$$

For example,

$$S_{55}(\mathbb{A}-x)\, S_{555}(\mathbb{A}) - S_{55}(\mathbb{A})\, S_{555}(\mathbb{A}-x) = x\, S_{66}(\mathbb{A}-x)\, S_{444}(\mathbb{A})\ .$$

Ex. 2.56. Let $\mathbb{A}, \mathbb{B}$ be alphabets; take on $\mathfrak{Sym}(\mathbb{A}) \otimes \mathfrak{Sym}(\mathbb{B})$ the tensor product of the usual scalar products.

Show that $\{\Psi^I(\mathbb{A}-\mathbb{B})\, \Psi^J(\mathbb{A}+\mathbb{B})\}$, (I, J) running over all pairs of partitions, is an orthogonal basis of $\mathfrak{Sym}(\mathbb{A}) \otimes \mathfrak{Sym}(\mathbb{B})$.

Ex. 2.57. Let $k, n \in \mathbb{N}$, $\mathbb{A}_0, \ldots, \mathbb{A}_{n-1}$ be n alphabets. Take on $\mathfrak{Sym}(\mathbb{A}_0) \otimes \cdots \otimes \mathfrak{Sym}(\mathbb{A}_{n-1})$ the tensor product of the usual scalar product on each $\mathfrak{Sym}(\mathbb{A}_i)$.

Show that

$$\prod_i S_{\nu^i}\left(\mathbb{A}_i - \Lambda^1([k])\mathbb{A}_{i-1} + \Lambda^2([k])\mathbb{A}_{i-2} - \cdots \pm \Lambda^k([k])\mathbb{A}_{i-k}\right)$$

is the adjoint basis of

$$\prod_i S_{\nu^i}\left(\mathbb{A}_i + S^1([k])\mathbb{A}_{i+1} + \cdots + S^{n-1-i}([k])\mathbb{A}_{n-1}\right)$$

the indexing set being all n-tuple of partitions $(\nu^0, \ldots, \nu^{n-1})$. The symbol $[k]$ means here $[k] := q + \cdots + q^k$, with q of rank 1; the alphabets of negative index are set equal to 0.

The case $k = 1$ is due to Leclerc and Miyachi [**122**]. For example, for $n = 4$, $k = 1$, the adjoint bases are the set of following elements, indexed by all possible quadruples of partitions :

$$\{\, S_{\nu^0}(\mathbb{A}_0) S_{\nu^1}(\mathbb{A}_1 - q\mathbb{A}_0) S_{\nu^2}(\mathbb{A}_2 - q\mathbb{A}_1) S_{\nu^3}(\mathbb{A}_3 - q\mathbb{A}_2)\,\}\ ,$$

$$\{\, S_{\nu^0}(\mathbb{A}_0 + q\mathbb{A}_1 + q^2\mathbb{A}_2 + q^3\mathbb{A}_3) S_{\nu^1}(\mathbb{A}_1 + q\mathbb{A}_2 + q^2\mathbb{A}_3) S_{\nu^2}(\mathbb{A}_2 + q\mathbb{A}_3) S_{\nu^3}(\mathbb{A}_3)\,\}\ .$$

CHAPTER 3

Euclidean Division

3.1. Euclid's Algorithm

Let $P(x)$ and $Q(x)$ be two polynomials in one variable, with respective alphabets of roots $\mathbb{B}$, $\mathbb{A}$: $P(x) = R(x, \mathbb{B})$, $Q(x) = R(x, \mathbb{A})$.

Algebraic operations on these two polynomials preserve the symmetry in the roots of each of the two polynomials, and thus produce symmetric functions in $\mathbb{A}$ and $\mathbb{B}$ separately.

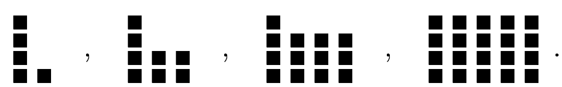
Let us look at the iterated division of P and Q, i.e.

$$
\begin{aligned}
P &= \star Q + \mathcal{R}_1 \ , deg(\mathcal{R}_1) < deg(Q) \\
Q &= \star \mathcal{R}_1 + \mathcal{R}_2 \ , deg(\mathcal{R}_2) < deg(\mathcal{R}_1) \\
\mathcal{R}_1 &= \star \mathcal{R}_2 + \mathcal{R}_3 \, , deg(\mathcal{R}_3) < deg(\mathcal{R}_2) \ \ldots
\end{aligned}
\tag{3.1.1}
$$

the polynomials $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \ldots$ being the successive remainders, the successive quotients denoted $\star$'s being polynomials of degree 1 in the generic case, except possibly for the first two steps. Notice that we treat symmetrically $P(x)$ and $Q(x)$. If Q is of degree striclty bigger than the degree of P, then $\mathcal{R}_1 = P$, and the second step is the division of Q by P.

We shall give a compact description of remainders in terms of Schur functions. Our answer, in (3.2.1), will be in short the following sequence when $dg(P) = 5$, $dg(Q) = 4$:

(3.1.2)



But first, lets us look at the classics. Euler found that the first remainder $\mathcal{R}_1$ can be characterized by the fact that it is, up to a scalar (we mean a function independent of x, but depending on $\mathbb{A}$ and $\mathbb{B}$), the only polynomial in the space linearly generated by $P(x), Q(x), xQ(x), \ldots, x^{m-n}Q(x)$ of degree $\leq n-1$, with $card(\mathbb{A}) = n$, $card(\mathbb{B}) = m$.

In other words, for any integer N, let $\mathfrak{Pol}_N(\mathbb{A}, \mathbb{B})$ be the linear span of

$$S^m(x - \mathbb{B}), \ldots, S^N(x - \mathbb{B})\, ;\ S^n(x - \mathbb{A}), \ldots, S^N(x - \mathbb{A}) \ .$$

Then $\mathcal{R}_1$ is the only polynomial of degree $\leq n - 1$ in $\mathfrak{Pol}_m(\mathbb{A}, \mathbb{B})$.

This result is clear when one eliminates the powers $x^m, x^{m-1}, \ldots, x^n$ in $S^m(x - \mathbb{B})$ by subtracting scalar multiples of $S^m(x - \mathbb{A}), \ldots, S^n(x - \mathbb{A})$: the process is triangular and the scalar by which to multiply the polynomials $S^m(x - \mathbb{A}), \ldots, S^n(x - \mathbb{A})$ are fixed at each step.

This process can conveniently be represented by a determinant

$$\mathcal{D}_1(\mathbb{A},\mathbb{B}) := \begin{vmatrix} S_0(-\mathbb{B}) & \cdots & S_{m-n}(-\mathbb{B}) & S_m(x-\mathbb{B}) \\ S_0(-\mathbb{A}) & \cdots & S_{m-n}(-\mathbb{A}) & S_m(x-\mathbb{A}) \\ S_{-1}(-\mathbb{A}) & \cdots & S_{m-n-1}(-\mathbb{A}) & S_{m-1}(x-\mathbb{A}) \\ \vdots & & \vdots & \vdots \\ S_{n-m}(-\mathbb{A}) & \cdots & S_0(-\mathbb{A}) & S_n(x-\mathbb{A}) \end{vmatrix} . \tag{3.1.3}$$

PROPOSITION 3.1.1. *The determinant $\mathcal{D}_1(\mathbb{A},\mathbb{B})$ is the first remainder of the division of $S^m(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$, with card$(\mathbb{A})=n$, card$(\mathbb{B})=m$.*

Proof. If we expand the determinant according to its last column, we see that it belongs to the space $\mathfrak{Pol}_m$. To find the degree of $\mathcal{D}_1$ in x, one expands each polynomial in the last column as follows (this expression is valid for any $k \leq m$; recall that complete functions of negative index are null):

$$S^k(x-C) = x^m S_{k-m}(-C) + x^{m-1} S_{k-m+1}(-C) + \cdots + x^0 S_k(-C) .$$

Now, subtracting column $1, 2, \ldots$, multiplied by the respective factors $x^m, x^{m-1}, \ldots$ to the last column, one sees that all terms of degree $> n-1$ disappear. □

It is now easy to infer the general theorem. We shall not repeat its proof which is exactly the same as for the first remainder, apart from different degrees.

Euler's characterization remains valid: the r-th remainder is, up to a scalar, the unique polynomial of degree $\leq n-r$, in the space $\mathfrak{Pol}_{m+r-1}$. It is easy to write a determinant which satisfy this characterization.

Let $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ be the following determinant of order $m-n+2r$, with $k = m-n+2r-2$:

$$\mathcal{D}_r(\mathbb{A},\mathbb{B}) := \begin{vmatrix} S_0(-\mathbb{B}) & \cdots & S_k(-\mathbb{B}) & S_{m+r-1}(x-\mathbb{B}) \\ \vdots & & \vdots & \vdots \\ S_{-r+1}(-\mathbb{B}) & \cdots & S_{k-r+1}(-\mathbb{B}) & S_m(x-\mathbb{B}) \\ S_0(-\mathbb{A}) & \cdots & S_k(-\mathbb{A}) & S_{m+r-1}(x-\mathbb{A}) \\ \vdots & & \vdots & \vdots \\ S_{n+1-m-r}(-\mathbb{A}) & \cdots & S_{r-1}(-\mathbb{A}) & S_n(x-\mathbb{A}) \end{vmatrix} . \tag{3.1.4}$$

THEOREM 3.1.2. *The determinant $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ is, up to a non-zero scalar, equal to the r-remainder in the Euclidean division of $S^m(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$.*

For example, with $m=5$, $n=4$, $S^5(x-\mathbb{B}) = x^5+2x^4+3x^3+4x^2+3x+2$, $S^4(x-\mathbb{A}) = (x^5-1)/(x-1)$, one gets the following successive remainders, up to sign :

$$\mathcal{R}_1 = \begin{vmatrix} 1 & 2 & 3x^3+4x^2+3x+2 \\ 1 & 1 & x^3+\ x^2+\ x \\ 0 & 1 & x^3+\ x^2+\ x+1 \end{vmatrix} = x^3+2x^2+x+1,$$

$$\mathcal{R}_2 = \begin{vmatrix} 1 & 2 & 3 & 4 & 3x^2+2x \\ 0 & 1 & 2 & 3 & 4x^2+3x+2 \\ 1 & 1 & 1 & 1 & x^2 \\ 0 & 1 & 1 & 1 & x^2+\ x \\ 0 & 0 & 1 & 1 & x^2+\ x+1 \end{vmatrix} = 2x^2+x+2,$$

$$\mathcal{R}_3 = \begin{vmatrix} 1&2&3&4&3&2&0 \\ 0&1&2&3&4&3&2x \\ 0&0&1&2&3&4&3x+2 \\ 1&1&1&1&1&0&0 \\ 0&1&1&1&1&1&0 \\ 0&0&1&1&1&1&x \\ 0&0&0&1&1&1&x+1 \end{vmatrix} = 3x+2,\ \mathcal{R}_4 = \begin{vmatrix} 1&2&3&4&3&2&0&0&0 \\ 0&1&2&3&4&3&2&0&0 \\ 0&0&1&2&3&4&3&2&0 \\ 0&0&0&1&2&3&4&3&2 \\ 1&1&1&1&1&0&0&0&0 \\ 0&1&1&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1&0&0 \\ 0&0&0&1&1&1&1&1&0 \\ 0&0&0&0&1&1&1&1&1 \end{vmatrix} = 5\ .$$

Remark If $S^m(x-\mathbb{B})$ and $S^n(x-\mathbb{A})$ have a greatest common divisor G of order $n-r$, then the successive remainders will be divisible by G. This implies that r-th remainder is equal to G (up to a scalar), and that the remainders of higher order are null. Conversely, if $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ is different from 0, and the determinants $\mathcal{D}_p(\mathbb{A},\mathbb{B})$ are all 0 for $p>r$, then the greatest common divisor of $S^m(x-\mathbb{B})$ and $S^n(x-\mathbb{A})$ is of degree $n-r$ and equal to $\mathcal{D}_r(\mathbb{A},\mathbb{B})$.

Let us notice that the determinant $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ also furnishes Euler's multiplicators, i.e. the polynomials $C_\mathbb{A}$, $C_\mathbb{B}$ such that

$$\mathcal{D}_r(\mathbb{A},\mathbb{B}) = C_\mathbb{A}\, R(x,\mathbb{A}) + C_\mathbb{B}\, R(x,\mathbb{B})\ .$$

Indeed, evaluating $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ modulo $R(x,\mathbb{A})$ consists in changing the last column into $[S_{m+r-1}(x-\mathbb{B}),\ldots,S_m(x-\mathbb{B}),0,\ldots,0]$. Subtracting x to the alphabets in the first $r-1$ rows, one gets, as a last column,

$$[\,S_{m+r-1}(-\mathbb{B}),\ldots,S_{m-1}(-\mathbb{B}),\ S_m(x-\mathbb{B}),0,\ldots,0\,]$$

that is, $[\,0,\ldots,0,R(x-\mathbb{B}),0,\ldots,0\,]$, because the $S_k(-\mathbb{B})$ are $\pm$ the elementary symmetric functions of an alphabet of cardinality m, and therefore are null for $k>m$.

Now the cofactor of $R(x-\mathbb{B})$ is the determinant

$$\begin{vmatrix} S_0(-x-\mathbb{B}) & \cdots & S_k(-x-\mathbb{B}) \\ \vdots & & \vdots \\ S_{-r+2}(-x-\mathbb{B}) & \cdots & S_{k-r+2}(-x-\mathbb{B}) \\ S_0(-\mathbb{A}) & \cdots & S_k(-\mathbb{A}) \\ \vdots & & \vdots \\ S_{n+1-m-r}(-\mathbb{A}) & \cdots & S_{r-1}(-\mathbb{A}) \end{vmatrix}$$

Expanding this last determinant according to the first $r-1$ rows, one recognize that it is equal to $S_\square(\mathbb{A}-x-\mathbb{B})$, with $\square=(m-n+r)^{r-1}$, $k=m-n+2r-2$.

By symmetry changing $\mathbb{A},\mathbb{B}$, one therefore gets

$$\text{(3.1.5)}\quad \mathcal{D}_r(\mathbb{A},\mathbb{B}) = \pm S_{(m-n+r)^{r-1}}(\mathbb{A}-x-\mathbb{B})R(x,\mathbb{B}) \pm S_{r^{m-n+r-1}}(\mathbb{B}-x-\mathbb{A})R(x,\mathbb{A})\ ,$$

with signs that specialists will know how to write. This can also be written

$$\text{(3.1.6)}\qquad \mathcal{D}_r(\mathbb{A},\mathbb{B}) = \pm S_{(m-n+r)^{r-1};m}(\mathbb{A}-\mathbb{B};x-\mathbb{B}) \pm S_{r^{m-n+r-1};n}(\mathbb{B}-\mathbb{A};x-\mathbb{A})\,.$$

In particular, when the two polynomials are relatively prime, then the last remainder is equal to the resultant and one has the identity

$$\text{(3.1.7)}\qquad 1 = \frac{S_{n^{m-1}}(\mathbb{B}-\mathbb{A}-x)}{S_{n^m}(\mathbb{B}-\mathbb{A})} S^n(x-\mathbb{A}) + \frac{S_{m^{n-1}}(\mathbb{A}-\mathbb{B}-x)}{S_{m^n}(\mathbb{A}-\mathbb{B})} S^m(x-\mathbb{B})\ .$$

This identity is easy to prove directly. Indeed, the RHS is a polynomial of degree $\leq m{+}n{-}1$ in x, that it is sufficient to test on $x \in \mathbb{A} \cup \mathbb{B}$. Let $a \in \mathbb{A}$, $\mathbb{A}' = \mathbb{A}{-}a$. Then the RHS becomes

$$S_{m^{n-1}}(\mathbb{A}'{+}a - \mathbb{B}{-}a)\, R(\mathbb{A},\mathbb{B})^{-1} R(a,\mathbb{B}) = R(\mathbb{A}',\mathbb{B})\, R(\mathbb{A}',\mathbb{B})^{-1} = 1 ,$$

and therefore the RHS is the constant polynomial equal to 1.

Let us parametrize the remainder by its degree d, to emphasize symmetry between $\mathbb{A}$ and $\mathbb{B}$. Then (3.1.5) shows that $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ can be interpreted as a first remainder :

LEMMA 3.1.3. *Let $\mathbb{A}$, $\mathbb{B}$ be two alphabets, card$(\mathbb{A}) = n$, card$(\mathbb{B}) = m$, d be an integer : $d < m, n$. Then the remainder of degree d in the Euclidean division of $R(x,\mathbb{B})$ by $R(x,\mathbb{A})$ is equal, up to a constant, to the remainder of*

$$S_{(m-d)^{n-d-1}}(\mathbb{A}{-}x{-}\mathbb{B})\, R(x,\mathbb{B}) = S_{(m-d)^{n-d-1},m}(\mathbb{A}{-}\mathbb{B}, x{-}\mathbb{B}) \tag{3.1.8}$$

modulo $R(x,\mathbb{A})$, and to the remainder of

$$S_{(n-d)^{m-d-1}}(\mathbb{B}{-}x{-}\mathbb{A})\, R(x,\mathbb{A}) = S_{(n-d)^{m-d-1},n}(\mathbb{B}{-}\mathbb{A}, x{-}\mathbb{A}) \tag{3.1.9}$$

modulo $R(x,\mathbb{B})$.

For example, for $m = 5$, $n = 4$, and $\mathbb{B} = 0$, the remainders

$$S_{111;2}(\mathbb{A}{-}x;\mathbb{A}),\ S_{11;33}(\mathbb{A}{-}x;\mathbb{A}),\ S_{1;444}(\mathbb{A}{-}x;\mathbb{A}),\ S_{5555}(\mathbb{A})$$

are equal to

$$S_1(\mathbb{A}{+}x)S^4(x{-}\mathbb{A}),\ S_{22}(\mathbb{A}{+}x)S^4(x{-}\mathbb{A}),\ S_{333}(\mathbb{A}{+}x)S^4(x{-}\mathbb{A}),\ S_{4444}(\mathbb{A}{+}x)S^4(x{-}\mathbb{A}),$$

modulo x^5, and equal, modulo $S^4(x{-}\mathbb{A})$, to

$$-x^5,\ S_3(\mathbb{A}{-}x)x^5,\ -S_{44}(\mathbb{A}{-}x)x^5,\ S_{555}(\mathbb{A}{-}x)x^5 .$$

3.2. Remainders as Schur functions

Forgetting about the preceding computations, let us see how to write remainders as Schur functions, avoiding manipulations of determinants.

The only non evident step is to start from

$$S_{\square}(\mathbb{A} - \mathbb{B} - x)S^m(x{-}\mathbb{B}) = S_{\square;\,m}(\mathbb{A} - \mathbb{B};\ x{-}\mathbb{B}) ,$$

that one evaluates modulo $S^n(x{-}\mathbb{A})$, with $\square = (m{-}n{+}r)^{r-1}$.

Writing $x{-}\mathbb{B} = x{-}\mathbb{A} + (\mathbb{A}{-}\mathbb{B})$, one can develop by linearity the last column :

$$S_{\square;\,m}(\mathbb{A} - \mathbb{B};\ x{-}\mathbb{B}) = \sum_{i=0}^{\infty} S_{\square,\,m-i}(\mathbb{A} - \mathbb{B})S^i(x{-}\mathbb{A}) .$$

This sum decomposes into three parts:

- $i \geq n$. The factors $S^i(x{-}\mathbb{A})$ are null modulo $S^n(x{-}\mathbb{A})$ in that case.
- $n{-}r < i < n$. The factors $S_{\square,\,m-i}(\mathbb{A} - \mathbb{B})$ are null, having two identical columns.
- $i \leq n{-}r$, giving a polynomial of degree $\leq n{-}r$.

Thus the only contribution comes from the subsum corresponding to $i \leq n{-}r$. Changing $S^i(x{-}\mathbb{A})$ into $(-1)^i S_{1^i}(\mathbb{A}{-}x)$, one recognizes in it the Laplace expansion, along the last column, of $\pm S_{1^{n-r};\,(m-n+r)^r}(\mathbb{A}{-}x;\mathbb{A}{-}\mathbb{B})$.

The Schur function thus obtained must be the r-th remainder, up to normalization, because it is of degree $\leq n-r$ and belongs to the space generated by the remainders of $S^m(x-\mathbb{B}), \ldots, S^{m+r-1}(x-\mathbb{B})$ modulo $S^n(x-\mathbb{A})$.

In final, one has the theorem

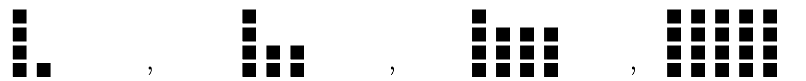
THEOREM 3.2.1. *The r-remainder in the division of $S^m(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$ is equal to*

$$S_{1^{n-r};\,(m-n+r)^r}(\mathbb{A}-x;\mathbb{A}-\mathbb{B}) \ . \tag{3.2.1}$$

In particular, the first remainder is equal to

$$S_{1^{n-1};\,m-n+1}(\mathbb{A}-x,\mathbb{A}-\mathbb{B}) \ . \tag{3.2.2}$$

We can now put the captions under (3.1.2), when $m=5$, $n=4$:



$S_{111;2}(\mathbb{A}-x;\mathbb{A}-\mathbb{B})$ $\quad$ $S_{11;33}(\mathbb{A}-x;\mathbb{A}-\mathbb{B})$ $\quad$ $S_{1;444}(\mathbb{A}-x;\mathbb{A}-\mathbb{B})$ $\quad$ $S_{5555}(\mathbb{A}-\mathbb{B})$

Let us however repeat that we have restricted to the generic case, where $\mathbb{A}$ and $\mathbb{B}$ are independent alphabets. One can freely specialize variables in the algebraic identities that we have written, if no disaster occurs, like a zero in denominator. For example, the determinantal expression (3.2.1) of the remainder of degree d is an expression of the G.C.D. of $R(x,\mathbb{A})$ and $R(x,\mathbb{B})$ when $\mathbb{A}\cap\mathbb{B}$ is of cardinality d.

3.3. Companion Matrix

One can use linear algebra to express the first remainder. Let $\mathbb{A}$ be still of cardinality n. Let $\mathfrak{Pol}_n$ be the space of polynomials of degree $\leq n-1$; it has basis $\{x^0, x^1, \ldots, x^{n-1}\}$. Let u be the morphism: $\mathfrak{Pol}_n \mapsto \mathfrak{Pol}_n$ "multiplication by x modulo $S^n(x-\mathbb{A})$".

Let $\mathcal{CC}(\mathbb{A})$ (resp. $\mathcal{C}(\mathbb{A})$) be the matrix the $n\times\infty$ (resp. $n\times n$) matrix expressing the remainders of $x^0, x^1, \ldots, x^\infty$ (resp. $x^1, \ldots, x^n$) modulo $S^n(x-\mathbb{A})$. These two matrices are called *companion matrices of* $S^n(x-\mathbb{A})$. The matrix expressing u in the basis of powers of x is precisely $C(\mathbb{A})$ and its powers are sub-matrices of $\mathcal{CC}(\mathbb{A})$. We know from (3.2.2) that the remainder of x^k is $S_{1^{n-1},\,k-n+1}(\mathbb{A}-x,\mathbb{A})$, and therefore the coefficients of the remainder of x^k are hook-Schur functions, up to sign. They can be written $S_{k,0^{n-1}}(\mathbb{A}), S_{k-1,0^{n-2}}(\mathbb{A}), \ldots, S_{k-n+1}(\mathbb{A})$. Finally

$$\mathcal{CC}(\mathbb{A}) = \Big[S_{k-i,0^{n-i-1}}(\mathbb{A})\Big]_{0\leq k,\,0\leq i\leq n-1} \ .$$

For example, for $n=3$, the infinite companion matrix is

$$\mathcal{CC}(\mathbb{A}) = \begin{bmatrix} S_{000} & S_{100} & S_{200} & S_{300} & S_{400} & S_{500} & \cdots \\ S_{-10} & S_{00} & S_{10} & S_{20} & S_{30} & S_{40} & \cdots \\ S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 & \cdots \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & S_{300} & S_{400} & S_{500} & \cdots \\ 0 & 1 & 0 & S_{20} & S_{30} & S_{40} & \cdots \\ 0 & 0 & 1 & S_1 & S_2 & S_3 & \cdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & S_{111} & S_{112} & S_{113} & \cdots \\ 0 & 1 & 0 & -S_{11} & -S_{12} & -S_{13} & \cdots \\ 0 & 0 & 1 & S_1 & S_2 & S_3 & \cdots \end{bmatrix}$$

and the finite one is

$$\mathcal{C}(A) = \begin{bmatrix} 0 & 0 & S_{111} \\ 1 & 0 & -S_{11} \\ 0 & 1 & S_1 \end{bmatrix} \ .$$

Suppose that all the elements of $\mathbb{A}$ be distinct. Let f be any polynomial. Then the composite of "multiplication by $f(x)$, and taking the remainder" is diagonal in the basis $\{S^{n-1}(x+a-\mathbb{A})\}_{a\in\mathbb{A}}$, with diagonal composed of $f(a) : a \in \mathbb{A}$. Indeed, since $f(x)-f(a)$ is a multiple of $(x-a)$, then $f(x)\, S^{n-1}(x+a-\mathbb{A})$ is congruent to $f(a)\, S^{n-1}(x+a-\mathbb{A})$ modulo $S^n(x-\mathbb{A})$. The *trace* of this morphism is therefore $\sum_{a\in\mathbb{A}} f(a)$, and each power sum $\psi_k(\mathbb{A}) = \sum_{a\in\mathbb{A}} a^k$ appears as the trace of u^k.

The resultant of two polynomials $R(x,\mathbb{A})$ and $R(x,\mathbb{B})$ can, of course, be expressed with the help of a companion matrix. Indeed, the determinant of the matrix $R\big(\mathcal{C}(\mathbb{A}),\mathbb{B}\big)$ factorizes into the product

$$\prod_{b\in\mathbb{B}} \det\left(\mathcal{C}(\mathbb{A})-b\right) = \prod_{b\in\mathbb{B}} R(\mathbb{A},b) = R(\mathbb{A},\mathbb{B}) \ ,$$

and the result applies to any pair of polynomials, not necessarily in factorized form.

For example, for $|\mathbb{A}|=4$, $|\mathbb{B}|=3$, writing e_i for $\Lambda^i(\mathbb{A})$ and f_i for $\Lambda^i(\mathbb{B})$, one has the matrix identity

$$\prod_{b\in\mathbb{B}} \begin{bmatrix} -b & 0 & 0 & -e_4 \\ 1 & -b & 0 & e_3 \\ 0 & 1 & -b & -e_2 \\ 0 & 0 & 1 & e_1-b \end{bmatrix} =$$

$$\begin{bmatrix} -f_3 & -e_4 & f_1e_4-e_4e_1 & e_4e_2-e_4e_1^2+f_1e_4e_1-f_2e_4 \\ f_2 & -f_3+e_3 & -f_1e_3-e_4+e_3e_1 & -e_3e_2+e_3e_1^2-e_4e_1-f_1(-e_4+e_3e_1)+f_2e_3 \\ -f_1 & f_2-e_2 & -f_3+f_1e_2-e_2e_1+e_3 & -e_4-e_2e_1^2+e_3e_1+e_2^2+f_1(e_2e_1-e_3)-f_2e_2 \\ 1 & -f_1+e_1 & f_2-f_1e_1-e_2+e_1^2 & -2e_2e_1+e_1^3+e_3-f_1(-e_2+e_1^2)+f_2e_1-f_3 \end{bmatrix},$$

both matrices having determinant $R(\mathbb{A},\mathbb{B}) = S_{3333}(\mathbb{A}-\mathbb{B})$.

3.4. Bézout's Matrix

Bézout [**16**] adopted another point of view about the elimination of an indeterminate x between two polynomials $S^m(x-\mathbb{B})$ and $S^n(x-\mathbb{A})$, $n \geq m$ (that is, about finding the remainder of degree $0 = R(\mathbb{A},\mathbb{B})$).

He first built n *Équations dérivées* $P_j := \begin{vmatrix} S^n(x-\mathbb{B}) & S^n(x-\mathbb{A}) \\ S^j(x-\mathbb{B}) & S^j(x-\mathbb{A}) \end{vmatrix}$, $0 \leq j \leq n--1$, and obtained

THEOREM 3.4.1. *Given two polynomials $S^m(x-\mathbb{B})$ and $S^n(x-\mathbb{A})$, $n \geq m$, let $Bez(\mathbb{A},\mathbb{B})$ be the $n\times n$ matrix of coefficients of $P_0,\ldots,P_{n-1}$, i.e. $Bez(\mathbb{A},\mathbb{B}) = \Big[coef(P_j,x^i)\Big]_{0\leq i,j\leq n-1}$. Then*

$$\det(Bez(\mathbb{A},\mathbb{B})) = R(\mathbb{A},\mathbb{B}) \ .$$

We shall prove Bézout's theorem by interpreting the matrix $Bez(\mathbb{A},\mathbb{B})$:

PROPOSITION 3.4.2. (1) *Bézout's matrix is the matrix of the linear morphism $u :=$ "multiplication by $S^n(x-\mathbb{B})$ modulo $S^n(x-\mathbb{A})$" from $\mathfrak{Pol}_n$ with basis $\{S^0(x-\mathbb{A}),\ldots,S^{n-1}(x-\mathbb{A})\}$ to $\mathfrak{Pol}_n$ with basis $\{x^0,\ldots,x^{n-1}\}$.*
(2) *It factorizes*

$$\mathfrak{Bez}(\mathbb{A},\mathbb{B}) = \mathbb{S}_{0^n}(-\mathbb{A})\ \mathbb{S}_{n^n}(\mathbb{A}-\mathbb{B})\ \mathbb{S}_{0^n}(-\mathbb{A}) \ .$$

(3) *In the case all $a \in \mathbb{A}$ are distinct, $\mathfrak{Bez}(\mathbb{A}, \mathbb{B})$ is equivalent to a diagonal matrix, with diagonal*

$$[\ldots,\ S^{n-1}(2a - \mathbb{A})\, S^n(a - \mathbb{B}), \ldots]_{a \in \mathbb{A}}\ .$$

Proof. Modulo $S^n(x - \mathbb{A})$, one has $P_j \equiv S^n(x - \mathbb{B})\, S^j(x - \mathbb{A})$, and this gives 1).

The matrix expressing u, from basis $\{x^0, \ldots, x^{n-1}\}$ to basis $\{S^0(x{-}\mathbb{A}), \ldots, S^{n-1}(x{-}\mathbb{A})\}$ is $\mathbb{S}_{n^n}(\mathbb{A}{-}\mathbb{B})$ because

$$x^j S^n(x{-}\mathbb{B}) = S^{n+j}(x{-}\mathbb{B}) = S^{n+j}\big((x{-}\mathbb{A}) + (\mathbb{A}{-}\mathbb{B})\big) \equiv \sum_{k=0}^{n-1} S^k(x{-}\mathbb{A})\, S^{n+j-k}(\mathbb{A}{-}\mathbb{B})$$

Using that the change of basis is given by the matrix $\mathbb{S}_{0^n}(-\mathbb{A})$, one obtains 2).

Take now any $a \in \mathbb{A}$. The product $\mathfrak{Bez}(\mathbb{A}, \mathbb{B})\ [a^{n-1}, \ldots, a^0]^t$ is by definition of $\mathfrak{Bez}$, the vector $[P_{n-1}(a), \ldots, P_0(a)]^t$, i.e.

$$[S^{n-1}(a - \mathbb{A}) S^n(a - \mathbb{B}), \ldots,\ S^0(a - \mathbb{A}) S^n(a - \mathbb{B})]^t\ .$$

This implies that

$$\mathfrak{Bez}(\mathbb{A}, \mathbb{B}) \begin{bmatrix} a_1^{n-1} & \cdots & a_n^{n-1} \\ \vdots & & \vdots \\ a_1^0 & \cdots & a_n^0 \end{bmatrix} =$$

$$\begin{bmatrix} S^{n-1}(a_1 - \mathbb{A}) & \cdots & S^{n-1}(a_n - \mathbb{A}) \\ \vdots & & \vdots \\ S^0(a_1 - \mathbb{A}) & \cdots & S^0(a_n - \mathbb{A}) \end{bmatrix} \begin{bmatrix} S^n(a_1{-}\mathbb{B}) & 0 \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & S^n(a_n{-}\mathbb{B}) \end{bmatrix}$$

and therefore, $\mathfrak{Bez}(\mathbb{A}, \mathbb{B})$ can be reduced to the diagonal form stated in 3) by multiplying it on both sides by invertible matrices. ☐

From property 2) follows that $\det(\mathfrak{Bez}(\mathbb{A}, \mathbb{B})) = \det(\mathbb{S}_{n^n}(\mathbb{A} - \mathbb{B})) = R(\mathbb{A}, \mathbb{B})$.

For example, Bézout matrices, for $n = m = 2, 3, 4$ are, writing e_i for $\Lambda^i(\mathbb{A})$, f_i for $\Lambda^i(\mathbb{B})$,

$$\begin{bmatrix} f_2 - e_2 & -e_1 f_2 + f_1 e_2 \\ -f_1 + e_1 & f_2 - e_2 \end{bmatrix} \qquad \begin{bmatrix} e_3 - f_3 & e_1 f_3 - f_1 e_3 & f_2 e_3 - e_2 f_3 \\ f_2 - e_2 & -f_3 - e_1 f_2 + e_3 + f_1 e_2 & e_1 f_3 - f_1 e_3 \\ -f_1 + e_1 & f_2 - e_2 & e_3 - f_3 \end{bmatrix}$$

$$\begin{bmatrix} f_4 - e_4 & -e_1 f_4 + f_1 e_4 & e_2 f_4 - f_2 e_4 & -e_3 f_4 + f_3 e_4 \\ e_3 - f_3 & e_1 f_3 + f_4 - e_4 - f_1 e_3 & -e_2 f_3 + f_2 e_3 + f_1 e_4 - e_1 f_4 & e_2 f_4 - f_2 e_4 \\ f_2 - e_2 & -f_3 - e_1 f_2 + e_3 + f_1 e_2 & e_1 f_3 + f_4 - e_4 - f_1 e_3 & -e_1 f_4 + f_1 e_4 \\ -f_1 + e_1 & f_2 - e_2 & e_3 - f_3 & f_4 - e_4 \end{bmatrix}$$

Suppose $m = n$. Cayley **[31]** found the following interesting interpretation of $\mathfrak{Bez}(\mathbb{A}, \mathbb{B})$, as the matrix of a quadratic form.

PROPOSITION 3.4.3. *Introduce two indeterminates x, y and define*

$$\sum B_{i,j}\, x^i y^j := \frac{S^n(x - \mathbb{A})\, S^n(y - \mathbb{B}) - S^n(x - \mathbb{B})\, S^n(y - \mathbb{A})}{x - y}\ .$$

Then the matrix $[B_{i,n-1-j}]_{0 \le i,j \le n-1}$ is equal to the Bézoutian $\mathfrak{Bez}(\mathbb{A}, \mathbb{B})$.

Proof. $S^n(x-\mathbb{A})\, S^n(y-\mathbb{B}) - S^n(x-\mathbb{B})\, S^n(y-\mathbb{A})$ can be written as the determinant of the product

$$\begin{bmatrix} x^0 & \cdots & x^n \\ y^0 & \cdots & y^n \end{bmatrix} \begin{bmatrix} S^n(-\mathbb{A}) & S^n(-\mathbb{B}) \\ \vdots & \vdots \\ S^0(-\mathbb{A}) & S^0(-\mathbb{B}) \end{bmatrix} .$$

If M is the matrix on the right, Binet-Cauchy expansion gives that the quadratic form is equal to $\sum_{0 \le I \le (n-1)^2} S_I(x+y)\, M_{0^2/I}$. Therefore, the coefficient of $x^i y^j$, $0 \le i \le j \le n-1$, is equal to $M_{0^2/[i,j]} + M_{0^2/[i-1,j+1]} + \cdots + M_{0^2/[0,i+j]}$, which is indeed an entry of $\mathfrak{Bez}(\mathbb{A}, \mathbb{B})$. $\square$

For example, for $n = 2, 3$, the quadratic forms are

$$[1\ x] \begin{bmatrix} f_2 - e_2 & -e_1 f_2 + f_1 e_2 \\ -f_1 + e_1 & f_2 - e_2 \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix} = -xyf_1 + xf_2 + xye_1 - xe_2 + f_1 e_2 - e_1 f_2 + f_2 y - e_2 y$$

$$[1\ x\ x^2] \begin{bmatrix} e_3 - f_3 & e_1 f_3 - f_1 e_3 & f_2 e_3 - e_2 f_3 \\ f_2 - e_2 & -f_3 - e_1 f_2 + e_3 + f_1 e_2 & e_1 f_3 - f_1 e_3 \\ -f_1 + e_1 & f_2 - e_2 & e_3 - f_3 \end{bmatrix} \begin{bmatrix} y^2 \\ y \\ 1 \end{bmatrix} =$$

$$x^2 f_2 y - x^2 f_1 y^2 + x^2 e_1 y^2 - x^2 e_2 y - x^2 f_3 + x^2 e_3 + x e_1 f_3 - x f_2 y e_1 - x f_1 e_3 + x e_2 y f_1 + x f_2 y^2 - x e_2 y^2 - x f_3 y + x e_3 y - e_2 f_3 + f_2 e_3 + y e_1 f_3 - y f_1 e_3 - f_3 y^2 + e_3 y^2$$

3.5. Sylvester's Summations

Let us take two alphabets $\mathbb{A}$ and $\mathbb{B}$, $|\mathbb{A}| = n$, $|\mathbb{B}| = m$ and let p, q be two positive integers such that $d := p + q < \min(m, n)$.

Sylvester obtained an expression of the remainder of degree d as a double summation on the roots of $S^n(x-\mathbb{A})$ and $S^m(x-\mathbb{B})$:

THEOREM 3.5.1. *The remainder of degree d of the division of $S^m(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$ is equal to*

$$G_d := \sum_{\mathbb{A}', \mathbb{B}'} S^p(x-\mathbb{A}') S^q(x-\mathbb{B}')\, \frac{R(\mathbb{A}', \mathbb{B}') R(\mathbb{A}-\mathbb{A}', \mathbb{B}-\mathbb{B}')}{R(\mathbb{A}', \mathbb{A}-\mathbb{A}') R(\mathbb{B}', \mathbb{B}-\mathbb{B}')} \,, \tag{3.5.1}$$

sum over all sub-alphabets $\mathbb{A}' \subset \mathbb{A}$, $\mathbb{B}' \subset \mathbb{B}$: $|\mathbb{A}'| = p$, $|\mathbb{B}'| = q$.

Proof. Let us notice that if $\mathbb{A} \cap \mathbb{B} = \mathbb{C}$, with $|\mathbb{C}| = d$, then the summation in G_d reduces to the terms such that $\mathbb{C}$ is the (disjoint) union of $\mathbb{A}'$, $\mathbb{B}'$. In that case G_d specializes to

$$\binom{d}{p} S^d(x-\mathbb{C})\, R(\mathbb{A}-\mathbb{C}, \mathbb{B}-\mathbb{C}) \ .$$

However, such a specialization is not sufficient to characterize the remainder of degree d, and it is not evident to check that G_d belongs to the space $\mathfrak{Pol}_{m+n-d-1}(\mathbb{A}, \mathbb{B})$. Without an appropriate characterization, we are forced to compute explicitly G_d as a symmetric function in $\mathbb{A}$ and $\mathbb{B}$.

This computation is a little intricate (except when p, or q is equal to 0. Summations on subsets of a set essentially reduce to Lagrange interpolation) and we prefer referring to [**104**], where a proof is given by induction, by specializing x to a letter in $\mathbb{A}$ or $\mathbb{B}$.

Let us notice that, using $R(\mathbb{A},\mathbb{B}) = \Delta(\mathbb{A}+\mathbb{B})\Delta(\mathbb{A})^{-1}\Delta(\mathbb{B})^{-1}$ (we order the alphabet $\mathbb{A}+\mathbb{B}$ by taking letters of $\mathbb{A}$ first) we can rewrite Sylvester's summation as

$$G_d := \frac{1}{\Delta(\mathbb{A})\Delta(\mathbb{B})} \sum_{\mathbb{A}',\mathbb{B}'} \pm\Delta(x+\mathbb{A}'+\mathbb{B}')\,\Delta((\mathbb{A}-\mathbb{A}')+(\mathbb{B}-\mathbb{B}'))\ . \tag{3.5.2}$$

Now, all the functions that we have to deal with are minors of a Vandermonde matrix; obtaining the required identity by using identities on minors is however more intricate than reasoning on specializations.

For example, for $p = q = 1$, and $m = n = 3$, one obtains twice the first remainder with each of the following summations

$$\sum_{a\in\mathbb{A},b\in\mathbb{B}} (x-a)(x-b)\frac{(a-b)S_{22}(\mathbb{A}-\mathbb{B}-a+b)}{S_2(2a-\mathbb{A})S_2(2b-\mathbb{B})}\ ,$$

$$\frac{1}{\Delta(\mathbb{A})\Delta(\mathbb{B})}\sum_{i,j=0}^{3}(-1)^{i+j}\Delta(x,a_i,a_j)\,\Delta(a_1,\ldots,\widehat{a_i},\ldots,a_3,\,b_1,\ldots,\widehat{b_j},\ldots,b_3)\,.$$

3.6. Sturm Sequence

The special case where one divides a polynomial by its derivative has been more specially studied by Sturm [**164**]. Let us first look at the remainder of degree 0, that is the resultant of the polynomial $S^n(x-\mathbb{A})$ and its derivative, that we shall write $n\,S^{n-1}(x-\mathbb{A}^{der})$: $\mathbb{A}^{der}$ is the alphabet of cardinality $n-1$ such that $\Lambda^i(\mathbb{A}^{der}) = (1-\frac{i}{n})\,\Lambda^i(\mathbb{A})$.

Now, according to Ex. 2.30, one has that $\frac{1}{n}\psi_j(\mathbb{A}-\mathbb{A}^{der}) = S^j(\mathbb{A}-\mathbb{A}^{der})$, $j\geq 0$. Therefore the resultant of $S^n(x-\mathbb{A})$ and $S^{n-1}(x-\mathbb{A}^{der})$, which is equal to $S_{(n-1)^n}(\mathbb{A}-\mathbb{A}^{der})$, can also be written as a $n\times n$ determinant of power sums

$$\Big|\psi_{i+j}(\mathbb{A})\Big|_{0\leq i,j\leq n-1}\ .$$

However the above matrix factorizes into the product

$$\begin{bmatrix} 1 & \cdots & 1\\ a_1 & \cdots & a_n\\ \vdots & & \vdots\\ a_1^{n-1} & \cdots & a_n^{n-1}\end{bmatrix}\ \begin{bmatrix} 1 & a_1 & \cdots & a_1^{n-1}\\ \vdots & \vdots & & \vdots\\ 1 & a_n & \cdots & a_n^{n-1}\end{bmatrix}$$

and therefore the discriminant is equal to

$$\Delta(\mathbb{A})^2 = \Big|\psi_{i+j}(\mathbb{A})\Big|_{0\leq i,j\leq n-1}\ . \tag{3.6.1}$$

Sturm found an expression which generalizes the expression (3.6.1) to the successive remainders. He defined

$$\left\{\begin{array}{lcl} f_0 & = & S^n(x-\mathbb{A})\\ f_1 & = & \sum_a S^{n-1}(x+a-\mathbb{A})\\ \vdots & & \vdots\\ f_r & = & \sum_{\mathbb{B}\subset\mathbb{A}} S^{n-r}(x+\mathbb{B}-\mathbb{A})\,\Delta(\mathbb{B})^2 \quad 1\leq r\leq n\\ \vdots & & \vdots \end{array}\right. \tag{3.6.2}$$

the sum for f_r being over all sub-alphabets $\mathbb{B}$ of $\mathbb{A}$ of cardinality r.

Let us rewrite these equations, following Cayley [**32**], art.[65], by factoring out f_0, and using that $f_0 = S^{n-r}(x-(\mathbb{A}-\mathbb{B}))\, S^r(x-\mathbb{B})$:

$$f_r = f_0 \sum_{\mathbb{B}} \frac{\Delta(\mathbb{B})^2}{S^r(x-\mathbb{B})} = f_0 \sum_{\mathbb{B}} \sum_{k=0}^{\infty} x^{-r-k}\, \Delta(\mathbb{B})^2\, S^k(\mathbb{B}) \;. \tag{3.6.3}$$

Now, $\Delta(\mathbb{B})\, S^k(\mathbb{B})$ is equal to the determinant

$$\left| 1,\, b\, \ldots,\, b^{r-2},\, b^{r-1+k} \right| .$$

Thanks to Binet-Cauchy theorem, the sum $\sum_{\mathbb{B}} \Delta(\mathbb{B})\ \Delta(\mathbb{B}) S^k(\mathbb{B})$ is equal to the determinant

$$\begin{vmatrix} \psi_0(\mathbb{A}) & \cdots & \psi_{r-2}(\mathbb{A}) & \psi_{r-1+k}(\mathbb{A}) \\ \vdots & & \vdots & \vdots \\ \psi_{r-1}(\mathbb{A}) & \cdots & \psi_{2r-3}(\mathbb{A}) & \psi_{2r-2+k}(\mathbb{A}) \end{vmatrix} .$$

Because $\sum_k x^{-k}\, \psi_{k+j}(\mathbb{A}) = \sum_a a^j/(1-a/x)$, one finally gets

$$f_r = f_0\, x^{-r} \begin{vmatrix} \psi_0(\mathbb{A}) & \cdots & \psi_{r-2}(\mathbb{A}) & \sum a^{r-1}/(1-a/x) \\ \vdots & & \vdots & \vdots \\ \psi_{r-1}(\mathbb{A}) & \cdots & \psi_{2r-3}(\mathbb{A}) & \sum a^{2r-2}/(1-a/x) \end{vmatrix} ,$$

that one can also write, by adding to the last column a linear combination of the first ones,

$$f_r = f_0 \begin{vmatrix} \psi_0(\mathbb{A}) & \cdots & \psi_{r-2}(\mathbb{A}) & \sum 1/(x-a) \\ \vdots & & \vdots & \vdots \\ \psi_{r-1}(\mathbb{A}) & \cdots & \psi_{2r-3}(\mathbb{A}) & \sum a^{r-1}/(x-a) \end{vmatrix} . \tag{3.6.4}$$

However, for every $k \geq 0$, $a^k = x^k + (a-x)S^{k-1}(x+a)$. It implies that f_r is congruent, modulo f_0, to

$$\begin{aligned} f_r \equiv f_0 &\begin{vmatrix} \psi_0(\mathbb{A}) & \cdots & \psi_{r-2}(\mathbb{A}) & \sum 1/(x-a) \\ \vdots & & \vdots & \vdots \\ \psi_{r-1}(\mathbb{A}) & \cdots & \psi_{2r-3}(\mathbb{A}) & \sum x^{r-1}/(x-a) \end{vmatrix} \\ &= \begin{vmatrix} \psi_0(\mathbb{A}) & \cdots & \psi_{r-2}(\mathbb{A}) & 1 \\ \vdots & & \vdots & \vdots \\ \psi_{r-1}(\mathbb{A}) & \cdots & \psi_{2r-3}(\mathbb{A}) & x^{r-1} \end{vmatrix} f_1 \;. \end{aligned} \tag{3.6.5}$$

This last expression belongs to the linear span of $\{f_1,\, x\, f_1, \ldots,\, x^{r-1}\, f_1\}$ and from (3.6.2), is of degree $\leq n-r$. Therefore it is the r-th remainder of the division of $S^n(x-\mathbb{A})$ by its derivative. We can now state Sturm's theorem [**164**] :

THEOREM 3.6.1. *Given a polynomial $f_0 = S^n(x-\mathbb{A})$ of degree n, the Sturm sequence described in (3.6.2) is the sequence of successive remainders in the division of f_0 by its derivative.*

3.7. Wroński's Algorithm

Wroński's method is different: he uses division by a single polynomial $S^n(x-\mathbb{A})$, instead of dividing by the successive remainders.

Let us call *normalized difference* of two polynomials of degree $m, n : m \geq n$, any non-zero linear combination of them which is of degree $\leq m-1$ (thus it is defined up to multiplication by a non-zero scalar).

Wroński's first step is to take the remainders of $S^m(x-\mathbb{B}), S^{m+1}(x-\mathbb{B}), \ldots$ modulo $S^n(x-\mathbb{A})$. Then he builds a triangular table with these remainders in the first row, the entries of the successive rows being the normalized differences of the two entries next to to it in the row just above (our definition of normalized difference is imprecise when two consecutive polynomials on row r are of degree $< n-r$, because any linear combination of them can be taken to fill the next row — we shall not be interested by possible degeneracies). The first entry of row r is a linear combination of the remainders of

$$S^m(x-\mathbb{B}), \ldots, S^{m+r-1}(x-\mathbb{B})$$

and it is not difficult to see that it is in fact a linear combination of

$$S^m(x-\mathbb{B}), \ldots, S^{m+r-1}(x-\mathbb{B});\ S^n(x-\mathbb{A}), \ldots, S^{m+r-1}(x-\mathbb{A})\ .$$

Therefore, it is the r-th remainder.

For example, for the pair $S_5(x-\mathbb{B}) = x^5+2x^4+3x^3+4x^2+3x+2$, $S_4(x-\mathbb{A}) = x^4+x^3+x^2+x+1$ seen above, Wroński's algorithm produces the following table of normalized differences :

$$\begin{array}{ccccccc}
x^3+2x^2+x+1 & & x^3-1 & & -x^3-x^2-2x-1 & & -x^2+1 \\
& 2x^2+x+2 & & x^2+2x+2 & & x^2-1 & \\
& & 3x+2 & & 2x+3 & & \\
& & & 5 & & &
\end{array}$$

The first row contains the remainders of $S_5(x-\mathbb{B}), \ldots, S_8(x-\mathbb{B})$ modulo $S_4(x-\mathbb{A})$; the successive remainders are the first entries of each row.

3.8. Division and Continued Fractions

Let us take $m = n+1$, and perform the Euclidean division of $S^{n+1}(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$, supposing $\mathbb{A}$ and $\mathbb{B}$ to be generic, and normalizing the successive remainders by imposing them to be monic: $\mathcal{R}_r = x^{n-r} + \cdots$.

The iterated division produces a sequence of monic polynomials of degree 1, $\mathcal{Q}_r$, as well as constants c_r, such that $\mathcal{R}_{-1} = S^{n+1}(x-\mathbb{B})$, $\mathcal{R}_0 = S^n(x-\mathbb{A})$,

$$\mathcal{R}_{r-1} = \mathcal{Q}_r\mathcal{R}_r - c_{r+1}\mathcal{R}_{r+1}\ . \tag{3.8.1}$$

From the expression of the remainders seen above, one deduce the quotients $Q_r = x + \zeta_r$, with

$$\zeta_r = \frac{S_{1,(r+1)^r}(\mathbb{A}, \mathbb{A}-\mathbb{B})}{S_{(r+1)^r}(\mathbb{A}-\mathbb{B})} - \frac{S_{1,r^{r-1}}(\mathbb{A}, \mathbb{A}-\mathbb{B})}{S_{r^{r-1}}(\mathbb{A}-\mathbb{B})}\ .$$

However, these coefficients can be written using only $\mathbb{A}-\mathbb{B}$,

$$\zeta_r = \frac{S_{1(r+1)^r}(\mathbb{A}-\mathbb{B})}{S_{(r+1)^r}(\mathbb{A}-\mathbb{B})} - \frac{S_{1r^{r-1}}(\mathbb{A}-\mathbb{B})}{S_{r^{r-1}}(\mathbb{A}-\mathbb{B})}\ . \tag{3.8.2}$$

because

$$S_{1,(r+1)^r}(\mathbb{A},\mathbb{A}-\mathbb{B}) = S_{1,(r+1)^r}(\mathbb{A}-\mathbb{B}+\mathbb{B},\mathbb{A}-\mathbb{B}) = \\ S_{1,(r+1)^r}(\mathbb{A}-\mathbb{B}) + S^1(\mathbb{B})\, S_{(r+1)^r}(\mathbb{A}-\mathbb{B}) \ ,$$

and therefore the factors of $S^1(\mathbb{B})$ eliminate.

For example, writing J for $S_J(\mathbb{A}-\mathbb{B})$, one has

$$\zeta_0 = [1],\ \zeta_1 = \frac{[12]}{[2]} - [1],\ \zeta_2 = \frac{[133]}{[33]} - \frac{[12]}{[2]},\ \zeta_3 = \frac{[1444]}{[444]} - \frac{[133]}{[33]}, \ldots$$

Following Wroński, one can put recursions (3.8.1) in a continued fraction form (we shall slightly change conventions in later sections) :

PROPOSITION 3.8.1. *Let n be an integer, $\mathbb{A},\mathbb{B}$ be two alphabets, $|\mathbb{A}| = n$, $|\mathbb{B}| = n+1$. Let $y = 1/x$. Let*

$$\beta_r := S_{(r-1)^{r-2}}(\mathbb{A}-\mathbb{B})\, S_{(r+1)^r}(\mathbb{A}-\mathbb{B}) / S_{r^{r-1}}(\mathbb{A}-\mathbb{B})^2 \ ,\ r \geq 1 \ . \tag{3.8.3}$$

Let $(x+\zeta_0)$, $(x+\zeta_1), \ldots$ be the successive quotients in the Euclidean division of $S^{n+1}(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$.

Then the continued fraction expansion of the rational function

$$S^{n+1}(x-\mathbb{A}) \,/\, S^{n+1}(x-\mathbb{B}) = \prod(1-ay) \,/\, \prod(1-by) = \sum y^j\, S^j(\mathbb{B}-\mathbb{A})$$

is

$$\cfrac{1}{(1+\zeta_0 y) - \cfrac{y^2\beta_1}{(1+\zeta_1 y) - \cfrac{y^2\beta_2}{(1+\zeta_2 y) - \cfrac{y^2\beta_3}{\ddots}}}}$$

$$= \cfrac{1}{(1+[1]y) + \cfrac{y^2[2]}{1 + (\frac{[12]}{[2]} - [1])y + \cfrac{y^2[33]/[2]^2}{1 + (\frac{[133]}{[33]} - \frac{[12]}{[2]})y + \cfrac{y^2[2][444]/[33]^2}{\ddots}}}}$$

where J denotes $S_J(\mathbb{A}-\mathbb{B})$.

The above continued fraction expansion of $\sigma_y(\mathbb{B}-\mathbb{A})$ stops because $S_J(\mathbb{A}-\mathbb{B}) = 0$ for all $J \supseteq (n+2)^{n+1}$. However, letting n tends towards infinity, and putting $\mathbb{B} = 0$, one gets the continued fraction expansion of $\sigma_y(-\mathbb{A}) = \lambda_{-y}(\mathbb{A})$ for any $\mathbb{A}$ (but now the link with Euclidean division has become obscure !).

For example, for $\mathbb{A} := \{a\}$, one gets

$$1 - ay = \cfrac{1}{1 + ay + \cfrac{a^2y^2}{1-ay}}$$

For $\mathbb{A}$ of cardinality 2, it is still possible to check directly that

$$1 - y\Lambda^1(\mathbb{A}) + y^2\Lambda^2(\mathbb{A}) = \cfrac{1}{1 + S_1 y + \cfrac{y^2 S_2}{1 + (\frac{S_{12}}{S_2} - S_1)y + \cfrac{y^2 S_{33}/S_2^2}{1 - \frac{S_{12}}{S_2} y}}}$$

The Euclidean division of x^3 by $x^2 - x\Lambda^1 + \Lambda^2$ produces indeed the two successive quotients $x + \Lambda^1$, $x - \frac{(\Lambda^1)^3 - 2\Lambda^1\Lambda^2}{(\Lambda^1)^2 - \Lambda^2} = x - \frac{S^3}{S^2} = x + \left(\frac{S_{12}}{S_2} - S_1\right)$.

Wroński's continued fraction is related to many others that one can find in the mathematical literature. In fact, Wroński used it in relation with another problem, the "Universal factorization of polynomials" (cf. Ex. 8.38).

In the next chapter, we shall approach continued fractions from a broader point of view than only Euclidean division.

Exercises

Ex. 3.1. Let $f(x)$ be a polynomial and $\mathbb{A}$ be an alphabet of cardinality n. Show that $f(x)$ is congruent, modulo $S^n(x - \mathbb{A})$, to :

$$\begin{vmatrix} f(a) & 1 & a & a^2 & \cdots & a^{n-1} \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 1 & x & x^2 & \cdots & x^{n-1} \end{vmatrix} \frac{1}{\Delta(\mathbb{A})} .$$

Ex. 3.2. Find back the first remainder of the division of $S^m(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$, applying Plücker relations to the following matrix of complete functions (denoted only by their exponents) :

$$\begin{array}{r} \\ \\ \\ \\ \textit{alphabets} \end{array} \begin{bmatrix} 0 & \cdots & n & m & n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -n & \cdots & 0 & m-n & 0 \\ \mathbb{A} & \cdots & \mathbb{A} & \mathbb{A} - \mathbb{B} + x & x \end{bmatrix}$$

Ex. 3.3. Use Plücker relations to recover, following Wroński, that the $r+1$-th remainder in the division of $S^m(x - \mathbb{B})$ by $S^n(x - \mathbb{A})$ is the normalized difference of the r-th remainders of $S^m(x - \mathbb{B})$ and $S^{m+1}(x - \mathbb{B})$.

Ex. 3.4. Let $\mathbb{A}, \mathbb{B}$ be of respective cardinalities $n, n+1$, and $f_k(x)$, $k = 0, \ldots, n$ be the k-th remainder in the Euclidean division of $S^{n+1}(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$. Show, after Kronecker [**89**], the vanishing of the sums

$$\sum_{b \in \mathbb{B}} \frac{f_j(b) f_k(b)}{R(b, \mathbb{A}) R(b, \mathbb{B} \setminus b)} , \quad 0 \le j < k \le n .$$

Ex. 3.5. Let $\mathbb{A}$ be of cardinality n, x of rank 1. Give the successive remainders in the Euclidean division of x^{n+1} by $S^n(1 - x\mathbb{A})$.

Ex. 3.6. Let $\mathbb{A}$ be an alphabet, with $|\mathbb{A}| = n$. Extract from the companion matrix of $\mathbb{A}$ the (infinite) Vandermonde matrix. Deduce from it that any Schur function can be expressed as determinant of hook functions $S_{1^i j}(\mathbb{A})$.

Ex. 3.7. Let $\mathbb{A}$ be an alphabet, with $|\mathbb{A}| = n$. Let $\mathbb{X} := \{x_1, \dots, x_k\}$ be an alphabet, $k \le n$. Show that the remainder of any minor of order k of the infinite Vandermonde matrix $\mathbb{V}(\mathbb{X})$ modulo the polynomials $S^n(x_i - \mathbb{A})$, is a sum of products of minors of the finite Vandermonde matrix of width n, by Schur functions in $\mathbb{A}$.

Ex. 3.8. Let $\mathbb{A}, \mathbb{B}$ be two alphabets, $|\mathbb{A}| = |\mathbb{B}| = n$. Express the matrix of Bézout's morphism

"Multiplication by $S^n(x - \mathbb{B})$ modulo $S^n(x - \mathbb{A})$"

in the basis $\{S^0(x - \mathbb{A}), \dots, S^{n-1}(x - \mathbb{A})\}$, and find back the expression of Bézout's matrix in terms of $\mathbb{S}_{n^n}(\mathbb{A} - \mathbb{B})$, using that

$$S^i(x - \mathbb{A}) S^n(x - \mathbb{B}) = (-1)^i S_{1^i, n}(\mathbb{A}, x - \mathbb{B}) = (-1)^i S_{1^i, n}(\mathbb{A}, (x - \mathbb{A}) + (\mathbb{A} - \mathbb{B})) \ .$$

Ex. 3.9. Let $\mathbb{A}$: $|\mathbb{A}| = n$ be such that all $a \in \mathbb{A}$ are distinct, and let $Q_a := S^{n-1}(x - \mathbb{A} + a)$. Show that the morphism u, defined in the section about Bézout's matrix, is diagonal in the basis $\{Q_a\}_{a \in \mathbb{A}}$, with diagonal $\{S^n(a - \mathbb{B})\}$.

Ex. 3.10. The coefficients of the successive remainders in Sturm sequence (for the division of a polynomial $R(x, \mathbb{A})$ by its derivative) are symmetric functions of $\mathbb{A}$. Find them (expressions (3.6.5) contain an extra factor f_0 or f_1 that needs to be eliminated).

Ex. 3.11. Relate the expressions given by Sturm for the successive remainders of a polynomial by its derivative, to the expression of the remainders (for a general pair of polynomials) as Schur functions.

Ex. 3.12. (*Hypergeometric alphabet*). Let a, b be elements of binomial type, $a, b \notin \mathbb{N}$, and x be rank-1 element. Define the alphabet $\mathbb{A}$ by $\Lambda^i(\mathbb{A}) := \Lambda^i(a)/\Lambda^i(b)$, $i = 1, 2, \dots$. Let, for any $n \in \mathbb{N}$, $\phi(n, a, b) = \Lambda^n(nx - \mathbb{A})$.

Show that the derivative of $\phi(n, a, b)$ with respect to x is equal to $n\phi(n-1, a, b)$, and that the respective remainders of $\phi(n, a, b)$ with respect to $\phi(n-1, a+k, c+\ell)$, with $[k, l] = [1, 0], [0, 0], [0, -1], [-1, 0], [-1, -1]$ or $[-1, -2]$, are proportional to some $\phi(n-2, a', b')$.

In particular, show after Stieltjes [**163**], that the Sturm sequence associated to $\phi(n, a, b)$ (i.e. the Euclidean division of $\phi(n, a, b)$ by $\phi(n-1, a, b)$) is the sequence

$$\phi(n - i - 1, a - i, b - 2i), \ i = 1, 2, \dots \ ,$$

up to factors to be determined.

Show in particular that the discriminant of $\phi(n, a, b)$ factorizes into factors which are linear in a and b.

For another method to prove this last point, see next exercise.

Ex. 3.13. Let a, b be elements of binomial type, $a, b \notin \mathbb{N}$, and let $n \in \mathbb{N}$. Define $\mathbb{A}$ and $\mathbb{B}$ by the equations

$$\Lambda^i(\mathbb{A}) := \binom{n}{i} \Lambda^i(a)/\Lambda^i(b) \quad \& \quad \Lambda^i(\mathbb{B}) := \binom{n-1}{i} \Lambda^i(a)/\Lambda^i(b), \ i = 1, 2, \dots$$

Compute the Schur functions $S_{k^n}(\mathbb{A} - \mathbb{B})$, for $k \ge n - 1$.

Ex. 3.14. Let $\mathbb{A}$, $\mathbb{B}$ be two alphabets: $|\mathbb{A}| = n$, $|\mathbb{B}| = m$. Let $\mathbb{S}$ be the $n \times \infty$ matrix $\mathbb{S} := \left[S^{m+j-i}(\mathbb{A} - \mathbb{B}) \right]_{0 \le i \le n-1, 0 \le j}$. Let $\mathcal{C}(\mathbb{A})$ be the infinite companion matrix of $S^n(x - \mathbb{A})$. Show that $\mathbb{S}$ factorizes into

$$\mathbb{S} = \mathbb{S}_{m^n}(\mathbb{A} - \mathbb{B}) \, \mathcal{C}(\mathbb{A}) \ .$$

Deduce, for every $J \in \mathbb{N}^n$ the following factorization :

$$S_{m^n+J}(\mathbb{A}-\mathbb{B}) = R(\mathbb{A},\mathbb{B})\, S_J(\mathbb{A}) \ .$$

Deduce also the expression of a Schur function $S_J(\mathbb{A})$ as a determinant of hook-functions $S_{1^i j}(\mathbb{A})$.

For example, for $n=3$, $m=5$, writing the alphabets below the matrices, and truncating them to width 7 as a first approximation to ∞, one has

$$\underset{\mathbb{A}-\mathbb{B}}{\begin{bmatrix} S^5 & S^6 & S^7 & S^8 & S^9 & S^{10} & S^{11} \\ S^4 & S^5 & S^6 & S^7 & S^8 & S^9 & S^{10} \\ S^3 & S^4 & S^5 & S^6 & S^7 & S^8 & S^9 \end{bmatrix}} = \underset{\mathbb{A}-\mathbb{B}}{\begin{bmatrix} S^5 & S^6 & S^7 \\ S^4 & S^5 & S^6 \\ S^3 & S^4 & S^5 \end{bmatrix}} \ \underset{\mathbb{A}}{\begin{bmatrix} 1 & \cdot & \cdot & S_{300} & S_{400} & S_{500} & S_{600} \\ \cdot & 1 & \cdot & S_{20} & S_{30} & S_{40} & S_{50} \\ \cdot & \cdot & 1 & S_1 & S_2 & S_3 & S_4 \end{bmatrix}}$$

CHAPTER 4

Reciprocal Differences and Continued Fractions

4.1. Euler's Recursions

Fractions $\frac{a}{b}$ were known to Egyptian and Chinese mathematicians some thousand years ago. It is natural to iterate them and consider expressions like $a_1 + \dfrac{b_2}{a_2}$, $a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3}}$ ... Putting the dots inside the fraction, one gets a continued fraction

$$a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{\ddots}}} \tag{4.1.1}$$

Stopping the continued fraction at step n, and reducing it to a fraction, one gets the *n-th convergent* of the continued fraction. The French terminology *Les Réduites* is more appropriate, because convergence is not always *au rendez-vous*. We refer to Brezinsky [**24**] for an extensive bibliography about continued fractions.

Euler [**40**] wrote a fundamental treatise on continued fractions and found recursions between consecutive convergents which can conveniently be written in matrix form :

PROPOSITION 4.1.1. *The convergents of a continued fraction*

$$\frac{P_n}{Q_n} := a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{\ddots + \cfrac{b_n}{a_n}}}}$$

can be expressed through multiplication of 2×2 *matrices:*

$$\begin{bmatrix} P_n \\ Q_n \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 1 \\ b_2 & 0 \end{bmatrix} \begin{bmatrix} a_3 & 1 \\ b_3 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & 1 \\ b_{n-1} & 0 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} \tag{4.1.2}$$

Proof. Write

$$\begin{bmatrix} a_3 & 1 \\ b_3 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{n-1} & 1 \\ b_{n-1} & 0 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Suppose the theorem be true for

$$a_2 + \frac{b_3}{\cdots + \frac{b_n}{a_n}} = \frac{a_2\alpha + \beta}{\alpha} \ .$$

On the other hand,

$$\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 1 \\ b_2 & 0 \end{bmatrix} \begin{bmatrix} a \\ \beta \end{bmatrix} = \begin{bmatrix} a_1 a_2 \alpha + b_2 \alpha + a_1 \beta \\ a_2\alpha + \beta \end{bmatrix}$$

The two entries of this last vector are exactly the numerator and denominator of

$$a_1 + \frac{b_2}{\dfrac{a_2\alpha + \beta}{\alpha}} = \frac{a_1(a_2\alpha + \beta) + b_2\alpha}{a_2\alpha + \beta}$$

□

Continued fraction can be used in interpolation of one variable functions. There are two approaches : either one wants the interpolant to coincide with the function in a given point up to a certain order, or one wants the interpolant to coincide with the function in a set of distinct points (of course, one can mix the two approaches).

4.2. Continued Fraction Expression of a Formal Series

Let us first look at the approximation of a formal series that one can choose to be $\sigma_z(\mathbb{A})$.

Choosing all $a_i = 1$, and replacing the b_i's by zb_i, one gets a continued fraction of the type

$$1 + \cfrac{zb_2}{1 + \cfrac{zb_3}{1 + \cfrac{zb_4}{\ddots}}}$$

For generic b_i's, one sees by induction that the degrees in z of the successive convergents $P_n(z)/Q_n(z)$ are

$$d^o(P_{2n}) = n,\ d^o(Q_{2n}) = n - 1,\ d^o(P_{2n+1}) = n,\ d^o(Q_{2n+1}) = n \ ,$$

starting with $P_2/Q_2 = (1 + zb_2)/1$. Choosing appropriate (and unique) $b_2, \ldots, b_n$ allows to approximate $\sigma_z(\mathbb{A})$ up to order $n-1$ included, and the rational fraction $P_n(z)/Q_n(z)$ is uniquely determined (up to multiplication of both numerator and denominator by a common scalar). The explicit expression of the b_i's for the expansion of a rational function of the type $\sigma_z(\mathbb{A} - \mathbb{B})$ was first found by Heilerman [**68**] (Muir II, p.361-365) and can be rewritten in the following theorem usually ascribed to Stieltjes [**162**] :

THEOREM 4.2.1. (1) *The series $\sigma_z(\mathbb{A})$ admits a continued fraction expansion* $1 + \cfrac{zb_2}{1 + \cfrac{zb_3}{\ddots}}$ *with*

$$b_{2n} = \frac{S_{n^n} S_{(n-1)^{n-2}}}{S_{n^{n-1}} S_{(n-1)^{n-1}}} \quad \& \quad b_{2n+1} = \frac{S_{(n+1)^n} S_{(n-1)^{n-1}}}{S_{n^n} S_{n^{n-1}}} \ . \tag{4.2.1}$$

(2) *The convergents are*

$$\frac{P_{2n}}{Q_{2n}} = (-1)^{n-1} z \frac{S_{n^n}(\mathbb{A}+\frac{1}{z})}{S_{(n-1)^{n-1}}(\mathbb{A}-\frac{1}{z})} \;\&\; \frac{P_{2n+1}}{Q_{2n+1}} = (-1)^n \frac{S_{n^{n+1}}(\mathbb{A}+\frac{1}{z})}{S_{(n+1)^n}(\mathbb{A}-\frac{1}{z})} \tag{4.2.2}$$

The first parameters are

$$b_2 = S_1,\ b_4 = S_{22}/S_2S_1,\ b_6 = S_{333}S_2/S_{33}S_{22},\ \ldots$$

$$b_3 = -S_2/S_1,\ b_5 = S_{33}S_1/S_{22}S_2,\ b_7 = S_{444}S_{22}/S_{333}S_{33},\ \ldots$$

while the first convergents are

$$\frac{P_2}{Q_2} = zS_1(\mathbb{A}+\frac{1}{z}) \ ,\ \frac{P_4}{Q_4} = -z\frac{S_{22}(\mathbb{A}+\frac{1}{z})}{S_3(\mathbb{A}-\frac{1}{z})} \ ,\ \frac{P_6}{Q_6} = z\frac{S_{333}(\mathbb{A}+\frac{1}{z})}{S_{44}(\mathbb{A}-\frac{1}{z})} \ ,\ldots$$

$$\frac{P_1}{Q_1} = 1 \ ,\ \frac{P_3}{Q_3} = -\frac{S_{11}(\mathbb{A}+\frac{1}{z})}{S_2(\mathbb{A}-\frac{1}{z})} \ ,\ \frac{P_5}{Q_5} = \frac{S_{222}(\mathbb{A}+\frac{1}{z})}{S_{33}(\mathbb{A}-\frac{1}{z})} \ ,\ldots$$

To prove the theorem, we need to rewrite some Schur functions :

LEMMA 4.2.2. *Let $k, n \in \mathbb{N}$, $\mathbb{A}$ be arbitrary, and z be an indeterminate. Then*

$$(-1)^n z^{n-k+1} S_{k^n}(\mathbb{A}-\frac{1}{z})\left(S_0(\mathbb{A}) + \cdots + z^{k+n-1}S_{k+n-1}(\mathbb{A})\right) \equiv S_{(k-1)^{n+1}}(\mathbb{A}+\frac{1}{z})$$

modulo terms of degree $\geq k(n+1)$ in $\mathbb{A}$.

Proof. Up to a power of z and a sign, the left hand side is equal to

$$S_{k^n;\,0}(\mathbb{A};\ \frac{1}{z})S_{k+n-1}(\mathbb{A}+\frac{1}{z}) = \begin{vmatrix} S_k(\mathbb{A}) & \cdots & S_{k+n-1}(\mathbb{A}) & z^0 S_{k+n-1}(\mathbb{A}+\frac{1}{z}) \\ \vdots & & \vdots & \vdots \\ S_{k-n}(\mathbb{A}) & \cdots & S_{k-1}(\mathbb{A}) & z^n S_{k+n-1}(\mathbb{A}+\frac{1}{z}) \end{vmatrix}$$

$$= \begin{vmatrix} S_k(\mathbb{A}) & \cdots & S_{k+n-1}(\mathbb{A}) & z^0 S_{k+n-1}(\mathbb{A}+\frac{1}{z}) \\ \vdots & & \vdots & \vdots \\ S_{k-n+1}(\mathbb{A}) & \cdots & S_k(\mathbb{A}) & S_k(\mathbb{A}+\frac{1}{z}) + zS_{k+1}(\mathbb{A}) + z^2S_{k+2}(\mathbb{A}) + \cdots \\ S_{k-n}(\mathbb{A}) & \cdots & S_{k-1}(\mathbb{A}) & S_{k-1}(\mathbb{A}+\frac{1}{z}) + zS_k(\mathbb{A}) + z^2S_{k+1}(\mathbb{A}) + \cdots \end{vmatrix}$$

Up to terms of degree $\geq kn+k$ in $\mathbb{A}$, this determinant coincides with $S_{k^n;\,k-1}(\mathbb{A};\ \mathbb{A}+\frac{1}{z}) = \pm S_{k-1+n;\,(k-1)^n}(\mathbb{A}+\frac{1}{z};\ \mathbb{A})$. Its expansion in terms of powers of $1/z$ is a sum of determinants which, up to zero terms, is the same as the expansion of

$$z^{-n}S_{k-1;\,(k-1)^n}(\mathbb{A}+\frac{1}{z};\ \mathbb{A}) = z^{-n}S_{(k-1)^{n+1}}(\mathbb{A}+\frac{1}{z}) \ .$$

Checking signs and powers of z, one gets the lemma. □

Proof of theorem. The convergents are the only rational functions with Taylor's expansion coinciding with $\sigma_z(\mathbb{A})$ up to degree n. However, thanks to the preceding lemma, the expressions written for P_n/Q_n have also the same property.

As for the explicit expression of the b_i's, let us check by induction that they give the right convergents. Recall that the induction on convergents is $\begin{bmatrix} P_n \\ Q_n \end{bmatrix} = \begin{bmatrix} P_{n-1} \\ Q_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ b_n & 0 \end{bmatrix}$; the case $n = 7$ will be convincing enough. Thus one has $P_7 =$

$P_6 + zb_7\, P_5$, and this gives for b_7 the following expression (do not forget to normalize the P_i's !) :

$$b_7 = \frac{P_7 - P_6}{P_5} = \Big(\frac{z^4 S_{3333}(\mathbb{A}+\frac{1}{z})}{S_{333}(\mathbb{A})} - \frac{z^3 S_{333}(\mathbb{A}+\frac{1}{z})}{S_{33}(\mathbb{A})}\Big) \Big/ \frac{z^3 S_{222}(\mathbb{A}+\frac{1}{z})}{S_{22}(\mathbb{A})}$$

As usual, the looked for relation is afforded by a matrix

$$\begin{array}{c|cccccc}
\textit{indexing} & 0 & 1 & 2 & 4 & 5 & 6 \\
\hline
 & \cdot & S^0 & z^3 S^3 & S^4 & S^5 & S^6 \\
 & \cdot & \cdot & z^2 S^2 & S^3 & S^4 & S^5 \\
 & \cdot & \cdot & z S^1 & S^2 & S^3 & S^4 \\
 & S^0 & \cdot & S^0 & S^1 & S^2 & S^3 \\
\hline
\textit{alphabets} & 0 & 0 & \mathbb{A}+\frac{1}{z} & \mathbb{A} & \mathbb{A} & \mathbb{A}
\end{array}$$

The reader will recognize that the products of minors :

$$[2,4,5,6]\,[0,1,4,5] - [0,2,4,5]\,[1,4,5,6] = [0,4,5,6]\,[1,2,4,5]$$

indeed leads to $b_7 = -S_{444}S_{22}/S_{333}S_{33}$. □

A simpler proof of Theorem 4.2.1 will be obtained with Pad approximants, identifying the Schur functions in the numerators and denominators of the succesive convergents to some resultants.

4.3. Interpolation of a Function by a Continued Fraction

Given a function $f(x_1, x_2, \ldots)$, one uses the following reciprocal of differences associated to a pair x_i, x_j

$$\frac{x_i - x_j}{f(\ldots x_i \ldots x_j \ldots) - f(\ldots x_j \ldots x_i \ldots)} \,.$$

One can start from a function of x_1 (considered as a function of the x_i's, of degree 0 in $x_2, x_3, \ldots$) and iterate these reciprocal of differences, at least while no zero arises in denominator. However, one prefers to modify slightly the preceding operations.

Given a function $f(x)$ and an alphabet $x_1, x_2, \ldots$, define recursively functions $[i, j, \ldots, h]$ of $x_i, x_j, \ldots, x_h$, which are called *reciprocal differences of* f, by

$$[1] = f(x_1) \;;\; [12] = \frac{x_1 - x_2}{f(x_1) - f(x_2)}$$

$$[123] = \frac{x_2 - x_3}{[12] - [13]} + [1]$$

$$\cdots$$

$$[12\cdots n] = \frac{x_{n-1} - x_n}{[1\cdots n-2, n-1] - [1\cdots n-2, n]} + [1\cdots n-2]$$

Associate to this sequence the continued fraction

$$g(x\,,\,1234\ldots) := [1] + \cfrac{x - x_1}{[12] + \cfrac{x - x_2}{[123] - [1] + \cfrac{x - x_3}{[1234] - [12] + \cdots}}}$$

THEOREM 4.3.1. *(Thiele Interpolation) The continued fraction $g(x\,,1\ldots n)$ takes values $f(x_i)$ at points $x = x_1, \ldots, x_n$.*

Proof. Take $n = 4$ for simplicity of indices. One has a symmetry in the last two interpolation points: $g(x, 1234) = g(x, 1243)$. Indeed, $[123] - [1] + \frac{x - x_3}{[1234]-[12]} = [124] - [1] + \frac{x-x_4}{[1234]-[12]}$ because $[1234] = [1243]$ (a divided difference, or a reciprocal difference in x_3, x_4 furnishes a function symmetrical in x_3, x_4).

Now, $g(x, 1234)$ coincides with $g(x, 123)$ for $x = x_1, x_2, x_3$, and $g(x, 1243)$ coincides with $g(x, 124)$ for $x = x_1, x_2, x_4$. Supposing the proposition true for the previous order, one thus obtains it for the next one. □

REMARK 4.3.2. The simpler reciprocal differences, recursively defined by

$$[1 \dots n]' := \frac{x_{n-1} - x_n}{[1 \cdots n-2, n-1]' - [1 \cdots n-2, n]'}$$

are not symmetrical in $x_1, \dots, x_n$, contrary to the $[1 \dots n]$, and are more difficult to characterize as functions of $x_1, \dots, x_n$.

The continued fraction $g(x\,,\,1 \dots n)$ is of the type (4.1.1), with $b_2, b_3, \dots$ of degree 1 in x. As for convergents, for even n, $deg(P_n) = n/2$, $deg(Q_n) = n/2 - 1$; for odd n, $deg(P_n) = deg(Q_n) = (n-1)/2$. Now, since P_n/Q_n interpolates $f(x)$ in n points, counting the number of parameters, we see that $P_n(x)$ and $Q_n(x)$ are unique, up to multiplication of both by the same scalar. This uniqueness will allow us in the next theorem to express the convergents of Thiele's continued fraction as Schur functions.

THEOREM 4.3.3. *Given an alphabet $\mathbb{Y}$ of cardinality k, let $f(x)$ be the polynomial $f(x) := S^k(x - \mathbb{Y})$. Then the convergents of Thiele continued fraction for $f(x)$ are expressible as Schur functions:*

$$\left\{ \begin{array}{ll} P_{2n} = & S_{1^n,\,(k-n)^n}(\mathbb{X}_{2n} - x,\; X_{2n} - \mathbb{Y}) \\ Q_{2n} = & S_{(k-n)^{n-1}}(\mathbb{X}_{2n} - x - \mathbb{Y}) \end{array} \right. \tag{4.3.1}$$

$$\left\{ \begin{array}{ll} P_{2n+1} = & S_{1^n,\,(k-n)^{n+1}}(\mathbb{X}_{2n+1} - x,\; \mathbb{X}_{2n+1} - \mathbb{Y}) \\ Q_{2n+1} = & S_{(k-n)^n}(\mathbb{X}_{2n+1} - x - \mathbb{Y}) \end{array} \right. \tag{4.3.2}$$

Proof. Because of the uniqueness of a rational interpolant at n points with sum of degrees of the numerator and denominator $\leq n-1$, we have only to check that P_n/Q_n coincides with $f(x)$ at points $x = x_1, \dots, x_n$.

Let us reason on an example, and take $n = 6$, $k = 7$. Then

$$P_6 = S_{111,\,444}(\mathbb{X}_6 - x,\; X_6 - \mathbb{Y})$$

$$= \begin{vmatrix} 1 & 2 & 3 & 7 & 8 & 9 \\ 0 & 1 & 2 & 6 & 7 & 8 \\ \cdot & 0 & 1 & 5 & 6 & 7 \\ \cdot & \cdot & 0 & 4 & 5 & 6 \\ \cdot & \cdot & \cdot & 3 & 4 & 5 \\ \cdot & \cdot & \cdot & 2 & 3 & 4 \\ \mathbb{X}_6 - x & \mathbb{X}_6 - x & \mathbb{X}_6 - x & \mathbb{X}_6 - \mathbb{Y} & \mathbb{X}_6 - \mathbb{Y} & \mathbb{X}_6 - \mathbb{Y} \end{vmatrix}$$

Take any $i \leq 6$ and subtract $(\mathbb{X}_6 - x_i)$ to the alphabets in the top row. Since degrees are increasing by 1 when passing from column 4 to 5, 5 to 6, one can similarly subtract x_i to columns 5 and 6. Alphabets in the determinant are now, with $\mathbb{B} = \mathbb{X}_6 - x_i - \mathbb{Y}$,

$$\begin{vmatrix} 0 & 0 & 0 & x_i - \mathbb{Y} & -\mathbb{Y} & -\mathbb{Y} & -\mathbb{Y} \\ \mathbb{X}_6 - x_i & \mathbb{X}_6 - x_i & \mathbb{X}_6 - x_i & \mathbb{X}_6 - \mathbb{Y} & \mathbb{B} & \mathbb{B} & \mathbb{B} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mathbb{X}_6 - x_i & \mathbb{X}_6 - x_i & \mathbb{X}_6 - x_i & \mathbb{X}_6 - \mathbb{Y} & \mathbb{B} & \mathbb{B} & \mathbb{B} \end{vmatrix}$$

Because $S^j(-\mathbb{Y}) = 0$ for $j > 6$, the elements of the top row vanish all, except for $S^7(x_i - \mathbb{Y})$. The cofactor of this non-zero element factorizes into $S_{000}(\mathbb{X}_6 - x_i) = 1$, and $S_{44}(\mathbb{X}_6 - x_i - \mathbb{Y})$. On the other hand $Q_6(x) = S_{44}(\mathbb{X}_6 - x - \mathbb{Y})$, and therefore the specialization of $P_6(x)/Q_6(x)$ in x_i is equal to $S^7(x_i - \mathbb{Y})$. □

One can notice that the numerators P_n of Thiele's convergents can be obtained in the Euclidean division of $f(x)$ by $S^n(x - x_1 - \cdots - x_n)$ (cf. expression 3.2.1).

Reciprocal differences can themselves be expressed as Schur functions :

PROPOSITION 4.3.4. *Given an alphabet $\mathbb{Y}$ and an integer k, let $f(x)$ be the polynomial $f(x) := S^k(x - \mathbb{Y})$. Then the reciprocal difference of f $[1, 2, \ldots, n]$, $n = 1, 2, \ldots$, of order n, is a quotient of two Schur functions of $x_1 + \cdots + x_n - \mathbb{Y}$:*

$$[1, \ldots, 2m+1] = S_{(k-m)^{m+1}}/S_{(k-m-1)^m} \ , \tag{4.3.3}$$
$$[1, \ldots, 2m] = S_{(k-m-1)^{m-1}}/S_{(k-m)^m} \ . \tag{4.3.4}$$

For example,

$$[1] = S_k, \ [123] = S_{(k-1)^2}/S_{k-2}, \ [12345] = S_{(k-2)^3}/S_{(k-3)^2} \ \ldots$$
$$[12] = 1/S_{k-1}, \ [1234] = S_{k-3}/S_{(k-2)^2}, \ [123456] = S_{(k-4)^2}/S_{(k-3)^3} \ \ldots$$

Proof. Once more, we need only Plücker relations, but twice! Let us see how to obtain $[1, \ldots, 7]$, knowing $[1, \ldots, 6]$ and $[1, \ldots, 5]$. Take $k = 9$ for example, and write $S_J(\mathbb{A})$ for $S_J(\mathbb{A} + x_1 + \cdots + x_5 - \mathbb{Y})$. From the exercises, we know that

$$S_{55}(x_6)/S_{666}(x_6) - S_{55}(x_7)/S_{666}(x_7) = (x_7 - x_6) S_{555}(x_6 + x_7) S_{66}()/S_{666}(x_6) S_{666}(x_7) \ ,$$

and therefore

$$[1, \ldots, 7] = -\frac{S_{666}(x_6)S_{666}(x_7)}{S_{555}(x_6+x_7)S_{66}()} + \frac{S_{777}()}{S_{66}()} \ .$$

However, one has also that

$$S_{666}(x_6)S_{666}(x_7) - S_{555}(x_6+x_7)S_{777}() = S_{6666}(x_6 + x_7)S_{66}() \ ,$$

whence the required expression

$$[1, \ldots, 7] = S_{6666}(x_1 + \cdots + x_7 - \mathbb{Y})/S_{555}(x_1 + \cdots + x_7 - \mathbb{Y}) \ .$$

□

4.4. Relation between Stieltjes and Wroński Continued Fractions

Stieltjes [**162**] prefers developing series at $x=\infty$, taking $x^{-1}\sigma_{1/x}(\mathbb{A})$ instead of $\sigma_z(\mathbb{A})$. He very often uses continued fractions of the type :

$$(4.4.1)\qquad \cfrac{1}{x+\cfrac{1}{-1/\alpha_1+\cfrac{1}{x/\alpha_2+\cfrac{1}{-1/\alpha_3+\cfrac{1}{x/\alpha_4+\ddots}}}}} = x^{-1}+\alpha_1 x^{-2}+$$

$$+(\alpha_1^2\alpha_2+\alpha_1^2)x^{-3}+(2{\alpha_1}^3\alpha_2+{\alpha_1}^2{\alpha_2}^2\alpha_3+{\alpha_1}^3{\alpha_2}^2+{\alpha_1}^3)x^{-4}+\cdots\,,$$

where the indeterminate $x=1/z$ appears only at half of the levels.

It is easy to transform the above continued fraction into

$$(4.4.2)\qquad \cfrac{1}{x-\alpha_1-\cfrac{\alpha_1^2\alpha_2}{x-\alpha_1\alpha_2-\alpha_2\alpha_3-\cfrac{\alpha_2\alpha_3^2\alpha_4}{x-\alpha_3\alpha_4-\alpha_4\alpha_5-\cfrac{\alpha_4\alpha_5^2\alpha_6}{x-\alpha_5\alpha_6-\alpha_6\alpha_7}}}}$$

and therefore, the relations between Stieltjes' parameters and Wroński's ones (with normalization fixed by Eq.8.3.2) are :

$$(4.4.3)\qquad \alpha_1=S_1,\ \alpha_2=-S_{11}/S_1^2,\ \alpha_3=-S_1S_{22}/S_{11}^2,\ \alpha_4=S_{11}S_{222}/S_{22}^2\,,$$

$$\alpha_{2n+1}=(-1)^n\frac{S_{n^n}S_{(n+1)^{n+1}}}{(S_{n^{n+1}})^2}\quad\&\quad \alpha_{2n}=(-1)^n\frac{S_{(n-1)^n}S_{n^{n+1}}}{(S_{n^n})^2}$$

$$\alpha_{2n}(\alpha_{2n-1}+\alpha_{2n+1})=\zeta_n\quad\&\quad \alpha_{2n-2}\alpha_{2n-1}^2\alpha_{2n}=-\beta_n$$

4.5. Jacobi's Tridiagonal Matrix

The recursions between the denominators of the successive convergents

$$Q_{n+1}=(x-\zeta_n)Q_n+\beta_nQ_{n-1}\ ,\ n\geq 0\ ;\ Q_0=1,\ Q_{-1}=0\ ,$$

can be written instead

$$(4.5.1)\qquad x\,Q_n=-\beta_nQ_{n-1}+\zeta_nQ_n+Q_{n+1}\ ,$$

in which case they are to be interpreted as describing the morphism " multiplication by x" in the basis $Q_0,Q_1,Q_2,\ldots$.

The matrix expressing it is an infinite tridiagonal matrix due to Jacobi :

$$(4.5.2)\qquad \mathfrak{Jac}:=\begin{bmatrix}\zeta_0 & 1 & 0 & \cdots & 0 & \cdots\\ -\beta_1 & \zeta_1 & 1 & \ddots & 0 & \cdots\\ & \ddots & \ddots & \ddots & \vdots & \\ & & \ddots & \zeta_{n-2} & 1 & \ddots\\ & & & -\beta_{n-1} & \zeta_{n-1} & \cdots\\ & & & & \ddots & \ddots\end{bmatrix}.$$

The following proposition,due to Whittaker [**179**], shows how to recover the series $\sigma_{1/x}(\mathbb{A})$ from $\mathfrak{Jac}$.

PROPOSITION 4.5.1. *For any positive integer n, the first entry of the n-th power of Jacobi's matrix is equal to $S^n(\mathbb{A})$.*

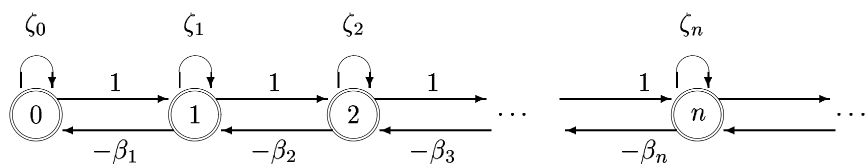
Proof. We shall use the linear functional $\mu : x^n \to \mu(x^n) := S^n(\mathbb{A})$, $n \geq 0$ (which will be motivated later by the theory of orthogonal polynomials). With respect to this functional, it is clear that every polynomial $S_{n^n}(\mathbb{A} - x) = S_{n^n;0}(\mathbb{A}; x)$, $n > 0$, is sent to 0. Therefore, the image of a decomposition $f = c_0 + c_1 Q_1 + c_2 Q_2 + \cdots$ under μ is $\mu(f) = c_0$. Thus, the first entry of $\mathfrak{Jac}^n$, which is the coefficient of the expansion of $x^n Q_0$ in the basis $Q_0, Q_1, \ldots$, is equal to $\mu(x^n) = S^n(\mathbb{A})$. $\square$

4.6. Motzkin Paths

A direct application of the use of Jacobi's matrix is to furnish a combinatorial interpretation of the expansion of continued fractions.

Indeed, one can interpret powers of a matrix with an automaton: edges are labelled by the entries of the matrix, and the entries of its k-th power are obtained by enumerating paths of length k.

Jacobi's matrix gives the following automaton :



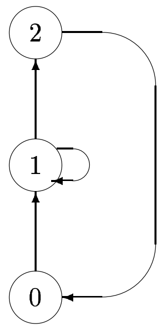
The first entry of $\mathfrak{Jac}^n$ will now be equal to the sum[1] of all paths of length n from 0 to 0.

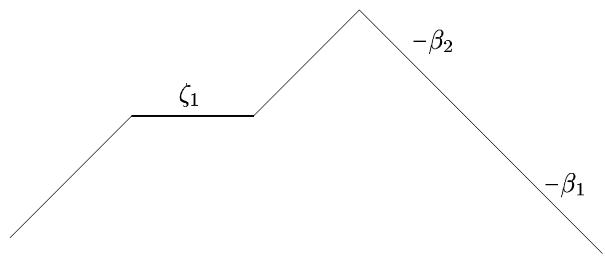
But there is another convenient combinatorial object replacing the paths in the Jacobi's automaton.

DEFINITION 4.6.1. *Motzkin paths* are paths in the positive integral plane, with East, North-East and South-East steps, starting and finishing at level 0.. A *weight* is attributed to each step: ζ_i for an horizontal step at level i, $-\beta_i$ for a South-East step between level i and $i-1$, North-East steps having weight 1. The *weight* of the path itself is the product of the weights of its steps.

Passing from paths in an automaton to Motzkin paths amounts to adding a time coordinate as an horizontal axis, and recording levels as function of time (uniform horizontal speed!) :

[1]interpreting a path as the product of all the labels of the edges composing it.





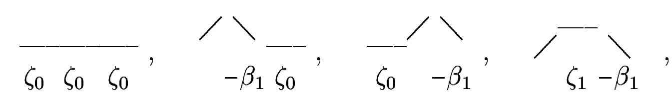
The path in the automaton is obtained by projecting a Motzkin path on the horizontal axis, the Motzkin path is obtained by drawing edges between the successive positions occupied in the automaton.

Therefore, Proposition 4.5.1 can be rewritten in a form due to Flajolet [**42**] :

THEOREM 4.6.2. *The coefficients of the formal series $\sigma_n(\mathbb{A})$ can be written in terms of the parameters in Wroński's continued fraction from an enumeration of Motzkin paths.*

Explicitly, $S^n(\mathbb{A})$ is the sum of the weights of all Motzkin paths of length n.

For example, there are 4 Motzkin paths of length 3,



and, indeed,

$$S^3 = S^{111} - S_{11}S^1 - S^1 S_{11} - (S_{12}/S_{11} - S^1)S_{11} \ .$$

4.7. Dyck Paths

We have encountered another continued fraction (4.4.1), that we can use to obtain another combinatorial interpretation of the series $\sigma_{1/x}(\mathbb{A})$.

DEFINITION 4.7.1. *Dyck paths* are paths in the positive integral plane, with North-East and South-East steps,starting and finishing at level 0 (i.e. they are Motzkin paths without horizontal steps). The *weight* of a Dyck path w is $\theta^{\mathcal{D}}(w) := \prod_i \alpha_i^{m_i}$, where m_i is the number of points at level i of the path. A *Dyck word* is a word in $1, 2$ obtained by replacing North-East steps by the letter 1, and South-East steps by 2.

THEOREM 4.7.2. *Given any $\mathbb{A}$, the complete functions $S_n(\mathbb{A})$ are obtained by enumerating all Dyck paths of length $2n$:*

$$S_n(\mathbb{A}) = \sum_{w\ Dyck,\ \ell(w)=2n} \theta^{\mathcal{D}}(w) \ . \qquad (4.7.1)$$

Proof. The reader will not be surprised to have to consider a tridiagonal matrix with null main diagonal :

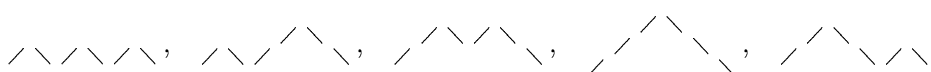
$$M := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ \alpha_1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & \alpha_2\alpha_1 & 0 & 1 & 0 & \\ 0 & 0 & \alpha_3\alpha_2 & 0 & 1 & \ddots \\ 0 & 0 & 0 & \alpha_4\alpha_3 & 0 & \ddots \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

and its square

$$M^2 = \begin{bmatrix} \boxed{\alpha_1} & 0 & \boxed{1} & 0 & \boxed{0} & \cdots \\ 0 & \alpha_1+\alpha_2\alpha_1 & 0 & 1 & 0 & \cdots \\ \boxed{\alpha_2\alpha_1^2} & 0 & \boxed{\alpha_2\alpha_1+\alpha_3\alpha_2} & 0 & \boxed{1} & \\ 0 & \alpha_3{\alpha_2}^2\alpha_1 & 0 & \alpha_3\alpha_2+\alpha_4\alpha_3 & 0 & \ddots \\ \boxed{0} & 0 & \boxed{\alpha_4\alpha_3^2\alpha_2} & 0 & \boxed{\alpha_4\alpha_3+\alpha_5\alpha_4} & \ddots \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

One recognizes that $\mathfrak{Jac}$ is a sub-matrix of M^2. A little more work shows that, as announced, the first entry of he n-th power of M^2 is equal to $S^n(\mathbb{A})$. However, it is also equal to the sum of all Motzkin paths of length $2n$. Moreover, one can put the weights on the levels instead of the edges. □

For example, there are 5 Dyck paths of length 6 (whose sum is equal to $S^3(\mathbb{A})$) :



${\alpha_1}^3{\alpha_0}^4$ ${\alpha_1}^3{\alpha_0}^3\alpha_2$ ${\alpha_1}^3{\alpha_2}^2{\alpha_0}^2$ ${\alpha_1}^2{\alpha_2}^2\alpha_3{\alpha_0}^2$ ${\alpha_1}^3{\alpha_0}^3\alpha_2$

4.8. Link between Enumeration of Motzkin and Dyck Paths

Having found two combinatorial interpretations of $S^n(\mathbb{A})$ as an enumeration of paths, we are forced to explicit how to pass directly from one interpretation to the other.

Viennot [**177**] does it by some chirurgy on paths, let us adopt a softer method. We first need to slightly extend the definition of Motzkin and Dyck paths by allowing points of negative levels.

DEFINITION 4.8.1. A *Super-Motzkin path* is a path from level 0 to level 0 with East, North-East and South-East steps, with no horizontal step of negative level. The *weight* $\theta^{\mathcal{M}}(w)$ of a path is the product of the weight of its steps, defining $\beta_{-i} := 1/\beta_i$.

A *Super-Dyck path* is a path from level 0 to level 0, with North-East and South-East steps. Defining $\alpha_{-i} = 1/\alpha_i$, the weight is still given by counting the number of points at level i : $\theta^{\mathcal{D}}(w) := \prod_i \alpha_i^{m_i}$.

Given $k \geq 0$, and two Super-Motzkin (resp. Super-Dyck) paths $w'2^k$ and $1^k w''$, one notices that the transformation

$$w'\,2^k\,\bar{2}^k\,\bar{1}^k\,1^k\,w'' \longrightarrow w'\,w'' \tag{4.8.1}$$

preserves the weight $\theta^{\mathcal{M}}$ (resp. $\theta^{\mathcal{D}}$).

Defining now *elementary paths* to be $[1^n\, 2^n]$, $[\bar{1}^n\, \bar{2}^n]$, $[1^n\, 0\, 2^n]$, then relation (4.8.1) shows that every Super-Motzkin (resp. Super-Dyck) path can be transformed into a path which is a concatanation of elementary paths and has the same weight.

For example

$$[111\,22\,0\,1\,22] \longrightarrow [111222\;\bar{2}\bar{1}\,1\,0\,2\,\bar{2}\bar{1}\,1122]$$

Let $\mathfrak{Motzkin}$ be the algebra generated by Super-Motzkin paths, and $\mathfrak{Dyck}$ be the algebra generated by Super-Dyck paths. Define φ to be the morphism from $\mathfrak{Motzkin}$ to $\mathfrak{Dyck}$ induced by the morphism from the algebra generated by $\{1,\, \bar{1}, 2,\, \bar{2},\, 0\}$ to itself :

$$1 \mapsto 11,\ \bar{1} \mapsto \bar{1}\bar{1},\ 2 \mapsto 22,\ \bar{2} \mapsto \bar{2}\bar{2},\ 0 \mapsto 21 + 12\ .$$

LEMMA 4.8.2. *The morphism φ preserves weights.*

Proof. One first checks that φ is compatible with transformation (4.8.1).

Now, if $w = [1^n\, 2^n]$, which has weight

$$\theta^{\mathcal{M}}(w) = (-\beta_n)\cdots(-\beta_1) = (-1)^n S_{n^{n+1}}/S_{(n-1)^n}\ ,$$

then its image is $\varphi(w) = [1^{2n}\, 2^{2n}]$ which has weight

$$\theta^{\mathcal{D}}(\varphi(w) = \alpha_1^2\alpha_2^2\cdots\alpha_{2n-1}^2\alpha_{2n} = (-1)^n S_{n^{n+1}}/S_{(n-1)^n}\ .$$

With $w = [\bar{1}^n\, \bar{2}^n]$ instead, one has inverse weights and still equality.

Finally, a truncated pyramid $w = [1^n\, 0\, 2^n]$ has weight

$$(-\beta_n)\cdots(-\beta_1)\zeta_n = \alpha_1^2\cdots\alpha_{2n-1}^2\alpha_{2n}\zeta_n\ .$$

Its image is the sum $[1^n\, 21\, 2^n] + [1^n\, 12\, 2^n]$, which has weight

$$\alpha_1^2\cdots\alpha_{2n-2}^2\,\left(\alpha_{2n-1}^3\alpha_{2n}^2 + \alpha_{2n-1}^2\alpha_{2n}^2\alpha_{2n+1}\right)\ ,$$

but the two weights are still equal because of relations (4.4.3). □

Since the image under φ of the sum of all Motzkin paths of length n is equal to the sum of all Dyck paths of length $2n$, lemma (4.8.2) and theorem (8.8.1) imply

$$S_n(\mathbb{A}) = \sum_{\substack{w\ Motzkin,\\ \ell(w)=n}} \theta^{\mathcal{M}}(w) = \sum_{\substack{w\ Motzkin,\\ \ell(w)=n}} \theta^{\mathcal{D}}(\varphi(w)) = \sum_{\substack{w\ Dyck,\\ \ell(w)=2n}} \theta^{\mathcal{D}}(w) \tag{4.8.2}$$

Let us say that a Motzkin or Dyck path is *irreducible* if it has no point at level zero, except the two end points. Since any (Motzkin or Dyck) path can be uniquely factorized into a product of irreducible paths, the generating function of all (Motzkin or Dyck) paths $\sum z^{\ell(w)}\, w$ is the inverse of $\sum_{w\ irr}(-z)^{\ell(w)}\, w$. Taking weights, it implies the following corollary :

COROLLARY 4.8.3. *Given $\mathbb{A}$ and a positive integer n, then*

$$(-1)^{n-1}\Lambda_n(\mathbb{A}) = \sum_{\substack{w\ Motzkin,\ irr\\ \ell(w)=n}} \theta^{\mathcal{M}}(w) = \sum_{\substack{w\ Dyck,\ irr\\ \ell(w)=2n}} \theta^{\mathcal{D}}(w) \tag{4.8.3}$$

A similar interpretation of powers sums or hook Schur functions in term of paths is given in [**100**].

Exercise (4.2) shows that the expressions of $S_n(\mathbb{A})$ or $\Lambda_n(\mathbb{A})$, in terms of $\alpha_1, \alpha_2, \ldots$, can be obtained by enumerating the compositions of n, the parameters b_i of Rogers being equal to $b_i = \alpha_i\alpha_{i-1}$ (cf. [**61**], prop. 5.2.4).

For a commentary about how to express symmetric functions in terms of the parameters ζ_i, β_j, see [**100**], [**183**].

A systematic study of continued fractions from the point of view of enumeration will be found in the book of Goulden and Jackson [**61**].

Exercises

Ex. 4.1. Let $a_1 + \frac{b_2}{a_2+b_3/\cdots}$ be a continued fraction. Check from the matrix expression that the parameters can be recovered from the convergents :

$$a_n = \frac{P_{n-2}Q_n - P_nQ_{n-2}}{P_{n-2}Q_{n-1} - P_{n-1}Q_{n-2}} \quad \& \quad b_n = \frac{P_nQ_{n-1} - P_{n-1}Q_n}{P_{n-2}Q_{n-1} - P_{n-1}Q_{n-2}}$$

Ex. 4.2. (Rogers [**154**]). Give the Taylor expansion of a continued fraction

$$\cfrac{1}{1 - \cfrac{b_1 z}{1 - \cfrac{b_2 z}{1 - \ddots}}}\,.$$

Ex. 4.3. Given $\alpha \in \mathbb{C}$, show that $(1+z)^\alpha$ has a continued fraction expansion $a_1 + \frac{b_2}{a_2+b_3/\cdots}$ with $a_1 = 1, a_{2n} = 2n-1$, $a_{2n+2} = 2$, $b_{2n} = (n-1+\alpha)z$, $b_{2n+1} = (n-\alpha)z$. Obtain, as a limit, the continued fraction for the exponential: $a_1 = 1, a_{2n} = 2n-1$, $a_{2n+2} = 2$, $b_{2n} = z$, $b_{2n+1} = -z$. Do not forget the logarithm: $\log(1+z)$ has a continued fraction expansion with $a_n = n$, $n = 0, 1, \ldots$, and $b_2 = z$, $b_{2n+1} = b_{2n+2} = z\,n^2$, $n = 1, 2, \ldots$

Ex. 4.4. Continued fraction expansions can also be found for divergent series. Check, after Euler [**41**], that the series $\sum z^n\, n!$ has a continued fraction expansion $1 + \frac{z}{1-b_2z/\cdots}$ with $[b_2, b_3, \ldots] = [2, 1, 3, 2, \ldots, n{+}1, n, \ldots]$. Introduce an extra parameter $\beta \in \mathbb{C}$ and show that $\sum z^n\, n! S^n(\beta)$ admits a continued fraction expansion

$$1 + z\beta + z^2\beta(\beta{+}1) + \cdots = 1 + \cfrac{z\beta}{1 - \cfrac{z(\beta{+}1)}{1 - \cfrac{z}{1 - \cfrac{z(\beta{+}2)}{1 - \cfrac{2z}{1 - \cfrac{z(\beta{+}3)}{1 - \ddots}}}}}}\,,$$

with parameters $\ldots$, $\beta{+}n$, n, $\ldots$.

Ex. 4.5. Let $a, b, \mathbb{A}$ be such that $S^i(\mathbb{A}) = (a-b)(a-bq)\cdots(a-bq^{i-1})$, $i \geq 0$. Give a continued fraction for $\sigma_z(\mathbb{A})$.

Ex. 4.6. Given two infinite sequences $[a_1, a_2, \ldots]$, $[b_2, \ldots]$, Sylvester [**166**], I, p.609-629 and 641-644, defines

$$K_n([a_1, \ldots, a_n], [b_2, \ldots, b_n])$$

to be be the $n \times n$ following determinant (with only three non zero diagonals):

$$K_n([a_1,\ldots,a_n],[b_2,\ldots,b_n]) := \begin{vmatrix} a_1 & b_2 & 0 & \cdots & & \\ -1 & a_2 & b_3 & 0 & \cdots & \\ 0 & \ddots & \ddots & \ddots & & \\ \vdots & & & & a_{n-1} & b_n \\ & & & & -1 & a_n \end{vmatrix}$$

Check directly, without using our previous considerations about Jacobi's matrix, but by mere manipulation of determinants as Sylvester did, that the convergents of the fraction $a_1 + \frac{b_2}{a_2+b_3/\cdots}$ are

$$P_n/Q_n = K([a_1,\ldots,a_n],[b_2,\ldots,b_n])/K([a_2,\ldots,a_n],[b_3,\ldots,b_n]) \ .$$

Ex. 4.7. Let $[a_1,\ldots,a_n]$, $[b_2,\ldots,b_n]$ be two sequences. Let the Jacobi matrix $J([a_1,\ldots,a_n],[b_2,\ldots,b_n])$ be the tridiagonal symmetric matrix with main diagonal $[a_1,\ldots,a_n]$, its two neighbours being $[b_2,\ldots,b_n]$ (convention different from (4.5.2)). Jacobi's continued fraction is $\cfrac{1}{u-a_1-\cfrac{{b_2}^2}{u-a_2-\cfrac{{b_3}^2}{u-a_3-\cdots}}}$. Show that its n-th convergent P_n/Q_n is such that P_n is the characteristic polynomial of $J([a_2,\ldots,a_n],[b_3,\ldots,b_n])$ and Q_n, of $J([a_1,\ldots,a_n],[b_2,\ldots,b_n])$.

Ex. 4.8. Let $\mathbb{A}$ be an alphabet of cardinality 2, and let $\Lambda^1 = \Lambda^1(\mathbb{A})$, $\Lambda^2 = \Lambda^2(\mathbb{A})$. Show that the n-th convergent of $\Lambda^1 - \frac{\Lambda^2}{\Lambda^1-\Lambda^2/\cdots}$ is $S^n(\mathbb{A})/S^{n-1}(\mathbb{A})$.

Ex. 4.9. (Contraction of a continued fraction). Since

$$1 + \frac{az}{1+bz/X} = 1 + az - \frac{abz^2}{bz+X}$$

one can contract Stieltjes' continuous fraction for $\sigma_z(\mathbb{A})$, or its inverse $\lambda_{-z}(\mathbb{A})$ by grouping terms two by two. There are two manners to do it:

$$\cfrac{1}{1+\cfrac{za_1}{1+\cfrac{za_2}{1+\cfrac{za_3}{\ddots}}}} = \cfrac{1}{1+za_1+\cfrac{z^2a_1a_2}{1+z(a_2+a_3)+\cfrac{z^2a_3a_4}{1+z(a_4+a_5)+\cfrac{z^2a_5a_6}{\ddots}}}}$$

$$1+\cfrac{za_1}{1+\cfrac{za_2}{1+\cfrac{za_3}{\ddots}}} = 1+\cfrac{za_1}{1+za_2-\cfrac{z^2a_2a_3}{1+z(a_3+a_4)-\cfrac{z^2a_4a_5}{1+z(a_5+a_6)+\cfrac{z^2a_6a_7}{\ddots}}}}$$

Show that the first contraction gives back Wroński's continuous fraction for $\lambda_{-z}(\mathbb{A})$.

For the second contraction, show that one can rewrite the coefficients

$$a_3 + a_4 = \frac{S_{222} - S_{33}}{S_1 S_{22}} \ , \ a_5 + a_6 = \frac{S_2 S_{3333} - S_{11} S_{444}}{S_{22} S_{33}} \ ,$$

$$a_{2n-1} + a_{2n} = \frac{S_{(n-1)^{n-2}} S_{n^{n+1}} - S_{(n-2)^{n-1}} S_{(n+1)^n}}{S_{(n-1)^{n-1}} S_{n^n}} .$$

Ex. 4.10. Let $f(x)$ be a function. Let $T_r(f)$ be the limit of Thiele's reciprocal difference $[x_1, \ldots, x_{r+1}]$, for $x_1 = \cdots = x_{r+1} = x$. Show that $T_0(f) = f$, $T_1(f) = 1/D(f)$, where D is the derivative with respect to x, and that

$$T_r(f) = T_{r-2}(f) + \frac{r}{D(T_{r-1}(f))}$$

Deduce Thiele's expansion of f in the neighbourhood of x :

$$f(x+\epsilon) = f(x) + \cfrac{\epsilon}{T_1(f) + \cfrac{\epsilon}{2T_1(T_1(f)) + \cfrac{\epsilon}{3T_1(T_2(f)) + \cfrac{\epsilon}{4T_1(T_2(f)) + \cfrac{\epsilon}{\ddots}}}}}$$

Ex. 4.11. Put $f(x) = e^x$ in the preceding exercise. Show after Perron [**146**] that $T_{2n}(e^x) = (-1)^n e^x$ and $T_{2n+1}(e^x) = (-1)^n (n+1) e^{-x}$. Transform Thiele's expression into

$$e^\epsilon = 1 + \cfrac{\epsilon}{1 - \cfrac{\epsilon}{2 + \cfrac{\epsilon}{3 - \cfrac{\epsilon}{2 + \cfrac{\epsilon}{5 - \cfrac{\epsilon}{\ddots}}}}}}$$

Ex. 4.12. Compute the generating function of Motzkin paths according to their length $\ell(w)$ and number of horizontal steps $h(w)$:

$$F = \sum_{w\, Motzkin} x^{\ell(w)} z^{h(w)} .$$

CHAPTER 5

Division, encore

5.1. Derived Alphabets

Instead of dividing polynomials, one can divide formal series in one variable. Of course, one wants to iterate division, and there are several possible definitions of Euclidean division for series.

In the case of generic polynomials in one variable, Euclidean division is a process which produces a sequence of linear factors[1], the successive quotients. Therefore, given two unitary series f, g in one variable z, by "division", we shall mean finding the unique constants ζ, β, and the unique unitary series h such that

$$f(z) = (1 - \zeta z)g(z) + \beta\, z^2 h \ . \tag{5.1.1}$$

Factorizing formally f and g : $f(z) = \sigma_z(\mathbb{B})$, $g = \sigma_z(\mathbb{A})$, the unknown coefficients are seen to be

$$\zeta = -S_1(\mathbb{B} - \mathbb{A}) \quad \& \quad \beta = S_2(\mathbb{B} - \mathbb{A}) \ . \tag{5.1.2}$$

One notices that the two coefficients ζ, β depend only upon the difference $\mathbb{B}-\mathbb{A}$. Starting with the pair of series $(\sigma_z(\mathbb{B} - \mathbb{A}),\, 1)$ means, in fact, *"dividing by 1"* (cf. [**101**]), and, contrary to the case of polynomials[2], gives the general case, because it simply consists into dividing all equations by $\sigma_z(\mathbb{A})$ without modifying coefficients.

We shall rather take as a starting pair $(1,\, \sigma_z(\mathbb{A} - \mathbb{B}))$, to have more regular indices.

Let us write $f = f_{-1}$, $g = f_0 = \sigma_z(\mathbb{A})$, $h = f_1$, and define the successive quotients and remainders in the Euclidean division of f_{-1} by f_0 by the equations

$$\begin{aligned} f_{-1}(z) &= (1 - z\zeta_0)\, f_0(z) + \beta_1 z^2\, f_1(z) \ , \\ f_0(z) &= (1 - z\zeta_1)\, f_1(z) + \beta_2 z^2\, f_2(z) \ , \\ f_1(z) &= (1 - z\zeta_2)\, f_2(z) + \beta_3 z^3\, f_3(z) \, , \ldots , \\ f_{i-1}(z) &= (1 - z\zeta_i)\, f_i(z) + \beta_{i+1} z^2\, f_{i+1}(z) \, , \ldots \end{aligned} \tag{5.1.3}$$

As we have just said, starting with the pair $(1, f_0/f_{-1})$ has the effect of dividing all equations by f_{-1}. Thus, without loss of generality, we shall take $f_{-1} = 1$, $f_0 = \sigma_z(\mathbb{A})$.

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[1] apart, possibly, from the first quotient, which is of degree the difference $m-n$ of degrees of the two input polynomials, when $m \geq n$.

[2] the quotient P/Q of two polynomials is not in general a polynomial, but the quotient f/g of two series is a series.

Define the *derived alphabets* $\mathbb{A}^i$ of $\mathbb{A} = \mathbb{A}^0$ by $f_i(z) = \sigma_z(\mathbb{A}^i)$, $i = 1, 2, \ldots$. Therefore, with $\mathbb{A}^{-1} = 0$, one has the system of equations

$$\sigma_z(\mathbb{A}^{i-1}) = (1 - z\zeta_i)\,\sigma_z(\mathbb{A}^i) + \beta_{i+1}\,z^2\sigma_z(\mathbb{A}^{i+1})\ , \quad i \geq 0\ . \tag{5.1.4}$$

Checking the terms of degree 1 and 2 :

$$1 + zS^1(\mathbb{A}^{i-1}) + z^2S^2(\mathbb{A}^{i-1}) + \cdots = (1 - z\zeta_i)\big(1 + zS^1(\mathbb{A}^i) + z^2S^2(\mathbb{A}^i) + \cdots\big) + z^2\beta_{i+1}$$

gives, as in (5.1.2),

$$\zeta_i = S_1(\mathbb{A}^i - \mathbb{A}^{i-1}) \quad \& \quad \beta_i = S_2(\mathbb{A}^{i-1} - \mathbb{A}^i)\ . \tag{5.1.5}$$

As in the case of polynomials, the remainder $\sigma_z(\mathbb{A}^k)$ of order k can be directly characterized. It is the unique unitary series f_k such that there exists scalars $\alpha_1^k, \ldots, \alpha_k^k$, $\beta_1^k, \ldots, \beta_{k-1}^k$, γ^k :

$$z^{2k}\gamma^k\, f_k = (1 + \alpha_1^k z + \cdots + \alpha_k^k z^k)\,\sigma_z(\mathbb{A}) - (1 + \beta_1^k z + \cdots + \beta_{k-1}^k z^{k-1})\,1\ . \tag{5.1.6}$$

We have already met similar coefficients in (3.1.5). Let us obtain them anew, thanks to the following property :

LEMMA 5.1.1. *The sum*

$$z^k S_{k^k}(\mathbb{A} - \frac{1}{z})\,\sigma_z(\mathbb{A}) + (-z)^{k-1} S_{(k-1)^{k+1}}(\mathbb{A} + \frac{1}{z}) \tag{5.1.7}$$

is equal to $z^{2k}S_{k^{k+1}}(\mathbb{A})$ *modulo* z^{2k+1}.

Proof. Indeed, $z^kS_{k^k}(\mathbb{A} - \frac{1}{z})\,\sigma_z(\mathbb{A})$ can be written as $z^kS_{k^k;k}(\mathbb{A}; \frac{1}{z})\,\sigma_z(\mathbb{A})$. Using the factor $\sigma_z(\mathbb{A})$ as a multiplier of the last column of $z^kS_{k^k;k}(\mathbb{A}; \frac{1}{z})$, one can now expand the determinant according to powers of z. Powers $z^k, \ldots, z^{2k-1}$ do not appear, because they are factors of determinants having two identical columns. The sum $z^{k-1}S_{k^k,-1}(\mathbb{A}) + \cdots + z^0S_{k^k,-k+1}(\mathbb{A})$ is equal to the expansion of $(-1)^k z^{k-1}$ $S_{(k-1)^{k+1}}(\mathbb{A}+z^{-1})$. Finally, the coefficient of z^{2k} is $S_{k^{k+1}}(\mathbb{A})$, the lemma is checked. □

The expression (5.1.7) is therefore equal to $z^{2k}S_{(k+1)^k}(\mathbb{A})\,f_k$. Recognizing that the coefficients of the powers z^i, $i \geq 2k$, are the Schur functions $S_{k^k,i-k}(\mathbb{A})$, one obtains the following description of the k-th derived alphabet :

THEOREM 5.1.2. *Given a generic alphabet* $\mathbb{A}$, *and a positive integer* k, *then the derived alphabet* $\mathbb{A}^k$ *of order* k *of* $\mathbb{A}$ *is such that*

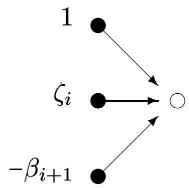
$$\sigma_z(\mathbb{A}^k) = \sum_{j=0}^{\infty} S_{k^k,\,k+j}(\mathbb{A})/S_{k^{k+1}}(\mathbb{A})\ . \tag{5.1.8}$$

The recursions between the successive series $z^i\sigma_z(\mathbb{A}^i)$ can be easily visualized when writing their coefficients into a matrix which is due to Stieltjes (writing S_I for $S_I(\mathbb{A})$, and $\cdot$ for 0:

$\sigma_z(\mathbb{A}^0)$	1	S_1	S_2	S_3	S_4	S_5	$\cdots$
$z\sigma_z(\mathbb{A}^1)$	$\cdot$	1	S_{12}/S_{11}	S_{13}/S_{11}	S_{14}/S_{11}	S_{15}/S_{11}	$\cdots$
$z^2\sigma_z(\mathbb{A}^2)$	$\cdot$	$\cdot$	1	S_{223}/S_{222}	S_{224}/S_{222}	S_{225}/S_{222}	$\cdots$
$z^3\sigma_z(\mathbb{A}^3)$	$\cdot$	$\cdot$	$\cdot$	1	S_{3334}/S_{3333}	S_{3335}/S_{3333}	$\cdots$
$\vdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	1	$\cdot$	$\cdots$

writing in row i, column j, $i, j = 0, 1, \ldots$ the coefficient of z^j in $z^i\sigma_z(\mathbb{A}^i)$.

Each entry is the weighted sum of three of its neighbours on the column on its left :



$$\Leftrightarrow \quad \frac{S_{(i-1)^{i-1}\,j}}{S_{(i-1)^i}} + \zeta_i \frac{S_{i^i\,j}}{S_{i^{i+1}}} - \beta_{i+1} \frac{S_{(i+1)^{i+1}\,j}}{S_{(i+1)^{i+2}}} = \frac{S_{i^i\,j+1}}{S_{i^{i+1}}}$$

For example, for $i = 2$, $j = 3$, one has

$$\frac{S_{13}}{S_{11}} + \zeta_2 \frac{S_{223}}{S_{222}} - \beta_3 \frac{S_{3333}}{S_{3333}} = \frac{S_{224}}{S_{222}} \ ,$$

with $\zeta_2 = S_{223}S_{222}^{-1} - S_{12}S_{11}^{-1}$, $\beta_3 = S_{3333}S_{11}S_{222}^{-2}$.

The recursions (5.1.4) can also be put into a matrix form

(5.1.9)

$$\begin{bmatrix} \zeta_0 & -\beta_1 & 0 & 0 & \cdots \\ 1 & \zeta_1 & -\beta_2 & 0 & \cdots \\ 0 & 1 & \zeta_2 & -\beta_3 & \cdots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} S_0(\mathbb{A}^0) & S_1(\mathbb{A}^0) & \cdots \\ S_{-1}(\mathbb{A}^1) & S_0(\mathbb{A}^1) & \cdots \\ S_{-2}(\mathbb{A}^2) & S_{-1}(\mathbb{A}^2) & \cdots \\ \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} S_1(\mathbb{A}^0) & S_2(\mathbb{A}^0) & \cdots \\ S_0(\mathbb{A}^1) & S_1(\mathbb{A}^1) & \cdots \\ S_{-1}(\mathbb{A}^2) & S_0(\mathbb{A}^2) & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

In other words, left multiplication by Jacobi's tridiagonal matrix of Stieltjes' matrix amounts to suppressing its first column (we have better see it as a shift of columns towards the right).

5.2. Normalized Differences

Instead of dividing, one could just use normalized differences[3], i.e. define unique constants γ_i, $i = 1, 2, \ldots$, and series $g_0 = \sigma_z(\mathbb{A})$, $g_1, g_2, \ldots$ such that

$$g_{i-1} = g_i - z\gamma_{i+1}g_{i+1} \quad , \quad i \geq 1 \ . \tag{5.2.1}$$

It is still appropriate to write the series as generating functions $\sigma_z(\mathbb{B})$ for some alphabets, putting them in two families $\{\mathbb{A}^n\}$, $\{\mathbb{A}^n_+\}$ according to parity. A minor adaptation of the preceding computations gives :

PROPOSITION 5.2.1. *According to parity, one has*

$$g_{2n} = f_n \quad = \quad \frac{1}{S_{n^{n+1}}(\mathbb{A})} \sum z^i S_{n^n\,,\,n+i}(\mathbb{A}) = \sigma_z(\mathbb{A}^n)\,, \tag{5.2.2}$$

$$g_{2n-1} \quad = \quad \frac{1}{S_{n^n}(\mathbb{A})} \sum z^i S_{n^{n-1}\,,\,n+i}(\mathbb{A}) = \sigma_z(\mathbb{A}^{n-1}_+) \ . \tag{5.2.3}$$

Let us call the alphabets $\mathbb{A}^0_+$, $\mathbb{A}^1_+$ ... the *secondary derived alphabets.*

Stieltjes's tableau, for the coefficients of the two families of series, becomes

[3]Replacing division by subtraction was also used by Wroński, as we have seen in Chapter 3. However, he needed first to evaluate the succesive powers of x modulo a given polynomial. We do not need this non-trivial first step here.

$$
\begin{array}{|cccccccccccc}
\hline
1 & 1 & S_1 & \frac{S_2}{S_1} & S_2 & \frac{S_3}{S_1} & S_3 & \frac{S_4}{S_1} & S_4 & \frac{S_5}{S_1} & S_5 & \cdots \\
\cdot & \cdot & 1 & 1 & \frac{S_{12}}{S_{11}} & \frac{S_{23}}{S_{22}} & \frac{S_{13}}{S_{11}} & \frac{S_{24}}{S_{22}} & \frac{S_{14}}{S_{11}} & \frac{S_{25}}{S_{22}} & \frac{S_{15}}{S_{11}} & \cdots \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \frac{S_{223}}{S_{222}} & \frac{S_{334}}{S_{333}} & \frac{S_{224}}{S_{222}} & \frac{S_{335}}{S_{333}} & \frac{S_{225}}{S_{222}} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \frac{S_{3334}}{S_{3333}} & \frac{S_{4445}}{S_{4444}} & \frac{S_{3335}}{S_{3333}} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \vdots & \cdots
\end{array} .
$$

In row n, $n = 0, 1, \ldots$, one finds the coefficients of $\sigma_z(\mathbb{A}^n)$ interlaced with those of $\sigma_z(\mathbb{A}^n_+)$. Writing $\square := n^n$, $\blacksquare = (n{+}1)^n$, one has :

$$
(5.2.4) \qquad 1,\ 1,\ \frac{S_{\square\, n+1}}{S_{\square\, n}},\ \frac{S_{\blacksquare\, n+2}(\mathbb{A})}{S_{\blacksquare\, n+2}(\mathbb{A})},\ \frac{S_{\square\, n+2}}{S_{\square\, n}},\ \frac{S_{\blacksquare\, n+3}(\mathbb{A})}{S_{\blacksquare\, n+2}(\mathbb{A})},\ \frac{S_{\square\, n+3}}{S_{\square\, n}},\ \frac{S_{\blacksquare\, n+4}(\mathbb{A})}{S_{\blacksquare\, n+2}(\mathbb{A})}, \ldots
$$

5.3. Associated Continued Fractions

Equations (5.1.3) can be written

$$
(5.3.1) \qquad f_{i-1}/f_i = (1 - \zeta_i z) + \beta_{i+1} z^2\, f_{i+1}/f_i
$$

and, therefore, translate into the continued fraction (with $\mathbb{A}^{-1} = 0$) :

$$
(5.3.2) \quad \sigma_z(\mathbb{A}) = \cfrac{1}{1 - \zeta_0 z + \cfrac{\beta_1 z^2}{1 - \zeta_1 z + \cfrac{\beta_2 z^2}{\ddots}}} =
$$

$$
\cfrac{1}{1 + zS_1(\mathbb{A}^{-1} - \mathbb{A}^0) + \cfrac{z^2 S_2(\mathbb{A}^{-1} - \mathbb{A}^0)}{1 - zS_1(\mathbb{A}^0 - \mathbb{A}^1) + \cfrac{z^2 S_2(\mathbb{A}^0 - \mathbb{A}^1)}{1 - zS_1(\mathbb{A}^1 - \mathbb{A}^2) + \cfrac{z^2 S_2(\mathbb{A}^1 - \mathbb{A}^2)}{\ddots}}}}
$$

Equations (5.2.1) can be rewritten

$$
(5.3.3) \qquad g_{k-1}/g_k = 1 - z\gamma_{k+1}\, g_{k+1}/g_k
$$

and give rise to the continued fraction

$$
(5.3.4) \qquad \sigma_z(\mathbb{A}) = \cfrac{1}{1 - \cfrac{\gamma_1 z}{1 - \cfrac{\gamma_2 z}{1 - \cfrac{\gamma_3 z}{\ddots}}}}
$$

with explicit parameters in terms of Schur functions of $\mathbb{A}$:

$$
(5.3.5) \quad \gamma_1 = S_1,\ \gamma_2 = -\frac{S_{1,1}}{S_1},\ \gamma_3 = \frac{S_{22}}{S_1 S_{11}},\ \gamma_4 = -\frac{S_{222} S_1}{S_{11} S_{22}},\ \gamma_5 = \frac{S_{333} S_{11}}{S_{22} S_{222}}
$$

$$
\gamma_{2n} = -\frac{S_{(n-1)^{n-1}} S_{n^{n+1}}}{S_{n^n} S_{(n-1)^n}} \qquad \& \qquad \gamma_{2n+1} = \frac{S_{(n+1)^{n+1}} S_{(n-1)^n}}{S_{n^n} S_{n^{n+1}}} .
$$

Notice the following relations, for any $n \geq 1$,

$$\beta_1^n \cdots \beta_n^1 = S_{n^{n+1}} \ \& \ \gamma_1^{2n-1} \cdots \gamma_{2n-1}^1 = S_{(n-1)^n} S_{n^n} \ \& \ \gamma_1^{2n} \cdots \gamma_{2n}^1 = -S_{n^n} S_{n^{n+1}} \,.$$

All these fractions have been considered by Stieltjes [**162**], who also examined problemes of convergence. For our part, we shall not touch upon the *analytic* theory of continued fractions, and will refer instead to Wall [**178**], apart from the original work of Stieltjes.

5.4. Hankel Forms

Stieltjes found that the problem of decomposing Hankel forms is closely related to division of series and continued fractions.

DEFINITION 5.4.1. The Hankel form corresponding to an alphabet $\mathbb{A}$ is

$$H(\mathbb{A}; x) = \sum_{i,j=0}^{\infty} S_{i+j}(\mathbb{A})\, x_i x_j \ . \tag{5.4.1}$$

The problem, solved by Stieltjes, is to write $H(\mathbb{A}; x)$ as a linear combination of squares :

$$H(\mathbb{A}; x) = (x_0 + \star x_1 + \star x_2 + \cdots)^2 + \star(x_1 + \star x_2 + \star x_3 + \cdots)^2 + \star(x_2 + \star x_3 + \star x_4 + \cdots)^2 + \cdots \,,$$

the $\star$'s designating unknown coefficients.

First one polarizes the form :

$$H(\mathbb{A}; x, y) = \sum_{i,j=0}^{\infty} S_{i+j}(\mathbb{A}) x_i y_j \ .$$

Now, as Rogers [**154**] noticed, it is equivalent to decompose the simpler form in two variables x, y, considered as rank-1 elements :

$$K(\mathbb{A};\ x, y) = H(\mathbb{A}; x^0, x^1, x^2, \ldots, \ y^0, y^1, y^2, \ldots) = \sum_{i=0}^{\infty} S_i(x+y)\, S_i(\mathbb{A}) \ , \tag{5.4.2}$$

recovering the previous case under the "umbral operator"[4] $x^i \to x_i$, $y^i \to y_i$.

We can now formulate Stieltjes' result.

THEOREM 5.4.2. *(Stieltjes). Given an alphabet $\mathbb{A}$, and its derived aphabets*

$$\mathbb{A}^0 = \mathbb{A},\ \mathbb{A}^0_+,\ \mathbb{A}^1,\ \mathbb{A}^1_+,\ \mathbb{A}^2, \ldots \ ,$$

then the form $K(\mathbb{A};\ x, y)$ decomposes as

$$\sum_{i,j=0}^{\infty} S_{i+j}(\mathbb{A})\, x^i y^j = \sigma_1\big((x+y)\mathbb{A}^0\big) - xyS_{11}(\mathbb{A})\, \sigma_1\big((x+y)\mathbb{A}^1\big) + x^2y^2 \frac{S_{222}(\mathbb{A})}{S_{11}(\mathbb{A})} \cdot \\ \cdot \sigma_1\big((x+y)\mathbb{A}^2\big) - \cdots + (-xy)^n \frac{S_{n^{n+1}}(\mathbb{A})}{S_{(n-1)^n}(\mathbb{A})} \sigma_1\big((x+y)\mathbb{A}^n\big) + \cdots \tag{5.4.3}$$

[4] The restriction that $x^0 = y^0 = 1$ is overcome by homogeneity, it is not necessary to shift the indices and start by $i = j = 1$ to be generic.

and the form $\sum_{i,j=0}^{\infty} S_{i+j+1}(\mathbb{A})x^i y^j$ *decomposes as*

$$(5.4.4)\qquad \sum_{i,j=0}^{\infty} S_{i+j+1}(\mathbb{A})\, x^i y^j = S_1(\mathbb{A})\,\sigma_1\big((x{+}y)\mathbb{A}_+^0\big) - xy\frac{S_{22}(\mathbb{A})}{S_1(\mathbb{A})}\,\sigma_1\big((x{+}y)\mathbb{A}_+^1\big)$$
$$+\cdots+(-xy)^n\frac{S_{n^n}(\mathbb{A})}{S_{(n-1)^{n-1}}(\mathbb{A})}\sigma_1\big((x{+}y)\mathbb{A}_+^n\big)+\cdots$$

Proof. One could check that the recursions underlying the two tableaux of Stieltjes displayed above are the same as for decomposing recursively the Hankel form.

Let us however give a more illuminating proof, which contrary to all pedagogical principles, uses decompositions which will be explained later (decomposing a polynomial in a basis of orthogonal polynomials).

From the theory of orthogonal polynomials, which can be written $S_{n^n}(\mathbb{A}{-}z)$ when the moments are $S_0(\mathbb{A})$, $S_1(\mathbb{A}),\ldots$, one knows how to decompose any polynomial in z, in particular, powers z^i :

$$(5.4.5)\qquad z^i = S_i(\mathbb{A})S_0(\mathbb{A}-z) - \frac{S_{1i}(\mathbb{A})}{S_{11}(\mathbb{A})}S_1(\mathbb{A}-z) + \frac{S_{22i}(\mathbb{A})}{S_{222}(\mathbb{A})}S_{22}(\mathbb{A}-z)$$

(identity which could be checked directly, using determinantal identities).

Multiply (5.4.5) by $1/(1{-}yz)$ and take its image under the operator $z^i \to S_i(\mathbb{A})$ (which send $S_{n^n}(\mathbb{A}-z)z^j = S_{n^n;\,j}(\mathbb{A};\,z)$ to $S_{n^n;\,j}(\mathbb{A})$). Then the equation becomes

$$(5.4.6)\qquad \sum_j y^j S_{i+j}(\mathbb{A}) = S_i(\mathbb{A})\sum y^j S_j(\mathbb{A}) - \frac{S_{1i}(\mathbb{A})}{S_{11}(\mathbb{A})}\sum y^j S_{1j}(\mathbb{A})$$
$$+\frac{S_{22j}(\mathbb{A})}{S_{222}(\mathbb{A})}\sum y^j S_{22j}(\mathbb{A}) - \cdots\ ,$$

which is the coefficient of x^i in (5.4.3) .

The second statement of the theorem results from the first one by a shift (by 1) of indices. ☐

Rogers [**154**] interprets Stieltjes' decomposition as an addition formula.

Exercises

Ex. 5.1. Let $\mathbb{A}$ be arbitrary, m, n be two positive integers. Find, after Frobenius [**47**] the polynomials $P(x)$ of degree $\le n$, $Q(x)$ of degree $\le m{-}1$ such that $\sigma_x(\mathbb{A})\,P(x) - Q(x)$ be divisible by x^{m+n}.

Ex. 5.2. Let $\mathbb{A}$ be an alphabet, and y be such that $-y$ be of rank 1.
Write a continued fraction for $\sum z^n S^n(\mathbb{A}{+}ny)$ (for example, of the type (5.3.4)).

Ex. 5.3. Let $\mathbb{A}$ be of cardinality n, $\Omega = \{\zeta\}$ be the set of n+1-roots of unity. Writing S^i for $S^i(-\mathbb{A})$, show that the circulant $\begin{vmatrix} S^n & S^0 & \cdots & S^{n-1} \\ S^{n-1} & S^n & \cdots & S^{n-2} \\ \vdots & \vdots & & \vdots \\ S^0 & S^1 & \cdots & S^n \end{vmatrix}$ is equal to

$$P := \prod_{\zeta\in\Omega}(S^0 + \zeta S^1 + \cdots + \zeta^n S^n)\ .$$

Ex. 5.4. Recall that the Hermite polynomials $H_n(x)$ are defined by the generating function $\exp(2zx - z^2) = \sum z^n H_n(x)/n!$. Write $\sum z^n H_n(x)$ as a continued fraction of the type (5.3.4): $\dfrac{1}{1 - \dfrac{\gamma_1 z}{1 - \dfrac{\gamma_2 z}{\ddots}}}$.

Ex. 5.5. Give a continued fraction expansion of the series $\sum z^i(1+2i)^{-1}$.

Ex. 5.6. Give a continued fraction expansion of $f_\alpha(z) = \left(\sum(-z)^i\alpha/i\right)^{-1}$, where α is a parameter.

Ex. 5.7. Let $\mathbb{A}$ be such that

$$S^i(\mathbb{A}) = \frac{1\cdot 3\cdots(2i-1)}{2\cdot 4\cdots 2i\ (2i+1)}\ , i \geq 1\ .$$

Give a continuous fraction expansion of $\sigma_z(\mathbb{A} + 1/2)$.

Ex. 5.8. Give the Taylor expansion of the continued fractions

$$\cfrac{\gamma_1}{1 - \cfrac{\gamma_2 z}{1 - \cfrac{\gamma_3 z}{1 - \cfrac{\gamma_4 z}{\ddots}}}} \qquad \& \qquad \cfrac{-\gamma_1\gamma_2}{1 - \cfrac{\gamma_1\gamma_3 z}{1 - \cfrac{\gamma_1\gamma_4 z}{1 - \cfrac{\gamma_1\gamma_5 z}{\ddots}}}}\ .$$

obtained by shift of indices from Stieltjes' continued fraction (5.3.4).

Ex. 5.9. Bell numbers B_n are usually defined through their exponential generating function

$$\exp(e^z - 1) = 1 + z + 2z^2/2! + 5z^3/3! + 15z^4/4! + \cdots + B_n z^n/n! + \cdots$$

Show that

$$\cfrac{1}{1 - x - \cfrac{x^2}{1 - 2x - \cfrac{2x^2}{\ddots\ 1 - nx - \cfrac{nx^2}{\ddots}}}} = 1 + x + 2x^2 + \cdots + B_n x^n + \cdots$$

Ex. 5.10. Recall that the q-exponential (Eq. 2.3.5) is equal to $\sigma_z\big(1/(1-q)\big)$. Define the q-logarithm to be $\sum_{i\geq 1} z^i(1 + q + \cdots + q^{i-1})^{-1}$.

Write continued fractions for both these functions.

CHAPTER 6

Padé Approximants

6.1. Recovering a Rational Function from a Taylor Series

Given a formal series $f(z) := \sum z^n S_n$, with $S_0 = 1$, one wants a rational function $\prod_{b\in\mathbb{B}}(1 - zb)/\prod_{a\in\mathbb{A}}(1 - za)$ (with no factor common to the numerator and the denominator) whose Taylor expansion coincides with $f(z)$ up to the order $card(\mathbb{A})+card(\mathbb{B})$ (which is the maximum that one can expect, counting the number of parameters). In other words, one wants the first coefficients S_i to be equal to the complete functions $S_i(\mathbb{A} - \mathbb{B})$.

Writing the rational function, up to a power of z, as follows

$$\frac{R(z^{-1}, \mathbb{B})}{R(z^{-1}, \mathbb{A})} = \pm\frac{R(z^{-1} + \mathbb{A}, \mathbb{B})}{R(z^{-1} + \mathbb{B}, \mathbb{A})} \,, \tag{6.1.1}$$

one could stop there, because the numerator and denominator of the last expression are functions of $\mathbb{A} - \mathbb{B}$, and therefore, as we have seen in the first chapter, are expressible in terms of the coefficients of the series.

However, it would not be polite towards Padé [**145**], and we shall provide some comments about the different expressions of the rational functions attached to his name.

Let us first remark that the extra factor that we have introduced, $R(\mathbb{A}, \mathbb{B}) = \prod_a \prod_b (a-b)$, is by hypothesis different from 0 (the rational function is reduced).

Following the usual convention, we shall write

$$[\beta/\alpha]$$

for the approximation of numerator of degree $\beta = card(\mathbb{B})$, and denominator of degree $\alpha = card(\mathbb{A})$.

Thanks to the different expressions of the resultant, the Padé approximant can be written :

$$\begin{aligned} [\beta/\alpha] &= (-1)^{\alpha\beta+\alpha+\beta}\, \frac{z^\beta S_{(\alpha+1)^\beta}(\mathbb{B}-\mathbb{A} - \frac{1}{z})}{z^\alpha S_{(\beta+1)^\alpha}(\mathbb{A}-\mathbb{B} - \frac{1}{z})} \\ &= (-1)^{\alpha}\, \frac{z^\beta S_{\beta^{\alpha+1}}(\mathbb{A}-\mathbb{B} + \frac{1}{z})}{z^\alpha S_{(\beta+1)^\alpha}(\mathbb{A}-\mathbb{B} - \frac{1}{z})} = (-1)^{\alpha\beta}\, \frac{z^\beta \Lambda_{(\alpha+1)^\beta}(\mathbb{A}-\mathbb{B} + \frac{1}{z})}{z^\alpha \Lambda_{(\beta+1)^\alpha}(\mathbb{B}-\mathbb{A} + \frac{1}{z})} \end{aligned} \tag{6.1.2}$$

Do not forget that, for any $\mathbb{A}$, any x of rank 1, on has the decompositions $S^k(\mathbb{A}+x) = \sum_i x^i S^{k-i}(\mathbb{A})$, $S^k(\mathbb{A}-x) = S^k(\mathbb{A}) - xS^{k-1}(\mathbb{A})$, $\Lambda^k(\mathbb{A}+x) = \Lambda^k(\mathbb{A}) - x\Lambda^{k-1}(\mathbb{A})$. This shows how to expand the preceding functions in terms of x.

In fact, using our favorite transformations on Schur functions, we can restrict $1/z$ to only one column and write :

$$[\beta/\alpha] = (-1)^{\alpha} z^{\beta-\alpha}\, S_{\beta;\,\beta^{\alpha}}(\mathbb{A}-\mathbb{B}+\frac{1}{z};\mathbb{A}-\mathbb{B})\,/\,S_{(\beta+1)^{\alpha};\,0}(\mathbb{A}-\mathbb{B};\frac{1}{z}) \tag{6.1.3}$$

These expressions can be found in the literature: (6.1.3) is due to Jacobi [**74**], while Sylvester [**165**] gave the second expression of (6.1.2).

The explicit expansion of the numerator and the denominator of the Padé approximant, in terms of the Schur functions of $\mathbb{A}-\mathbb{B}$, is

$$[\beta/\alpha] = \frac{S_{\beta^{\alpha}} + zS_{1\beta^{\alpha}} + z^2 S_{2\beta^{\alpha}} + \cdots + z^{\beta} S_{\beta\beta^{\alpha}}}{S_{\beta^{\alpha}} - zS_{\beta^{\alpha-1},\beta+1} + z^2 S_{\beta^{\alpha-2},(\beta+1)^2} + \cdots + (-z)^{\alpha} S_{(\beta+1)^{\alpha}}} \,. \tag{6.1.4}$$

For example, for $\beta=1, \alpha=1$, one gets

$$[1/1] = \frac{S_1 + zS_{11}}{S_1 - zS_2} = 1 + zS^1 + z^2S^2 + z^3 S^{22}/S^1 + z^4 S^{222}/S^{11} + \cdots\,,$$

for $\beta=1, \alpha=2$,

$$[1/2] = \frac{S_{11} + zS_{111}}{S_{11} - zS_{12} + z^2 S_{22}} = 1 + zS^1 + z^2S^2 + z^3S^3 + z^4\frac{S_{114}+S_{15}-S_{222}}{S_{11}} + \cdots\,,$$

for $\beta=2, \alpha=1$,

$$[2/1] = \frac{S_2 + zS_{12} + z^2 S_{22}}{S_2 - zS_3} = 1 + zS^1 + z^2S^2 + z^3S^3 + z^4\frac{S^{33}}{S^2} + \cdots$$

and for $\beta=2, \alpha=2$,

$$\frac{S_{22} + zS_{122} + z^2 S_{222}}{S_{22} - zS_{23} + z^2 S_{33}} = 1 + zS^1 + z^2S^2 + z^3S^3 + z^4S^4 + \frac{S^{144} - 2S^{234} + S^{333}}{S_{22}} + \cdots$$

Let us repeat that we have supposed that $R(\mathbb{A},\mathbb{B})$ be different from 0, that is, we have assumed the non-vanishing of the Hankel determinant $S_{\beta^{\alpha}}(\mathbb{A}-\mathbb{B})$ of coefficients of the initial series. Specialists know what to do in the degenerate cases [**145**], [**11**].

6.2. Padé Table

Usually, the $[\beta/\alpha]$ are displayed in a two-dimensional table (let us take $\beta = 0, 1, .. =$ horizontal axis), and one is interested in giving relations between neighbours. Now, the alphabets $\mathbb{A}$ and $\mathbb{B}$ are variable and we shall avoid using them. Say that the function to be interpolated is $f(z) = \sigma_z(\mathbb{D})$.

Because $[\beta{-}1/\alpha]$ and $[\beta/\alpha]$ are approximations of consecutive orders of the same function, one sees that $[\beta/\alpha]-[\beta{-}1/\alpha]$ is a rational function with a numerator reduced to $cz^{\alpha+\beta}$, with $c \in \mathfrak{Sym}$. In fact, we have already encountered the relation

$$S_{k^n}(\mathbb{A}+x)S_{k^{n-1}}(\mathbb{A}-x) - xS_{(k-1)^n}(\mathbb{A}+x)S_{(k+1)^{n-1}}(\mathbb{A}-x) = S_{k^n}(\mathbb{A})S_{k^{n-1}}(\mathbb{A})\,,$$

and its image under $\mathbb{A} \mapsto -\mathbb{A}$.

This relation gives us

$$[\beta/\alpha] - [\beta{-}1/\alpha] = (-1)^{\alpha} z^{\beta-\alpha} \frac{S_{\beta^{\alpha}}(\mathbb{D})S_{\beta^{\alpha+1}}(\mathbb{D})}{S_{\beta^{\alpha}}(\mathbb{D}-1/z)S_{(\beta+1)^{\alpha}}(\mathbb{D}-1/z)}\,, \tag{6.2.1}$$

as well as

$$[\beta/\alpha] - [\beta/\alpha{-}1] = (-1)^{\alpha} z^{\beta+1-\alpha} \frac{S_{\beta^{\alpha}}(\mathbb{D})S_{(\beta+1)^{\alpha}}(\mathbb{D})}{S_{(\beta+1)^{\alpha-1}}(\mathbb{D}-1/z)S_{(\beta+1)^{\alpha}}(\mathbb{D}-1/z)}\,. \tag{6.2.2}$$

Now, one can relate the inverse of differences between $[\beta/\alpha]$ and its North, East, South, West neighbours, as shows the following lemma, due to Wynn.

LEMMA 6.2.1. *(Wynn) The elements of a generic Padé table satisfy :*

$$\frac{1}{[\beta/\alpha{+}1]-[\beta/a]}+\frac{1}{[\beta/\alpha{-}1]-[\beta/a]}=\frac{1}{[\beta{-}1/\alpha]-[\beta/a]}+\frac{1}{[\beta{+}1/\alpha]-[\beta/a]} \tag{6.2.3}$$

Proof. Recall Bazin's relation :

$$\begin{vmatrix} [\mathcal{C}12] & [\mathcal{C}13] & [\mathcal{C}14] \\ [\mathcal{C}32] & [\mathcal{C}33] & [\mathcal{C}34] \\ [\mathcal{C}25] & [\mathcal{C}35] & [\mathcal{C}45] \end{vmatrix} = 0$$

for the minors of order n of a matrix $n \times \infty$, $\mathcal{C}$ being a set of integers of cardinality $n-2$.

Apply this relation to the matrix obtained from $\mathbb{S}_{(\beta+1)^\alpha}(\mathbb{D}-x)$ by repeating the first column, with the alphabet $\mathbb{D}$, and adding the two columns $[1,0,\ldots,0]$ and $[0,\ldots,0,1]$. Explicitly, with $\alpha=3$, $\beta=5$, the matrix is

$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 & S^6(\mathbb{D}) & S^6(\mathbb{D}-x) & S^7(\mathbb{D}-x) & S^8(\mathbb{D}-x) & S^9(\mathbb{D}-x) \\ 0 & 0 & S^5(\mathbb{D}) & S^5(\mathbb{D}-x) & S^6(\mathbb{D}-x) & S^7(\mathbb{D}-x) & S^8(\mathbb{D}-x) \\ 0 & 0 & S^4(\mathbb{D}) & S^4(\mathbb{D}-x) & S^5(\mathbb{D}-x) & S^6(\mathbb{D}-x) & S^7(\mathbb{D}-x) \\ 0 & 1 & S^3(\mathbb{D}) & S^3(\mathbb{D}-x) & S^4(\mathbb{D}-x) & S^5(\mathbb{D}-x) & S^6(\mathbb{D}-x) \end{bmatrix} \\ \textit{columns} & \begin{matrix} 1 & 2 & 3 & 4 & \mathcal{C} & \mathcal{C} & 5 \end{matrix} \end{array} .$$

Using that $[34\mathcal{C}] = S_{6,555}(\mathbb{D},\mathbb{D}{-}x) = S_{65,55}(\mathbb{D},\mathbb{D}{-}x) - xS_{64,55}(\mathbb{D},\mathbb{D}{-}x) = xS_{55,55}(\mathbb{D},\mathbb{D}{-}x) = xS_{5555}(\mathbb{D})$, one gets the relation

$$\begin{vmatrix} S_{66}(\mathbb{D}{-}x) & S_{555}(\mathbb{D}) & S_{555}(\mathbb{D}{-}x) \\ -S_{666}(\mathbb{D}) & 0 & xS_{5555}(\mathbb{D}) \\ S_{777}(\mathbb{D}{-}x) & S_{6666}(\mathbb{D}) & S_{6666}(\mathbb{D}{-}x) \end{vmatrix} = 0 ,$$

which expands into

$$S_{777}(\mathbb{D}{-}x)S_{555}(\mathbb{D})xS_{5555}(\mathbb{D}) + S_{666}(\mathbb{D})S_{555}(\mathbb{D})S_{6666}(\mathbb{D}{-}x) -$$
$$S_{666}(\mathbb{D})S_{6666}(\mathbb{D})S_{555}(\mathbb{D}{-}x) - S_{66}(\mathbb{D}{-}x)S_{6666}(\mathbb{D})xS_{5555}(\mathbb{D}) = 0 .$$

This is Wynn's identity (6.2.3), reduced to the same denominator (up to the common factor $S_{(\beta+1)^\alpha}(\mathbb{D}{-}x)$). □

See in Exercises 6.5,6.6 a step by step proof of Wynn's identity.

6.3. Euclid and Padé

Euclidean division can be put into use to provide recursions between Padé entries. Let $[\beta/\alpha]$ denote a Padé approximant of the function $\lambda_{-z}(\mathbb{A})$. Then the following lemma, due to Kronecker, relates the approximants $[n-j/j]$, $j=0,\ldots,n$, for any fixed n.

LEMMA 6.3.1. *(Kronecker) Let n be an integer. Define $P_{-1}=z^{n+1}$, $P_0 = S^n(1-z\mathbb{A})$, $Q_{-1}=0$, $Q_0=1$, and for $j=0,\ldots,n-1$, and P_j, Q_j, ρ_j be such that they satisfy the following equations*

$$P_{j+1} = -P_{j-1} + \rho_j P_j \tag{6.3.1}$$

$$Q_{j+1} = -Q_{j-1} + \rho_j Q_j , \tag{6.3.2}$$

with P_j of degree $n-j$ in z.

In other words, $-P_{j+1}$ is the remainder of the division of P_{j-1} by P_j, and ρ_j is the quotient.

Then P_j/Q_j is the Padé approximant of type $[n-j/j]$ of $\lambda_{-z}(\mathbb{A})$, and

$$P_jQ_{j+1} - P_{j+1}Q_j = z^{n+1} .$$

Proof. The successive remainders have been described in the third chapter (we need the version of Ex. 3.5). Up to normalization, they are

$$z^{1-n}S_{2^{n-1}}(\mathbb{A}-z),\ z^{2-n}S_{3^{n-2}}(\mathbb{A}-z),\ z^{3-n}S_{4^{n-3}}(\mathbb{A}-z),\ \ldots$$

On the other hand, Equations (6.3.1) implies that the determinant $\begin{vmatrix} P_j & P_{j+1} \\ Q_j & Q_{j+1} \end{vmatrix}$ be independent of j, and this determines the Q_j's knowing the P_j's.

Recognizing the multiplicators (3.1.9) in the Euclidean division, one gets that $Q_1, Q_2, \ldots$ are, up to the same normalization as the P_j's,

$$Q_1 = S_1(\mathbb{A}+z),\ Q_2 = S_{22}(\mathbb{A}+z),\ \ldots,\ Q_j = S_{j^j}(\mathbb{A}+z), \ldots .$$

Checking values in $z=0$, one indeed recognizes that P_j/Q_j is the $[n-j/j]$-approximant. □

For example, for $n=4$, the approximants [4/0], [3/1], [2/2], [1/3], [0/4] are

$$\frac{z^4S_{1111}(\mathbb{A}-z^{-1})}{1},\ \frac{-z^3S_{222}(\mathbb{A}-z^{-1})}{zS_{1111}(\mathbb{A}+z^{-1})},\ \frac{z^2S_{33}(\mathbb{A}-z^{-1})}{z^2S_{222}(\mathbb{A}+z^{-1})},\ \frac{-zS_4(\mathbb{A}-z^{-1})}{z^3S_{33}(\mathbb{A}+z^{-1})},\ \frac{1}{z^4S_4(\mathbb{A}+z^{-1})}.$$

The following congruences modulo z^5, show that, indeed, $Q_1, \ldots, Q_4$ are the multiplicators of $\lambda_{-z}(\mathbb{A})$ in the division of z^5 by $\lambda_{-z}(\mathbb{A})$:

$$\begin{aligned}
zS_{1111}(\mathbb{A}+z^{-1}) &\equiv -z^3S_{222}(\mathbb{A}-z^{-1}) ,\\
z^2S_{222}(\mathbb{A}+z^{-1}) &\equiv z^2S_{33}(\mathbb{A}-z^{-1}) ,\\
z^3S_{33}(\mathbb{A}+z^{-1}) &\equiv -zS_4(\mathbb{A}-z^{-1}) ,\\
z^4S_4(\mathbb{A}+z^{-1}) &\equiv 1 .
\end{aligned}$$

6.4. Rational Interpolation

Padé approximants are rational interpolants of $f(z)$. But in general interpolation theory, one wants to produce expressions depending upon the values of $f(z)$ at specific points $y_1, y_2, \ldots \in \mathcal{Y}$. The preceding sections can be considered as describing the case where all y_i's are concentrated in 0, and replacing the values $f(x_i)$ by the Taylor expansion of $f(z)$ in $z=0$.

Already Cauchy had solved the rational interpolation problem in 1820, writing a summation generalizing Lagrange interpolation. We shall rather use the following determinantal expression, and give its relation with Padé expressions.

LEMMA 6.4.1. *Let $f(z)$ be a function of one indeterminate, and let $\alpha, \beta \in \mathbb{N}$. Let $P(z)/Q(z)$, $deg(P) \le \beta$, $deg(Q) \le \alpha$, coincides with $f(z)$ at points $y \in \mathcal{Y}$, $card(\mathcal{Y}) = \alpha+\beta+1$, . Then P and Q are determined, up to a factor, by the following identity (the determinant is of order $\alpha+\beta+1$; each row, except the last* *one, depends upon a single $y \in \mathcal{Y}$ in turn).*

$$\mathcal{Y}\left\{\begin{vmatrix} \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 1 & y & \cdots & y^\beta & y^\alpha f(y) & \cdots & yf(y) & f(y) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ Q(z) & zQ(z) & \cdots & z^\beta Q(z) & z^\alpha P(z) & \cdots & zP(z) & P(z) \end{vmatrix}\right. = 0\,. \tag{6.4.1}$$

Proof. Multiply row y by the corresponding factor $Q(y)$. The new determinant is an alternating function in $y_1, \ldots, y_{\alpha+\beta+1}, z$ of degree $\leq \alpha+\beta$ in each letter. Being divisible by the Vandermonde in $\mathcal{Y}+z$, it must be null. □

To get more compact expressions of P and Q, let us suppose that $f(z)$ is a polynomial that we shall write $f(z) = S^N(z - \mathbb{D})$.

The denominator $Q(z)$ is proportional to

$$\begin{vmatrix} \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 1 & y & \cdots & y^\beta & y^\alpha f(y) & \cdots & yf(y) & f(y) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & z^\alpha & \cdots & z & 1 \end{vmatrix}\,.$$

Extracting the Vandermonde $\Delta(\mathcal{Y})$ from the upper part, one transforms the preceding determinant into a determinant of complete functions :

$$\begin{vmatrix} S^0(\mathcal{Y}) & \cdots & S^\beta(\mathcal{Y}) & S^{N+\alpha}(\mathcal{Y}-\mathbb{D}) & \cdots & S^N((\mathcal{Y}-\mathbb{D}) \\ \vdots & & \vdots & \vdots & & \vdots \\ S^{-\alpha-\beta}(\mathcal{Y}) & \cdots & S^{-\alpha}(\mathcal{Y}) & S^{N-\beta}(\mathcal{Y}-\mathbb{D}) & \cdots & S^{N-\alpha-\beta}((\mathcal{Y}-\mathbb{D}) \\ 0 & \cdots & 0 & z^\alpha & \cdots & z^0 \end{vmatrix}\,.$$

The determinant factorizes into its North-West part (equal to $\mathbb{S}_{0^{\beta+1}}(\mathcal{Y}) = 1$) and its South-East part in which one recognizes, up to transposition,

$$S_{(N-\beta)^\alpha,0}(\mathcal{Y}-\mathbb{D}, z) = S_{(N-\beta)^\alpha}(\mathcal{Y}-\mathbb{D}-z)\ .$$

Similarly, $P(z)$ is proportional to

$$\begin{vmatrix} S^0(\mathcal{Y}) & \cdots & S^\beta(\mathcal{Y}) & S^{N+\alpha}(\mathcal{Y}-\mathbb{D}) & \cdots & S^N((\mathcal{Y}-\mathbb{D}) \\ \vdots & & \vdots & \vdots & & \vdots \\ S^{-\alpha-\beta}(\mathcal{Y}) & \cdots & S^{-\alpha}(\mathcal{Y}) & S^{N-\beta}(\mathcal{Y}-\mathbb{D}) & \cdots & S^{N-\alpha-\beta}((\mathcal{Y}-\mathbb{D}) \\ z^0 & \cdots & z^\beta & 0 & \cdots & 0 \end{vmatrix}\,.$$

Subtracting z to alphabets in columns $2, \ldots, \alpha+1$, the bottom row becomes $[1, 0, \ldots, 0]$ and what remains of the determinant after suppressing the first column and bottom row (and permuting the last columns) is the Schur function $S_{1^\beta,(N-\beta)^{\alpha+1}}(\mathcal{Y}-z, \mathcal{Y}-\mathbb{D})$. This function is in fact the remainder of degree β in the Euclidean division of $S^N(z-\mathbb{D})$ by $R(z, \mathcal{Y})$, as one could have realized beforehand without computation!

In summary, one has

LEMMA 6.4.2. *The approximant of type $[\beta/\alpha]$ of the function $S^N(z-\mathbb{D})$, obtained from the interpolation set $\mathcal{Y}$ of cardinality $\alpha+\beta+1$, is*

$$S_{1^b,(N-\beta)^{\alpha+1}}(\mathcal{Y}-z, \mathcal{Y}-\mathbb{D})\ /\ S_{(N-\beta)^\alpha}(\mathcal{Y}-\mathbb{D}-z)\ . \tag{6.4.2}$$

Questions of convergence are at the heart of the Padé approximation paradigm, and we have only touched upon the relevant algebraic identities. This algebraic part is only part of the story. However for what concerns it proper, we disagree with most of the classics.

First, the theory is full of quadratic relations between minors, and no reference is usually given to Plücker relations[1], which are more general and more understandable that the 3 terms or 5 terms relations commonly used.

Secondly, many functions or coefficients can be identified with Schur functions. It is true that if one restricts to Schur functions indexed by rectangular partitions, one can dispense with symmetric function theory altogether. However, inductions involve other partitions than the rectangular ones, and become cumbersome when only Hankel or Toeplitz determinants are allowed, and not their general minors.

Thirdly, using partitions is the only way of easily seeing that a function appearing in a certain process is the same Schur function as the one pertaining to another theory. Even to identify some coefficients in the Euclidean division of two polynomials (when they are not the leading terms) would be cumbersome without reference to Schur functions.

In the fourth place, this is a text about the beauties of symmetric functions and their generalizations.

Exercises

EX. 6.1. Let $f(z) := \lambda_z(\mathbb{B})/\lambda_z(\mathbb{A})$ be a rational fraction. Given two integers m, n, write the numerator and the denominator of the Padé approximant P/Q of $f(z)$ of degree $[\beta/\alpha]$ as determinants with entries the elementary symmetric functions in $\mathbb{A}$ or $\mathbb{B}$.

EX. 6.2. Let $\mathbb{A}$ be an alphabet of infinite cardinality, m, n be two positive integers, $K \subset [1, \ldots, m{+}n]$ be a subset of cardinality n. Find a polynomial $Q(z) = S^n(1 - z\mathbb{B})$ of degree n such that

$$S^k(\mathbb{A} - \mathbb{B}) = 0 \ \forall k \in K \ .$$

Thus $P(z) := S^{m+n}(1 + z(\mathbb{A}{-}\mathbb{B}))$ is a polynomial of degree $\leq m + n$ such that

$$P/Q \equiv \sigma_z(\mathbb{A}) \qquad \mathrm{mod}\ (x^{m+n+1}) \ ,$$

but, contrary to the case of a Padé approximant, the numerator is of degree $\leq m{+}n$, having n coefficients null.

EX. 6.3. Let α, β be two integers, and $[\beta/\alpha]$ the Padé approximant of $\sigma_z(\mathbb{D})$. Give the coefficients in the Taylor expansion of $[\beta/\alpha]$ as functions of $\mathbb{D}$. In particular describe explicitly the coefficient of $z^{\alpha+\beta+1}$.

For example,

$$[1/1] = 1{+}zS^1{+}z^2S^2{+}z^3S^{22}/S^1{+}z^4S^{222}/S^{11}{+}\cdots = 1{+}zS^1{+}z^2S^2\left(1 - zS^2/S^1\right)^{-1} \ .$$

EX. 6.4. ([**11**], p.339). Let $f(z)$ be a function of one variable, α, β two positive integers, $\mathcal{Y} = \{y_0, \ldots, y_{\alpha+\beta}\} \subset \mathbb{C}$. Write $\mathcal{Y}_n$ for $y_0, \ldots, y_n$, $n \geq 0$, and $\mathcal{Y}_{-1}$ for the void set. Write $[i \ldots j]$ for the image of $f(y_i)$ under the product of divided differences $\partial_{y_i,y_{i+1}} \cdots \partial_{y_{j-1},y_j}$, and $[i \ldots j, z]$ for the image of $[i \ldots j]$ under $\partial_{y_j,z}$. When $i > j$, put $[i \ldots j] = 0$.

[1]which were already known in the $XVIII^e$ century, cf. the first volume of Muir.

Let moreover $P(z)/Q(z)$ be the rational interpolant of $f(z)$ of type $[\beta/\alpha]$ at points $\mathcal{Y}$.

Show that $Q(z)$ is proportional to

$$\begin{vmatrix} [\alpha\ldots(\beta{+}1)] & \cdots & [\alpha\ldots(\alpha{+}\beta)] & R(z,\mathcal{Y}_{\alpha-1}) \\ \vdots & & \vdots & \vdots \\ [1\ldots(\beta{+}1)] & \cdots & [1\ldots(\alpha{+}\beta)] & R(z,\mathcal{Y}_0) \\ [0\ldots(\beta{+}1)] & \cdots & [0\ldots(\alpha{+}\beta)] & R(z,\mathcal{Y}_{-1}) \end{vmatrix} .$$

Show that $\big(f(z)Q(z)-P(z)\big)/R(z,\mathcal{Y})$ is equal to the determinant

$$\begin{vmatrix} [\alpha\ldots(\beta{+}1)] & \cdots & [\alpha\ldots(\alpha{+}\beta)] & [\alpha\ldots(\alpha{+}\beta),z] \\ \vdots & & \vdots & \vdots \\ [1\ldots(\beta{+}1)] & \cdots & [1\ldots(\alpha{+}\beta)] & [1\ldots(\alpha{+}\beta),z] \\ [0\ldots(\beta{+}1)] & \cdots & [0\ldots(\alpha{+}\beta)] & [0\ldots(\alpha{+}\beta),z] \end{vmatrix} . \qquad (\star)$$

Ex. 6.5. Compute the vertical and horizontal distance 2 differences in the Padé table for $\sigma_z(\mathbb{A})$, i.e. compute $[\beta/\alpha]-[\beta{-}2/\alpha]$ and $[\beta/\alpha]-[\beta/\alpha{-}2]$.

Ex. 6.6. Let $\mathbb{A}$ be arbitrary, k,n be integers, x be a rank-1 element. Show that

$$S_{k^n}(\mathbb{A}{-}x)\,S_{(k+1)^n}(\mathbb{A}) = S_{k^n}(\mathbb{A})\,S_{(k+1)^n}(\mathbb{A}{-}x) - xS_{k^{n+1}}(\mathbb{A})S_{(k+1)^{n-1}}(\mathbb{A}{-}x)\ .$$

Use this formula, in conjunction with the preceding exercise, to reprove Wynn's formula.

Ex. 6.7. (Montsessus) Let $f(z)$ be a meromorphic function in the unit disk. Assume it has n poles (counting multiplicities), all different from 0.

Show that the sequence of Padé approximants $[m/n]$, $m\to\infty$, converges towards $f(z)$ in any point in the disk, different from a pole.

The reader, supposed to have only algebraic tools at his disposal, can take $f(z)=\sigma_z(\mathbb{A}-\mathbb{B})$, with $card(\mathbb{A})=n$, $card(\mathbb{B})<\infty$, and $|a|>1, \forall a\in\mathbb{A}$.

CHAPTER 7

Symmetrizing operators

7.1. Divided Differences

Divided differences ∂_σ have been introduced by Newton to solve the interpolation problem in one variable. Apart from them, we shall also need *isobaric divided differences* π_σ.

In the case where σ is a simple transposition s_i exchanging a_i and a_{i+1}, writing then ∂_i, π_i, the operators (**acting on their left**) are :

$$\begin{cases} \partial_i &= (a_i - a_{i+1})^{-1}(1+s_i) = (1-s_i)(a_i-a_{i+1})^{-1} \\ \pi_i &= (1-a_{i+1}/a_i)^{-1}(1+s_i) = a_i\partial_i \end{cases} \tag{7.1.1}$$

More explicitly, ∂_i and π_i are the operators on functions $f(a_1,\ldots,a_n)$ of several variables, acting only on the pair (a_i, a_{i+1}):

$$\begin{cases} f\,\partial_i &= \Big(f(\ldots a_i, a_{i+1}\ldots) - f(\ldots a_{i+1}, a_i\ldots)\Big)(a_i-a_{i+1})^{-1} \\ f\,\pi_i &= \Big(a_i\, f(\ldots a_i, a_{i+1}\ldots) - a_{i+1}\, f(\ldots a_{i+1}, a_i\ldots)\Big)(a_i-a_{i+1})^{-1} \end{cases} \tag{7.1.2}$$

In a more compact manner, we can write

$$\partial_i :\ f \to f\,\partial_i := \frac{f - f^{s_i}}{a_i - a_{i+1}} \qquad \& \qquad \pi_i :\ f \to f\,\pi_i := \frac{a_i f - a_{i+1} f^{s_i}}{a_i - a_{i+1}}\,. \tag{7.1.3}$$

We shall check below that each of the families of operators $\{s_i\}$, $\{\partial_i\}$, $\{\pi_i\}$ separately satisfies the Moore/Coxeter braid relations (writing $\{D_i\}$ in each of these three cases) :

$$D_iD_j = D_jD_i \text{ if } |i-j| \geq 2 \quad , \quad D_iD_{i+1}D_i = D_{i+1}D_iD_{i+1} \tag{7.1.4}$$

One can notice that, for what concerns squares, one has

$$\partial_i\partial_i = 0\ ,\ \pi_i\pi_i = \pi_i\ ,\ s_is_i = 1\ . \tag{7.1.5}$$

Both divided differences ∂_i and isobaric divided differences π_i are special cases of local rational operators of the type

$$D_i = P(a_i,\, a_{i+1})\mathbf{1} + Q(a_i,\, a_{i+1})\, s_i\ , \tag{7.1.6}$$

that is, of a linear combination of the two permutations $\mathbf{1}$, s_i, the coefficients $P(x,y)$ and $Q(x,y)$ being two rational functions of two variables independent of i.

In [**108**] [**112**], one determines the conditions on P and Q which ensures that the D_i's satisfy braid relations and preserve polynomials. These operators happen to satisfy an *Hecke relation*, i.e. there exists constants q_1, q_2 such that

$$(D_i - q_1)(D_i - q_2) = 0\ ,\ \forall i \geq 1\ . \tag{7.1.7}$$

To represent the Hecke algebra, one usually takes

$$T_i = \pi_i(q_1 + q_2) - q_2\, s_i \ , \ i \geq 1. \tag{7.1.8}$$

It is convenient to display the symmetric group as its *Cayley graph*, called *permutohedron*, which describes how permutations are successively generated by products of simple transpositions, starting from the identity permutation. Paths in this oriented graph, starting from the identity, are called *reduced decompositions*, i.e. decompositions of permutations into products of simple transpositions of minimum length.

One can use the same graph to describe operators, the edges now being interpreted as simple divided differences, or isobaric divided differences instead of simple transpositions[1].

Here follows $\mathfrak{S}_3$, the edges being divided differences; on the right figure, we have written the images of $a_1^2 a_2$ under the operators displayed on the left permutohedron, to illustrate that it codes a family of operators on polynomials :

$$\begin{array}{ccccc} & & 123 & & \\ \partial_1 \;/\!/ & & & & \diagdown \; \partial_2 \\ 213 & & & & 132 \\ \partial_2 \Big| & & & & \Big\| \, \partial_1 \\ 231 & & & & 312 \\ \partial_1 \diagdown\!\diagdown & & & & / \; \partial_2 \\ & & 321 & & \end{array} \qquad\qquad \begin{array}{ccccc} & & a_1^2 a_2 & & \\ \partial_1 \;/\!/ & & & & \diagdown \; \partial_2 \\ a_1 a_2 & & & & a_1^2 \\ \partial_2 \Big| & & & & \Big\| \, \partial_1 \\ a_1 & & & & a_1{+}a_2 \\ \partial_1 \diagdown\!\diagdown & & & & / \; \partial_2 \\ & & 1 & & \end{array}$$

Divided differences satisfy Leibnitz formulas[2], as easily seen from the definition:

$$fg\partial_i = f\,(g\partial_i) + f\partial_i\, g^{s_i} = g\,(f\partial_i) + g\partial_i\, f^{s_i} \ . \tag{7.1.9}$$

In particular, symmetric functions in a_i, a_{i+1} are scalars for ∂_i and π_i:

$$g = g^{\sigma_i} \Rightarrow f\,g\,\partial_i = f\partial_i\, g \quad \text{and} \quad f\,g\,\pi_i = f\pi_i\, g \ . \tag{7.1.10}$$

When a_{i+1} tends towards a_i, the divided difference ∂_i becomes the usual derivative with respect to a_i. However, in that limit one loses the fact that symmetric functions in a_i, a_{i+1} commute with the operator, a property which greatly simplify computations. Indeed, to compute the image of a function under ∂_i, one will try to write it as a function in a_i, a_{i+1} as simple as possible, the complications being pushed to the coefficients (which are symmetrical in a_i, a_{i+1} and involve the other variables). Finite differences also give a discrete version of differential calculus on functions of one variable, but there is no invariant sub-ring playing the role of symmetric functions. We shall illustrate in the sequel how the use of the symmetric

[1] We have already mentioned that there are more general operators satisfying the braid relations, but one can also add parameters (according to *Yang-Baxter equations*) on the edges of the permutohedron. We shall be more precise later, when introducing Yang-Baxter bases.

[2] Notice that formulas are disymmetrical in f, g, one has two expressions for the image of a product.

group simplifies the computations with divided differences, and renders this discrete differential calculus simpler than the usual one[3].

7.2. Compatibility with Complete Functions

Complete functions behave "nicely" with respect to divided differences. Indeed, let $\mathbb{A}_i = \{a_1, \ldots, a_i\}$, and let $\mathbb{B}$ be a second alphabet symmetrical[4] in a_i, a_{i+1}. Then one has

$$S^k(\mathbb{A}_i - \mathbb{B})\,\partial_i = S^{k-1}(\mathbb{A}_{i+1} - \mathbb{B}) \tag{7.2.1}$$

and similarly

$$S^k(\mathbb{A}_i - \mathbb{B})\,\pi_i = S^{k}(\mathbb{A}_{i+1} - \mathbb{B})\ , \tag{7.2.2}$$

properties which are immediate to check on generating function. Indeed :

$$\sigma_z(\mathbb{A}_i - \mathbb{B}) = \frac{\prod_{b\in\mathbb{B}}(1-zb)}{\prod_{j=1}^{j=i}(1-za_j)} = \left(\frac{\prod_{b\in\mathbb{B}}(1-zb)}{\prod_{j=1}^{j=i+1}(1-za_j)}\right)(1-za_{i+1})$$

and, the factor inside parentheses being symmetrical in a_i, a_{i+1}, one is left to check that the image of $(1-za_{i+1})$ is z under ∂_i, and 1 under π_i.

Therefore,

$$\frac{\prod_{b\in\mathbb{B}}(1-zb)}{1-za_1}\,\partial_1\partial_2\cdots\partial_n\, z^{-n} = \frac{\prod_{b\in\mathbb{B}}(1-zb)}{1-za_1}\,\pi_1\pi_2\cdots\pi_n = \frac{\prod_{b\in\mathbb{B}}(1-zb)}{(1-za_1)\cdots(1-za_{n+1})} \tag{7.2.3}$$

7.3. Braid Relations

We can check now the braid relations. Divided differences which involve two disjoint pairs of variables clearly commute, thus we have only to show $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$, and $\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}$. To simplify indices, let us check it for $i=1$, and only for the divided differences, the computation being similar for the π_i's.

Instead of applying three times the definition of ∂_i to expand $\partial_2\partial_1\partial_2$, one makes it act on monomials, that one writes as Schur functions according to (1.4.10):

$$a_1^\alpha a_2^\beta a_3^\gamma = S_{\gamma;\beta;\alpha}(\mathbb{A}_3\,;\mathbb{A}_2\,;\mathbb{A}_1)$$

Now, the first and third column of $S_{\gamma\beta\alpha}$ are symmetrical in a_2, a_3, and thus the image of the determinant is a similar determinant, because we have to operate only on the middle column:

$$S_{\gamma;\beta;\alpha}(\mathbb{A}_3;\ \mathbb{A}_2;\ \mathbb{A}_1) \xrightarrow{\partial_2} S_{\gamma;\beta-1;\alpha}(\mathbb{A}_3\,;\mathbb{A}_3;\ \mathbb{A}_1)\ .$$

Now, the new determinant has its first columns symmetrical in a_1, a_2 and becomes $S_{\gamma;\beta-1;\alpha-1}(\mathbb{A}_3;\ \mathbb{A}_3;\ \mathbb{A}_2)$ under ∂_1, and finally, under ∂_2 again :

$$S_{\gamma;\beta-1;\alpha-2}(\mathbb{A}_3;\ \mathbb{A}_3;\ \mathbb{A}_3) = S_{\gamma,\beta-1,\alpha-2}(\mathbb{A}_3)$$

[3]Unfortunately, it was almost forgotten during three hundred years, its only application being more or less the original application of Newton to the interpolation of functions of one variable.

[4]i.e. invariant under s_i, which is the case if, for example, it contains p copies of a_i, and of a_{i+1} at the same time. Usually, $\mathbb{B}$ will be disjoint from $\mathbb{A}$, but sometimes, to get rid of denominators for example, one needs to transform $\mathbb{B}$ into $\mathbb{B}\pm\mathbb{A}$.

which is a usual Schur function in three variables. The result being symmetrical in a_1, a_2, a_3, the image of the same monomial under $\partial_1\partial_2\partial_1$ produces the same Schur function (however, to apply $\partial_1\partial_2\partial_1$, one writes the starting monomial as a Schur function in $a_1 + a_2 + a_3$; $a_2 + a_3$; a_3).

Because monomials are a linear basis of polynomials, this computation implies the braid relation $\partial_1\partial_2\partial_1 = \partial_2\partial_1\partial_2$.

The preceding computation was not really necessary. Indeed, we did not use that ∂_1, ∂_2 commute with multiplication by symmetric functions in a_1, a_2, a_3. Knowing that the space $\mathbb{Z}[a_1, a_2, a_3]$ is a module (in fact a free module) over $\mathfrak{Sym}(a_1, a_2, a_3)$, with basis

$$1,\ a_1,\ a_2\ ,\ a_1^2,\ a_1a_2,\ a_1^2a_2$$

we see that both $\partial_1\partial_2\partial_1$ and $\partial_2\partial_1\partial_2$ decrease degree by 3, and thus send the basis to 0, apart from the last element which is sent to a constant (necessarily non-zero, because none of the operators is identically null). We shall see later a good reason why the constant is in both cases equal to 1.

The braids relations imply the following lemma[5].

LEMMA 7.3.1. *Given a permutation σ and a reduced decomposition $\sigma = s_is_j\cdots s_k$, then the product $\partial_i\partial_j\cdots\partial_k$ (resp. $\pi_i\pi_j\cdots\pi_k$) is independent of the choice of the reduced decomposition of σ. One denotes it ∂_σ (resp. π_σ).*

Let now the cardinality of $\mathbb{A}$ be n, and let ω be the maximal element of $\mathfrak{S}_n$. Then writing any monomial in $\mathbb{A}$ as a Schur function, the same type of computation as in the case of three variables shows that the image of a monomial under ∂_ω or π_ω is a Schur function:

$$a_n^{i_1}\cdots a_1^{i_n}\,\pi_\omega = S_I(\mathbb{A}_n) \quad , \quad a_n^{i_1}\cdots a_1^{i_n}\,\partial_\omega = S_{I-\rho}(\mathbb{A}_n)\ , \tag{7.3.1}$$

where $I = [i_1, \ldots, i_n]$, $\rho = [0, \ldots, n-1]$, $I - \rho = [i_1 - 0, \ldots, i_n - n + 1]$.

Instead of taking the group algebra of $\mathfrak{S}(\mathbb{A})$ with constant coefficients, one can use the algebra $\mathfrak{Diff}$ generated by rational functions in $\mathbb{A}$, and permutations. This is the proper way of combining multiplication by elements of $\mathfrak{Rat}(\mathbb{A})$ and symmetrizing operators. Indeed, M.P. Schützenberger and I called this algebra *algebra of divided differences* to stress the fact that divided differences, or permutations act on the coefficients, that one can choose to write on the left or on the right. Beware that a rational function P/Q must now be interpreted as the operator "multiplication by P/Q". The algebra $\mathfrak{Diff}$ is nowadays called the *affine Hecke algebra*[6].

This algebra contains all our symmetrizing operators, for example :

$$\partial_i = (1 - s_i)\frac{1}{x_i - x_{i+1}} = \frac{1}{x_i - x_{i+1}}(1 + s_i)\ , \tag{7.3.2}$$

but also, for any q_1, q_2, generators

$$T_i^{q_1,q_2} = (q_1a_i + q_2a_{i+1})\partial_i + q_2 = \frac{q_1a_i + q_2a_{i+1}}{a_i - a_{i+1}}(1 + s_i) + q_2\ , \tag{7.3.3}$$

---

[5]knowing that two reduced decompositions of a permutation differ by a sequence of transformations $s_is_j = s_js_i$ or $s_{i+1}s_is_{i+1} = s_is_{i+1}s_i$.

[6]Cherednik [**36**] uses double affine Hecke algebras, starting with the *affine symmetric group* instead of the symmetric group.

that we had written differently in (7.1.8), and which satisfy the braid relations, together with the Hecke relations

$$(T_i^{q_1,q_2} - q_1)\,(T_i^{q_1,q_2} - q_2) = 0 \ .$$

It has linear bases : $\{\sigma \in \mathfrak{S}(\mathbb{A})\}$, $\{\partial_\sigma,\ \sigma \in \mathfrak{S}(\mathbb{A})\}$, $\{\pi_\sigma,\ \sigma \in \mathfrak{S}(\mathbb{A})\}$, $\{T_\sigma^{q_1,q_2},\ \sigma \in \mathfrak{S}(\mathbb{A})\}$, and many others. An interesting problem is to write explicit matrices of changes of bases.

For example, even to pass from the $T_\sigma^{q_1,q_2}$ to the $T_\sigma^{r_1,r_2}$, that is just to let the pair of eigenvalues be different, is not as straightforward as it would seem.

Indeed, by specializing the parameters q_1, q_2, r_1, r_2, one covers all the known change of bases. We shall encounter later Schubert polynomials, which appear for both changes of bases $\sigma \mapsto \partial_\sigma$ and $\partial_\sigma \mapsto \sigma$. Let us just say a word about how to express any element of the affine Hecke algebra in the basis of permutations.

7.4. Decomposing in the Basis of Permutations

We first need a straightforward lemma, introducing a second alphabet $\mathbb{B}$ of cardinality n :

LEMMA 7.4.1. *Let* $\mathbf{K}_n = \mathbf{K}_n(\mathbb{A},\mathbb{B}) = \prod_{1\le i,j\le n}(b_i - a_j)$. *Then* $\mathbf{K}_n(\mathbb{A}^\sigma,\mathbb{A}) = 0$ *if* σ *is not the identity permutation.*

Using these vanishing properties, one obtains expansions by just specializing the images of $\mathbf{K}$, sums restricting to a single term :

PROPOSITION 7.4.2. *Let*

$$\hbar = \sum_{\sigma\in\mathfrak{S}(\mathbb{A})} \sigma\, h_\sigma, \quad \textit{with} \quad h_\sigma \in \mathfrak{Rat}(\mathbb{A},\ \mathbb{B}) \ .$$

Then

$$\mathbf{K}_n\, \hbar\big|_{\mathbb{B}=\mathbb{A}^\sigma} = (-1)^{\ell(\sigma)}\, h_\sigma \prod_{1\le i,j\le n} (a_i - a_j) \tag{7.4.1}$$

This proposition says, for example, that the expansion of a divided difference ∂_σ in the basis of permutations is obtained by specializing the image of $\mathbf{K}$ under ∂_σ. We shall see later that the images, under divided differences, of $\mathbf{K}$ themselves[7] constitute a fundamental basis of the ring of polynomials in $\mathbb{A}$, as a module over $\mathfrak{Sym}(\mathbb{A})$.

We shall also need the explicit change of bases between the Hecke algebra with parameters $(1,0)$ and the Hecke algebra with parameters $(-1,0)$ (both are *0-Hecke algebras*). The π_i, $i = 1,\dots,n{-}1$, are algebraic generators of the first one, the $\widehat{\pi}_i := \pi_i - 1$, $i = 1,\dots,n{-}1$, are generators of the second.

First, recall that the *Ehresmann-Bruhat order* on the symmetric group is defined as follows.

DEFINITION 7.4.3. Let σ be a permutation, $I = [i_1,\dots,i_k] \in \mathbb{N}^k$ be such that $s^I := s_{i_1}\cdots s_{i_k}$ be a reduced decomposition of σ. Then the permutations *smaller than* σ *with respect to the Ehresmann-Bruhat order* are all the permutations obtained by evaluating in the symmetric group the subwords of s^I.

[7] one starts with $\prod_{i,j:\, i+j<n}(a_i - b_j)$ rather than $\mathbf{K}$.

Bourbaki[**22**] gives an "Exchange lemma" which can be reformulated in terms of intervals for the Ehresmann-Bruhat order. Let us write $[\cdot \leq \sigma]$ for the interval of permutations between the identity and σ.

LEMMA 7.4.4. *Let σ be a permutation, s be a simple transposition such that $\ell(\sigma s) > \ell(\sigma)$. Then $[\cdot \leq \sigma]$ decomposes into two subsets, A containing permutations η such that $\eta, \eta s \in [\cdot \leq \sigma]$, and B containing the permutations ν such that $\nu s \notin [\cdot \leq \sigma]$. The interval $[\cdot \leq \sigma s]$ is the disjoint union of A, B, Bs, where Bs is the set $\{\nu s, \nu \in B\}$.*

PROPOSITION 7.4.5. *The π_σ and $\widehat{\pi}_\sigma$, $\sigma \in \mathfrak{S}(\mathbb{A})$ are related by the relations*

$$\begin{aligned} \pi_\sigma &= \sum_{\nu \leq \sigma} \widehat{\pi}_\nu \\ \widehat{\pi}_\sigma &= \sum_{\nu \leq \sigma} (-1)^{\ell(\sigma)-\ell(\nu)}\, \pi_\nu \ , \end{aligned} \quad . \tag{7.4.2}$$

Proof. Let use prove the result by induction on the length of σ, using the notations of lemma 7.4.4. One supposes that $\pi_\sigma = \sum_{\eta \in A} \widehat{\pi}_\eta + \sum_{\nu \in B} \widehat{\pi}_\nu$. By definition, $\pi_{\sigma s} = \pi_\sigma \pi_s = \pi_\sigma(1 + \widehat{\pi}_s)$. The sum over A is invariant under multiplication by π_s, because it is composed of terms $\widehat{\pi}_\eta + \widehat{\pi}_{\eta s} = \widehat{\pi}_\eta(1+\widehat{\pi}_s)$ (taking η in the pair $\eta, \eta s$ such that $\ell(\eta) < \ell(\eta s)$), and because $(1 + \widehat{\pi}_s)\pi_s = (1 + \widehat{\pi}_s)$. The sum over B becomes a sum over $B \cup Bs$. In total one obtains the full interval of permutations $\leq \sigma s$. The first assertion is proved, by involution $\pi_i \to -\pi_i$, one gets the second. □

Notice that this lemma determines the Möbius function of the symmetric group (with respect to the Ehresmann-Bruhat order), cf. Verma[**174**].

7.5. Generating Series by Symmetrization

We have seen in (7.2.3) that divided differences allow to obtain rational functions having denominator $\prod(1 - za_i)$ starting from a rational function with denominator $(1 - za_1)$. Let us illustrate this method to generating functions of Tchebychef polynomials of the first and second kind, $T_n(x)$ and $U_n(x)$.

These polynomials are defined by the generating series

$$\sum_{n=1}^{\infty} \frac{z^n}{n} T_n(x) = -\frac{1}{2} \log(1 - 2zx + z^2) \quad \& \quad \sum_{n=0}^{\infty} z^n U_n(x) = \frac{1}{1 - 2zx + z^2} \ . \tag{7.5.1}$$

In other words, let $\mathbb{A}$ be the alphabet of cardinality 2 such that $\Lambda^1(\mathbb{A}) = 2x$, $\Lambda^2(\mathbb{A}) = 1$. Then

$$T_n(x) = \Psi_n(\mathbb{A})/2 \qquad \& \qquad U_n(x) = S_n(\mathbb{A}) \ . \tag{7.5.2}$$

Since $a_1^n(a_1 - a_2)\,\partial_1 = a_1^n + a_2^n$ and $a_1^n \partial_1 = S_{n-1}(a_1 + a_2)$, one can obtain $T_n(x)$ and $U_n(x)$ by symmetrization. More generally, let $\mathbb{B}$ be an alphabet of cardinality 2, with $\Lambda^2(\mathbb{B}) = 1$. Put $\Lambda^1(\mathbb{B}) = 2y$. Then [**157**], [**43**] :

LEMMA 7.5.1.

$$\sum_{n=0}^{\infty} z^n U_n(x)U_n(y) = \frac{1-z^2}{1-4zxy+z^2(4x^2+4y^2-2)-4z^3xy+u^4} \tag{7.5.3}$$

$$\sum_{n=0}^{\infty} z^n U_n(x)T_n(y) = \frac{1-2zxy+z^2(2y^2-1)}{1-4zxy+z^2(4x^2+4y^2-2)-4z^3xy+u^4} \tag{7.5.4}$$

$$\sum_{n=0}^{\infty} z^n T_n(x)T_n(y) = \frac{1-3zxy+z^2(2x^2-1+2y^2)-z^3xy}{1-4zxy+z^2(4x^2+4y^2-2)-4z^3xy+u^4} \tag{7.5.5}$$

Proof. The three series are the images, under $\partial_{a_1,a_2}\partial_{b_1,b_2}$ of $a_1b_1/(1-za_1b_1)$, $a_1(b_1-b_2)/2(1-za_1b_1)$, $(a_1-a_2)(b_1-b_2)/4(1-za_1b_1)$, respectively. They all have the denominator

$$\prod(1-za_ib_j) = \lambda_{-z}(\mathbb{AB}) = 1 - zS_1(\mathbb{A})S_1(B) + z^2S_{11}(\mathbb{A})S_2(B) + \\ z^2S_2(\mathbb{A})S_{11}(B) - z^3S_{12}(\mathbb{A})S_{12}(\mathbb{B}) + z^4S_{22}(\mathbb{A})S_{22}(\mathbb{B}) \ .$$

The numerator is obtained by applying twice a divided difference. For example,

$$\frac{a_1(b_1-b_2)}{1-za_1b_1} \xrightarrow{\partial_{a_1a_2}} \frac{b_1-b_2}{1-2zxb_1+z^2b_1^2} \xrightarrow{\partial_{b_1b_2}} \frac{2-2x\Psi_1(\mathbb{B})+z^2\Psi_2(\mathbb{B})}{\prod(1-za_ib_j)}$$

and one concludes, using that $\Psi_2(\mathbb{B}) = \Lambda_1(\mathbb{B})^2 - 2\Lambda_2(\mathbb{B}) = 4y^2-2$. The other cases are obtained by a similar computation. □

The same method could be applied to generating functions of products of three or more Tchebychef polynomials, but in that case, only the denominator would be easy to write, and one would have recourse to ACE [**9**] for the numerator.

The reader will certainly find in the mathematical literature other similar examples of rational functions which can be obtained from a function having only one pole. We prefer in these notes illustrate other uses of divided differences.

7.6. Maximal Symmetrizers

The operators corresponding to the maximal permutation ω play a fundamental role. Let us give a different expression of them involving the full symmetric group, instead of defining them by products involving only $\mathfrak{S}_2$ at a time.

First, we need to extend the property which allowed us to check the braid relations, and show that any polynomial can be written as a linear combination of monomials of small degree, taking coefficients in $\mathfrak{Sym}$.

LEMMA 7.6.1. *The ring of polynomials in $\mathbb{A} = \{a_1, \dots, a_n\}$ is a module*[8] *over $\mathfrak{Sym}(\mathbb{A})$ with generators $a^I : I \le \rho : [n-1, \dots, 0]$.*

Proof. Writing $\mathbb{A}_k = \{a_1, \dots, a_k\}$, one sees that the equations

$$S^{n-k+1}(\mathbb{A}_k - \mathbb{A}_n) = 0 \ , \ 1 \le k \le n$$

allow to eliminate the powers of a_{n-i} strictly bigger than i, $0 \le i \le n-1$. There remains only the $n!$ monomials of exponent majorized by ρ. □

[8] It is in fact a free module, but we shall prove this fact and provide different bases in later sections.

PROPOSITION 7.6.2. *Given $\mathbb{A}$ of cardinality n, the divided differences ∂_ω and π_ω can be expressed as a summation on $\mathfrak{S}_n(\mathbb{A})$:*

$$\partial_\omega = \sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)}\sigma \frac{1}{\Delta} , \tag{7.6.1}$$

$$\pi_\omega = a^\rho \sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)}\sigma \frac{1}{\Delta} , \tag{7.6.2}$$

writing Δ for the Vandermonde $\prod_{1\le i<j\le n}(a_i - a_j)$.

Proof. We need to check the action of the preceding operators on our generating set $\{a^I : I \le \rho\}$. We notice that each of these monomials, except the last one, has at least one repeated exponent. In other words, there is at least one pair i, j such that the monomial is symmetrical in a_i, a_j. In that case, it will be annihilated by $\sum \pm\sigma$, because each monomial in $a^I \sum \pm\sigma$ will be accompanied by the same one with a different sign. In the case $I = \rho$, then $\sum \pm(a^\rho)^\sigma$ is the expansion of the Vandermonde, and a^ρ is sent to 1 by $\sum \pm\sigma/\Delta$.

On the other hand, ∂_ω decreases degrees by $\ell(\omega) = \binom{n}{2}$, and therefore sends all $a^I : I < \rho$ to 0 for degree reason. To compute the image of a^ρ (which is of degree 0 in $\mathbb{A}$), one writes

$$\omega = (\partial_1 \cdots \partial_{n-1})(\partial_1 \cdots \partial_{n-2}) \cdots (\partial_1) .$$

Using this product of divided differences, one sees that each step consists of computing the image of some monomial $a^J a_i$ by ∂_i, where a^J is symmetrical in a_i, a_{i+1}. At the end, one finds again that a^ρ is sent to 1, and therefore $\sum \pm\sigma\Delta^{-1}$ and ∂_ω coincide.

One gets the case of π_ω by using $a^{I-\rho}$ as a generating set $\big($taking the space of *Laurent polynomials*, i.e. polynomials in $a_1^{\pm1}, \ldots, a_n^{\pm1}\big)$. □

7.7. Schur Functions and Bott's Theorem

Recall that, for two finite alphabets $\mathbb{A}, \mathbb{B}$, the resultant $R(\mathbb{A}, \mathbb{B})$ is the product $\prod_{a\in\mathbb{A},\, b\in\mathbb{B}}(a - b)$. Recall also that for $\mathbb{A} = \{a_1, \ldots, a_n\}$, and $K = [k_1, \ldots, k_n] \in \mathbb{N}^n$, $\lambda = [\lambda_1, \ldots, \lambda_n] = K^\omega = [k_n, \ldots, k_1]$, the Schur function $S_K(\mathbb{A})$ can be defined as a quotient :

$$S_K(\mathbb{A}) = \left|a_i^{\lambda_j+j-1}\right|_{1\le i,j\le n} / \Delta(\mathbb{A}) = \left|a_i^{\lambda_j+j-1}\right| \left|a_i^{j-1}\right|^{-1} . \tag{7.7.1}$$

Expanding both determinants, one recognize them to be obtained from a single monomial by either ∂_ω or π_ω, and the preceding formula becomes now the *Weyl character formula* for type A_{n-1} :

$$S_K(\mathbb{A}) = a^{\lambda+\rho}\,\partial_\omega = \frac{\sum_\sigma (-1)^{\ell(\sigma)} \left(a^{\lambda+\rho}\right)^\sigma}{\sum_\sigma (-1)^{\ell(\sigma)} \left(a^\rho\right)^\sigma} = a^\lambda\,\pi_\omega . \tag{7.7.2}$$

Schur functions of a difference of two alphabets $\mathbb{A}$ and $\mathbb{B}$ can also be obtained by a double symetrization involving the groups $\mathfrak{S}(\mathbb{A})$ and $\mathfrak{S}(\mathbb{B})$. The precise statement is given by the *Sergeev-Pragacz* formula, for which we refer to [**149**]. This formula can also be obtained by symmetrizing any vexillary Schubert polynomial [**134**], using the product of the two maximal symmetrizers π_ω relative to $\mathbb{A}$ and $\mathbb{B}$.

Proposing to generalize the Lagrange's interpolation, that we shall detail in the next section, Jacobi and Sylvester described which summations to take when the input function has already partial symmetries. For any decomposition $n = n' + n''$, define[9] $\mathfrak{Sym}(n'|n'')$ to be the ring of polynomials of $\mathbb{A}$ which are symmetrical in $a_1, \ldots, a_{n'}$, and also symmetrical in $a_{n'+1}, \ldots, a_n$. Jacobi and Sylvester's summation defines a morphism from $\mathfrak{Sym}(n'|n'')$ to $\mathfrak{Sym}(n)$. Because of its application to the cohomology theory of Grassmannians, we shall attribute to Bott the following formula[10]. Write $\partial_{n'|n''}$ for $\partial_{\omega_{n'|n''}}$, with $\omega_{n'|n''} = [n'+1, \ldots, n, 1, \ldots, n']$, and similarly $\pi_{n'|n''}$ for $\pi_{\omega_{n'|n''}}$.

PROPOSITION 7.7.1. *Let $\mathbb{A}$ be of cardinality $n = n'+n''$, $J \in \mathbb{N}^{n'}$, $I \in \mathbb{N}^{n''}$. Let $\square = (n'')^{n'}$. Then*

$$(7.7.3) \quad S_{I,J}(\mathbb{A}) = \sum_{\mathbb{A}=\mathbb{A}'+\mathbb{A}''} \frac{S_I(\mathbb{A}'')\, S_{J+\square}(\mathbb{A}')}{R(\mathbb{A}', \mathbb{A}'')}$$
$$= S_I(a_{n'+1} + \cdots + a_n) S_{J+\square}(a_1 + \cdots + a_{n'})\, \partial_{n'|n''}$$
$$= S_I(a_{n'+1} + \cdots + a_n) S_J(a_1 + \cdots + a_{n'})\, \pi_{n'|n''} \ ,$$

sum over all partitions of $\mathbb{A}$ into two sub-alphabets $\mathbb{A}', \mathbb{A}''$ of respective cardinalities n', n''.

Proof. Each Schur function is the image of a monomial under respectively $\pi_{\omega'}$ and $\pi_{\omega''}$, with $\omega' = [n', \ldots, 1, n'+1, \ldots, n]$, and $\omega'' = [1, \ldots, n', n, \ldots, n'+1]$. Therefore, $S_I(a_{n'+1} + \cdots + a_n) S_J(a_1 + \cdots + a_{n'})$ is the image of the monomial

$$a^\lambda = a_n^{i_1} \cdots a_{n'+1}^{i_{n''}}\, a_{n'}^{j_1} \cdots a_1^{j_{n'}}$$

under the product $\pi_{\omega'}\pi_{\omega''}$. Since $\omega = \omega'\omega''\, \omega_{n'|n''}$, the image of the product of two Schur functions is equal to the image of a^λ under π_ω, which is $S_{I,J}(\mathbb{A})$.

Taking $\partial_{n'|n''}$ instead of $\pi_{n'|n''}$ involves the extra factor $(a_1 \cdots a_{n'})^{n''} = S_\square(a_1 + \cdots + a_{n'})$. Indeed

$$\pi_\omega = a^\rho \partial_\omega = a^\rho \partial_{\omega'} \partial_{\omega''} \partial_{n'|n''} = (a_1 \cdots a_{n'})^{n''} \pi_{\omega'} \pi_{\omega''} \partial_{n'|n''} = \pi_{\omega'} \pi_{\omega''} (a_1 \cdots a_{n'})^{n''} \partial_{n'|n''} \ .$$

The summation on subsets will be obtained by factorizing the symmetric group. Write $\mathfrak{S}_{n'|n''}$ for the Young subgroup preserving the two sets $\{1, \ldots, n'\}$ and $\{n'+1, \ldots, n\}$. Write also $\mathfrak{S}^{n'|n''}$ for $\mathfrak{S}_{n'|n''} \backslash \mathfrak{S}_n$ (taking coset representatives of minimal length; explicitly, $\mathfrak{S}^{n'|n''}$ is the set of permutations which, considered as words, have subwords[11] $[1, \ldots, n']$ and $[n'+1, \ldots, n]$). Now $\sum_{\sigma\in\mathfrak{S}_n} \sigma = \sum_{\nu\in\mathfrak{S}_{n'|n''}} \nu \sum_{\zeta\in\mathfrak{S}^{n'|n''}} \zeta$. Noticing that

$$\Delta(\mathbb{A}) = \Delta(a_1, \ldots, a_{n'}) \Delta(a_{n'+1}, \ldots, a_n)\, R\left(\{a_1 \ldots, a_{n'}\}, \{a_{n'+1} \ldots, a_n\}\right) \ ,$$

---

[9]When several alphabets will be involved, we shall precise the alphabet used by writing $\mathfrak{Sym}(\mathbb{A} : n'|n'')$

[10]which describes the class, in K theory, of the cohomology spaces of an homogeneous vector bundle over a Grassmannian. I gave this theorem in [**96**] without reference to the work of Bott which was unknown to me at that time. In a sense, it was fortunate, because I was forced to find a reason why concatanation of partitions could be interesting, and I used it to study determinantal ideals.

[11]one also call them *shuffles* of $[1, \ldots, n']$ and $[n'+1, \ldots, n]$.

one can factorize Cauchy-Jacobi summation into

$$\frac{1}{\Delta(\mathbb{A})}\sum_{\sigma\in\mathfrak{S}_n}\sigma =$$
$$\frac{1}{\Delta(a_1,\ldots,a_{n'})\Delta(a_{n'+1},\ldots,a_n)}\frac{1}{R(\{a_1\ldots,a_{n'}\},\{a_{n'+1}\ldots,a_n\})}\sum_{\nu\in\mathfrak{S}_{n'|n''}}\nu\sum_{\zeta\in\mathfrak{S}^{n'|n''}}\zeta$$
$$=\partial_{\omega'}\partial_{\omega''}\frac{1}{R(\{a_1\ldots,a_{n'}\},\{a_{n'+1}\ldots,a_n\})}\sum_{\zeta\in\mathfrak{S}^{n'|n''}}\zeta\,.$$

Applying now this operator to the monomial $a^{\lambda+\rho}$, one recognizes the first equality in the proposition. □

Notice that it has become evident that the operator

$$\mathfrak{Sym}(n'|n'')\ni f\to\sum_{\mathbb{A}=\mathbb{A}'+\mathbb{A}''}\frac{f(\mathbb{A}',\mathbb{A}'')}{R(\mathbb{A}',\mathbb{A}'')}$$

preserves polynomials, only after recognizing that it is a factor of the operator $\Delta^{-1}\sum_\sigma\sigma=\sum_\sigma\pm\sigma\,\Delta^{-1}$.

The preceding proposition can be merely seen as the translation, in precise terms, of the idea that to symmetrize a function which has already partial symmetries, one does not need a summation on the full symmetric group. Let us give an explicit example. Let $n'=2, n''=3$, $I=[024]$, $J=[15]$. Then $\omega'=[21\,345]$, $\omega''=[12\,543]$, $\omega^{2|3}=[345\,12]$. The product $S_{15}(a_1+a_2)S_{024}(a_3+a_4+a_5)$ is the image of $a^{51\,420}$ under $\pi_{21\,543}$. It is sent onto $S_{02415}(\mathbb{A})=-S_{02235}(\mathbb{A})$ under π_{34512}, as well as under π_{54321}. Similarly, but with a shift in the exponents, $S_{48}(a_1+a_2)S_{024}(a_3+a_4+a_5)$ is the image of $a^{94\,630}=a^{51\,420}a^{43210}$ under $\partial_{21543}=d_1\partial_3\partial_4\partial_3$. Applying $\partial_{34512}=\partial_2\partial_3\partial_4\partial_1\partial_2\partial_3$ gives the same image as the image of $a^{94\,630}$ under ∂_{54321}, that is, $S_{03649-01234}(\mathbb{A})$.

Finally, the different $S_{15+33}(\mathbb{A}')S_{024}(\mathbb{A}'')R(\mathbb{A}',\mathbb{A}'')^{-1}$, for variable $\mathbb{A}',\mathbb{A}''$, are the images under the permutations $\zeta\in\mathfrak{S}^{2|3}$ of

$$\frac{(a_1a_2)^3S_{15}(a_1+a_2)S_{024}(a_3+a_4+a_5)}{R(\{a_1,a_2\},\{a_3,a_4,a_5\})}\,.$$

On the other hand, $S_{48}(a_1+a_2)S_{024}(a_3+a_4+a_5)$ is equal to the image of

$$\frac{a_1^9a_2^4\;a_3^6a_4^3a_5^0}{\Delta(a_1,a_2)\Delta(a_3,a_4,a_5)}$$

under the sum $\sum_{\nu\in\mathfrak{S}_{2|3}}\nu$. Notice that there is no sign in the sum, because we divided by the Vandermonde before applying the summation.

Bott's theorem can be interepreted in the following two different ways, as furnishing a pair of adjoint bases of $\mathfrak{Sym}(n'|n'')$ (as a free $\mathfrak{Sym}(n)$-module, [**96**], Cor. 2.2), or as giving a reproducing kernel on a space of symmetric functions.

COROLLARY 7.7.2. *Let $\mathbb{A}$ be of cardinality $n=n'+n''$. Let $\square=(n'')^{n'}$.*

(1) *Let I,J be partitions, $I,J\le\square$. Then*

$$S_{\square/I}\big(-(a_{n'+1}+\cdots+a_n)\big)\,S_J(a_1+\cdots+a_{n'})\,\partial_{n'|n''}=\delta_{I,J}\,. \tag{7.7.4}$$

(2) *Let f be a symmetric function belonging to the linear span of Schur functions of index $J \le \square$. Let $\mathbb{B}$ be arbitrary. Then*

$$f(a_1+\cdots+a_{n'})\, S_\square\big(\mathbb{B}-(a_{n'+1}+\cdots+a_n)\big)\,\partial_{n'|n''} = f(\mathbb{B}) \ . \tag{7.7.5}$$

Proof. Multiplying by $(a_1\cdots a_n)^{n''}$, one can apply (7.7.3). But the image of $\pm S_H(a_{n'+1}+\cdots+a_n)S_{J+\square}(a_1+\cdots+a_{n'})$, with $H = n^{n''} - I^{\sim}$, is $\pm S_{H,J}(\mathbb{A})$, and is null, or equal to $S_0(\mathbb{A})$ due to the restriction on I, J. Lemma 1.2.5 describes the non vanishing pairs, and imply the first statement. One gets the second by expanding $S_\square\big(\mathbb{B}-(a_{n'+1}+\cdots+a_n)\big)$ in terms of $\mathbb{B}$ and $(a_{n'+1}+\cdots+a_n)$. $\square$

For example, for $n = 2+3$, one has

$$S_{33/02}(-a_3-a_4-a_5)S_{02}(a_1+a_2)\,\partial_{2|3} = S_{112}(a_3+a_4+a_5)S_{02}(a_1+a_2)\,\partial_{2|3}$$
$$= S_{1,1,2,0-3,2-3}(\mathbb{A}) = S_{00000}(\mathbb{A}) = 1 \ .$$

7.8. Lagrange Interpolation

Let us detail independently the special case $n' = 1$ of the preceding section.

Given a function $f(x)$ of a single variable x, and a finite set $\mathbb{B} = \{b\}$ of m "distinct interpolation points", the problem solved by Lagrange was to find universal polynomials $P_b(x)$ such that $f(x) = \sum_{b\in\mathbb{B}} f(b)\,P_b(x)$ whenever $\deg(f) < m$. First recall that, for two alphabets $\mathbb{A},\mathbb{B}$, $R(\mathbb{A},\mathbb{B})$ denotes the product $\prod_{a\in\mathbb{A},\, b\in\mathbb{B}}(a-b)$.

PROPOSITION 7.8.1. *For any polynomial $f(x)$ one has*

$$f(x) \equiv \sum_{b\in\mathbb{B}} f(b)\frac{R(x,\mathbb{B}\backslash b)}{R(b,\mathbb{B}\backslash b)} \quad \mathrm{mod}\ R(x,\mathbb{B}) \ . \tag{7.8.1}$$

When $f(x)$ is equal to $S^k(x-\mathbb{E})$ for some $k\in\mathbb{N}$, some $\mathbb{E}$, x being considered as a rank-1 element, then the interpolation polynomial can be written

$$\sum_{b\in\mathbb{B}} f(b)\frac{R(x,\mathbb{B}\backslash b)}{R(b,\mathbb{B}\backslash b)} = (-1)^{m-1}S_{1^{m-1};k-m+1}(\mathbb{B}-x;\mathbb{B}-\mathbb{E}) \ . \tag{7.8.2}$$

Proof. The first formula is linear, and true on the basis $\{R(x,\mathbb{B}\backslash b) : b\in\mathbb{B}\} \cup \{R(x,\mathbb{B}\backslash b),\ x\,R(x,\mathbb{B}\backslash b),\ x^2\,R(x,\mathbb{B}\backslash b),\ldots\}$. Indeed, for each polynomial in the first family, there is only one non-zero term equal to $(f(b)R(b,\mathbb{B}\backslash b)/R(b,\mathbb{B}\backslash b)$; in the case of the second family, the sum is identically null.

Similarly, $S^k(x-\mathbb{E}) \to (-1)^{m-1}S_{1^{m-1};k-m+1}(\mathbb{B}-x;\mathbb{B}-\mathbb{E})$ is a linear morphism from the space $\mathfrak{Pol}(x)$ to the space of polynomials in x of degree $< m$. In the case $k = m-1$, $\mathbb{E} = \mathbb{B}-b$, then by subtracting b to the the first $m-1$ rows of $(-1)^{m-1}S_{1^{m-1};k-m+1}(\mathbb{B}-x;\mathbb{B}-(\mathbb{B}-b))$, one gets $(-1)^{m-1}S_{1^{m-1}}(\mathbb{B}-x-b) = S^{m-1}(x-\mathbb{B})$, and therefore the image of any polynomial under this morphism is Lagrange's interpolation polynomial $\sum f(b)S^{m-1}(x+b-\mathbb{B})R(b,\mathbb{B}\backslash b)^{-1}$. $\square$

Thus, the Lagrange universal coefficients are

$$R(x,\mathbb{B}\backslash b) = S^{m-1}(x+b-\mathbb{B}) \ .$$

In fact, there is a symmetry between x and the $b\in\mathbb{B}$. To preserve it, it is desirable to use $\mathbb{A} = \{a\} := \{x\}\cup\mathbb{B}$ instead of $\mathbb{B}$, putting $n := m+1$. Dividing by $R(x,\mathbb{B})$, one rewrites Lagrange's summation as

$$f \to \sum_{a\in\mathbb{A}} \frac{f(a)}{R(a,\mathbb{A}\backslash a)} \ . \tag{7.8.3}$$

Instead of taking as in input a function of a single variable, one can as well operate on the ring $\mathfrak{Sym}(1|n-1)$ of n variables, which are symmetrical in the last $n-1$ ones, as is the denominator in of (7.8.3) :

$$\mathfrak{Sym}(1|n-1) \ni f \longrightarrow \sum_{a\in\mathbb{A}} \frac{f(a,\mathbb{A}\backslash a)}{R(a,\mathbb{A}\backslash a)} \tag{7.8.4}$$

Let us note $f \to L_{\mathbb{A}}(f)$ or $f \to f\partial_{\mathbb{A}}$ this functional.

THEOREM 7.8.2. *Let $\mathbb{A} = \{a_1,\ldots,a_n\}$, and let $\partial_1,\ldots,\partial_{n-1}$ be the divided differences relative to $\mathbb{A}$. Then for any $f \in \mathfrak{Sym}(1|n-1)$, one has*

$$L_{\mathbb{A}}(f) = f(a_1,\, a_2,\ldots,a_n)\,\partial_1\cdots\partial_{n-1}\ . \tag{7.8.5}$$

Proof. Both members of (7.8.5) commute with multiplication with symmetric functions in n variables :

$$g \in \mathfrak{Sym}(n) \quad \text{implies} \quad L_{\mathbb{A}}(fg) = f\,\partial_1\cdots\partial_{n-1}\,g(\mathbb{A})\ .$$

It is easy to see that the ring $\mathfrak{Sym}(x_1 \mid x_2,\ldots x_n)$ is a module over $\mathfrak{Sym}(x_1,\ldots,x_n)$, generated by $x_1^0, x_1^1,\ldots,x_1^{n-1}$ (we shall see a more precise statement in a later section). Therefore, it is sufficient to test $L_{\mathbb{A}}$ and $\partial_1\cdots\partial_{n-1}$ on the powers of x_1 of degree no more than $n-1$.

But both operators decrease degree by $n-1$, and thus send x_1^k to 0 for $k < n-1$, and x_1^{n-1} to a constant. We have seen that for any $k \geq 0$, $a_1^k\,\partial_1\cdots\partial_{n-1} = S^{k-n+1}(\mathbb{A})$; thus $a_1^{n-1}\,\partial_1\cdots\partial_{n-1} = S^0 = 1$.

On the other hand,

$$\sum_{a\in\mathbb{A}} \frac{a^{n-1}}{R(a,\mathbb{A}\backslash a)} = \frac{1}{\Delta(\mathbb{A})}\sum_i (-1)^i a_i^{n-1}\Delta\big(a_1,\ldots,\widehat{a_i},\ldots,a_n\big) = 1\ ,$$

and therefore $L_{\mathbb{A}}$ and $\partial_1\cdots\partial_{n-1}$ are equal, as operators on $\mathfrak{Sym}(1|n-1)$. □

Coming back to the original case of functions of one variable, with $\mathbb{B} := \{a_1,\ldots,a_{n-1}\}$, $x = a_n$, equation (7.8.5) becomes

$$\frac{f(x)}{R(x,\mathbb{B})} + \sum_{b\in\mathbb{B}} \frac{f(b)}{(b-x)R(b,\mathbb{B}\backslash b)} = f(a_1)\partial_1\cdots\partial_{n-1}\ , \tag{7.8.6}$$

Now, it is to be interpreted as decomposing a rational function $f(x)R(x,\mathbb{B})^{-1}$ into its principal part and its simple poles. Multiplying the equation by $R(x,\mathbb{B})$,

$$f(x) - \sum_{b\in\mathbb{B}} f(b)\frac{R(x,\mathbb{B}\backslash b)}{R(b,\mathbb{B}\backslash b)} = f(a_1)\partial_1\cdots\partial_{n-1}R(x,\mathbb{B})\ , \tag{7.8.7}$$

one gets now on the right the *quotient* in the division of $f(x)$ by $R(x,\mathbb{B})$, the sum $\sum_b f(b)S^{m-1}(x+b-\mathbb{B})/S^{m-1}(2b-\mathbb{B})$ being the *remainder* of the division. Indeed, all the computations in this section till now are closely connected to our study of the first remainder in the division of two polynomials. Let us display the case $n = 4$:

$$\begin{aligned} f(x) - f(a_1)\frac{R(x,a_2+a_3)}{R(a_1,a_2+a_3)} - f(a_2)\frac{R(x,a_1+a_3)}{R(a_2,a_1+a_3)} - f(a_3)\frac{R(x,a_1+a_2)}{R(a_3,a_1+a_2)} \\ = f(a_1)\partial_{a_1a_2}\partial_{a_2a_3}\partial_{a_3x}\,R(x,a_1+a_2+a_3)\ . \end{aligned}$$

Breaking symmetry is a way to disguise Lagrange formula, and to seemingly move away from Euclid. For example, instead of a single alphabet, let us cut the alphabet into two pieces and use $\mathbb{A} = \{a_1, \ldots, a_n\}$, $\mathbb{B} = \{a_{n+1}, \ldots, a_{n+m}\}$. Then the summation $\sum f(a) R(x, \mathbb{A}\backslash a) R(a, \mathbb{A}\backslash a)^{-1}$ is replaced by

$$(7.8.8)\quad \sum_{a\in\mathbb{A}} f(a) \frac{R(x,\mathbb{A}\backslash a)R(x,\mathbb{B})}{R(a,\mathbb{A}\backslash a)R(a,\mathbb{B})} + \sum_{b\in\mathbb{B}} f(b)\frac{R(x,\mathbb{B}\backslash b)R(x,\mathbb{A})}{R(b,\mathbb{B}\backslash b)R(b,\mathbb{A})}$$
$$= \frac{1}{R(\mathbb{A},\mathbb{B})}\sum_{a\in\mathbb{A}} f(a)\frac{R(x,\mathbb{A}\backslash a)R(x+\mathbb{A}\backslash a,\mathbb{B})}{R(a,\mathbb{A}\backslash a)}$$
$$+\frac{1}{R(\mathbb{B},\mathbb{A})}\sum_{b\in\mathbb{B}} f(b)\frac{R(x,\mathbb{B}\backslash b)R(x+\mathbb{B}\backslash b,\mathbb{A})}{R(b,\mathbb{B}\backslash b)}\ .$$

Each of the two subsums can still be expressed with a product of divided differences, and in final, one gets

$$(7.8.9)\quad f(a_1)R(a_2+\cdots+a_n, x+\mathbb{B})\partial_1\cdots\partial_{n-1}\frac{R(x,\mathbb{B})}{R(\mathbb{A},\mathbb{B})}+$$
$$f(a_{n+1})R(a_{n+2}+\cdots+a_{n+m}, x+\mathbb{A})\partial_{n+1}\cdots\partial_{n+m-1}\frac{R(x,\mathbb{A})}{R(\mathbb{B},\mathbb{A})}$$
$$= f(a_1)\partial_1\cdots\partial_{n+m-1}\ .$$

Explicitly, with $f(x) = S^k(x-\mathbb{E})$, remembering that $R(a_2+\cdots+a_n, \mathbb{B}+x)S^k(a_1-\mathbb{E}) = S_{(m+1)^{n-1};k}(\mathbb{A}-x-\mathbb{B}; a_1-\mathbb{E})$, one gets from (7.8.9)

$$(7.8.10)\quad \frac{R(x,\mathbb{B})}{R(\mathbb{A},\mathbb{B})}S_{(m+1)^{n-1};k-n+1}(\mathbb{A}-x-\mathbb{B};\mathbb{A}-\mathbb{E})$$
$$+\frac{R(x,\mathbb{A})}{R(\mathbb{B},\mathbb{A})}\cdot S_{(n+1)^{m-1};k-m+1}(\mathbb{B}-x-\mathbb{A};\mathbb{B}-\mathbb{E})$$
$$= S_{1^{m+n-1};k-m-n+1}(\mathbb{A}+\mathbb{B}-x;\mathbb{A}+\mathbb{B}-\mathbb{E})\ ,$$

which is no other than the remainder of $S^k(x-\mathbb{E})$ modulo $R(x, \mathbb{A}+\mathbb{B})$. More generally, one could break the expression of a remainder by factorizing the polynomial $R(x, \mathbb{A})$ into pieces corresponding to sub-alphabets.

Let us further disguise Lagrange by specializing all a_i to a, all b_i to b. The expression (7.8.10) becomes

$$(7.8.11)\quad \frac{(x-b)^m}{(a-b)^{mn}}S_{(m+1)^{n-1};k-n+1}(na-x-mb;na-\mathbb{E})+$$
$$\frac{(x-a)^n}{(b-a)^{mn}}S_{(n+1)^{m-1};k-m+1}(mb-x-na;nb-\mathbb{E})$$
$$= S_{1^{m+n-1};k-m-n+1}(na+mb-x;na+mb-\mathbb{E})$$
$$\equiv S^k(x-\mathbb{E}) \quad \bmod\ (x-a)^n(x-b)^m\ .$$

Since the product $\partial_1\cdots\partial_r$ tends towards $\frac{d^r}{r!\,da^r}$ when $a_1 = a$, $a_2 = a+\epsilon, \ldots, a_r = a+(r-1)\epsilon$, and ϵ tends towards 0, one can rewrite the limit of Lagrange summation with derivatives, and obtain the following proposition, due to Jacobi in his thesis [**71**].

Proposition 7.8.3. *Let $\mathbb{A}$ be a multialphabet, writing m_a for the multiplicity of the letter a. Let $n = \sum m_a$, and $f(x)$ be a polynomial of degree strictly less than* *n. Then*

$$\frac{f(x)}{\prod(x-a)^{m_a}} = \sum_a \frac{1}{(m_a-1)!}\frac{d^{m_a-1}}{da^{m_a-1}}\left(\frac{f(a)}{(x-a)R(a,\mathbb{A}-m_a a)}\right) . \tag{7.8.12}$$

For example, in the case of $\mathbb{A} = na + mb$ as before, Jacobi gives the following identity, for any polynomial of degree $< n+m$:

$$\frac{f(x)}{(x-a)^n(x-b)^m} = \frac{1}{(n-1)!}\frac{d^{n-1}}{da^{n-1}}\left(\frac{f(a)}{(x-a)(a-b)^m}\right) + \frac{1}{(m-1)!}\frac{d^{m-1}}{db^{m-1}}\left(\frac{f(b)}{(x-b)(b-a)^n}\right) . \tag{7.8.13}$$

Interpolation with points having multiplicities is usually attached to the name of Hermite. One has no problem to specialize, inside a symmetric function, a generic alphabet to an alphabet with multiplicities. Mixing divided differences and derivatives require some more care, but would lead us too far from our present topic. We intend to treat it in more details in some notes on interpolation.

7.9. Finite Derivation

Given $\mathbb{A} = \{a_1, \dots, a_n\}$, the Lagrange interpolation allows to express the image of any polynomial $f(x)$ modulo $S^n(x-\mathbb{A})$, as a summation over the values $f(a_i)$:

$$f(x) \equiv \sum_i \frac{S^{n-1}(x+a_i-\mathbb{A})}{S^{n-1}(2a_i-\mathbb{A})}\, f(a_i) . \tag{7.9.1}$$

Supposing $a_1, \dots, a_n$ to be different elements of $\mathbb{C}$, one can attach to the Lagrange interpolation an isomorphism φ :

$$\mathbb{C}[x]/\big(S^n(x-\mathbb{A})\big) \ni f(x) \leftrightarrow [f(a_1), \dots, f(a_n)] \in \mathbb{C}^n$$

and write it as a scalar product :

$$\Big([\dots, f(a_i), \dots]\, , [\dots, \frac{S^{n-1}(x+a_i-\mathbb{A})}{S^{n-1}(2a_i-\mathbb{A})}, \dots]\Big) = f(x)\ , \deg(f(x)) \le n-1 .$$

The next proposition determines what are the images of the powers of the derivative $\frac{d}{dx}$ under this isomorphism.

PROPOSITION 7.9.1. *For any integer $k \ge 0$, the image of the restriction of the multiplication by x^k in $End(V)$ is represented by the diagonal matrix $diag(a_1^k, \dots, a_n^k)$.*

The matrix expressing $\frac{1}{k!}\frac{d^k}{dx^k}$ is

$$\mathcal{D}^k := \left[\frac{S^{n-1-k}((k+1)a_i + a_j - \mathbb{A})}{S^{n-1}(2a_j-\mathbb{A})}\right]_{1\le i,j\le n} .$$

Proof. Pointwise evaluation is, of course, compatible with products of polynomials, hence the first assertion.

The second assertion is equivalent to

$$\sum_j \frac{S^{n-1-k}((k+1)a_i + a_j - \mathbb{A})}{S^{n-1}(2a_j-\mathbb{A})}\, f(a_j) = g(a_i) ,$$

with $k!\, g = d^k(f)/dx^k$, $\deg(f) \le n-1$. Let us test this identity in the basis of Lagrange polynomials, say for $f(x) = S^{n-1}(x+a_1-\mathbb{A})$.

The sum reduces to the single term

$$\frac{S^{n-1-k}((k+1)a_i + a_1 - \mathbb{A})}{S^{n-1}(2a_1 - \mathbb{A})\, S^{n-1}(2a_1 - \mathbb{A})} = S^{n-1-k}((k+1)a_i + a_1 - \mathbb{A}) \ ,$$

which is, indeed, the value of the derivative $S^{n-1-k}((k+1)x + a_1 - \mathbb{A})$ in $x = a_i$. □

The reader may check by himself that the square of $\mathcal{D}^1$ is $\mathcal{D}^2$.

Explicitly, each diagonal entry $[i,i]$ of $\mathcal{D}^1$ is

$$\frac{S^{n-2}(3a_i - \mathbb{A})}{S^{n-1}(2a_i - \mathbb{A})} = \frac{d}{dx}\log\Big(S^{n-1}(x + a_i - \mathbb{A})\Big) = \sum_{j\neq i} \frac{1}{a_i - a_j} \tag{7.9.2}$$

and each off-diagonal entry $[i,j]$, $j \neq i$, is

$$\frac{S^{n-2}(2a_i + a_j - \mathbb{A})}{S^{n-1}(2a_i - \mathbb{A})} = \frac{1}{a_i - a_j} \ . \tag{7.9.3}$$

Let us show now how to use Lagrange's operator to express derivatives without writing matrices.

Given $\mathbb{A} = \{a_1, \ldots, a_n\}$, for any $i : 1 \le i \le n$, any $k \ge 0$, let $\mathbb{B}_i^k := (k+1)a_i - \mathbb{A}$. For any symmetric function g, any $f \in \mathfrak{Sym}(1|n-1)$, let us write $L_{\mathbb{A}}(g(\mathbb{B}_i^k)f)$ for the specialization $\mathbb{B} = \mathbb{B}_i^k$ of $L_{\mathbb{A}}(g(\mathbb{B})f)$.

PROPOSITION 7.9.2. *Let $f(x)$ be a polynomial in x, k be a positive integer. Then for any $i : 1 \le i \le n$, the $i-th$ component of the image of $\frac{1}{k!}\frac{d^k}{dx^k}(f)$ under φ is equal to*

$$L_{\mathbb{A}}\big(f(x)\, S^{n-k-1}(x + \mathbb{B}_i^k)\big) = f(a_1)\, S^{n-k-1}(a_1 + \mathbb{B}_i^k)\, \partial_1 \cdots \partial_{n-1} \ . \tag{7.9.4}$$

Explicitly, when $f(x) = S^j(x - \mathbb{E})$ for some j, $\mathbb{E}$, then

$$L_{\mathbb{A}}\big(S^j(x - \mathbb{E})\, S^{n-k-1}(x + \mathbb{B}_i^k)\big) = S_{1^{n-k-1};j-n+1}(\mathbb{A} - (k+1)a_i\, ;\, \mathbb{A} - \mathbb{E}) \ . \tag{7.9.5}$$

Proof. We test (7.9.4) on the basis $\{x^0, \ldots, x^{n-1}, S^n(x - \mathbb{A}), S^{n+1}(x - \mathbb{A}), \ldots\}$. Since

$$x^j S^{n-k-1}(x + \mathbb{B}_i^k) = S^{n-k-1+j}(x + \mathbb{B}_i^k) + \Big(x^{j-1} S^{n-k}(\mathbb{B}_i^k) + x^{j-2} S^{n-k+1}(\mathbb{B}_i^k) + \cdots\Big) \ ,$$

for any $j \le n-1$, one has

$$L_{\mathbb{A}}\big(x^j S^{n-k-1}(x + \mathbb{B}_i^k)\big) = L_{\mathbb{A}}\big(S^{n-k-1+j}(x + \mathbb{B}_i^k)\big) = S^{j-k}(\mathbb{A} + \mathbb{B}_i^k) = S^{j-k}((k+1)a_i) \ ,$$

and this is indeed the value of $\frac{1}{k!}\frac{d^k x^j}{dx^k}$ in a_i. Moreover, the polynomials $S^{n+j}(x - \mathbb{A})$, $j \ge 0$ are null under φ, since they vanish in any a_i and this finish the proof of (7.9.4).

Writing

$$(-1)^{n-k-1} S^{n-k-1}(a_1 + \mathbb{B}_i^k) S^j(a_1 - \mathbb{E}) = S_{1^{n-k-1};j}(-\mathbb{B}_i^k\, ;\, a_1 - \mathbb{E})$$

one gets that the image under $\partial_1 \cdots \partial_{n-1}$ is equal to

$$S_{1^{n-k-1};j-n+1}(-\mathbb{B}_i^k\, ;\, \mathbb{A} - \mathbb{E}) = S_{1^{n-k-1};j-n+1}(\mathbb{A} - (k+1)a_i\, ;\, \mathbb{A} - \mathbb{E})$$

□

For example, for $k = 1$, the first component of the image of the derivative under φ is the morphism

$$f(x) \to \frac{d}{dx}\left(\overline{f}(x)\right)\Big|_{x=a_1} \ ,$$

where $\overline{f}(x)$ is the remainder of $f(x)$ modulo $S^n(x-\mathbb{A})$, and (7.9.4) states that it is equal to

$$\begin{aligned} &\partial_1, && n=2\,,\\ a_1\partial_1\partial_2+&\partial_1\partial_2(2a_1-\Lambda^1\mathbb{A}), && n=3\,,\\ a_1^2\partial_1\partial_2\partial_3+a_1\partial_1\partial_2\partial_3\,(2a_1-\Lambda^1\mathbb{A})+&\partial_1\partial_2\partial_3\,(3a_1^2-2a_1\Lambda^1\mathbb{A}+\Lambda^2\mathbb{A}), && n=4\,. \end{aligned}$$

Indeed, for $n=4$, the polynomial $x^4 \equiv x^3\Lambda^1\mathbb{A}-x^2\Lambda^2\mathbb{A}+x\Lambda^3\mathbb{A}-\Lambda^4\mathbb{A}$ has derivative $3x^2\Lambda^1\mathbb{A}-2x\Lambda^2\mathbb{A}+\Lambda^3\mathbb{A}$, while the image of a_1^4 is

$$\begin{aligned} &S^3\mathbb{A}+S^2\mathbb{A}(2a_1-\Lambda^1\mathbb{A})+S^1\mathbb{A}(3a_1^2-2a_1\Lambda^1\mathbb{A}+\Lambda^2\mathbb{A})\\ &\qquad = S^3(\mathbb{A}+2a_1-\mathbb{A})-S^3(2a_1-\mathbb{A}) = 3a_1^2\Lambda^1\mathbb{A}-2a_1\Lambda^2\mathbb{A}+\Lambda^3\mathbb{A}. \end{aligned}$$

7.10. Calogero's Raising and Lowering Operators

The space of polynomials in x is an infinite dimensional space, on which x and $\frac{d}{dx}$ act as *raising* and *lowering operators* :

$$x\cdot x^k = x^{k+1} \qquad \& \qquad \frac{d}{dx}x^k = kx^{k-1}\,. \tag{7.10.1}$$

Given n, the space $\mathfrak{Pol}_n$ of polynomials in x of degree $\le n-1$ is still a representation of SL_2, putting $x\cdot x^{n-1}=0$. In other words, one defines raising and lowering operators R, L on $\mathfrak{Pol}_n$ by

$$R(x^k)=x^{k+1},\ k\le n-2,\ R(x^{n-1})=0 \qquad \& \qquad L(x^k)=kx^{k-1}\,. \tag{7.10.2}$$

Calogero has described the matrices representing the raising and lowering operators in the basis of Lagrange polynomials.

THEOREM 7.10.1. *Given an alphabet $\mathbb{A}$ of n distinct letters, let*
(7.10.3)
$$Z=\left[\frac{S^{n-2}(2a_i+a_j-\mathbb{A})}{S^{n-1}(2a_i-\mathbb{A})}\right] \ \& \ X = diag(\dots a_i\dots)-diag\left(\dots,\frac{a_i^n}{S^{n-1}(2a_i-\mathbb{A})},\dots\right)\Big[\mathbf{1}\Big]\,,$$

where $\Big[\mathbf{1}\Big]$ is the constant matrix of order n with entries equal to 1.

Then Z, X represent L, R in the basis $\{S^{n-1}(x+a_1-\mathbb{A}),\dots,S^{n-1}(x+a_n-\mathbb{A})\}$.

Proof. We have already computed in the preceding section the expression of the derivative in the basis of Lagrange polynomials. Some caution is needed for X, because it does not represent the multiplication by x, contrary to the infinite dimensional case. The matrix expressing the Lagrange polynomials in the basis $\{x^0,\dots,x^{n-1}\}$, and its inverse are :

$$\Big[S^{n-i}(a_j-\mathbb{A})\Big]_{1\le i,j\le n-1}\,,\ diag(\dots(S^{n-1}(2a_i-\mathbb{A}))^{-1}\dots)\Big[a_i^{j-1}\Big]_{1\le i,j\le n-1}\,. \tag{7.10.4}$$

With these explicit matrices of change of basis, it is immediate to recognize that X represents R in the Lagrange basis. □

Let us detail the expressions of L, R, $[R,L]$ in the successive bases $\{x^0,\dots,x^{n-1}\}$, $\{S^0(x-\mathbb{A}),\dots,S^{n-1}(x-\mathbb{A})\}$, $\{S^{n-1}(x+a_1-\mathbb{A}),\dots,S^{n-1}(x+a_n-\mathbb{A})\}$, for $n=5$, writing S for $S(\mathbb{A})$ and $\cdot$ for 0.

$$L=\begin{bmatrix}\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&2&\cdot&\cdot\\ \cdot&\cdot&\cdot&3&\cdot\\ \cdot&\cdot&\cdot&\cdot&4\\ \cdot&\cdot&\cdot&\cdot&\cdot\end{bmatrix},\quad\begin{bmatrix}\cdot&1&\Psi^1&\Psi^2&\Psi^3\\ \cdot&\cdot&2&\Psi^1&\Psi^2\\ \cdot&\cdot&\cdot&3&\Psi^1\\ \cdot&\cdot&\cdot&\cdot&4\\ \cdot&\cdot&\cdot&\cdot&\cdot\end{bmatrix},$$

$$\begin{bmatrix}\frac{S^3(3a_1-\mathbb{A})}{S^4(2a_1-\mathbb{A})}&\frac{1}{a_1-a_2}&\frac{1}{a_1-a_3}&\frac{1}{a_1-a_4}&\frac{1}{a_1-a_5}\\ \frac{1}{a_2-a_1}&\frac{S^3(3a_2-\mathbb{A})}{S^4(2a_2-\mathbb{A})}&\frac{1}{a_2-a_3}&\frac{1}{a_2-a_4}&\frac{1}{a_2-a_5}\\ \frac{1}{a_3-a_1}&\frac{1}{a_3-a_2}&\frac{S^3(3a_3-\mathbb{A})}{S^4(2a_3-\mathbb{A})}&\frac{1}{a_3-a_4}&\frac{1}{a_3-a_5}\\ \frac{1}{a_4-a_1}&\frac{1}{a_4-a_2}&\frac{1}{a_4-a_3}&\frac{S^3(3a_4-\mathbb{A})}{S^4(2a_4-\mathbb{A})}&\frac{1}{a_4-a_5}\\ \frac{1}{a_5-a_1}&\frac{1}{a_5-a_2}&\frac{1}{a_5-a_3}&\frac{1}{a_5-a_4}&\frac{S^3(3a_5-\mathbb{A})}{S^4(2a_5-\mathbb{A})}\end{bmatrix}$$

$$R=\begin{bmatrix}\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot\end{bmatrix},\quad\begin{bmatrix}\Lambda^1&-\Lambda^2&\Lambda^3&-\Lambda^4&\Lambda^5-S^5\\ 1&\cdot&\cdot&\cdot&-S^4\\ \cdot&1&\cdot&\cdot&-S^3\\ \cdot&\cdot&1&\cdot&-S^2\\ \cdot&\cdot&\cdot&1&-S^1\end{bmatrix},$$

$$\begin{bmatrix}a_1-\frac{a_1^5}{S^4(2a_1-\mathbb{A})}&-\frac{a_1^5}{S^4(2a_1-\mathbb{A})}&-\frac{a_1^5}{S^4(2a_1-\mathbb{A})}&-\frac{a_1^5}{S^4(2a_1-\mathbb{A})}&-\frac{a_1^5}{S^4(2a_1-\mathbb{A})}\\ -\frac{a_2^5}{S^4(2a_2-\mathbb{A})}&a_2-\frac{a_2^5}{S^4(2a_2-\mathbb{A})}&-\frac{a_2^5}{S^4(2a_2-\mathbb{A})}&-\frac{a_2^5}{S^4(2a_2-\mathbb{A})}&-\frac{a_2^5}{S^4(2a_2-\mathbb{A})}\\ -\frac{a_3^5}{S^4(2a_3-\mathbb{A})}&-\frac{a_3^5}{S^4(2a_3-\mathbb{A})}&a_3-\frac{a_3^5}{S^4(2a_3-\mathbb{A})}&-\frac{a_3^5}{S^4(2a_3-\mathbb{A})}&-\frac{a_3^5}{S^4(2a_3-\mathbb{A})}\\ -\frac{a_4^5}{S^4(2a_4-\mathbb{A})}&-\frac{a_4^5}{S^4(2a_4-\mathbb{A})}&-\frac{a_4^5}{S^4(2a_4-\mathbb{A})}&a_4-\frac{a_4^5}{S^4(2a_4-\mathbb{A})}&-\frac{a_4^5}{S^4(2a_4-\mathbb{A})}\\ -\frac{a_5^5}{S^4(2a_5-\mathbb{A})}&-\frac{a_5^5}{S^4(2a_5-\mathbb{A})}&-\frac{a_5^5}{S^4(2a_5-\mathbb{A})}&-\frac{a_5^5}{S^4(2a_5-\mathbb{A})}&a_5-\frac{a_5^5}{S^4(2a_5-\mathbb{A})}\end{bmatrix}$$

$$[R,L]=\begin{bmatrix}-1&\cdot&\cdot&\cdot&\cdot\\ \cdot&-1&\cdot&\cdot&\cdot\\ \cdot&\cdot&-1&\cdot&\cdot\\ \cdot&\cdot&\cdot&-1&\cdot\\ \cdot&\cdot&\cdot&\cdot&4\end{bmatrix},\quad\begin{bmatrix}-1&\cdot&\cdot&\cdot&5S^4\\ \cdot&-1&\cdot&\cdot&5S^3\\ \cdot&\cdot&-1&\cdot&5S^2\\ \cdot&\cdot&\cdot&-1&5S^1\\ \cdot&\cdot&\cdot&\cdot&4S^0\end{bmatrix},$$

$$\begin{bmatrix}\frac{5a_1^4}{S^4(2a_1-\mathbb{A})}-1&\frac{5a_1^4}{S^4(2a_1-\mathbb{A})}&\frac{5a_1^4}{S^4(2a_1-\mathbb{A})}&\frac{5a_1^4}{S^4(2a_1-\mathbb{A})}&\frac{5a_1^4}{S^4(2a_1-\mathbb{A})}\\ \frac{5a_2^4}{S^4(2a_2-\mathbb{A})}&\frac{5a_2^4}{S^4(2a_2-\mathbb{A})}-1&\frac{5a_2^4}{S^4(2a_2-\mathbb{A})}&\frac{5a_2^4}{S^4(2a_2-\mathbb{A})}&\frac{5a_2^4}{S^4(2a_2-\mathbb{A})}\\ \frac{5a_3^4}{S^4(2a_3-\mathbb{A})}&\frac{5a_3^4}{S^4(2a_3-\mathbb{A})}&\frac{5a_3^4}{S^4(2a_3-\mathbb{A})}-1&\frac{5a_3^4}{S^4(2a_3-\mathbb{A})}&\frac{5a_3^4}{S^4(2a_3-\mathbb{A})}\\ \frac{5a_4^4}{S^4(2a_4-\mathbb{A})}&\frac{5a_4^4}{S^4(2a_4-\mathbb{A})}&\frac{5a_4^4}{S^4(2a_4-\mathbb{A})}&\frac{5a_4^4}{S^4(2a_4-\mathbb{A})}-1&\frac{5a_4^4}{S^4(2a_4-\mathbb{A})}\\ \frac{5a_5^4}{S^4(2a_5-\mathbb{A})}&\frac{5a_5^4}{S^4(2a_5-\mathbb{A})}&\frac{5a_5^4}{S^4(2a_5-\mathbb{A})}&\frac{5a_5^4}{S^4(2a_5-\mathbb{A})}&\frac{5a_5^4}{S^4(2a_5-\mathbb{A})}-1\end{bmatrix}$$

Given any differential operator $\sum_0^m c_k(x)\frac{d^k}{dx^k}$ with polynomial coefficients, Calogero **[26]** approximate it by the operator

$$\sum_0^m c_k(R)L^k$$

in the basis of Lagrange, choosing $n > m$. This transforms differential calculus into pure linear algebra, but, of course, it remains to describe for which differential operators this approximation is interesting. We shall not comment this topic any further here.

Exercises

Ex. 7.1. Extend Leibnitz formula to the following two cases (any $i, n \in \mathbb{N}$, any functions f_j):

$$f_1 \cdots f_n\, \partial_i \qquad \& \qquad f g\, \partial_1 \cdots \partial_n \ .$$

Ex. 7.2. Let x, y be two letters, $\partial = \partial_{xy}$, $\pi = \pi_{xy}$ be the associated divided differences. For a word w in $\{\pi, \partial\}^*$ of length r, and any $f_i \in \mathbb{Z}[x, y]$, $1 \le i \le r$, one notes $f_1 \cdots f_r\, w$ the product $f_1 w_1\, f_2 w_2 \cdots f_r w_r$.

Show that

$$f_1 \cdots f_r\, \pi = \sum\nolimits_w (-1)^{k-1} f_1 \cdots f_r\, w\, S_{1,k-1}(x+y)$$

sum over all words of length r, k being the number of occurrences of ∂ in w.

Ex. 7.3. Let $\mathbb{A} = \{a_1, \ldots, a_n\}$. Define δ_i, $i = 1, \ldots, n-1$, to be the operator

$$f \to f\, \delta_i = \Big(f(\ldots a_i, a_{i+1} \ldots) - f(\ldots q a_{i+1}, a_i/q \ldots)\Big)\, (a_i - q a_{i+1})^{-1} \ .$$

Show that the operators δ_i satisfy the braid relations.

Therefore, for any permutation $\sigma \in \mathfrak{S}_n$, there is an operator δ_σ. Give the image of any monomial under δ_ω.

Ex. 7.4. Let $\mathbb{A} = \{a_1, a_2, \ldots\}$. Let $\delta_i := a_i \partial_i \frac{1}{a_i}$, $i = 1, 2, \ldots$.

Show that the operators δ_i satisfy the braid relations. Beware that they do not preserve polynomials, and therefore do not belong to the family mentioned in (7.1.6).

Compute the images of 1 under all δ_σ, $\sigma \in \mathfrak{S}_4$. Compute the images of 1 under $\sum_1^n \delta_i$, under $\sum_{1 \le i \le j \le n} \delta_i \delta_j$, under $\sum_{1 \le i < j \le n} \delta_j \delta_i$.

Ex. 7.5. Let $\mathbb{A} = \{a_1, a_2, \ldots\}$. Let $\delta_i := \partial_i\, a_i a_{i+1}$, $i = 1, 2, \ldots$.

Show that the δ_i's satisfy the braid relations. Compute δ_ω.

Ex. 7.6. Let p, q be two parameters. Show that the operators

$$\delta_i := \big(1 + s_i\big)\, \frac{q a_i + p}{a_i - a_{i+1}}$$

satisfy the braid relations, then explicit the maximal operator δ_ω.

Ex. 7.7. Let $\mathbb{A}$ be finite, f be a function of one variable. Check again that the operators $\widehat{\pi}_i := \pi_i - 1 = \partial_i\, a_{i+1}$ satisfy the braid relations, but that the operators $\delta_i^c := \pi_i - c$ do not, when $c \ne 0, 1$.

Express $f(a_1)\pi_\omega$ in terms of the non-zero images of $f(a_1)$ under products of $\widehat{\pi}_i$'s.

Ex. 7.8. Show how to obtain the first remainder in the Euclidean division of two polynomials, using divided differences.

Ex. 7.9. Let $\mathbb{A}, \mathbb{B}$ be two finite alphabets having a single letter ξ in common: $\mathbb{A} = \mathbb{A}' + \xi$, $\mathbb{B} = \mathbb{B}' + \xi$, $\mathbb{A}' \cap \mathbb{B}' = \emptyset$. In other words, the G.C.D. of the polynomials $R(x-\mathbb{A})$ and $R(x-\mathbb{B})$ is of degree 1.

Following Abel [**1**], vol. 1, art. 13, find the value of any rational function in ξ (of course, in terms of $\mathbb{A}$ and $\mathbb{B}$, not of ξ, which is unknown).

Ex. 7.10. Given two alphabets $\mathbb{A}, \mathbb{B}$, of respective cardinalities n, m, let $0 < r \le m, n$. Find the remainder of $S_{(r-1)^{m-n+r}}(\mathbb{A}-\mathbb{B}-x)\, S^m(x-\mathbb{B})$ modulo $S^n(x-\mathbb{A})$.

Ex. 7.11. Let $\mathbb{A} = \{a_1, \ldots, a_n\}$, x be an indeterminate. Compute

$$(x - a_1)(x + a_2) \cdots (x + a_n)(a_1 + a_2) \cdots (a_1 + a_n)\, \partial_1 \cdots \partial_{n-1} \ .$$

Ex. 7.12. Let $\mathbb{A} = \{a_1, \ldots, a_n\}$. Show that

$$L_{\mathbb{A}} \left(\frac{R(1, x\mathbb{A})}{1+x} \right) = (-1)^{n-1}(1 - a_1 \cdots a_n) \ .$$

Ex. 7.13. Let $\mathbb{A}, \mathbb{B}$ be of the same cardinality n. Show that

$$1 - a_1 \cdots a_n = \sum\nolimits_j (1 - a_j) \prod\nolimits_{i: i \neq j} \frac{b_j - a_i}{b_j - b_i} \ .$$

Ex. 7.14. Let $\mathbb{A}, \mathbb{B}$ be of the same cardinality n, and $\partial_1, \ldots, \partial_{n-1}$ be the divided differences relative to $\mathbb{B}$.

Show that

$$(1-a_1)(1 + a_n + a_n a_{n-1} + \ldots + a_n \cdots a_2) = (b_1 - b_1 a_1) \cdots (b_1 - b_n a_n) b_1^{-1}\, \partial_1 \cdots \partial_{n-1} \ .$$

Ex. 7.15. Let $\mathbb{A} = \{a_1, \ldots, a_n\}$, $n \ge 2$, ω be the maximal element of $\mathfrak{S}(\mathbb{A})$.

Show that

$$(1-a_1^2)^{-1}(1-a_2^2)^{-1}\, \pi_\omega = (1-a_1^2)^{-1} \cdots (1-a_n^2)^{-1} \ .$$

Deduce from it that

$$\sum_n S^n(\Psi^2(\mathbb{A})) = \sum_{i \le j:\, i+j\ even} (-1)^i S_{ij}(\mathbb{A}) \ .$$

Ex. 7.16. Let $\mathbb{A} = \{a_1, \ldots, a_n\}$, ω be the maximum element of $\mathfrak{S}(\mathbb{A})$. Show that

$$\begin{aligned}\big(1 - a_1 x\big)^{-1}\big(1 - (a_1{+}a_2)x\big)^{-1} \cdots \big(1 - (a_1{+} \cdots {+}a_{n-1})x\big)^{-1}\, \pi_\omega \\ = \big(1 + x a_1 - x S^1(\mathbb{A})\big)^{-1} \cdots \big(1 + x a_n - x S^1(\mathbb{A})\big)^{-1} \ .\end{aligned}$$

Ex. 7.17. Let $n, m \in \mathbb{N} : n > m$, $\mathbb{X} = \{x_1, \ldots, x_m\}$, $\mathbb{A}$ be of cardinality n. Let $f_1(x), \ldots, f_m(x)$ be polynomials of degree $\le n-1$ in a variable x.

Show, after Borchardt [**20**], that

$$\det \Big| f_i(x_j) \Big|_{1 \le i,j \le n} = \sum_{\mathbb{A}' + \mathbb{A}'' = \mathbb{A}} \det \Big| f_i(a) \Big|_{a \in \mathbb{A}'} \frac{\Delta(\mathbb{X} + \mathbb{A}'')}{\Delta(\mathbb{A}' + \mathbb{A}'')} \ ,$$

sum over all decompositions of $\mathbb{A} = \mathbb{A}' + \mathbb{A}''$, with $\mathbb{A}'$ of cardinality m (one chooses an arbitrary total order on $\mathbb{A}'$ and $\mathbb{A}''$ to write the determinant, and the two Vandermonde).

Ex. 7.18. Let $\mathbb{A}$ be of cardinality n, $\mathbb{B}$ be of cardinality m, with $n \geq m$. Evaluate, after Nanson [**139**], the sum

$$\sum_{a\in\mathbb{A}} \frac{R(a,\mathbb{B})}{R(a,\mathbb{A}-a)} \Lambda_I(\mathbb{A}-a) \ ,$$

where I is a partition of length ≤ 2.

Ex. 7.19. Let $\mathbb{A}$ be finite, x, y be two indeterminates, $\mathbb{A}^\diamond$ be the alphabet $\mathbb{A}^\diamond := \{(a-x)^{-1} : a \in \mathbb{A}\}$. Show that

$$\frac{R(x,\mathbb{A})}{R(y,\mathbb{A})} = \sigma_{y-x}\left(\mathbb{A}^\diamond\right) := \prod_{a\in\mathbb{A}} \left(1 - \frac{y-x}{a-x}\right) \ .$$

Ex. 7.20. Let $\mathbb{A}, \mathbb{B}$ be two disjoint alphabets, $|\mathbb{A}| = \alpha$, $|\mathbb{B}| = \beta$, and let $f(x)$ be a polynomial of degree γ. Express $f(x)$ as a combination

$$f(x) = P_1(x)R(x,\mathbb{A}) + P_2(x)R(x,\mathbb{B}) + P_3(x)R(x,\mathbb{A}+\mathbb{B}) \ ,$$

with $\deg P_1 \leq \beta - 1$, $\deg P_2 \leq \alpha - 1$, $\deg P_3 \leq \gamma - \alpha - \beta$.

Ex. 7.21. Let $a_1, \ldots, a_{2n}$ be indeterminates. Prove that

$$\sum_{j=0}^{n} \prod_{i=1}^{j} \left(1 - \frac{a_{2i-1}}{a_{2i-1} + \cdots a_{2j}}\right) \prod_{i=j+1}^{n} \left(1 - \frac{a_{2i}}{a_{2j+1} + \cdots a_{2i}}\right) = 1 \ .$$

Ex. 7.22. Let $k \in \mathbb{C}$, $n \in \mathbb{N}$, $\mathbb{A}$ be arbitrary. After Sylvester (*Development of an idea of Eisenstein*, [**166**], art. [21]), Glaisher and Cayley (*Addition to Mr Glaisher's note on Sylvester's paper...*,[**32**], art. [649]), write $S^n(k\mathbb{A})$ in terms of $S^n(0\mathbb{A}), \ldots, S^n(n\mathbb{A})$.

Ex. 7.23. The Lagrange interpolation involves a summation on all letters of an alphabet. In his thesis [**71**], Jacobi gave the following generalization, as a summation over subalphabets of fixed cardinality k of $\mathbb{A}$, for any polynomial $f(x)$ of degree $\leq n-k$ (with $n = |\mathbb{A}|$) :

$$\frac{f(x)}{R(x,\mathbb{A})} = \sum_{\mathbb{A}=\mathbb{A}'+\mathbb{A}''} \frac{S_{(n-k)^k}(\mathbb{A}')}{R(x,\mathbb{A}')\, R(\mathbb{A}',\mathbb{A}'')} \sum_{a'\in\mathbb{A}'} f(a')\, {a'}^{k-n} \ ,$$

sum over all disjoint decompositions of $\mathbb{A}$ into two subalphabets, with $|\mathbb{A}'| = k$.

Prove Jacobi's formula.

Explicit it for $n = 3$, $k = 2$, $f(x) = x$.

Settle the contradiction by computing the right member for any f without restricting the degree.

Ex. 7.24. Let $\mathbb{A}$ be of cardinality n, f be a function of one variable. For every $j \geq 0$, let $\psi(j) = \sum a^j f(a)$. Let $r < j$. Show, after Aitken [**2**], that the determinant

$$\begin{vmatrix} \psi(j) & \cdots & \psi(j+n-1) \\ \vdots & & \vdots \\ \psi(j-n+1) & \cdots & \psi(j) \end{vmatrix}$$

is equal to

$$\sum\nolimits_{\mathbb{B}} \Delta(\mathbb{B})^2 \prod\nolimits_{\in\mathbb{B}} f(b) b^{j-n+1} \ ,$$

sum over all subsets $\mathbb{B}$ of $\mathbb{A}$ of cardinality r.

Ex. 7.25. Let $\mathbb{A} = \{a_1, \dots, a_n\}$, $\mathbb{B} = \{b_1, \dots, b_n\}$, f a function of one variable, g be a symmetrical function of n variables. Let ω be the maximum element of $\mathfrak{S}(\mathbb{A})$, and let the divided differences be relative to $\mathbb{A}$.

Show, after Jacobi, that $f\partial_1 \cdots \partial_{n-1}$ is the coefficient of z^{-1} in $f(z)/R(z, \mathbb{A})$ (when expanding $1/R(z, \mathbb{A})$ as a function of z^{-1}), and that $g\partial_\omega$ be the coefficient of $(b_1 \cdots b_n)^{-1}$ in $\Delta(\mathbb{B})g(\mathbb{B})/R(\mathbb{B}, \mathbb{A})$.

Ex. 7.26. Decompose $(x-a)^{-n}(x-b)^{-n}$ and $(x-a)^{-3}(x-b)^{-3}(x-c)^{-3}$ into simple fractions (relative to x).

Ex. 7.27. Let $\mathbb{A}$ be a totally ordered set of n+1 distinct complex numbers, and $\partial_1, \dots, \partial_n$ be the divided differences associated to $\mathbb{A}$. Let Γ be a simple curve enclosing $\mathbb{A}$, $f(z)$ be a function analytic in a disk containing $\mathbb{A}$.

Show, after Hermite, that

$$f(a_1)\partial_1 \cdots \partial_n = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \frac{f(x)}{R(x, \mathbb{A})} dx \ .$$

Ex. 7.28. Given $r, n, J, K \in \mathbb{N}^n$, and $\mathbb{A}$ of cardinality $n + r$, show that

$$\sum_L (-1)^{|I|} S_{L,I}(\mathbb{A}) S_{J,L^\sim}(\mathbb{A}) = \left| S_{i_h + j_k + k - h}(\mathbb{A}) \right|_{1 \le h,k \le n} ,$$

sum over all partitions $L \in \mathbb{N}^r$.

Ex. 7.29. Let n be an integer, f a function of x, $D := \frac{d}{dx}$. Evaluate the determinant having first row $1, x, \dots, x^n$, and other rows

$$D^i f/i!,\ D^i f^2/i!,\ \dots,\ D^i f^{n+1}/i! \ ,$$

for $0 \le i \le n-1$.

Ex. 7.30. Let M be a matrix with minimal polynomial[12] $(z-a)^3(z-b)^2$, $a, b \in \mathbb{C}$. Compute $\exp(M)$.

[12]minimal polynomial: $(M-a)^3(M-2)^2 = 0$, and no polynomial of lower degree vanishes on M

CHAPTER 8

Orthogonal Polynomials

8.1. Orthogonal Polynomials as Symmetric Functions

To any "generic" linear functional $\int$ on $\mathfrak{Pol}(x)$, with $\int 1 = 1$, is associated a (unique) family of orthogonal polynomials:

$$\int P_m(x)P_n(x) = 0 \text{ if } m \neq n \ , \quad \int P_n(x)P_n(x) = 1 \ . \tag{8.1.1}$$

We shall treat this subject having only in mind to show algebraic identities. The reader will find a broader point of view in the book of Andrews, Askey, Roy [**5**], and the one of Szegö [**167**].

One can formally suppose that there exists an alphabet $\mathbb{A}$ such that the *moments* $\int x^n$ be the complete functions of $\mathbb{A}$, i.e.

$$\int x^n = S^n(\mathbb{A}) \ , \ n \geq 0 \ .$$

Now $\int$ is a linear functional, that we shall note $\int_{\mathbb{A}}$, with values in symmetric functions:

$$\int_{\mathbb{A}} : \mathfrak{Pol}(x) \mapsto \mathfrak{Sym}(\mathbb{A}) \ .$$

The linear functional can be thought as a quadratic form on the space of polynomials in x, compatible with product :

$$\big(f(x)\,,\, g(x)\big) := \int_{\mathbb{A}} f(x)g(x) = (f(x)g(x)\,,\, 1) \ . \tag{8.1.2}$$

As could be expected, the orthogonal polynomials $P_n(x)$ are Schur functions, as shown by the following theorem.

THEOREM 8.1.1. *Given a generic alphabet $\mathbb{A}$, then there exists non zero-constants $c_n \in \mathfrak{Sym}(\mathbb{A})$ such that the orthonormal polynomials associated to $\int_{\mathbb{A}}$ are*

$$P_n(x) = c_n S_{n^n}(\mathbb{A} - x) \ , \ n = 0, 1, \ldots \tag{8.1.3}$$

with normalization

$$\Big(S_{n^n}(\mathbb{A}-x)\,,\, S_{n^n}(\mathbb{A}-x)\Big) = c_n^{-2} = (-1)^n S_{(n-1)^n}(\mathbb{A})\, S_{n^{n+1}}(\mathbb{A}) \tag{8.1.4}$$

Proof. On can write the orthogonality relations as $\int_{\mathbb{A}} P_n(x)x^m = 0$, $m < n$. However, $S_{n^n}(\mathbb{A} - x)\, x^m = S_{n^n, m}(\mathbb{A}, x)$ and its image under $\int_{\mathbb{A}}$ is obtained by replacing the last column of powers of x by complete functions of $\mathbb{A}$, i.e. one has

$$\int_{\mathbb{A}} S_{n^n, m}(\mathbb{A}, x) = S_{n^n, m}(\mathbb{A}) \ .$$

This determinant vanishes for $m < n$, having two identical columns. Moreover,

$$(8.1.5)\qquad \int_{\mathbb{A}} S_{n^n}(\mathbb{A}-x)\, S_{n^n}(\mathbb{A}-x) = \int_{\mathbb{A}} S_{n^n}(\mathbb{A}-x)\,(-x)^n\, S_{(n-1)^n}(\mathbb{A})$$
$$= S_{n^n,n}(\mathbb{A})\, S_{(n-1)^n}(\mathbb{A})\ .$$

The notation $S_{n^n}(\mathbb{A}-x)$ encodes the classical determinantal expressions of orthogonal polynomials in terms of moments [**23**] :

$$S_{333}(\mathbb{A}-x) = \begin{vmatrix} S^3(\mathbb{A}-x) & S^4(\mathbb{A}-x) & S^5(\mathbb{A}-x) \\ S^2(\mathbb{A}-x) & S^3(\mathbb{A}-x) & S^4(\mathbb{A}-x) \\ S^1(\mathbb{A}-x) & S^2(\mathbb{A}-x) & S^3(\mathbb{A}-x) \end{vmatrix} ,$$

$$x^m S_{333}(\mathbb{A}-x) = \begin{vmatrix} S^3(\mathbb{A}) & S^4(\mathbb{A}) & S^5(\mathbb{A}) & x^{m+3} \\ S^2(\mathbb{A}) & S^3(\mathbb{A}) & S^4(\mathbb{A}) & x^{m+2} \\ S^1(\mathbb{A}) & S^2(\mathbb{A}) & S^3(\mathbb{A}) & x^{m+1} \\ S^0(\mathbb{A}) & S^1(\mathbb{A}) & S^2(\mathbb{A}) & x^m \end{vmatrix} .$$

Notice that the functional $\int_{\mathbb{A}}$ can also be interpreted as a symmetrizing operator. Indeed, when $\mathbb{A}$ is of finite cardinality n, let ω be the maximal permutation in $\mathfrak{S}_n$. Then

$$a_1^k\, \pi_\omega = S_k(\mathbb{A})\ ,\ \ k = 0, 1, 2, \dots\ ,$$

and thus, for any polynomial $f(x)$, one has

$$(8.1.6)\qquad \int_{\mathbb{A}} f(x) = f(a_1)\, \pi_\omega\ .$$

Since $a^J\, \pi_\omega = S_J(\mathbb{A})$, $J \in \mathbb{N}$, there is no difficulty in extending the definition of π_ω to an alphabet of infinite cardinality, as is needed in the theory of orthogonal polynomials.

8.2. Reproducing Kernels

Instead of handling orthogonality relations, one can introduce *reproducing kernels*, i.e. functions $K_n(x, y)$ of two variables having the property that $\int_{\mathbb{A}} K_n(x, y)\, f(x) = f(y)$ for all $f(x)$ of degree $\le n - 1$ in x.

Darboux-Christoffel theorem gives the explicit expression of these kernels:

THEOREM 8.2.1. *Let x, y be rank 1 elements, and*

$$(8.2.1)\qquad K_n(x, y) := (-1)^{n-1} \frac{S_{n^{n-1}}(\mathbb{A} - x - y)}{S_{(n-1)^n}(\mathbb{A})}\ .$$

Then $K_n(x, y)$ is a reproducing kernel, with respect to $\int_{\mathbb{A}}$, for polynomials of degree $\le n - 1$. Moreover

$$(8.2.2)\qquad K_n(x, y) = \frac{(-1)^n}{(S_{(n-1)^n}(\mathbb{A}))^2}\, \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{x - y}\ .$$

Proof. Write, for any $i \ge 0$, $x^i\, K_n(x, y) = S_{n^{n-1};\, i}(\mathbb{A} - y;\, x)$. When $i \le n - 1$, one has

$$\int_{\mathbb{A}} x^i\, K_n(x, y) = S_{n^{n-1};\, i}(\mathbb{A} - y;\, \mathbb{A}) = (-y)^i S_{(n-1)^i, n^{n-1-i}, i}(\mathbb{A})$$
$$= (-1)^{n-1} y^i S_{(n-1)^n}(\mathbb{A})$$

and this gives the reproducing property.

The second statement can be recognized as a quadratic relation, which is once more, an instance of the simplest Plücker relations ($\mathcal{C}$ being a set of common indices) :

$$[a,b,\mathcal{C}]\cdot[c,d,\mathcal{C}] - [a,c,\mathcal{C}]\cdot[b,d,\mathcal{C}] = [a,d,\mathcal{C}]\cdot[b,c,\mathcal{C}]\ .$$

For simplicity of indices, let us take $n=3$. The two determinants expressing $P_3(x) = S_{333}(\mathbb{A}-x) = S_{3330}(\mathbb{A},x)$ and $P_2(x) = S_{22}(\mathbb{A}-x) = S_{220}(\mathbb{A},x)$ are minors of the following matrix (writing k instead of $S_k()$; alphabets are constant in each column and written at the bottom) :

$$\begin{array}{r|cccccc}
 & 0 & 3 & 4 & 5 & 3 & 3\\
 & \cdot & 2 & 3 & 4 & 2 & 2\\
 & \cdot & 1 & 2 & 3 & 1 & 1\\
 & \cdot & 0 & 1 & 2 & 0 & 0\\
\hline
\textit{alphabets} & \mathbb{A} & \mathbb{A} & \mathbb{A} & \mathbb{A} & x & y\\
\textit{column index} & 0 & 3 & 4 & 5 & x & y
\end{array}$$

Taking for a,b,c,d the indices $0,5,x,y$, with $\mathcal{C}=\{3,4\}$, Plücker relation reads

$$\frac{[0,3,4,y]}{S_{022,0}(\mathbb{A},y)}\ \frac{[3,4,5,x]}{S_{333,0}(\mathbb{A},x)} - \frac{[0,3,4,x]}{S_{022,0}(\mathbb{A},y)}\ \frac{[3,4,5,x]}{S_{333,0}(\mathbb{A},x)} = \frac{[0,3,4,5]}{S_{0222}(\mathbb{A})}\ \frac{[3,4,x,y]}{S_{3,3,1,0}(\mathbb{A},x,y)}$$

Now, subtracting $x+y$ to all rows, but the last two ones, of S_{3310}, one gets

$$S_{33,1,0}(\mathbb{A},x,y) = S_{33}(\mathbb{A}-x-y)\,S_{1,0}(x,y)$$

and finally

$$\big(P_2(y)P_3(x) - P_2(x)P_3(y)\big)\frac{1}{x-y} = S_{222}(\mathbb{A})\,S_{33}\,(\mathbb{A}-x-y)$$

Writing

$$(8.2.3)\qquad S_{n^{n-1}}(\mathbb{A}-x-y) = \begin{vmatrix} x^{n-1} & S_{n-1}(\mathbb{A}) & \cdots & S_{2n-2}(\mathbb{A})\\ \vdots & \vdots & & \vdots\\ x^0 & S_0(\mathbb{A}) & \cdots & S_{n-1}(\mathbb{A})\\ 0 & y^0 & \cdots & y^{n-1}\end{vmatrix}$$

(one recovers $S_{n^{n-1}}(\mathbb{A}-x-y)$ by subtracting x in the first $n-1$ rows, and y in the last $n-1$ columns), one can also prove (8.2.2) by using Sylvester's identity with pivot $S_{n^{n-1}}(\mathbb{A})$.

The explicit expansion of the kernel is, writing S_I for $S_I(\mathbb{A})$ in denominator :

$$(8.2.4)\quad K_{n+1}(x,y) = 1 - \frac{1}{S_{11}}S_1(\mathbb{A}-x)S_1(\mathbb{A}-y) + \frac{1}{S_{11}S_{222}}S_{22}(\mathbb{A}-x)S_{22}(\mathbb{A}-y) + \cdots + (-1)^n\frac{1}{S_{(n-1)^n}S_{(n+1)^n}}S_{n^n}(\mathbb{A}-x)S_{n^n}(\mathbb{A}-y)\ .$$

8.3. Continued Fractions

The reader will have noticed that the functions $S_{n^n}(\mathbb{A}-x)$ already appeared many times. Indeed, orthogonal polynomials can be obtained from Euclidean division and continued fractions, as shows the following statement, which is a rewriting of the continued fractions of Wroński and Stieltjes (we have given the convergents in (4.2.2) and various parametrizations of continued fractions in (4.4.3)) :

THEOREM 8.3.1. *Let $\mathbb{A}$ be a generic alphabet, x be an indeterminate. Let*

$$x^{-1}\sigma_{1/x}(\mathbb{A}) = \cfrac{1}{(x-\zeta_0) + \cfrac{\beta_1}{(x-\zeta_1) + \cfrac{\beta_2}{(x-\zeta_2) + \cfrac{\beta_3}{\ddots}}}} \tag{8.3.1}$$

be the continued fraction expansion for the series $x^{-1}\sigma_{1/x}(\mathbb{A})$ at $x=\infty$. Then the parameters ζ_i, β_j are the Wroński parameters $\zeta_0 = S_1(\mathbb{A})$,

$$\zeta_n = \frac{S_{n^n,n+1}}{S_{n^{n+1}}} - \frac{S_{(n-1)^{n-1},n}}{S_{(n-1)^n}} \quad \& \quad \beta_n = \frac{S_{n^{n+1}}(\mathbb{A})\, S_{(n-2)^{n-1}}(\mathbb{A})}{(S_{(n-1)^n}(\mathbb{A}))^2} \tag{8.3.2}$$

and the orthogonal polynomials $P_n = (-1)^n S_{n^n}(\mathbb{A}-x)$ are the denominators of the convergents. The numerators are the "adjoint polynomials" defined by $Q_n := S_{(n-1)^{n+1}}(\mathbb{A}+x)$.

Moreover, $(-1)^n \frac{S_{(n-1)^{n+1}}(\mathbb{A}+x)}{S_{n^n}(\mathbb{A}-x)}$ is the Padé approximant of degree $[n-1,n]$ at $x=\infty$ of $x^{-1}\sigma_{1/x}(\mathbb{A})$, i.e.

$$\frac{S_{(n-1)^{n+1}}(\mathbb{A}+x)}{(-1)^n S_{n^n}(\mathbb{A}-x)} = x^{-1} + x^{-2}S_1(\mathbb{A}) + \cdots + x^{-2n}S_{2n-1}(\mathbb{A}) + \mathcal{O}(x^{-2n-1}) \tag{8.3.3}$$

For example, writing S_I for $S_I(\mathbb{A})$, the first terms are

$$\frac{1}{x-S_1} = \frac{Q_1}{P_1}\ ; \qquad \cfrac{1}{x - S_1 + \cfrac{S_{11}}{x - (\frac{S_{12}}{S_{11}} - S_1)}} = \frac{Q_2}{P_2} = \frac{S_{111}(x+\mathbb{A})}{S_{22}(\mathbb{A}-x)}$$

$$\cfrac{1}{x - S_1 + \cfrac{S_{11}}{x - (\frac{S_{12}}{S_{11}} - S_1) + \cfrac{S_{222}/S_{11}^2}{x - (\frac{S_{223}}{S_{222}} - \frac{S_{12}}{S_{11}})}}} = \frac{Q_3}{P_3} = -\frac{S_{2222}(\mathbb{A}+x)}{S_{333}(\mathbb{A}-x)}\,.$$

In presenting the work of Wroński in the third chapter, we had interpolated the series $\lambda_x(\mathbb{A})$ instead of the series $\sigma_{1/x}(\mathbb{A})$ which is now more convenient. This accounts for the conjugation of the partitions indexing Schur functions in the ζ_n, β_n, and the slight changes in conventions.

8.4. Higher Order Kernels

Monic orthogonal polynomials, that we shall note $\widetilde{P}_n(x) = x^n + \cdots$, satisfy *three-term recurrences*, which are equivalent to the continued fraction expansion given above, with $\beta_0 = 1$, $\widetilde{P}_0 = 1$, $\widetilde{P}_{-1} = 0$:

$$\widetilde{P}_1 = S_1(x-\mathbb{A}) = (x - S_1(\mathbb{A}))\,1\ ,$$

$$\widetilde{P}_2 = \frac{S_{22}(x-\mathbb{A})}{S_{11}(\mathbb{A})} = \left(x - \frac{S_{12}(\mathbb{A})}{S_{11}(\mathbb{A})} + S_1(\mathbb{A})\right)(x - S_1(\mathbb{A})) + S_{11}(\mathbb{A}),\ \ldots$$

$$\widetilde{P}_{n+1} = (x-\zeta_n)\widetilde{P}_n + \beta_n \widetilde{P}_{n-1}\ . \tag{8.4.1}$$

The adjoint polynomials $\widetilde{Q}_n$ satisfy the same recursion, but with initial conditions $\widetilde{Q}_0 = 0, \widetilde{Q}_1 = 1$.

We have preferred giving similar expressions to P_n and Q_n. However, one can also write $P_n = (-1)^n S_{n^n,0}(\mathbb{A}, x)$ and $Q_n = (-1)^n S_{n^n,-1}(\mathbb{A}, \mathbb{A}+x)$ to break up the symmetry in P and Q. Exchanging the P_n's and Q_n's mostly consists in the involution $\mathbb{A} \mapsto -\mathbb{A}$, apart from some shifts in indices.

All formulas satisfied by the polynomials P_n can be adapted to the adjoint polynomials Q_n. One still has a Darboux-Christoffel kernel :

$$\frac{Q_n(x)Q_{n-1}(y) - Q_n(y)Q_{n-1}(x)}{x-y} = \Lambda_{n^{n-1}}(\mathbb{A})\,\Lambda_{(n+1)^{n-2}}(\mathbb{A}+x+y) \ .$$

For example,

$$\big(Q_2(y)Q_3(x) - Q_2(x)Q_3(y)\big)\frac{1}{x-y} = \Lambda_{33}(\mathbb{A})\,\Lambda_4(\mathbb{A}+x+y) \ .$$

Recursions (8.4.1) can be put into matrix form, as with Euler recursions (4.1.2) :

$$\text{(8.4.2)} \qquad \begin{bmatrix} \widetilde{Q}_{n-1} & \widetilde{Q}_n \\ \widetilde{P}_{n-1} & \widetilde{P}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & x-\zeta_0 \end{bmatrix} \begin{bmatrix} 0 & \beta_1 \\ 1 & x-\zeta_1 \end{bmatrix} \begin{bmatrix} 0 & \beta_2 \\ 1 & x-\zeta_2 \end{bmatrix} \cdots \begin{bmatrix} 0 & \beta_{n-1} \\ 1 & x-\zeta_{n-1} \end{bmatrix}$$

One can "shift" the linear functional $\int_{\mathbb{A}}$ by a finite alphabet $\mathbb{B}$, defining

$$\text{(8.4.3)} \qquad \forall f \in \mathfrak{Pol}(x) \ , \ \int_{\mathbb{A},\mathbb{B}} f := \int_{\mathbb{A}} f(x)\, R(x,\mathbb{B}) \ .$$

Christoffel obtained the associated orthogonal polynomials. A remarkable feature of his result, stated in the following proposition, is that it connects two determinants of different orders (n and k+1).

PROPOSITION 8.4.1. *Let $\mathbb{B} = \{b_1, \dots, b_k\}$. Then the orthogonal polynomials relative to $\int_{\mathbb{A},\mathbb{B}}$ are*

$$P_{n,k}(x) = S_{(n+k)^n}(\mathbb{A} - \mathbb{B} - x) \ ,$$

and $P_{n,k}(x)\, R(x,\mathbb{B})$ is proportional, up to a factor independent of x and $\mathbb{B}$, to the Christoffel determinant

$$\big| P_{n-1+j}(b_i) \big|_{1 \le i,j \le k+1} \ ,$$

with $b_{k+1} := x$.

Proof. The verification that $P_{n,k}(x)$ is orthogonal to $x^0, \dots, x^{n-1}$ is the same as in the case of $P_n(x)$ and $\int_{\mathbb{A}}$, apart from changing $\mathbb{A}$ into $\mathbb{A}-\mathbb{B}$, and shifting indices.

The determinant is divisible by the Vandermonde $\Delta(\mathbb{B}+x)$. Evaluating the image of the quotient multiplied by a function of x under $\int_{\mathbb{A},\mathbb{B}}$ is the same as computing the image of the last row, multiplied by the same function, under $\int_{\mathbb{A}}$. Therefore $P_{n,k}(x)$ is orthogonal (with respect to $\int_{\mathbb{A},\mathbb{B}}$) to $x^0, \dots, x^{n-1}$, while being of degree n in x. It must be proportional to $S_{(n+k)^n}(\mathbb{A}-\mathbb{B}-x)$. The explicit factor is explained by the Bazin formula and is equal to

$$\pm S_{(n-k+1)^{n-k+2}}(\mathbb{A}) \cdots S_{(n-2)^{n-1}}(\mathbb{A}) S_{(n-1)^n}(\mathbb{A}) \ .$$

□

In other words, $P_{n,k}(x)$ can be interpreted as the value of the discrete Wrońskian of $P_n, P_{n+1}, \ldots, P_{n+k}$ in the points $b_1, \ldots, b_k$, x (cf. Prop. 9.3.1).

The limit case of Christoffel's determinant, when $b_1 = \cdots = b_k = b$ is obtained by taking successive derivatives, e.g.

$$\begin{bmatrix} P_n(x) & P_{n+1}(x) & P_{n+2}(x) & P_{n+3}(x) \\ P_n(b) & P_{n+1}(b) & P_{n+2}(b) & P_{n+3}(b) \\ P'_n(b) & P'_{n+1}(b) & P'_{n+2}(b) & P'_{n+3}(b) \\ P''_n(b) & P''_{n+1}(b) & P''_{n+2}(b) & P''_{n+3}(b) \end{bmatrix}$$

is proportional to $S_{(n+3)^3}(\mathbb{A} - 3b - x)$.

The limit case where all b_i's tends towards x can be interpreted differently, as stated by the following proposition.

PROPOSITION 8.4.2. *Let* $\mathbb{A}$ *be generic,* x *be of rank* 1, $P_n(x) = S_{n^n}(\mathbb{A}-x)$, $q_n(x) = S_n(\mathbb{A} - nx)$, $n \geq 0$.

Given two positive integers k, n, *then the Wrońskian of* $P_n(x), \ldots, P_{n+k-1}(x)$ *is proportional to the determinant* $\left| q_{k+i+j} \right|_{0 \leq i,j \leq n-1}$, *the factor being*

$$\pm 1! \cdots (k-1)!\, S_{(n-k+2)^{n-k+3}}(\mathbb{A}) \cdots S_{(n-2)^{n-1}}(\mathbb{A}) S_{(n-1)^n}(\mathbb{A}) \ .$$

Proof. The limit case of Christoffel's determinant, divided by $\Delta(b_1, \ldots, b_{k-1}, x)$, when $b_1, \ldots, b_{k-1}$ tends towards x, is the Wrońskian, divided by $0!\,1! \cdots (k-1)!$. On the other hand, the Schur function $S_{(n+k-1)^n}(\mathbb{A} - kx)$ can be transformed, by subtraction of $0, x, 2x, \ldots, (k-1)x$ in successive rows, then columns, into the required determinant in the q_i's. □

For example, for $n = 3$, $k = 5$, the Wrońskian of $P_3(x), \ldots, P_7(x)$ is proportional to

$$\begin{aligned} &S_{777}(\mathbb{A}-5x) \\ &= \begin{vmatrix} S^7(\mathbb{A}-5x) & S^8(\mathbb{A}-5x) & S^9(\mathbb{A}-5x) \\ S^6(\mathbb{A}-5x) & S^7(\mathbb{A}-5x) & S^8(\mathbb{A}-5x) \\ S^5(\mathbb{A}-5x) & S^6(\mathbb{A}-5x) & S^7(\mathbb{A}-5x) \end{vmatrix} \begin{vmatrix} S^7(\mathbb{A}-7x) & S^8(\mathbb{A}-7x) & S^9(\mathbb{A}-7x) \\ S^6(\mathbb{A}-6x) & S^7(\mathbb{A}-6x) & S^8(\mathbb{A}-6x) \\ S^5(\mathbb{A}-5x) & S^6(\mathbb{A}-5x) & S^7(\mathbb{A}-5x) \end{vmatrix} \\ &= \begin{vmatrix} S^7(\mathbb{A}-7x) & S^8(\mathbb{A}-8x) & S^9(\mathbb{A}-9x) \\ S^6(\mathbb{A}-6x) & S^7(\mathbb{A}-7x) & S^8(\mathbb{A}-8x) \\ S^5(\mathbb{A}-5x) & S^6(\mathbb{A}-6x) & S^7(\mathbb{A}-7x) \end{vmatrix} = \begin{vmatrix} q_7 & q_8 & q_9 \\ q_6 & q_7 & q_8 \\ q_5 & q_6 & q_7 \end{vmatrix} . \end{aligned}$$

Leclerc [**121**] uses the preceding identity to give a simple proof of identities of Karlin and Szegö.

One can notice that Christoffel polynomials generalize the Darboux-Christoffel kernels, and indeed, they have reproducing properties that I explained with Shi He in the note [**117**] (see Ex.(8.11)).

Instead of taking determinants of orthogonal polynomials, let us go one step further, and consider determinants of Darboux-Christoffel kernels. For two alphabets $\mathbb{X} = \{x_1, \ldots, x_r\}$, $\mathbb{Y} = \{y_1, \ldots, y_r\}$ of the same cardinality r, writing $\Delta(\mathbb{X}) = \prod_{1 \leq i < j \leq r}(x_i - x_j)$, $\Delta(\mathbb{Y}) = \prod_{1 \leq i < j \leq r}(y_i - y_j)$, define

$$K_n^r(\mathbb{A};\, \mathbb{X}, \mathbb{Y}) := \frac{1}{\Delta(\mathbb{X})\Delta(\mathbb{Y})} \det\big(K_n(x_i, y_j)\big) \ . \tag{8.4.4}$$

Schur functions come again :

PROPOSITION 8.4.3. *For any pair of integers* $1 \le r \le n$, *one has*

$$K_n^r(\mathbb{A};\ \mathbb{X},\mathbb{Y}) = S_{(n+r-1)^{n-r}}(\mathbb{A}-\mathbb{X}-\mathbb{Y})\, S_{(n-1)^n}(\mathbb{A})^{-1}\ , \tag{8.4.5}$$

and $K_n^r(\mathbb{A};\ \mathbb{X},\mathbb{Y}) = 0$ *for* $r > n$.

Proof. The most remarkable feature of (8.4.5) is that its right side depends only upon $\mathbb{X}+\mathbb{Y}$, and not on $\mathbb{X}$ and $\mathbb{Y}$ separately.

Take the $(n+r)\times(n+r)$ determinant

$$\begin{vmatrix} x_r^{n-1} & \cdots & x_0^{n-1} & S^{n-1}(\mathbb{A}) & \cdots & S^{n+r-1}(\mathbb{A}) & \cdots & S^{2n-2}(\mathbb{A}) \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ x_r^r & \cdots & x_0^r & S^r(\mathbb{A}) & \cdots & S^{2r}(\mathbb{A}) & \cdots & S^{n+r-1}(\mathbb{A}) \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ x_r^0 & \cdots & x_0^0 & S^0(\mathbb{A}) & \cdots & S^r(\mathbb{A}) & \cdots & S^{n-1}(\mathbb{A}) \\ 0 & \cdots & 0 & y_1^0 & \cdots & y_1^r & \cdots & y_1^{n-1} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & y_r^0 & \cdots & y_r^r & \cdots & y_r^{n-1} \end{vmatrix}$$

Subtracting $\mathbb{X}$ in the first $n-r$ rows and $\mathbb{Y}$ in the last $n-r$ columns, one factorizes this determinant into $\Delta(\mathbb{X})\Delta(\mathbb{Y})S_{(n+r-1)^{n-r}}(\mathbb{A}-\mathbb{X}-\mathbb{Y})$. However, from Sylvester's identity with pivot $S_{(n-1)^n}(\mathbb{A})$, it is equal to

$$\det(S_{n^{n-1}}(\mathbb{A}-x_i-y_j))\,\big(S_{(n-1)^n}(\mathbb{A})\big)^{-r+1}\ .$$

When now $r > n$, since $K_n^r(\mathbb{A};\ \mathbb{X},\mathbb{Y})$ is of degree $\le n-1$ in each of the variables x_i, and since it is divisible by $\Delta(\mathbb{X})$, then it must be null. □

One can make the function $K_n^r(\mathbb{A};\ \mathbb{X},\mathbb{Y})$ less recognizable by breaking symmetry, and taking $\mathbb{X}=\{x,b_2,\dots,b_k\}$, $\mathbb{Y}=\{y,b_2,\dots,b_k\}$ [**54**].

For example, with $k=3$, $\square = n^{n-1}$, one obtains from (8.4.5) the following identity :

$$\begin{vmatrix} S_\square(\mathbb{A}-x-y) & S_\square(\mathbb{A}-x-b_2) & S_\square(\mathbb{A}-x-b_3) \\ S_\square(\mathbb{A}-b_2-y) & S_\square(\mathbb{A}-2b_2) & S_\square(\mathbb{A}-b_2-b_3) \\ S_\square(\mathbb{A}-b_3-y) & S_\square(\mathbb{A}-b_3-b_2) & S_\square(\mathbb{A}-2b_3) \end{vmatrix} =$$

$$\Delta(x,b_2,b_3)\Delta(y,b_2,b_3)\Big(S_{(n-1)^n}(\mathbb{A})\Big)^2 S_{(n+2)^{n-3}}(\mathbb{A}-x-y-2b_2-2b_3)\,.$$

8.5. Even Moments

The case when all moments of odd order are null is met with several classical families like Hermite, Legendre polynomials. We show in this section how to simplify the formulas previously given.

By hypothesis, $\mathbb{A}$ is such that $S^{2i+1}(\mathbb{A}) = 0$, $\forall i \ge 0$. Let us define $\mathbf{A}^\diamond$ by $S^i(\mathbf{A}^\diamond) := S^{2i}(\mathbb{A})$, $i = 1,2,\dots$. Equivalently, one can define $\mathbf{A}^\diamond$ by $\Psi^i(\mathbf{A}^\diamond) := \Psi^{2i}(\mathbb{A})$, $i=1,2,\dots$ (the odd powers sums in $\mathbb{A}$ are null).

For every n, the determinant $S_{n^n;0}(\mathbb{A};x)$ factorizes into two determinants, with only one containing x. For example,

$$S_{333;0}(\mathbb{A};x) = \begin{vmatrix} \cdot & S^4(\mathbb{A}) & \cdot & x^3 \\ S^2(\mathbb{A}) & \cdot & S^4(\mathbb{A}) & x^2 \\ \cdot & S^2(\mathbb{A}) & \cdot & x^1 \\ S^0(\mathbb{A}) & \cdot & S^2(\mathbb{A}) & x^0 \end{vmatrix} = \begin{vmatrix} S^4(\mathbb{A}) & x^3 \\ S^2(\mathbb{A}) & x^1 \end{vmatrix} \begin{vmatrix} S^2(\mathbb{A}) & S^4(\mathbb{A}) \\ S^0(\mathbb{A}) & S^2(\mathbb{A}) \end{vmatrix}\ .$$

One easily checks the following expression of the orthogonal polynomials in terms of the new alphabet $\mathbf{A}^\Diamond$.

PROPOSITION 8.5.1. *Let $\mathbb{A}$ be such that $S^i(\mathbb{A}) = 0$ for all odd i, and let $\mathbf{A}^\Diamond$ be such that $S^i(\mathbf{A}^\Diamond) := S^{2i}(\mathbb{A})$. Then*

$$1,\, x,\, S_1(\mathbf{A}^\Diamond - x^2),\, xS_2(\mathbf{A}^\Diamond - x^2),\, \ldots, S_{n^n}(\mathbf{A}^\Diamond - x^2),\, xS_{(n+1)^n}(\mathbf{A}^\Diamond - x^2), \ldots$$

are a family of orthogonal polynomials with respect to $\int_{\mathbb{A}}$, and

$$0,\, 1,\, x,\, S_{11}(\mathbf{A}^\Diamond + x^2),\, \ldots, xS_{(n-1)^{n+1}}(\mathbf{A}^\Diamond + x^2), S_{n^{n+1}}(\mathbf{A}^\Diamond + x^2), \ldots$$

are the adjoint polynomials.

In the recursion for the monic orthogonal polynomials

$$\widetilde{P}_{n+1} = x\widetilde{P}_n + \beta_n \widetilde{P}_{n-1} \;,$$

the parameter β_n can now be rewritten in terms of $\mathbf{A}^\Diamond$, starting from (8.3.2) :

$$\beta_1 = -S_1(\mathbf{A}^\Diamond),\ \beta_2 = \frac{S_{11}(\mathbf{A}^\Diamond)}{S_1(\mathbf{A}^\Diamond)},\ \beta_3 = \frac{S_{22}(\mathbf{A}^\Diamond)}{S_1(\mathbf{A}^\Diamond)S_{11}(\mathbf{A}^\Diamond)}, \ldots, \tag{8.5.1}$$
$$\beta_{2n} = \frac{S_{(n-1)^{n-1}}(\mathbf{A}^\Diamond)S_{n^{n+1}}(\mathbf{A}^\Diamond)}{S_{(n-1)^n}(\mathbf{A}^\Diamond)S_{n^n}(\mathbf{A}^\Diamond)},\ \beta_{2n+1} = -\frac{S_{(n-1)^n}(\mathbf{A}^\Diamond)S_{(n+1)^{n+1}}(\mathbf{A}^\Diamond)}{S_{n^n}(\mathbf{A}^\Diamond)S_{n^{n+1}}(\mathbf{A}^\Diamond)}$$

These expressions are compatible with the continuous fraction expansion (5.3.4)

$$\sum z^i S^i(\mathbb{A}) = \sum z^{2i} S^i(\mathbf{A}^\Diamond) = \cfrac{1}{1 - \cfrac{\gamma_1 z^2}{1 - \cfrac{\gamma_2 z^2}{1 - \cfrac{\gamma_3 z^2}{\ddots}}}} \;.$$

One still can shift $\mathbb{A}$ by a finite alphabet, but to preserve the vanishing of odd moments, one must take alphabets composed of pairs of letters $b, \bar{b}$ (such that $b + \bar{b} = 0$).

The simplest shift is thus

$$\int_{\mathbb{A}} \longrightarrow \int_{\mathbb{A};b,\bar{b}} : f \to \int_{\mathbb{A}} (x^2 - b^2) f(x) \;. \tag{8.5.2}$$

The orthogonal polynomials are expressed by the Christoffel determinants

$$\frac{1}{x^2 - b^2} \begin{vmatrix} P_n(x) & P_{n+1}(x) & P_{n+2}(x) \\ P_n(b) & P_{n+1}(b) & P_{n+2}(b) \\ P_n(-b) & P_{n+1}(-b) & P_{n+2}(-b) \end{vmatrix} \;.$$

However, the shift $\mathbb{A} \to \mathbb{A} - b - \bar{b}$ induces

$$\mathbf{A}^\Diamond \to \mathbf{A}^\Diamond - b^2 \;.$$

Therefore, the orthogonal polynomials relative to the functional $\int_{\mathbb{A};b,\bar{b}}$ can also be written

$$1,\, x,\, S_2(\mathbf{A}^\Diamond - b^2 - x),\, xS_3(\mathbf{A}^\Diamond - b^2 - x), S_{33}(\mathbf{A}^\Diamond - b^2 - x) \ldots, \tag{8.5.3}$$
$$S_{(n+1)^n}(\mathbf{A}^\Diamond - b^2 - x),\, xS_{(n+2)^n}(\mathbf{A}^\Diamond - b^2 - x), \ldots$$

The three-terms recursions

$$P^+_{n+1} = xP^+_n + \beta^+_n P^+_{n-1}$$

involve the parameters
(8.5.4)

$$\beta^+_{2n} = \frac{S_{n^{n-1}}(\mathbf{A}^\diamond - b^2)S_{(n+1)^{n+1}}(\mathbf{A}^\diamond - b^2)}{S_{n^n}(\mathbf{A}^\diamond - b^2)S_{(n+1)^n}(\mathbf{A}^\diamond - b^2)},\ \beta^+_{2n+1} = -\frac{S_{n^n}(\mathbf{A}^\diamond - b^2)S_{(n+2)^{n+1}}(\mathbf{A}^\diamond - b^2)}{S_{(n+1)^n}(\mathbf{A}^\diamond - b^2)S_{(n+1)^{n+1}}(\mathbf{A}^\diamond - b^2)}$$

in which one can recognize the values of the orthogonal polynomials in $x = b$. In summary, one has the following recursions, due to Barrucand [**12**] :

PROPOSITION 8.5.2. *Let $\mathbb{A}$ be such that $S^i(\mathbb{A}) = 0$ for odd i, and $\{P_n(x)\}$ be the monic orthogonal polynomials relative to $\int_{\mathbb{A}}$. Let b be a rank 1 element, or a complex number. Then the monic orthogonal polynomials P^+_n relative to the functional $f(x) \to \int_{\mathbb{A}}(x^2 - b^2)f(x)$ obey the recursion*

$$P^+_n(x) = xP^+_{n-1}(x) + \beta_{n-1}\frac{P_{n-2}(b)P_{n+1}(b)}{P_{n-1}(b)P_n(b)}P^+_{n-2}(x)\ . \tag{8.5.5}$$

Note that Barrucand's formula can be iterated, introducing each time a factor involving the values of the successive orthogonal polynomials in a new point b.

8.6. Zeros

We have not mentioned the zeros of orthogonal polynomials. We shall encroach upon this important topic only from the point of view of symmetric functions.

Write now

$$(-1)^n S_{n^n}(\mathbb{A}-x)/S_{(n-1)^n}(\mathbb{A}) = S_n(x - \mathbb{A}^n)\ ,$$

where $\mathbb{A}^n$ is the alphabet of zeros of P_n. We shall suppose all zeros distinct (i.e. zeros are simple and belong to a single alphabet $\mathbb{A}^n$).

The parameters ζ_n, β_n can be written in terms of successive zeros. Indeed, interpret equation (8.4.1) as a division :

$$\begin{aligned} S_n(x - \mathbb{A}^n) &= (x - \zeta_n)S_{n-1}(x - \mathbb{A}^{n-1}) + \beta_n S_{n-2}(x - \mathbb{A}^{n-2}) = \\ &\big(x - S_1(\mathbb{A}^{n-1} - \mathbb{A}^n)\big)S_{n-1}(x - \mathbb{A}^{n-1}) + S_2(\mathbb{A}^{n-1} - \mathbb{A}^n)S_{n-2}(x - \mathbb{A}^{n-2}) \end{aligned} \tag{8.6.1}$$

The most important properties of $\mathbb{A}^n$ is that it allows to evaluate the restriction of the functional $\int_{\mathbb{A}}$ to polynomials of degree not greater than $2n-1$.

We shall first give the specializations in $x = \alpha \in \mathbb{A}^n$ of some of the polynomials we have encountered so far.

LEMMA 8.6.1. *For every integer k, every alphabet $\mathbb{B}$, one has*

$$S_k(x + \mathbb{B} - \mathbb{A}^n) = S_{k;(n-1)^n}(x+\mathbb{B};\ \mathbb{A})\ /\ S_{(n-1)^n}(\mathbb{A})\ , \tag{8.6.2}$$

$$\int_{\mathbb{A}} S_k(x + \mathbb{B} - \mathbb{A}^n) = S_{k;(n-1)^n}(\mathbb{A}+\mathbb{B};\ \mathbb{A})\ /\ S_{(n-1)^n}(\mathbb{A})\ . \tag{8.6.3}$$

Proof. Developing the determinant expressing $S_{k;(n-1)^n}(x+\mathbb{B};\ \mathbb{A})$ along its first column, one recognizes as cofactors the coefficients of P_n. Moreover, the image of a linear combination (i.e. with coefficients independent of x) of complete functions $S_k(x + \mathbb{B})$ under $\int_A$ is obtained by replacing x by $\mathbb{A}$. □

LEMMA 8.6.2. *Let $\mathbb{A}^n$ be the alphabet of zeros of $S_{n^n}(\mathbb{A})$, and $\alpha \in \mathbb{A}^n$. Then one has*

$$S_{n-1}(\mathbb{A}+\alpha-\mathbb{A}^n) = \frac{S_{n-1;(n-1)^n}(\mathbb{A}+\alpha\,;\,\mathbb{A})}{S_{(n-1)^n}(\mathbb{A})} = \widetilde{Q}_n(\alpha)\ , \tag{8.6.4}$$

$$S_{2n-2}(\mathbb{A}+2\alpha-2\mathbb{A}^n) = S_{n-1}(\mathbb{A}+\alpha-\mathbb{A}^n)\,S_{n-1}(2\alpha-\mathbb{A}^n)\ , \tag{8.6.5}$$

$$S_{(n+1)^{n+1}}(\mathbb{A}-\alpha) = \Big(\frac{S_{n^{n+1}}(\mathbb{A})}{S_{(n-1)^n}(\mathbb{A})}\Big)^2 S_{(n-1)^{n-1}}(\mathbb{A}-\alpha) = \frac{(-1)^{n+1}S_{n^{n+1}}(\mathbb{A})^2}{S_{(n-1)^{n+1}}(\mathbb{A}+\alpha)} \tag{8.6.6}$$

$$\frac{S_{n^{n-1}}(\mathbb{A}-2\alpha)}{S_{n-1}(2\alpha-\mathbb{A}^n)} = (-1)^{n-1}S_{(n-1)^{n-1}}(\mathbb{A}-\alpha) \tag{8.6.7}$$

$$\frac{S_{(n+1)^n}(\mathbb{A}-2\alpha)}{S_{n-1}(2\alpha-\mathbb{A}^n)} = (-1)^n\frac{S_{(n-1)^n}(\mathbb{A})}{S_{n^{n+1}}(\mathbb{A})}\,S_{(n+1)^{n+1}}(\mathbb{A}-\alpha) \tag{8.6.8}$$

Proof. Taking $\mathbb{B}=\alpha$ in (8.6.3) gives (8.6.4). The polynomial $S_{2n-2}(x-2(\mathbb{A}^n-a))$ is congruent to $(-1)^{n-1}S_{1^{n-1};\,n-1}(\mathbb{A}^n-x;\ \mathbb{A}^n-2(\mathbb{A}^n-\alpha))$ modulo $S_n(x-\mathbb{A}^n)$. Subtracting α in the $n-1$ first rows, one gets

$$\begin{aligned} S_{2n-2}(x+2\alpha-2\mathbb{A}^n) &\equiv (-1)^{n-1}S_{1^{n-1}}(\mathbb{A}^n-x-\alpha)\,S_{n-1}(2\alpha-\mathbb{A}^n) \\ &= S_{n-1}(x+\alpha-\mathbb{A}^n)S_{n-1}(2\alpha-\mathbb{A}^n)\ , \end{aligned}$$

which gives (8.6.5) by applying $\int_{\mathbb{A}}$.

Evaluating the determinant in (8.4.2) for n or $n-1$ gives (8.6.6).

To get (8.6.7) or (8.6.8) , one evaluates $K_n(\alpha,\alpha)=K_{n+1}(\alpha,\alpha)$, noticing that

$$S_{n-1}(2\alpha-\mathbb{A}^n) = R(\alpha,\mathbb{A}^n-\alpha) = \widetilde{P}'_n(\alpha)\ .$$

□

We are now ready to state the *Gauss quadrature formula* (given by Gauss for Legendre polynomials).

THEOREM 8.6.3. *Let $\mathbb{A}$ be a generic alphabet. For any positive integer n, let $\mathbb{A}^n$ be the alphabet of zeros of $\widetilde{P}_n(x) := (-1)^n S_{n^n}(\mathbb{A}-x)/S_{(n-1)^n}(\mathbb{A})$.*

Then there exists constants λ_α, $\alpha\in\mathbb{A}_n$, such that for every polynomial $f(x)$ of degree $\le n-1$, one has

$$\int_{\mathbb{A}} f(x) = \sum_{\alpha\in\mathbb{A}^n} \lambda_\alpha\, f(\alpha)\ . \tag{8.6.9}$$

The coefficients can be expressed as

$$\begin{aligned} \lambda_\alpha &= S_{n-1}(\mathbb{A}+\alpha-\mathbb{A}^n)/S_{n-1}(2\alpha-\mathbb{A}^n) && (8.6.10)\\ &= S_{2n-2}(2\alpha+\mathbb{A}-2\mathbb{A}^n)/S_{2n-2}(3\alpha-2\mathbb{A}^n) && (8.6.11)\\ &= \frac{S_{(n-1)^{n+1}}(\alpha+\mathbb{A})}{S_{n-1}(2\alpha-\mathbb{A}^n)} = \frac{\widetilde{Q}_n(\alpha)}{\widetilde{P}'_n(\alpha)} && (8.6.12)\\ &= \frac{S_{(n-1)^n}(\mathbb{A})}{S_{n^{n-1}}(\mathbb{A}-2\alpha)} = \frac{S_{n^{n+1}}(\mathbb{A})}{S_{(n+1)^n}(\mathbb{A}-2\alpha)} && (8.6.13) \end{aligned}$$

Proof. Let us test the first statement on the $2n$ polynomials

$$\{x^kP_n(x): k=0,\dots,n-1\}\cup\{x^kP_{n-1}(x): k=0,\dots,n-2\}\cup\{\widetilde{P}^2_{n-1}(x)\}\ .$$

The functional $\int_{\mathbb{A}}$ vanishes on $\{x^k P_n(x)\}$ and $\{x^k P_{n-1}(x)\}$ because $P_n(x)$ and $P_{n-1}(x)$ are orthogonal polynomials. On the other hand, summation (8.6.9) is identically null in the first case; it is also null in the second case because $f(x)/P_{n-1}(x)$ is a polynomial of degree $\leq 2n-1$. There remains only one case to check: $f = \widetilde{P}^2_{n-1}$, which fixes the normalization.

Since $\widetilde{Q}_n(x)$ is of degree $n-1$, Lagrange tells us that

$$\frac{\widetilde{Q}_n(x)}{\widetilde{P}_n(x)} = \sum_{\alpha\in\mathbb{A}^n} \frac{\widetilde{Q}_n(\alpha)}{(x-\alpha)R(\alpha, \mathbb{A}^n - \alpha)} \ .$$

Therefore (8.6.12) is true, as well as the values of λ_α, which are equivalent thanks to Lemma (8.6.2). □

Notice the similarity between the Lagrange interpolation formula

$$f(x) = \sum_{\alpha\in\mathbb{A}^n} \frac{f(\alpha)\, S_{n-1}(x+\alpha-\mathbb{A})}{R(\alpha, \mathbb{A}^n - \alpha)} \tag{8.6.14}$$

and the Gauss quadrature

$$\int_{\mathbb{A}} f(x) = \sum_{\alpha\in\mathbb{A}^n} \int_{\mathbb{A}} \frac{f(\alpha)\, S_{n-1}(x+\alpha-\mathbb{A})}{R(\alpha, \mathbb{A}^n - \alpha)} \ . \tag{8.6.15}$$

The main difference is that the degree of $f(x)$ is $\leq 2n-1$ in the second case, contrary to the first case where it is $\leq n-1$. Choosing the interpolation set $\mathbb{A}^n$ instead of taking an arbitrary one has increased the domain of validity of the interpolation.

The fact that $\widetilde{Q}_n(x)/\widetilde{P}_n(x)$ is a Padé approximant of $x^{-1}\sigma_{1/x}(\mathbb{A})$ implies relations between $\mathbb{A}$ and $\mathbb{A}^n$. Define $\mathbb{B}$ to be the alphabet of zeros of $\widetilde{Q}_n(x)$. Then

$$S_{n-1}(x-\mathbb{B})/S_n(x-\mathbb{A}^n) \equiv x^{-1} + \cdots + x^{-2n} S_{2n-1}(\mathbb{A}) \mod (x^{-2n})$$

can be written

$$x^{-1}\sigma_{1/x}(-\mathbb{B}) \equiv x^{-1}\sigma_{1/x}(\mathbb{A}-\mathbb{A}^n) \mod (x^{-2n}) \ .$$

Therefore

$$\begin{aligned} \widetilde{Q}_n(x) = S_{n-1}(x-\mathbb{B}) &= S_{n-1}(x+\mathbb{A}-\mathbb{A}^n) \\ &\& \quad S_n(\mathbb{A}-\mathbb{A}^n) = 0 = \cdots = S_{2n-1}(\mathbb{A}-\mathbb{A}^n) \end{aligned} \tag{8.6.16}$$

Let us illustrate Gauss' theorem in the case of polynomials of degree no more than 5. It states that the integral $\int_0^1 (f(x)$, $d^o(f) \leq 5$, can be realized as a summation

$$\lambda_1 f(\alpha_1) + \lambda_2 f(\alpha_2) + \lambda_3 f(\alpha_3) \ ,$$

where $\alpha_1, \alpha_2, \alpha_3$ are the roots of $S_{333}(\mathbb{A}-x)$. However, the moments are $S^n(\mathbb{A}) = \int_0^1 x^n = 1/(n+1)$, and therefore

$$S_{333}(\mathbb{A}-x) = S_{333;0}(\mathbb{A};x) = \begin{vmatrix} 1/4 & 1/5 & 1/6 & x^3 \\ 1/3 & 1/4 & 1/5 & x^2 \\ 1/2 & 1/3 & 1/4 & x^1 \\ 1/1 & 1/2 & 1/3 & x^0 \end{vmatrix} = (-1+12x-30x^2+20x^3)/43200$$

has the three roots

$$\frac{1}{2} \ , \ \frac{1}{2} + \frac{1}{2}\sqrt{\frac{3}{5}} \ , \ \frac{1}{2} - \frac{1}{2}\sqrt{\frac{3}{5}} \ .$$

In conclusion, the integral will be approximated by

$$\frac{4}{9} f\left(\frac{1}{2}\right) + \frac{5}{18} f\left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3}{5}}\right) + \frac{5}{18} f\left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{3}{5}}\right) .$$

8.7. The Moment Generating Function

Given a positive function $\phi(x)$ on a real interval $[a, b]$, Tchebychef and Stieltjes have studied the continued fraction expansion of the integral

$$\int_a^b \frac{\phi(y)}{x-y} dy \tag{8.7.1}$$

around $x = \infty$.

Expanding it formally,

$$\int_a^b \frac{\phi(y)}{x-y} dy = \sum_{n=0}^{\infty} x^{-n-1} \int_a^b y^n \phi(y) dy \ , \tag{8.7.2}$$

one recognizes that it is the generating function of moments $S_n(\mathbb{A})$, for the linear functional

$$\int_{\mathbb{A}} \ : \ f \to \int_a^b f(y)\phi(y)dy \ .$$

Therefore, the continued fractions that Tchebychef and Stieltjes have studied are those that we have considered in Theorem 8.3.1 for the expansion of $x^{-1}\sigma_{1/x}(\mathbb{A})$.

The hypothesis that ϕ be positive has many interesting consequences, about the zeroes of the associated orthogonal polynomials in particular. We refer for these properties to the classical literature [**167**] [**5**], [**23**]. Many of these properties are easy to obtain using the algebraic identities that we have already written.

For example, the coefficients appearing in Gauss quadrature must be positive. Indeed, take the normalized orthogonal polynomial $\widetilde{P}_n(x) = S_n(x - \mathbb{A}^n)$, and let $\alpha \in \mathbb{A}^n$ be any root of it. Then the weight λ_α is equal, according to formula (8.6.11), to

(8.7.3)

$$\lambda_\alpha = \frac{S_{2n-2}(\mathbb{A}-2(\mathbb{A}^n-\alpha))}{S_{2n-2}(\alpha-2(\mathbb{A}^n-\alpha))} = \frac{1}{\big(S_{n-1}(\alpha-(\mathbb{A}^n-\alpha))\big)^2} \int_a^b \big(S_{n-1}(y-(\mathbb{A}^n-\alpha))\big)^2 \phi(y) d_y$$

which is positive.

To take into account the limits of integration, one has to consider some slight deformations of the linear functional, like

$$\int_a^b f(y)\,(y-a)\phi(y)dy \ , \ \int_a^b f(y)\,(b-y)\phi(y)dy \ , \tag{8.7.4}$$
$$\int_a^b f(y)\,(y-a)(b-y)\phi(y)dy \ .$$

Considering the limits a, b to be rank-1 elements, the associated moments are

$$S^{n+1}(\mathbb{A}-a) \quad , \quad -S^{n+1}(\mathbb{A}-b) \quad , \quad -S^{n+2}(\mathbb{A}-a-b) \tag{8.7.5}$$

and the unnormalized orthogonal polynomials are

$$S_{(n+1)^n}(\mathbb{A}-a-x) \ , \ S_{(n+1)^n}(\mathbb{A}-b-x) \ , \ S_{(n+2)^n}(\mathbb{A}-a-b-x) \ , \tag{8.7.6}$$

since it is a special case of Christoffel's shift (Proposition 8.4.1).

These polynomials still satisfy some positivity properties. Let us state a result due to Possé [**148**] :

PROPOSITION 8.7.1. *For any positive integer n, the integrals*

$$\int_a^b \frac{S_{(n+1)^n}(A{-}a-y)}{S_{(n+1)^n}(A-2a)}\phi(y)dy \ , \quad \int_a^b \frac{S_{(n+1)^n}(A{-}b-y)}{S_{(n+1)^n}(A-2b)}\phi(y)dy$$
$$\int_a^b \frac{S_{(n+2)^n}(A{-}a{-}b-y)}{S_{(n+2)^n}(A-2a-b)}(b-y)\phi(y)dy \ , \quad \int_a^b \frac{S_{(n+2)^n}(A{-}a{-}b-y)}{S_{(n+2)^n}(A-a-2b)}(y-a)\phi(y)dy$$

are positive.

Proof. $\big(S_{(n+1)^n}(A{-}a-y) - S_{(n+1)^n}(A{-}a-a)\big)/(y-a) = R(y)$ is a polynomial in y of degree $\le n-1$. Therefore,

$$\begin{aligned} \frac{S_{(n+1)^n}(A{-}a-y)}{S_{(n+1)^n}(A{-}2a)} &= 1+(y-a)R(y) \\ \left(\frac{S_{(n+1)^n}(A{-}a-y)}{S_{(n+1)^n}(A{-}2a)}\right)^2 &= \frac{S_{(n+1)^n}(A{-}a-y)}{S_{(n+1)^n}(A{-}2a)} + (y{-}a)R(y)\frac{S_{(n+1)^n}(A{-}a-y)}{S_{(n+1)^n}(A{-}2a)} \end{aligned}$$

Now, the integral $\int_a^b R(y)S_{(n+1)^n}(A{-}a-y)(y{-}a)\phi(y)dy$ is zero because $S_{(n+1)^n}(A{-}a-y)$ is an orthogonal polynomial. Therefore,

$$\int_a^b \frac{S_{(n+1)^n}(A{-}a-y)}{S_{(n+1)^n}(A{-}2a)}\phi(y)dy = \int_a^b \left(\frac{S_{(n+1)^n}(A{-}a-y)}{S_{(n+1)^n}(A{-}2a)}\right)^2 \phi(y)dy$$

is positive, and the other statements of the proposition could be similarly proven. □

Let us notice that the explicit value of the integral $\int_a^b S_{(n+1)^n}(A{-}a-y)\phi(y)dy = \int_a^b S_{(n+1)^n;0}(A{-}a;\, y)\phi(y)dy$ is

$$S_{(n+1)^n;\,0}(A{-}a;\, \mathbb{A}) = S_{n;n^n}(\mathbb{A};\, \mathbb{A}{-}a) = S_{n^{n+1}}(\mathbb{A}) \ .$$

8.8. Jacobi's Matrix and Paths

Instead of using 2×2 matrices to handle recursions defining orthogonal polynomials, one can take tridiagonal matrices

(8.8.1)
$$M_n := \begin{bmatrix} \zeta_0 & 1 & 0 & \cdots & 0 \\ -\beta_1 & \zeta_1 & 1 & \ddots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \zeta_{n-2} & 1 \\ & & & -\beta_{n-1} & \zeta_{n-1} \end{bmatrix}, \ N_n := \begin{bmatrix} \zeta_1 & 1 & 0 & \cdots & 0 \\ -\beta_2 & \zeta_2 & 1 & \ddots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \zeta_{n-2} & 1 \\ & & & -\beta_{n-1} & \zeta_{n-1} \end{bmatrix}$$

We have already met the Jacobi matrix $\mathfrak{Jac} = M_\infty$, which is the matrix of "multiplication by x" in the basis of monic orthogonal polynomials $\{\widetilde{P}_n\}$, $n = 0, 1, 2, \ldots$ (reading the matrix by rows : row $i{+}1$ gives the expansion of $x\,\widetilde{P}_i(x)$). Similarly, N_∞ expresses "multiplication by x" in the basis of adjoint polynomials $\{\widetilde{Q}_n\}$, $n = 1, 2, \ldots$.

Let $f(\mathfrak{Jac})$ be the evaluation of a polynomial $f(x)$ in $x = \mathfrak{Jac}$. Then it is the matrix of "multiplication by $f(x)$", and it can be interpreted as a matrix of scalar products:

$$f(\mathfrak{Jac}) = \begin{bmatrix} \frac{(f(x),\widetilde{P}_0)}{(\widetilde{P}_0,\widetilde{P}_0)} & \frac{(f(x),\widetilde{P}_1)}{(\widetilde{P}_1,\widetilde{P}_1)} & \frac{(f(x),\widetilde{P}_2)}{(\widetilde{P}_2,\widetilde{P}_2)} & \cdots \\ \frac{(f(x)\widetilde{P}_1,\widetilde{P}_0)}{(\widetilde{P}_0,\widetilde{P}_0)} & \frac{(f(x)\widetilde{P}_1,\widetilde{P}_1)}{(\widetilde{P}_1,\widetilde{P}_1)} & \frac{(f(x)\widetilde{P}_1,\widetilde{P}_2)}{(\widetilde{P}_2,\widetilde{P}_2)} & \cdots \\ \frac{(f(x)\widetilde{P}_2,\widetilde{P}_0)}{(\widetilde{P}_0,\widetilde{P}_0)} & \frac{(f(x)\widetilde{P}_2,\widetilde{P}_1)}{(\widetilde{P}_1,\widetilde{P}_1)} & \frac{(f(x)\widetilde{P}_2,\widetilde{P}_2)}{(\widetilde{P}_2,\widetilde{P}_2)} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{8.8.2}$$

Computing powers of $\mathfrak{Jac}$ or M_n can be described in terms of Motzkin paths, as we have seen in Theorem 4.6.2. Let us restate it in terms of orthogonal polynomials.

THEOREM 8.8.1. *Let $\mathbb{A}$ be a generic alphabet, $\widetilde{P}_n(x)$ and $\widetilde{Q}_n(x)$ be the monic orthogonal polynomials, and adjoint polynomials, associated to $\mathbb{A}$. Then*

(1) *The characteristic polynomial* $\det(x - M_n)$ *is equal to* $\widetilde{P}_n(x)$*; the characteristic polynomial* $\det(x - N_n)$ *is equal to* $\widetilde{Q}_n(x)$.
(2) *For any $k \le n \in \mathbb{N}$, the entry $[0,0]$ of M_n^k is equal to the coefficient of x^{-k-1} of $\widetilde{Q}_n(x)/\widetilde{P}_n(x)$ (expanded at $x = \infty$), i.e. to $S_k(\mathbb{A})$.*
(3) *For any $i, j, k \in \mathbb{N}$, the entry $[i,j]$ of $\mathfrak{Jac}^k$ is the weight of the sum of all Motzkin paths of length k, starting in $[i,0]$ and ending in $[k,j]$. It is also equal to the normalized scalar product*

$$(x^k\, \widetilde{P}_i\,,\, \widetilde{P}_j)/(\widetilde{P}_j\,,\, \widetilde{P}_j)\ .$$

Proof. Expanding $\det(x - M_n)$ along its last column provides the first point. Restricting to powers $k \le n$ imply that the entries of M_n^k are equal to the corresponding entries of $\mathfrak{Jac}^n$; the first entry is described in Theorem 4.6.2, but one can more generally start from any level and arrive at any level, instead of taking only level 0.

The interpretation in terms of scalar products comes from (8.8.2). In particular, one recovers that the entry $[0,0]$ of $\mathfrak{Jac}^k$ is $(x^k, 1) = S_n(\mathbb{A})$. □

For example,

$$\mathfrak{Jac}^3 = \begin{bmatrix} S_3 & \frac{S_{13}}{S_{11}} & \frac{S_{223}}{S_{222}} & \frac{S_{3333}}{S_{3333}} & \cdots \\ S_{13} & \frac{S_{14}-S_1S_{13}}{S_{11}} & \frac{S_{224}-S_1S_{223}}{S_{222}} & \frac{S_{3334}-S_1S_{3333}}{S_{3333}} & \cdots \\ \frac{S_{223}}{S_{11}} & \frac{S_1S_{223}-S_{224}}{S_{11}S_{11}} & \vdots & \vdots & \\ \vdots & \vdots & \vdots & & \end{bmatrix}$$

For what concerns the first entry $S_k(\mathbb{A})$ of $\mathfrak{Jac}^k$, it can be obtained by taking the truncated matrix M_n instead of $\mathfrak{Jac}$, as long as $k \le 2n-1$. Thus, to get $S_3(\mathbb{A})$, M_2 will suffice.

Recall that the description of $\widetilde{Q}_n(x)/\widetilde{P}_n(x)$ from the powers of M_n is due to Whittaker [**179**], and that the interpretation of continued fractions in terms of Motzkin paths is due to Flajolet [**42**]. Viennot [**175**] made an extensive use of Motzkin and Favard paths, in particular to describe classical families of orthogonal polynomials, and to give bijective proofs of classical identities in the theories of orthogonal polynomials, continued fractions, Padé approximants.

Viennot [**176**] states that the entry $[i,j]$ of the inverse matrix of the coefficients of the $\widetilde{P}_n$ is the weight of the sum of all Motzkin paths of length i, from 0 to j. But this is the coefficient of $\widetilde{P}_j/(\widetilde{P}_j\,,\,\widetilde{P}_j)$ in the expansion of x^i, and therefore the matrix considered by Viennot is made of the first rows of the successive powers of $\mathfrak{Jac}$.

8.9. Discrete Measures

Let us leave paths and go back to algebra. In all this chapter, we have used a generic alphabet $\mathbb{A}$ of infinite cardinality. With some care, on can use the algebraic identities obtained so far in more general situations (e.g. requiring only the non-vanishing of some Hankel determinants).

Following Tchebychef, let us treat the case of a finite and uniform discrete measure. Let $\mathbb{B} = \{b_1, \ldots, b_n\}$, the b_i's being indeterminates, let $\mathbb{B}^{der}$ be the alphabet of zeroes of the derivative of $S_n(x-\mathbb{B})$, and let $\oint_{\mathbb{B}}$ be the functional

$$\oint_{\mathbb{B}} x^i = \Psi^i(\mathbb{B}) = n\, S_i(\mathbb{B}-\mathbb{B}^{der}) \ , \ i \geq 0 \ . \tag{8.9.1}$$

The generating function of moments $x^{-1}\sigma_{1/x}(\mathbb{A})$ is replaced by

$$\sum_{b\in\mathbb{B}} \frac{1}{x-b} = \sum_{i\geq 0} x^{-1+i}\Psi^i(\mathbb{B}) \ . \tag{8.9.2}$$

All formulas given in this chapter remain valid when replacing $S_i(\mathbb{A})$ by $\Psi^i(\mathbb{B})/n$ *if they involve no* $S_i(\mathbb{A})$, *with* $i \geq 2n$ because

$$\widetilde{Q}_n(x)/\widetilde{P}_n(x) \equiv x^{-1} + \cdots + x^{-2n}S_{2n-1}(\mathbb{A}) \quad \mod x^{-2n-1} \ .$$

Let us define $\mathbb{A} = \mathbb{B} - \mathbb{B}^{der}$. The n-th convergent determines the moments $nS_0(\mathbb{A}), \ldots, nS_{2n-1}(\mathbb{A})$, as well as the convergents of order $< n$. The denominators of these are still an orthogonal basis with respect to $\oint_{\mathbb{B}}$, the last convergent being

$$\frac{n\widetilde{Q}_n(x)}{\widetilde{P}_n(x)} = \sum_{b\in\mathbb{B}} \frac{1}{x-b} = \sum_{i\geq 0} x^{-1+i}\Psi^i(\mathbb{B}) = \frac{S_{n-1}(2x-\mathbb{B})}{S_n(x-\mathbb{B})} \tag{8.9.3}$$

Up to normalization, writing Ψ^i instead of $\Psi^i(\mathbb{B})$, the orthogonal polynomials are $\oint_{\mathbb{B}} : \oint_{\mathbb{B}} x^i = \Psi^i(\mathbb{B})$, $i \geq 0$,

$$P_0(x) = \Psi^0, \begin{vmatrix} \Psi^1 & x \\ \Psi^0 & 1 \end{vmatrix}, \begin{vmatrix} \Psi^2 & \Psi^3 & x^2 \\ \Psi^1 & \Psi^2 & x^1 \\ \Psi^0 & \Psi^1 & x^0 \end{vmatrix}, \cdots, \ P_n(x) = \begin{vmatrix} \Psi^n & \cdots & \Psi^{2n-1} & x^n \\ \vdots & & \vdots & \vdots \\ \Psi^0 & \cdots & \Psi^{n-1} & x^0 \end{vmatrix}$$

(already encountered as a Sturm sequence (3.6.5)).

Every polynomial of degree $\leq n-1$ can be expanded in this basis, as shown by Tchebychef [**168**], who interpreted this expansion as an interpolation formula :

$$f(x) = \sum_{i=0}^{n-1} \frac{\oint_{\mathbb{B}} f(x)P_i(x)}{\oint_{\mathbb{B}} P_i(x)^2} \ . \tag{8.9.4}$$

Beware that $\oint_{\mathbb{B}} P_n(x)^2 = S_{(n-1)^n}(\mathbb{A})S_{n^{n+1}}(\mathbb{A})$ is null because $S_{n^{n+1}}(\mathbb{A}) = S_{n^{n+1}}(\mathbb{B}-\mathbb{B}^{der}) = 0$. In fact, the linear functional $f(x) \to \oint_{\mathbb{B}} f$ can be expressed in terms of the $f(b)$, $b \in \mathbb{B}$, and therefore it coincides on two polynomials congruent modulo $S_n(x-\mathbb{B})$.

Tchebychef found the Darboux-Christoffel kernel respective to $\oint$ (and because one can put weights ν_b, $b \in \mathbb{B}$, and let n tend towards ∞, the general theory should be attributed to Tchebychef[1]), and used it to produce a second expansion. Let

(8.9.5)
$$K(x,y) = \frac{(-1)^n}{(S_{(n-1)^n}(\mathbb{A}))^2} \frac{S_{n^n}(\mathbb{A}-x)S_{(n-1)^{n-1}}(\mathbb{A}-y) - S_{n^n}(\mathbb{A}-y)S_{(n-1)^{n-1}}(\mathbb{A}-x)}{x-y} .$$

Then

PROPOSITION 8.9.1. *(Tchebychef[**168**], I p. 416–430). For any polynomial f of degree $\le n-1$, one has*

$$f(y) = \oint_{\mathbb{B}} f(x)K(x,y) = \sum_{b\in\mathbb{B}} f(b)K(b,y) . \tag{8.9.6}$$

Proof. Tchebychef's formula is the discrete version of Theorem 8.2.1 and the same algebraic proof could be given. However a direct proof is even simpler, because (8.9.6) is just another way of writing the Lagrange interpolation formula. Indeed,

$$K(b,y) = \frac{S_{n^{n-1}}(\mathbb{A}-y-b)}{S_{(n-1)^n}(\mathbb{A})} = \frac{S_{n^{n-1}}((\mathbb{B}-b)-y)}{S_{(n-1)^n}(\mathbb{B}-\mathbb{B}^{der})} = \frac{R(y,\mathbb{B}-b)}{R(b,\mathbb{B}^{der})}$$

and therefore (8.9.6) is the interpolation of $f(y)$ at points $\mathbb{B}$:

$$f(y) = \sum_{b\in\mathbb{B}} f(b)\frac{R(y,\mathbb{B}-b)}{R(b,\mathbb{B}^{der})} = \sum_{b\in\mathbb{B}} f(b)\frac{S_{n-1}(y+b-\mathbb{B})}{S_{n-1}(2b-\mathbb{B})} .$$

□

One shall find in Ex.8.9 the case of a non-uniform measure, which is useful in probability theory. Let us now detail the case where $\mathbb{B}$ is symmetrical with respect to 0. Let $B = \{b\}$ be of cardinality m, let $n = 2m$, and let $\mathbb{B} = B + \bar{B}$, with $\bar{B} = \{\bar{b}\}$ defined by $\Psi^i(\bar{B}) = (-1)^i\Psi^i(B)$, $i = 0, 1, \ldots$ In other words, $\Psi^{2i+1}(\mathbb{B}) = 0$, $\Psi^{2i}(\mathbb{B}) = 2\Psi^i(B)$ (to cover the case n odd, one adds the element 0 to $\mathbb{B}$).

The determinants expressing the orthogonal polynomials factorize :

$$\begin{vmatrix} \Psi^2 & 0 & x^2 \\ 0 & \Psi^2 & x^1 \\ \Psi^0 & 0 & x^0 \end{vmatrix} = \Psi^2 \begin{vmatrix} \Psi^2 & x^2 \\ \Psi^0 & x^0 \end{vmatrix} , \quad \begin{vmatrix} 0 & \Psi^4 & 0 & x^3 \\ \Psi^2 & 0 & \Psi^4 & x^2 \\ 0 & \Psi^2 & 0 & x^1 \\ \Psi^0 & 0 & \Psi^2 & x^0 \end{vmatrix} = \begin{vmatrix} \Psi^2 & \Psi^4 \\ \Psi^0 & \Psi^2 \end{vmatrix} \begin{vmatrix} \Psi^4 & x^3 \\ \Psi^2 & x^1 \end{vmatrix}$$

Up to normalization, one finds for the general case

$$P_{2k} = \begin{vmatrix} \Psi^{2k} & \cdots & \Psi^{4k-2} & x^{2k} \\ \vdots & & \vdots & \vdots \\ \Psi^2 & \cdots & \Psi^{2k} & x^2 \\ \Psi^0 & \cdots & \Psi^{2k-2} & x^0 \end{vmatrix} , \quad P_{2k+1} = \begin{vmatrix} \Psi^{2k+2} & \cdots & \Psi^{4k} & x^{2k+1} \\ \vdots & & \vdots & \vdots \\ \Psi^4 & \cdots & \Psi^{2k+2} & x^3 \\ \Psi^2 & \cdots & \Psi^{2k} & x^1 \end{vmatrix} \tag{8.9.7}$$

In the case where $\mathbb{B}$ is specialized to a sequence of consecutive integers, Tchebychef wrote the explicit convergents :

PROPOSITION 8.9.2. *(Tchebychef[**168**], I p. 382). Let n be a positive integer,*

$$\phi(x) = \frac{1}{x-1} + \frac{1}{x-2} + \cdots + \frac{1}{x-n} .$$

[1]Thus, Darboux-Christoffel formula should be called *Darboux-Christoffel-Tchebychef* formula, in the reverse historical order.

Then $\phi(x)$ admit the following continued fraction expansion, with $y := 2x-n-1$

(8.9.8)
$$\cfrac{n}{y-\cfrac{1^2(n^2-1)}{3y-\cfrac{2^2(n^2-2^2)}{5y-\cfrac{3^2(n^2-3^2)}{7y-\cfrac{4^2(n^2-4^2)}{9y-\cdots}}}}}$$

The denominators of the successive convergents satisfy the recursion

$$P_k(y) = (2k-1)y\,P_{k-1}(y) - (k-1)^2\big(n^2-(k-1)^2\big)P_{k-2}(y)\ ,\ \ k \le n\ .$$

Proof. Taking y instead of x introduces a symmetry which shows that Tchebychef's case corresponds to $B = \{1/2, 3/2, \ldots\}$ or $B = \{0, 1, 2, \ldots\}$, according to the parity of n. The values of the Schur functions involved in Tchebychef's formulas are straightforward to determine. □

Exercises

Ex. 8.1. Let $\mathbb{A}$ be an alphabet and let $\int$ be the linear functional such that $\int t^k = S_k(\mathbb{A})$, $k \ge 0$. Let $P_n(x) = (-1)^n S_{n^n}(\mathbb{A}-x)$. Show that $\int \partial_{t,x}(P_n(t))$ is equal to the adjoint polynomial $Q_n(x) = S_{(n-1)^{n+1}}(\mathbb{A}+x)$.

Ex. 8.2. Let $\mathbb{A}$ be generic and let $Q_1(x), Q_2(x), \ldots$, be the adjoint polynomials for the linear functional $\int$ such that $\int(-x)^n = \Lambda^n(\mathbb{A})$. Find the functional with respect to which the Q_i's are the orthogonal polynomials.

Ex. 8.3. Let $\mathbb{A}$ be generic, x be of rank 1. Decompose any polynomial in x in the basis of adjoint polynomials $Q_n(x) := S_{(n-1)^{n+1}}(\mathbb{A}+x)$.

Ex. 8.4. Let k a positive integer, $\mathbb{A}$ be generic. Define the *associated polynomials* of order k to be

$$P_n^{(k)} = S_{(k-2)^k}(\mathbb{A}+x)\,S_{(n+k)^{(n+k)}}(\mathbb{A}-x)+(-1)^n S_{(k-1)^{(k-1)}}(\mathbb{A}-x)\,S_{(n+k-1)^{n+k+1}}(\mathbb{A}+x)\ .$$

Show that $\{P_0^{(k)}, P_1^{(k)}, P_2^{(k)}, \ldots\}$ is a family of orthogonal polynomials, and that the corresponding monic polynomials satisfy the recursions

$$\widetilde{P}^{(k)}_{n+1} = (x-\zeta_{n+k})\widetilde{P}^{(k)}_n + \beta_{n+k}\widetilde{P}^{(k)}_{n-1}\ ,$$

where ζ_n, β_n are the parameters relative to the orthogonal polynomials $\widetilde{P}^{(0)}_n = (-1)^n S_{n^n}(\mathbb{A}-x)/S_{(n-1)^n}(\mathbb{A})$ (cf. (8.4.1)).

Ex. 8.5. Find the basis $T_n(x)$ adjoint to $\{x^n\}$, with respect to the quadratic form (8.1.2). Write $T_0, \ldots, T_n$, modulo the span of $S_{(n+k)^{n+k}}(\mathbb{A}-x)$, $k > 0$, as Schur functions.

Ex. 8.6. Let $\widetilde{P}_n(x) = x^n + \cdots$ be the n-th orthogonal polynomial relative to $\mathbb{A}$. Find the coefficients of the expansion

$$\widetilde{P}_n(x) = x^n + \sum_{k<n} c_k \widetilde{P}_k(x)\ .$$

EX. 8.7. Given the orthogonal polynomials $\widetilde{P}_n$ and $\widetilde{Q}_n$, $n = 0, 1, \ldots$, associated to $\mathbb{A}$, find the convergents of

$$\lim_{n\to\infty} x\frac{d}{dx}\big(\widetilde{Q}_n(x)/\widetilde{P}_n(x)\big) \ .$$

EX. 8.8. Let n be an integer, $\mathbb{A}$ be generic. Evaluate

$$1 - \frac{S_1(\mathbb{A})^2}{S_{11}(\mathbb{A})} + \frac{S_{22}(\mathbb{A})^2}{S_{11}(\mathbb{A})S_{222}(\mathbb{A})} - \cdots + (-1)^n \frac{S_{n^n}(\mathbb{A})^2}{S_{(n-1)^n}(\mathbb{A})S_{n^{n+1}}(\mathbb{A})} \ .$$

EX. 8.9. Let $\mathbb{A}$ be of cardinality n, and ν_a, $a \in \mathbb{A}$ be constants such that $\sum_{a\in\mathbb{A}} \nu_a = 1$. Define $\oint$ to be the linear functional

$$k \geq 0 \Rightarrow \oint x^k = \sum_{a\in\mathbb{A}} \nu_a\, a^k \ .$$

Give the associated orthogonal polynomials, the Darboux-Christoffel kernel and an interpolation formula for polynomials of degree $\leq n{-}1$.

EX. 8.10. Let $\mathbb{A}$ be generic, with associated unitary orthogonal polynomials $\widetilde{P}_n(x)$. Given two rank-1 elements y, z, following [**148**] (formula 20) show that

$$\text{(8.9.9)} \qquad \frac{S_{n^n}(\mathbb{A}-z)S_{(n+1)^n}(\mathbb{A}-y-x)}{S_{n^n}(\mathbb{A}-y)S_{(n+1)^n}(\mathbb{A}-z-x)} = \frac{\Big(\widetilde{P}_{n-1}(x)/\widetilde{P}_n(x)\Big)\partial_{xy} - 1/\beta_n}{\Big(\widetilde{P}_{n-1}(x)/\widetilde{P}_n(x)\Big)\partial_{xz} - 1/\beta_n} \ ,$$

where β_n is the Wroński coefficient $S_{n^{n+1}}(\mathbb{A})\, S_{(n-2)^{n-1}}(\mathbb{A})/(S_{(n-1)^n}(\mathbb{A}))^2$.

EX. 8.11. Let $\mathbb{A}$ be generic and let $\mathbb{B} = \{b_1, \ldots, b_k)$. Define $P_{n,k}(x; \mathbb{B}) = S_{(n+k)^n}(\mathbb{A} - \mathbb{B} - x)$. Show that $P_{n,k}(x; \mathbb{B})$ has the following reproducing property, extending Darboux-Christoffel formula :

$$\int_{\mathbb{A}} f(x)P_{n,k}(x; \mathbb{B}) = (-1)^n f(b_1)P_{n+1,k-2}(b_2; b_3, \ldots, b_k)\,\partial_1 \cdots \partial_{k-1}$$

for all polynomial of degree $\leq k{+}n{-}1$, divided differences being relative to $\mathbb{B}$ (see [**117**]).

EX. 8.12. Let $\mathbb{A}$ be arbitrary, x be of rank 1, $n \in \mathbb{N}$. Show that $S^{2n+1}(\mathbb{A} - (2n{+}1)x)$ can be written as the following sum of $n{+}1$ powers:

$$S^{2n+1}(\mathbb{A} - (2n{+}1)x) = \sum_{b\in\mathbb{B}} (b{-}x)^{2n+1} \frac{S^n(b + \mathbb{A} - \mathbb{B})}{S^n(2b - \mathbb{B})} \ ,$$

where $\mathbb{B}$ is the alphabet of roots of $S_{(n+1)^{n+1}}(\mathbb{A}{-}x)$.

Such an expression is called the *canonical form of a binary form.*

In the simplest case, $n = 1$, this reduction allows to write the roots of a polynomial $S^3(\mathbb{A}{-}3x)$ by first computing the roots of $S_{22}(\mathbb{A}{-}x)$. The corresponding formulas are due to Eisenstein.

For the reduction of binary forms of even degree, cf. Kung, [**90**].

EX. 8.13. Let b be of rank 1, and $x = b + 1/b$. Show that the polynomials $H_n(x) := S^n(x/(1-q))/S^n(1/(1-q))$, $n \geq 0$, satisfy the recursions

$$H_n - xH_{n-1} + (1 - q^{n-1})H_{n-2} = 0$$

and therefore constitute a family of orthogonal polynomials.

Find the associated Darboux-Christoffel kernels.

Ex. 8.14. Define the *Legendre polynomials* $P_n(x)$ by the generating function

$$\sum z^n P_n(x) = \frac{1}{\sqrt{1-2xz+z^2}} .$$

Show that they are the denominators of the continued fraction expansion of $\frac{1}{2}\log(\frac{x+1}{x-1})$.

Ex. 8.15. Let $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\alpha \geq 0$. Evaluate the integral

$$\int_{-1}^{1} (1+x)^\alpha P_n(x)dx$$

($P_n(x)$ is the Legendre polynomial of degree n. Recall that Legendre polynomials are orthogonal with respect to the uniform measure on the interval $[-1,1]$).

Ex. 8.16. Let $\mathbb{A}$ be defined by $S_1(\mathbb{A}) = 1$,

$$\Psi_{2k}(\mathbb{A}) = \Lambda_{2k+1}(\mathbb{A}) \quad \& \quad \Psi_{2k-1}(\mathbb{A}) = -\Lambda_{2k}(\mathbb{A}) ,\ k \geq 1 .$$

Compute the $S_k(\mathbb{A})$.

Ex. 8.17. Define the *Laguerre polynomials* $L_n(x)$ by the recursion

$$(n+1)L_{n+1} = (2n+1-x)L_n(x) - nL_{n-1}(x) = 0 \ ,\ L_0 = 1 .$$

Find the moments. Show that the associated linear functional is

$$f(x) \mapsto \int_0^\infty e^{-x} f(x)dx \ .$$

Ex. 8.18. Compute the adjoint $L_n^*(x)$ of a Laguerre polynomial $L_n(x)$, using the linear functional associated to Legendre polynomials:

$$L_n^*(x) := \int_{-1}^{+1} \partial_{xy}(L_n(x))\, dy \ ,$$

and find the limit, for $n \mapsto \infty$, of $L_n^*(x)/L_n(x)$ (as a formal series in x^{-1}).

Ex. 8.19. Generalize the preceding computation by taking the *Laguerre polynomials* $L_n^\alpha(x)$, defined by

$$\sum_{n\geq 0} z^n L_n^\alpha(x) = (1-z)^{-\alpha-1} \exp(-xz/(1-z))$$

Ex. 8.20. Given a family of orthogonal polynomial $P_n(x) = S_{n^n}(\mathbb{A}-x)$ relative to some alphabet $\mathbb{A}$, and a linear functional $\int$ on a second variable y, compute the limit

$$\lim_{n\to\infty} \frac{\int \partial_{xy}(P_n(x))dy}{P_n(x)}$$

(developping the functions near $x = \infty$).

Ex. 8.21. Let x be of rank 1, and α be of binomial type. Define

$$P_n := \sum_0^n (-x)^j S^j(\alpha+n-1)\,/(n-j)! \quad , \quad n = 0,1,2,\ldots$$

Show that $\{P_n\}$ is a family of orthogonal polynomials, and find the moments $S^n(\mathbb{A})$. Compute the quotients $\Psi^n(\mathbb{A})/\Lambda^n(\mathbb{A})$, $n \geq 2$. Decompose x^n in the basis $S^n(\mathbb{A})$.

These polynomials are due to Krall and Frink [**86**]. The usual *Bessel polynomials* are for $\alpha = 2$ (Riordan ([**153**], p.77) takes the normalization $P_n(-x/2)$).

Leclerc and Thibon [**123**] generalize Bessel polynomials as symmetric functions: see next exercise.

Ex. 8.22. Define the b_i's by the generating function $\sum z^i b_i = z\big(\exp(z)-1\big)^{-1}$ ($i!\, b_i$ is the i-th *Bernoulli number*). For any positive integer n, use the preceding exercise to evaluate the following two determinants :

$$\begin{vmatrix} x^n & b_{2n} & \cdots & b_{4n-4} \\ x^{n-1} & b_{2n-2} & \cdots & b_{4n-6} \\ \vdots & \vdots & & \vdots \\ x^1 & b_2 & & b_{2n-2} \end{vmatrix} \quad \& \quad \begin{vmatrix} x^n & b_{2n+2} & \cdots & b_{4n-2} \\ x^{n-1} & b_{2n} & \cdots & b_{4n-4} \\ \vdots & \vdots & & \vdots \\ x^1 & b_4 & & b_{2n} \end{vmatrix} .$$

Ex. 8.23. Let $\mathbb{E}$ be generic. For any $n \geq 0$, define $\mathcal{P}_n := \sum_{i=0}^{n-1} S_{1^i, n-i}(\mathbb{E})$ (this is a P-Schur function).

Show, after Leclerc and Thibon [**123**], that the orthogonal polynomials with respect to the functional $\int x^n = \mathcal{P}_{n+1}$, are the *staircase Schur functions* $S_{1\ldots n}(\mathbb{E}-x)$. Deduce a continued fraction expansion of $\sum_1^\infty z^{-i}\mathcal{P}_i$ (exceptionally, we have taken $\int x^0 = \mathcal{P}_1$ and not $\int x^0 = 1$).

Ex. 8.24. Still writing $\mathcal{P}_i$ for the P-Schur functions, consider the morphism on $\mathfrak{Sym}$ defined by $S^i \to \mathcal{P}_{i+1}/\mathcal{P}_i$ ($\mathfrak{Sym}$ is here the ring generated by complete functions, with powers of S^1 in $\mathbb{Z}$ instead of $\mathbb{N}$).

Show that the power sums of odd degree are invariant under this morphism. The reader can use Cauchy's proposition that symmetric functions are rational functions in the odd power sums (Exercise 1.22), but must not conclude from it that all symmetric functions are invariant.

Notice that this invariance implies that the morphism is a projection of $\mathfrak{Sym}$, since P-Schur functions depends only on odd power sums.

Notice also, from the choice of the moments, that the image of the orthogonal polynomials $S_{n^n}(\mathbb{A}-x)$ are proportional to the Bessel polynomials of Leclerc and Thibon seen in Ex. 8.23.

Ex. 8.25. Let $\mathbb{E}$ be arbitrary, and $\{P_n(x) = S_{1\ldots n}(\mathbb{E}-x)\}$ be the symmetric analogs of Bessel polynomials defined by [**120**] and [**126**]. Given $x_1, \ldots, x_k$ of rank 1, and $n \geq 0$, evaluate the following Christoffel-type determinant :

$$\Big| P_{n+2j-2}(x_i) \Big|_{1\leq i,j\leq k} .$$

Ex. 8.26. Let q be a rank one element. Develop in continous fraction the q-exponential

$$E(x) := \sum_{i\geq 0} x^{-1-i} S_i(1/(1-q)) = \sum_{i\geq 0} x^{-1-i}/(1-q)\cdots(1-q^i) .$$

Find the convergents. Give the limit of $E(x(1-q))$ for $q=1$.

Ex. 8.27. Let k be a positive integer, $\mathbb{A}$ a generic alphabet, $\int$ the linear functional $x^{ik} \mapsto S_i(\mathbb{A})$, $x^i \mapsto 0$ if i is not divisible by k.

Find the orthogonal polynomials $P_n^{(k)}$ relative to $\int$. What about the subfamily $P_{nk}^{(k)}$, $n \in \mathbb{N}$?

Ex. 8.28. Let k be a positive integer, ζ a k-th primitive root of unity, $\{P_n(x)\}$ the orthogonal polynomials associated to a generic alphabet $\mathbb{A}$.

For any n, following Deruyts [**38**], expand

$$Q_n(x) := x^{-\binom{k}{2}} \begin{vmatrix} P_n(x) & P_n(\zeta x) & \cdots & P_n(\zeta^{k-1}x) \\ \vdots & \vdots & & \vdots \\ P_{n+k-1}(x) & P_{n+k-1}(\zeta x) & \cdots & P_{n+k-1}(\zeta^{k-1}x) \end{vmatrix}$$

as a polynomial in x.

EX. 8.29. Given a constant c, define the *associated Hermite polynomials* [**7**] by the recursions

$$H^c_{n+1}(x) = 2xH^c_n(x) - 2(n+c)H^c_{n-1}(x) \quad , \quad H^c_{-1} = 0, H^c_0 = 1$$

(the usual Hermite polynomials H_n are for $c = 0$).

Decompose them in the basis of Hermite polynomials. Determine $H^{-n}_n(x)$ and $H^{-n}_{n+k}(x)/H^{-n}_n(x)$ for $k \geq 0$.

EX. 8.30. Given an alphabet $\mathbb{A} = \{a_1, \ldots, a_n\}$, write $\sum_{1<i\leq n} 1/(a_1 - a_i)$ as a function of a_1 and $\mathbb{A}$.

Knowing that the Hermite polynomial $H_n(x)$ satisfy the differential equation

$$D^2 H_n - 2x\, DH_n + 2nH_n = 0 \ ,$$

with $D := \frac{d}{dx}$, use the preceding expression to prove that

$$a_1 = \sum\nolimits_{1<i\leq n} 1/(a_1 - a_i)$$

when $\mathbb{A}$ is specialized to the set of roots of H_n, a_1 being any root.

EX. 8.31. Let $f(x)$ be the *Jacobi polynomial* $f(x) = P^{2p-1,2q-1}_n(1-2x)$. Let $\mathbb{A} = \{\, a_1, \ldots, a_n\}$ be the alphabet of zeros of $f(x)$, arbitrarily ordered. Adapt the preceding exercise to show, after Stieltjes, that

$$\frac{p}{a_1} - \frac{q}{1-a_1} + \sum_{1<i\leq n} 1/(a_1 - a_i) = 0 \ ,$$

knowing that $f(x)$ satisfies the differential equation

$$x(1-x)D^2(f) + 2\big(p - (p+q)x\big)D(f) + n(n+2p+2q-1)f = 0 \ .$$

Show that

$$\sum_{1<i\leq n} 1/(a_1 - a_i) = \frac{1}{2} - \frac{\alpha+1}{2a_1}$$

in the case of the Laguerre polynomials satisfying the differential equation :

$$xD^2(f) + (\alpha + 1 - x)D(f) + nf = 0 \ .$$

EX. 8.32. Let $\mathbb{A}$ be an infinite alphabet, x an indeterminate, and Ψ_0 a parameter. Define a linear functional L on polynomials in x by

$$\int (x^0) = \psi_0 \, , \qquad \int (x^i) = \Psi_i(\mathbb{A}) \, , i > 0 \ .$$

Find the orthogonal polynomials associated to $\int$.

Ex. 8.33. Let $\mathbb{A}$ be an infinite alphabet, $\mathbb{X} = \{x_1, \dots, x_n\}$ be a finite one. Let $\int_j$, $j = 1, \dots, n$, be the linear functional on $\mathbb{C}[x_j]$ defined by $\int_j x_j^k = S^k(\mathbb{A})$, $k \geq 0$, and let $\iint$ be the product of the $\int_j$'s.

Show that, for every $J \in \mathbb{N}^n$,

$$\iint S_J(\mathbb{X})\, \Delta(\mathbb{X})^2 = (-1)^{n(n-1)/2} n!\, S_{J+(n-1)^n}(\mathbb{A}) \ .$$

Notice the corollary corresponding to $J = 1^n$, with z of rank 1 :

$$\iint R(\mathbb{X}, z)\Delta(\mathbb{X})^2 = \pm S_{n^n}(\mathbb{A} - z) \ .$$

Ex. 8.34. Let x be a rank-1 element, $\int_{\mathbb{A}}$ be the linear functional $x^j \mapsto S^j(\mathbb{A})$, k be a positive integer. Show that

$$\int_{\mathbb{A}} S_{(k+1)^k}(\mathbb{A} - 2x) = (-1)^k (k+1)\, S_{k^{k+1}}(\mathbb{A}) \ .$$

Deduce that

$$\int_{\mathbb{A}} S_{k^k}(\mathbb{A} - x)^2 = (-1)^k S_{k^{k-1}}(\mathbb{A}) S_{k^{k+1}}(\mathbb{A}) \ .$$

Ex. 8.35. (Stieltjes). Let $g(t)$ be a positive continuous function of $t \in [0,1]$, satisfying $\int_0^1 g(t)dt = 1$. Define $\mathbb{A}$ by $S^k(\mathbb{A}) := \int_0^1 g(t)t^k dt$, $k = 1, 2, \dots$.

Given any positive integer n, and indeterminates $x_1, \dots, x_n$, remark that the quadratic form

$$Q(x_1, \dots, x_n) := \int_0^1 g(t) \left(\sum x_i t^{i-1} \right)^2 dt$$

is positive, and deduce from it that

$$(-1)^{n(n-1)/2} S_{(n-1)^n}(\mathbb{A}) \geq 0 \ .$$

More generally, show that any diagonal minor $M_{J,J}$ of the Hankel matrix $M := \left[S_{i+j}(\mathbb{A}) \right]_{i,j \geq 0}$ is positive.

Consider the "generic" case $g(t) = \sigma_t(\mathbb{B})$, and express the Schur functions of $\mathbb{A}$ in terms of $\mathbb{B}$.

Given any k, evaluate the multiple integral

$$\int_0^1 x_1^k g(x_1) \int_0^1 \cdots \int_0^1 x_n^k g(x_n) \prod_{1 \leq i < j \leq n} ((x_i - x_j)^2 dx_1 \cdots dx_n$$

and conclude, once more, that the Hankel determinant $|S_{k+j-i}(\mathbb{A})|_{1 \leq i,j \leq n}$ is positive (Exercise (8.33) could have been used).

Ex. 8.36. With the same hypotheses than in the preceding exercise, show that for every $n, k \in \mathbb{N}$, one has the positivity

$$\sum_i (-1)^i \binom{k}{i} S^{n+i}(\mathbb{A}) \ .$$

Ex. 8.37. (Stieltjes). Let $\int_{\mathbb{A}}$ be the linear functional associated to $\mathbb{A}$, $\widetilde{P}_n$, the corresponding monic orthogonal polynomials. Let $R_n(z) := \int_{\mathbb{A}} \widetilde{P}_n(x)(z-x)^{-1}$. Write

$$\frac{1}{R_n(z)} = St_{n+1}(z) + z^{-1}(\cdots)) \ ,$$

taking the Laurent expansion of $1/R_n(z)$. Show that the polynomial $St_{n+1}(z)$ verifies

$$\int_{\mathbb{A}} \widetilde{P}_n(x)\, St_{n+1}(x) x^i = 0 \ , \ i = 0, \dots, n.$$

Ex. 8.38. Let $\mathbb{A}$, $\mathbb{B}$ be two finite sets of complex numbers, of respective cardinalities α, β, with $a \in \mathbb{A} \Rightarrow |a| > 1$, $b \in \mathbb{B} \Rightarrow |b| < 1$.

Given the polynomial $R(x, \mathbb{A}+\mathbb{B})$, Wroński [**180**] claims to be able to obtain $\mathbb{A}$ and $\mathbb{B}$ separately, i.e. to find the factor of $R(x, \mathbb{A}+\mathbb{B})$ comprising the roots of modulus less than 1.

Support Wroński's claim of a "universal factorization" by evaluating the limit of

$$S_{1^\beta;\, n^\alpha}(\mathbb{A}+\mathbb{B}-x;\ \mathbb{A}+\mathbb{B}) / S_{n^\alpha}(\mathbb{A}+\mathbb{B})$$

for $n \to \infty$.

Ex. 8.39. Let $\mathbb{A}$ be a finite set of complex numbers, that one orders according to decreasing modulus: $|a_1| \geq |a_2| \geq |a_3| \geq \cdots$. Let k be such that $|a_k| > |a_{k+1}|$. Writing S_I for $S_I(\mathbb{A})$, show that

$$\frac{1}{a_1 \cdots a_k} = \frac{1}{S_{1^k}} + \frac{S_{1^{k-1}} S_{1^{k+1}}}{S_{1^k} S_{2^k}} + \cdots + \frac{S_{n^{k-1}} S_{n^{k+1}}}{S_{n^k} S_{(n+1)^k}} + \cdots$$

Ex. 8.40. Let $\mathbb{A}$ be a finite set of complex numbers, $|a_1| > |a_2| > \cdots$. Knowing from the preceding exercise that $a_1 \sim S_{n+1}/S_n$, iterate, using $\mathbb{A} - a_1$ instead of $\mathbb{A}$, to conclude with Ostrowski ([**144**], formula J.37 p.280) that $1/a_2$ is the limit of

$$S_{k,k+1}(\mathbb{A}) / S_{k+1,k+1}(\mathbb{A}) \ .$$

Test the result on $\mathbb{A} = \{2, 1\}$ and withdraw your conclusion.

Ex. 8.41. Given a one variable function f, let a be a root of f and $x = a + \epsilon$. Newton shows that $x - f(x)/f'(x)$ is equal to a modulo ϵ^2. Define, for any integer n, $D_n(f)$ to be the determinant obtained from S_{1^n} by specializing $S_i \mapsto f^{(i)}(x)/(i! f(x))$, $i = 1, \dots, n$, where $f^{(i)}$ is the i-th derivative.

Generalize Newton's method by showing that

$$x - D_{n-1}(f)/D_n(f) \equiv a \mod \epsilon^{n+1} \ .$$

For example,

$$x - \frac{f'/f}{\begin{vmatrix} f'/f & f''/2f \\ 1 & f'/f \end{vmatrix}} \equiv a \mod \epsilon^3 \ .$$

Following Kalantari [**78**], take the special case $f(x) = 1 - 2\sin(x)$, $x = 0$, to approximate $\pi/6$ by a quotient of two determinants.

Ex. 8.42. (Stieltjes, letter 422 [**10**]). Compute the specialization of Schur functions indexed by partitions ν such that $\nu \supseteq (\ell-1)^\ell$, ℓ being the length of ν, when $S_i = q^{i^2}$, $i = 1, 2, \dots$.

Deduce a continued fraction expansion of

$$x^{-1} + q x^{-2} + q^4 x^{-3} + \cdots + q^{i^2} x^{-1-i} + \cdots$$

CHAPTER 9

Schubert Polynomials

9.1. Newton Interpolation Formula

We want to be able to perform algebraic computations in the ring of polynomials in $a_1, a_2, \ldots$, in a manner similar to what we did in the case of $\mathfrak{Sym}(\mathbb{A})$, that is, independently of the number of variables; in particular, we want to avoid expanding a polynomial in the basis of monomials, but rather use operations on combinatorial objects to replace algebraic computations on polynomials. Moreover, we want this combinatorics to extend the combinatorics of symmetric polynomials.

Of course, a generic polynomial of degree 5 in 6 variables depends on $\binom{10}{5}$ coefficients, whatever way one chooses to represent it. Any specific tool is unlikely to be interesting in the generic case, but will apply only to the description of spaces of polynomials satisfying some requirements.

Let us show how Newton's point of view can be used.

Given a function of one variable (the time t), let $\mathbb{B} = \{b_1, b_2, b_3, \ldots\}$, consider the divided differences with respect to $\mathbb{B}$ and let

$$f^{\partial}, f^{\partial\partial}, f^{\partial\partial\partial}, \ldots$$

denote

$$f(b_1)\partial_1, f(b_1)\,\partial_1\partial_2, f(b_1)\,\partial_1\partial_2\partial_3, \ldots$$

Then the formula of Newton to interpolate $f(t)$ from its values at time $b_1, b_2, \ldots$ is

$$(9.1.1) \quad f(t) = f(b_1) + f^{\partial}(t-b_1) + \\ f^{\partial\partial}(t-b_1)(t-b_2) + f^{\partial\partial\partial}(t-b_1)(t-b_2)(t-b_3) + \cdots$$

the expansion being exact if $n+1$ points b_i are used and f is a polynomial of degree $\leq n$.

In other words, Newton found that one could extend the already known case of finite differences, when $b_1, \ldots, b_i = b_1+i-1, \ldots$ form an arithmetical progression, to an arbitrary sequence $\mathbb{B}$, by normalizing the differences by the interval (of time) to which they correspond.

When $b_1, b_2, \ldots$ all collapse to 0, then equation (9.1.1) becomes *Taylor's formula*

$$f(t) = f(0) + f'(0)\,t + f''(0)/2!\,t^2 + \cdots ,$$

the factorials in the denominator being clear by putting $b_i = i\epsilon$ and letting ϵ tend to 0.

Since divided differences are the discrete analogues of derivatives, the question is :

What is the discrete analogue of Taylor's formula in several variables ?

In other words, what are the coefficients of the images of $f(a_1, a_2, \ldots)$ under the possible different divided differences (evaluated in $b_1, b_2, b_3, \ldots$) in the expansion of $f(a_1, a_2, \ldots)$?

The answer, that I obtained with Marcel-Paul Schützenberger some years ago (cf. [**106**]), is as simple as in the case of one variable, and will be presented in the next sections. First, let us make some comments about the formula of Newton, and prove it[1].

9.2. Newton and Euclid

Newton's formula is linear in $f(x)$. To prove it, it is sufficient to test it on powers of x, or equivalently, to test it on the single function $(z-x)^{-1}$.

Since $f\partial_1 \cdots \partial_{n-1} = (z-a_1)^{-1} \cdots (z-a_n)^{-1}$, one has to check that

$$\frac{1}{z-x} = \frac{1}{z-a_1} + \frac{x-a_1}{(z-a_1)(z-a_2)} + \frac{(x-a_1)(x-a_2)}{(z-a_1)(z-a_2)(z-a_3)} + \cdots \tag{9.2.1}$$

As we did for Lagrange interpolation, we insert x in the set of interpolating points to see what happens. Let us do it twice, in consecutive places, i.e. let us look at the case where $a_n = x$, or $a_{n+1} = x$. The equation becomes

$$\begin{aligned} \frac{1}{z-x} &= \frac{1}{z-a_1} + \cdots + \frac{(x-a_1)\cdots(x-a_{n-1})}{\cdots(z-a_{n-1})(z-x)} \\ &= \frac{1}{z-a_1} + \cdots + \frac{(x-a_1)\cdots(x-a_{n-1})}{\cdots(z-a_{n-1})(z-a_n)} + \frac{(x-a_1)\cdots(x-a_n)}{\cdots(z-a_n)(z-x)} . \end{aligned}$$

The difference is

$$\frac{(x-a_1)\cdots(x-a_{n-1})}{\cdots(z-a_{n-1})(z-x)} - \frac{(x-a_1)\cdots(x-a_{n-1})}{\cdots(z-a_{n-1})(z-a_n)} - \frac{(x-a_1)\cdots(x-a_n)}{\cdots(z-a_n)(z-x)} ,$$

which, indeed is null, and this proves (9.2.1), and therefore the Newton formula in general, by induction on n.

Interpolating $(1-zx)^{-1}$, instead of $(z-x)^{-1}$, one gets a formula equivalent to (9.2.1) :

$$\frac{1}{1-zx} = \frac{1}{1-za_1} + z\frac{x-a_1}{(1-za_1)(1-za_2)} + \cdots + z^k \frac{R(x, \mathbb{A}_k)}{R(1, z\mathbb{A}_{k+1})} + \cdots \tag{9.2.2}$$

that one often encounter with $\mathbb{A}$ specialized. For example, one has

$$\frac{1}{1-zx} = \frac{1}{1-z} + z\frac{x-1}{(1-z)(1-zq)} + \cdots + z\frac{(x-1)\cdots(x-q^{n-1})}{(1-z)\cdots(1-zq^n)} + \cdots$$

Let us pause a moment and ask the favourite question of Israël Gelfand: what was the problem?

Given $f(x)$ and $\mathbb{A} = \{a_1, \ldots, a_n\}$, Newton and Lagrange wanted to write a polynomial $g_{\mathbb{A}}(x)$ of degree $\leq n-1$ such that

$$f(a_i) = g_{\mathbb{A}}(a_i) \ , \ 1 \leq i \leq n \ . \tag{9.2.3}$$

[1]We could have given a proof using vanishing properties, but in that case, it would be almost the same as the one that we give for the multi-variate case in Theorem 9.6.1.

In that terms, the problem was solved much earlier by Euclid, by division, as we have seen in chapter 3 :

$$f(x) = g(x) + (\star)\, R(x, \mathbb{A}) \tag{9.2.4}$$

where $(\star)$ is the *quotient*, i.e. the integral part of the fraction $f(x)/R(x, \mathbb{A})$, or equivalently, where the RHS is determined by the condition that $g(x)$ be of degree $< n$.

Suppose that $f(x) = S^k(x - \mathbb{B})$, $k \geq n$. Then, Euclid's answer, put in λ-ring notation, consists in writing $x - \mathbb{B} = (x - \mathbb{A}) + (\mathbb{A} - \mathbb{B})$, and expand accordingly :

$$\begin{aligned}
&(9.2.5) \quad S^k(x-\mathbb{B}) \\
&= S^k(x-\mathbb{A}) + S^1(\mathbb{A}-\mathbb{B}) S^{k-1}(x-\mathbb{A}) + S^2(\mathbb{A}-\mathbb{B}) S^{k-2}(x-\mathbb{A}) + \cdots + S^{k-n}(\mathbb{A}-\mathbb{B}) S^n(x-\mathbb{A}) \\
&\qquad + S^{k-n+1}(\mathbb{A}-\mathbb{B}) S^{n-1}(x-\mathbb{A}) + \cdots + S^k(\mathbb{A}-\mathbb{B}) S^0(x-\mathbb{A}) \\
&\qquad = \Big(x^{k-n} + S^1(\mathbb{A}-\mathbb{B}) x^{k-n-1} + \cdots + S^{k-n}(\mathbb{A}-\mathbb{B}) \Big)\, R(x, \mathbb{A}) \\
&\qquad\qquad + S^{k-n+1}(\mathbb{A}-\mathbb{B}) S^{n-1}(x-\mathbb{A}) + \cdots + S^k(\mathbb{A}-\mathbb{B}) S^0(x-\mathbb{A})\,.
\end{aligned}$$

The factor of $R(x, \mathbb{A})$ can be rewritten $S^{k-n}(x + \mathbb{A} - \mathbb{B})$, and one sees once more that it is equal to $f(a_1)\partial_1 \cdots \partial_n = S^k(a_1 - \mathbb{B})\partial_1 \cdots \partial_n$, when defining a_{n+1} to be equal to x.

Therefore, the three equations of Lagrange, Newton, Euclid can be summarized into

$$f(x) - f\partial_1 \cdots \partial_n R(x, \mathbb{A}_n)$$

$$= \sum\nolimits_{a \in \mathbb{A}_n} f(a) \frac{R(x, \mathbb{A}_n \backslash a)}{R(a, \mathbb{A}_n \backslash a)} \tag{9.2.6}$$

$$= f(a_1) + \cdots + f\partial_1 \cdots \partial_{n-1}\, R(x - A_{n-1}) \tag{9.2.7}$$

$$= S^0(x-\mathbb{A}) + \cdots + S^{k-n+1}(\mathbb{A}-\mathbb{B}) S^{n-1}(x-\mathbb{A}) \tag{9.2.8}$$

The three mathematicians cited above have just chosen three different bases, triangularly equivalent, of polynomials of degree $\leq n-1$:

$$\{R(x, \mathbb{A}_n \backslash a) : a \in \mathbb{A}_n\}\ ,\ \{R(x - \mathbb{A}_j\} : 0 \leq j \leq n-1\ ,\quad \{S^j(x-\mathbb{A}) : 0 \leq j \leq n-1\}\ . \tag{9.2.9}$$

Notice that what Euclid calls the *remainder* is the principal part $(= g_{\mathbb{A}})$ for Newton and Lagrange.

The advantage of Newton's method is to show instantly the difference $f(x) - g_{\mathbb{A}}(x)$. Take notice that there are many ways to express $f\partial_1 \cdots \partial_n$, and disguise that it be a divided difference.

One can, for example, use residues and Cauchy integral $\big($cf. Ex.(7.27)$\big)$, we shall come back to this point later.

Comparing the different expressions (9.2.6), (9.2.7), (9.2.8) is a way to produce q-identities, as shown by Fu et al [**48**].

9.3. Discrete Wrońskian

Newton's formula is closely connected with a discrete analog of the Wrońskian, and could be checked using it. Here we shall on the contrary use the Newton formula to describe Wrońskians.

Indeed, given n one variable-functions f_i, we define their $\mathbb{A}$-Wrońskian to be

$$W_{\mathbb{A}}(f_1,\dots,f_n) := \Big| f_i(a_1)\partial_1\dots\partial_{j-1} \Big|_{1\le i,j\le n} \tag{9.3.1}$$

PROPOSITION 9.3.1. *The $\mathbb{A}$-Wrońskian of n functions $f_1,\dots,f_n$ is equal to the quotient $\left|f_i(a_j)\right|/\Delta(\mathbb{A})$, taking here the Vandermonde $\Delta(\mathbb{A})$ to be $\prod_{j>i}(a_j-a_i)$.*

It is also equal to the determinant $\left|f_i\partial_1\cdots\partial_{j-1}\pi_j\cdots\pi_{n-1}\right|$

Proof. Replace each $f(a_j)$ by its Newton's expansion: $f(a_j) = f(a_1)+\cdots+ f\partial_1\cdots\partial_{j-1}\,R(a_j,\mathbb{A}_{j-1})$. Then we see that the determinant has columns equal to linear combinations of the columns of the Wrońskian,

$$col_j = col_1 + col_2\,R(a_j,\mathbb{A}_1)+\cdots+col_j\,R(a_j,\mathbb{A}_{j-1})\ .$$

Therefore, the two determinants coincide, up to the factor

$$R(a_2,\mathbb{A}_1)R(a_3,\mathbb{A}_2)\cdots R(a_n,\mathbb{A}_{n-1}) = \Delta(\mathbb{A})\ .$$

Because $\left|f_i(a_j)\right|/\Delta(\mathbb{A})$ is a symmetric function, one does not change the value of the Wrońskian by taking its image under

$$\pi_\omega = (\pi_{n-1})(\pi_{n-2}\pi_{n-1})\cdots(\pi_1\cdots\pi_{n-1})\ .$$

Now, the Wrońskian has been written as a determinant such that the entries in the successive columns are symmetric functions of $\mathbb{A}_1,\ \mathbb{A}_2,\ \dots,\mathbb{A}_n$ respectively. The operator π_{n-1} operates non trivially only on column $n-1$, then the operator $\pi_{n-2}\pi_{n-1}$ operates on column $n-2$ of the new determinant, and finally the image of the Wrońskian under π_ω is the determinant where each of the operators (π_{n-1}), $(\pi_{n-2}\pi_{n-1})$, $\dots(\pi_1\cdots\pi_{n-1})$ have been applied to the appropriate column. The entries of the final determinant are now symmetric functions in $\mathbb{A}$. □

For example,

$$\begin{aligned}
&\begin{vmatrix} f_1(a_1) & f_1(a_2) & f_1(a_3)\\ f_2(a_1) & f_2(a_2) & f_2(a_3)\\ f_3(a_1) & f_3(a_2) & f_3(a_3)\end{vmatrix}\\
&= \begin{vmatrix} f_1(a_1) & f_1(a_1)+f_1\partial_1(a_2-a_1) & f_1(a_1)+f_1\partial_1(a_3-a_1)+f_1\partial_1\partial_2(a_3-a_1)(a_3-a_2)\\ f_2(a_1) & f_2(a_1)+f_2\partial_1(a_2-a_1) & f_2(a_1)+f_2\partial_1(a_3-a_1)+f_2\partial_1\partial_2(a_3-a_1)(a_3-a_2)\\ f_3(a_1) & f_3(a_1)+f_3\partial_1(a_2-a_1) & f_3(a_1)+f_3\partial_1(a_3-a_1)+f_3\partial_1\partial_2(a_3-a_1)(a_3-a_2)\end{vmatrix}\\
&= \begin{vmatrix} f_1(a_1) & f_1\partial_1(a_2-a_1) & f_1\partial_1\partial_2(a_3-a_1)(a_3-a_2)\\ f_2(a_1) & f_2\partial_1(a_2-a_1) & f_2\partial_1\partial_2(a_3-a_1)(a_3-a_2)\\ f_3(a_1) & f_3\partial_1(a_2-a_1) & f_3\partial_1\partial_2(a_3-a_1)(a_3-a_2)\end{vmatrix} = W_{\mathbb{A}}(f_1,f_2,f_3)\,\Delta(\mathbb{A})\ .
\end{aligned}$$

Moreover,

$$W_{\mathbb{A}}(f_1,f_2,f_3) = W_{\mathbb{A}}(f_1,f_2,f_3)\,\pi_2\pi_1\pi_2 = \begin{vmatrix} f_1(a_1)\pi_1\pi_2 & f_1\partial_1\pi_2 & f_1\partial_1\partial_2\\ f_2(a_1)\pi_1\pi_2 & f_2\partial_1\pi_2 & f_2\partial_1\partial_2\\ f_3(a_1)\pi_1\pi_2 & f_3\partial_1\pi_2 & f_3\partial_1\partial_2\end{vmatrix}\ .$$

In the special case where all the functions f_j are powers of a variable, we see that the Wrońskian interpolates between the expression of a Schur function as a determinant of complete functions, and as the quotient by the Vandermonde of a determinant of powers of variables. For example, with $\mathbb{A}$ of order 3, transposing matrices for commodity, one has :

$$W_{\mathbb{A}}(x^2, x^5, x^9) = \begin{vmatrix} S_2(\mathbb{A}_1) & S_5(\mathbb{A}_1) & S_9(\mathbb{A}_1) \\ S_1(\mathbb{A}_2) & S_4(\mathbb{A}_2) & S_8(\mathbb{A}_2) \\ S_0(\mathbb{A}_3) & S_3(\mathbb{A}_3) & S_7(\mathbb{A}_3) \end{vmatrix} = \begin{vmatrix} a_1^2 & a_1^5 & a_1^9 \\ a_2^1 & a_2^4 & a_2^8 \\ a_3^0 & a_3^3 & a_3^7 \end{vmatrix} / \Delta(\mathbb{A})$$

$$= \begin{vmatrix} a_1^2\pi_1\pi_2 & a_1^5\pi_1\pi_2 & a_1^9\pi_1\pi_2 \\ S_1(\mathbb{A}_2)\pi_2 & S_4(\mathbb{A}_2)\pi_2 & S_8(\mathbb{A}_2)\pi_2 \\ S_0(\mathbb{A}_3) & S_3(\mathbb{A}_3) & S_7(\mathbb{A}_3) \end{vmatrix} = \begin{vmatrix} S_2(\mathbb{A}) & S_5(\mathbb{A}) & S_9(\mathbb{A}) \\ S_1(\mathbb{A}) & S_4(\mathbb{A}) & S_8(\mathbb{A}) \\ S_0(\mathbb{A}) & S_3(\mathbb{A}) & S_7(\mathbb{A}) \end{vmatrix}$$

More generally, multi-Schur functions can be considered as Wrońskians, powers of x being replaced by some $S^k(x-\mathbb{B})$. For example, given any $i, j, k \in \mathbb{N}$, and any elements ξ_1, ξ_2, ξ_3 of a λ-ring, then

(9.3.2)

$$W_{\mathbb{A}}\big(S^i(x-\xi_1), S^j(x-\xi_2), S^k(x-\xi_1)\big) = \begin{vmatrix} S^i(a_1-\xi_1) & S^j(a_1-\xi_2) & S^k(a_1-\xi_3) \\ S^i(a_2-\xi_1) & S^j(a_2-\xi_2) & S^k(a_2-\xi_3) \\ S^i(a_3-\xi_1) & S^j(a_3-\xi_2) & S^k(a_3-\xi_3) \end{vmatrix} / \Delta(\mathbb{A})$$

$$= \begin{vmatrix} S^i(\mathbb{A}_1-\xi_1) & S^j(\mathbb{A}_1-\xi_2) & S^k(\mathbb{A}_1-\xi_3) \\ S^{i-1}(\mathbb{A}_2-\xi_1) & S^{j-1}(\mathbb{A}_2-\xi_2) & S^{k-1}(\mathbb{A}_2-\xi_3) \\ S^{i-2}(\mathbb{A}_3-\xi_1) & S^{j-2}(\mathbb{A}_3-\xi_2) & S^{k-2}(\mathbb{A}_3-\xi_3) \end{vmatrix} = S_{i;\,j-1;\,k-2}(\mathbb{A}-\xi_1\,;\, \mathbb{A}-\xi_2\,;\, \mathbb{A}-\xi_3)\,.$$

All expressions involving multi-Schur functions can now be rewritten. For example, the r-remainder in the division of $S^m(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$ given in (3.2.1) can be reformulated :

(9.3.3) $$S_{1^{n-r};\,(m-n+r)^r}(\mathbb{A}-x; \mathbb{A}-\mathbb{B}) = \Delta(\mathbb{A})^{-1} \times$$
$$\begin{vmatrix} S^1(a_1-x) & \cdots & S^{n-r}(a_1-x) & S^m(a_1-\mathbb{B}) & \cdots & S^{m+r-1}(a_1-\mathbb{B}) \\ \vdots & & \vdots & \vdots & & \vdots \\ S^1(a_n-x) & \cdots & S^{n-r}(a_n-x) & S^m(a_n-\mathbb{B}) & \cdots & S^{m+r-1}(a_n-\mathbb{B}) \end{vmatrix}.$$

9.4. Schubert Polynomials

Let us go back to the problem of writing a discrete analog of Taylor's formula, when replacing derivatives by divided differences.

The universal coefficients, called *Schubert polynomials*[2] due to their relevance to geometry, can be defined recursively as follows:

(1) They are polynomials $\mathbb{Y}_v(\mathbb{A}, \mathbb{B})$ in two infinite totally ordered sets of variables $\mathbb{A} = \{a_1, a_2, \ldots\}$, $\mathbb{B} = \{b_1, b_2, \ldots\}$, indexed by vectors $v \in \mathbb{N}^\infty$ having only a finite number of non-zero components.

(2) They are globally stable under divided differences in the a_i's, that is, the image of a Schubert polynomial is either 0 or a Schubert polynomial.

(3) When v is weakly decreasing (one says that v is *dominant*, or is a *decreasing partition*), then

(9.4.1) $$\mathbb{Y}_v = \prod_{(i,j)\in Diagr(v)} (a_i - b_j) \ ,$$

[2] It would have been justified to call them *Newton polynomials*, but this interpretation was obtained later, though it can be considered the simplest.

product on all boxes[3] in the diagram of v.

(4) Given $v \in \mathbb{N}^\infty$ and $i \in \mathbb{N}$ such that $v_i > v_{i+1}$, writing $v\partial_i$ for $[v_1, \ldots, v_{i-1}, v_{i+1}, v_i - 1, v_{i+2}, \ldots]$, one has

$$\mathbb{Y}_v\, \partial_i = \mathbb{Y}_{v\, \partial_i}\ . \tag{9.4.2}$$

One does not need to write what happens when $v_i \leq v_{i+1}$, because in that case $\mathbb{Y}_v$ is the image of some $\mathbb{Y}_{v'}$ under ∂_i, (precisely $v' = [v_1, \ldots, v_{i-1}, v_{i+1} + 1, v_i, v_{i+2}, \ldots]$). The vanishing $\partial_i^2 = 0$ implies then that $\mathbb{Y}_v\, \partial_i = 0$ in that case.

One has now to check that the above definition is consistent.

First, a Schubert polynomial can be obtained from another one by two different sequences (necessarily reduced) of divided differences. But, since divided differences satisfy the braid relations, the images of any Y_v under two different reduced products corresponding to the same permutation are equal.

A more serious problem is that in the orbit of a dominant polynomial under divided differences, one finds other dominant polynomials, and we must ensure that they are still equal to the product described by their diagram, as required by axiom (9.4.1).

For example, $\mathbb{Y}_{5543}$ is by definition the product

$$\mathbb{Y}_{5543} = \begin{array}{|c|c|c|c|c|} \hline a_4-b_1 & a_4-b_2 & a_4-b_3 & & \\ \hline a_3-b_1 & a_3-b_2 & a_3-b_3 & a_3-b_4 & \\ \hline a_2-b_1 & a_2-b_2 & a_2-b_3 & a_2-b_4 & a_2-b_5 \\ \hline a_1-b_1 & a_1-b_2 & a_1-b_3 & a_1-b_4 & a_1-b_5 \\ \hline \end{array}$$

but it can be obtained from $\mathbb{Y}_{987654321}$ by a (long!) sequence of divided differences, and one must ensure that the sequence produces at the end the polynomial indicated by a diagram of boxes.

Because the set of partitions is a lattice (with respect to intersection and union of diagrams), compatibility of dominant polynomials can be reduced to checking compatibility for pairs of the type: a staircase, and a partition contained in it.

We first check an elementary step, corresponding to a pair of consecutive rows in a diagram constituting a regular stair (with differences of lengths equal to 1). Let us write $\mathcal{D}_\lambda$ for the product $\prod_{(i,j)\in Diagr(\lambda)}(a_i - b_j)$, λ being a (decreasing) partition.

LEMMA 9.4.1. *Let ν be a partition, and let $i \in \mathbb{N}$ be such that $\nu_i = \nu_{i+1} + 1$. Then the image of $\mathcal{D}_\nu$ under ∂_i is equal to $\mathcal{D}_\eta$, with η obtained from ν by decreasing ν_i by 1.*

Proof. The polynomial $\mathcal{D}_\nu$ is equal to $\mathcal{D}_\eta\,(a_i - b_{\nu_i})$, and $\mathcal{D}_\eta$ is symmetrical in a_i, a_{i+1}. Thus the statement comes from the fact that the image of the linear factor under ∂_i is 1. □

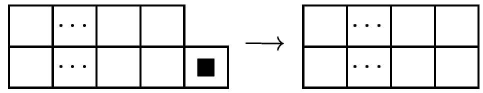
Now, we verify that one can find enough chains of partitions satisfying the hypothesis of the preceding lemma.

LEMMA 9.4.2. *Given an integer n, and a partition λ such that $\lambda \leq \rho := [n-1, n-2, \ldots, 1]$, then there exists a sequence of partitions from ρ to λ such that each step is of the type*

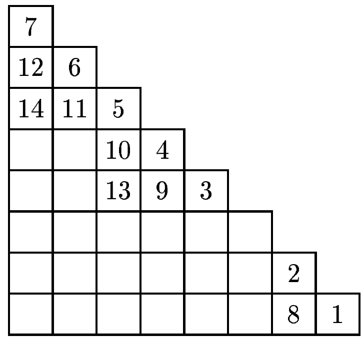
$$[\nu_1, \ldots, \nu_i = \nu_{i+1}+1, \nu_{i+1}, \ldots] \to [\nu_1, \ldots, \nu_i - 1, \nu_{i+1}, \ldots]\ .$$

[3] A box in position (i,j), $i,j = 1,2,\ldots$, gives a factor $(a_i - b_j)$.

Proof. We give a graphical proof. The elementary step concerns a pair of consecutive rows, say row i and row $i+1$, of lengths differing by 1. In that case, we have just seen that the effect of the divided difference ∂_i is to erase one box :

($\star$) 

If one numbers the boxes of the skew diagram ρ/λ by successive diagonals, as shown in the figure for $\lambda = [6, 6, 6, 2, 2]$,



then one easily observes that peeling off the boxes in the order they are numbered satisfy the configuration ($\star$) at each step. □

In final, if $\mathbb{Y}_v = \mathcal{D}_\lambda$, then all its images under products of divided differences which are indexed by a dominant vector satisfy property (9.4.1), and therefore the definition of Schubert polynomials is consistent.

9.5. Vanishing Properties

Newton's polynomials $(t-b_1)\cdots(t-b_n)$ could be characterized by their vanishing properties, they are even defined in terms of their roots $b_1, b_2, \ldots$.

In general, the knowledge that a polynomial in several variables vanishes at some points is difficult to use to reconstruct the polynomial. However, we shall see that in the case of Schubert polynomials, the situation is as simple as in the original set-up of Newton.

The next lemma describes *almost dominant* Schubert polynomials.

LEMMA 9.5.1. *Let v be such that $v_2 \geq v_3 \geq \cdots \geq v_n \geq 0 = v_{n+1} = v_{n+2} = \cdots$. Define k by the condition that $v_1 < v_k$, $v_1 \geq v_{k+1}$, and let λ, J be*

$$\lambda = [v_2{+}1, v_3{+}1, \ldots, v_k{+}1, v_1, v_{k+1}, \ldots, v_n] \ , \ J = [v_n, \ldots, v_{k+1}, v_1, v_k, \ldots, v_2] \ .$$

Then

$$\mathbb{Y}_v = S_J(\mathbb{A}_n - \mathbb{B}_{\lambda_n}; \ldots; \mathbb{A}_k - \mathbb{B}_{\lambda_k}; \mathbb{A}_k - \mathbb{B}_{\lambda_{k-1}}; \ldots; \mathbb{A}_2 - \mathbb{B}_{\lambda_1}) \ . \tag{9.5.1}$$

Proof. $\mathbb{Y}_v$ is the image of the dominant polynomial $\mathbb{Y}_\lambda$ under $\partial_{k-1}\cdots\partial_1$. But

$$\mathbb{Y}_\lambda = S^{\lambda_n}(a_n - \mathbb{B}_{\lambda_n})\cdots S^{\lambda_1}(a_1 - \mathbb{B}_{\lambda_1}) = S_{\lambda_n;\ldots;\lambda_1}(\mathbb{A}_n - \mathbb{B}_{\lambda_n}; \ldots; \mathbb{A}_1 - \mathbb{B}_{\lambda_1}) \ ,$$

the last expression being obtained by using, once more, the transformation lemma (1.4.1).

At each step in the sequence $\mathbb{Y}_\lambda, \mathbb{Y}_\lambda\,\partial_{k-1}, \mathbb{Y}_\lambda\,\partial_{k-1}\partial_{k-2}, \ldots$, one has a determinant with only one column which is not invariant under s_i, $1 \leq i \leq k{-}1$. Therefore

∂_i acts in that case by changing $\mathbb{A}_i$ into $\mathbb{A}_{i+1}$ and decreasing by 1 one component of the index of the current Schubert polynomial. $\square$

We can now study specializations. Let us see the effect of putting $a_1 = b_1$. Recall that we denoted $\mathbb{A}^+ = \{a_2, a_3, \ldots\}$ the alphabet starting with a_2.

LEMMA 9.5.2. *Let* $v \in \mathbb{N}^\infty$. *Then*

$$\mathbb{Y}_v\Big|_{a_1=b_1} = \begin{cases} \mathbb{Y}_{v_2,v_3,\ldots}(\mathbb{A}^+, \mathbb{B}^+) & \text{if } v_1 = 0\,, \\ 0 & \text{otherwise} \end{cases} \tag{9.5.2}$$

Proof. Because any $\mathbb{Y}_{v'}$ can be obtained from some Y_v, $v : v_2 \geq v_3 \geq \cdots$, using $\partial_2, \partial_3, \ldots$, but not ∂_1, one can suppose that v is such that $v_2 \geq v_3 \geq \cdots \geq v_n \geq 0 = v_{n+1} = \cdots$. In that case, $\mathbb{Y}_v$ has a determinantal expression (9.5.1). If $v_1 = 0$, then the partition indexing the determinant has a first part equal to 0, one can reduce the order by 1 and change all $\mathbb{A}_i - \mathbb{B}_j$ into $(\mathbb{A}^+)_{i-1} - (\mathbb{B}^+)_{j-1}$ without changing the value of the determinant if $a_1 = b_1$.

If $v_1 > 0$, then with λ as in (9.5.1), $\mathbb{Y}_\lambda$ is divisible by $(a_1-b_1)\cdots(a_k-b_1)$. This factor being symmetrical in $a_1, \ldots, a_k$, the image of $\mathbb{Y}_\lambda$ under $\partial_{k-1}\cdots\partial_1$ is still divisible by it. Therefore $\mathbb{Y}_v$ vanishes under the specialization $a_1 = b_1$. $\square$

If one now specializes $\mathbb{A}$ to $\mathbb{B}$ (i.e. $a_1 = b_1$, $a_2 = b_2, \ldots$), then the preceding lemma implies by recursion the following fundamental vanishing properties.

PROPOSITION 9.5.3. *All Schubert polynomials vanish under the specialization* $\mathbb{A} = \mathbb{B}$, *except* $\mathbb{Y}_0$ *(which is equal to* 1*).*

We shall see later that Schubert polynomials satisfy more vanishing properties related to the Ehresmann-Bruhat order on the symmetric group.

9.6. Newton Interpolation in Several Variables

Let us for a moment index products of divided differences by vectors, instead of permutations.

Given $K = [k_1, k_2, \ldots] \in \mathbb{N}^n$, let

$$\partial^K := \cdots (\partial_3 \cdots \partial_{k_3+2})\,(\partial_2 \cdots \partial_{k_2+1})\,(\partial_1 \cdots \partial_{k_1})\ ,$$

that is, ∂^K consists in products of blocks of consecutive divided differences, of respective lengths $k_1, k_2, \ldots$, starting with ∂_1, ∂_2, $\ldots$ respectively.

This product is chosen in such way that $\mathbb{Y}_K(\mathbb{A}, \mathbb{B})\,\partial^K = \mathbb{Y}_0(\mathbb{A}, \mathbb{B})$. For example, for $K = [2, 0, 4, 1]$, one has

$$\partial^{2041} = (\partial_4)(\partial_3\partial_4\partial_5\partial_6)(\ )(\partial_1 d_2)$$

and

$$\mathbb{Y}_{2041} \xrightarrow{\partial_4} \mathbb{Y}_{204} \xrightarrow{\partial_3\partial_4\partial_5\partial_6} \mathbb{Y}_{20} \xrightarrow{\partial_1\partial_2} \mathbb{Y}_0\ .$$

Each divided difference acts by pushing to the right, and decreasing it, the rightmost non-zero component of the index of the current Schubert polynomial.

One easily checks that $\mathbb{Y}_J(\mathbb{A}, \mathbb{B})\,\partial^K \neq \mathbb{Y}_0$ if $J \neq K$.

We can now state the *multivariate Newton interpolation formula.*

THEOREM 9.6.1. *For any polynomial in* $a_1, a_2, \ldots$, *one has :*

$$f(\mathbb{A}) = \sum_{K \in \mathbb{N}^\infty} f^{\partial^K}(\mathbb{B})\,\mathbb{Y}_K(\mathbb{A}, \mathbb{B})\ . \tag{9.6.1}$$

Proof. One has to test (9.6.1) on a linear basis of polynomials in $\mathbb{A}$ (with coefficients in anything, for example functions of $\mathbb{B}$). One takes the Schubert polynomials $\{\mathbb{Y}_J(\mathbb{A},\mathbb{B})\}$ as such a basis. Now, one has to check whether the following equations are true :

$$\mathbb{Y}_J(\mathbb{A},\mathbb{B}) = \sum_{K\in\mathbb{N}^\infty} (\mathbb{Y}_J)^{\partial^K}(\mathbb{B},\mathbb{B})\,\mathbb{Y}_K(\mathbb{A},\mathbb{B}) \ . \tag{9.6.2}$$

But the $\mathbb{Y}_J^{\partial^K}$ are either zero, or Schubert polynomials, and they all vanish under the specialization $\mathbb{A}=\mathbb{B}$, except for $\mathbb{Y}_{[0,0,\ldots]}=1$. This happens only in the case $K=J$, and the sum reduces to the single term $\mathbb{Y}_0(\mathbb{B},\mathbb{B})\,\mathbb{Y}_J(\mathbb{A},\mathbb{B})$. It proves equation (9.6.1) in full generality. □

The original Newton formula has not been lost. When f depends only on a_1, then f^{∂^K} is 0 if K is not of the type $K=[n,0,0,\ldots]$ for some integer n. In that case, $\mathbb{Y}_K$ is dominant, corresponding to a row diagram with n boxes, and indeed the Newton polynomial is

$$\mathbb{Y}_{[n,0,0,\ldots]} = (a_1-b_1)\,(a_1-b_2)\cdots(a_1-b_n) \ .$$

Let us give a case with two variables, interpolating $f=a_1^3a_2$. Formula (9.6.1) reads

$$\begin{aligned} f = f\,\mathbb{Y}_{00} + f\partial_2\,\mathbb{Y}_{01} + f\partial_1\,\mathbb{Y}_{10} + f\partial_1\partial_2\,\mathbb{Y}_{20} + f\partial_2\partial_1\,\mathbb{Y}_{11} \\ + f\partial_1\partial_2\partial_3\,\mathbb{Y}_{30} + f\partial_2\partial_1\partial_2\,\mathbb{Y}_{21} f\partial_2\partial_1\partial_2\partial_3\,\mathbb{Y}_{31} \ , \end{aligned}$$

that is, in expanded form,

$$\begin{aligned} a_1^3a_2 = b_1^3b_2\,\mathbb{Y}_{00} + b_1^3\,\mathbb{Y}_{01} + (b_1^2+b_1b_2+b_2^2)\,\mathbb{Y}_{10} + (b_1^2+b_1b_2+b_1b_3)\,\mathbb{Y}_{20} \\ + (b_1^2+b_1b_2+b_2^2)\,\mathbb{Y}_{11} + b_1\,\mathbb{Y}_{30} + (b_1+b_2+b_3)\,\mathbb{Y}_{21} + \mathbb{Y}_{31} \ . \end{aligned}$$

The interpolation (9.6.1) allows to expand permutations in terms of divided differences.

Indeed, for any permutation σ, one has

$$f(\mathbb{A}^\sigma) = \sum_K f^{\partial^K}\Big|_{\mathbb{A}=\mathbb{B}} \mathbb{Y}_K(\mathbb{A}^\sigma,\mathbb{B}) \ ,$$

and identifying $\mathbb{A}$ and $\mathbb{B}$ gives the following proposition.

PROPOSITION 9.6.2. *Any permutation σ expands, in the degenerate affine Hecke algebra, as a sum of divided differences with coefficients which are specializations of Schubert polynomials:*

$$\sigma = \sum_{K\in\mathbb{N}^\infty} \partial^K\,\mathbb{Y}_K(\mathbb{A}^\sigma,\mathbb{A}) \ . \tag{9.6.3}$$

We had given in (7.4.1) the expansion of any element of the affine Hecke algebra in the basis of permutations, and could have inverted the matrix expressing divided differences, which involved specializations of Schubert polynomials. In fact, we shall see in (10.2.10) that, more generally, Schubert polynomials occur as coefficients in the expansion of Schubert polynomials in other variables.

9.7. Interpolation of Symmetric Functions

The interpolation of symmetric functions is a natural step after interpolation in one variable. It can be studied directly, using only properties of symmetric functions [**33**], we shall prefer obtaining it as a by product of the Newton's formula (9.6.1).

Let $f(\mathbb{A}_n)$ be a function symmetrical in $\{a_1, \dots, a_n\}$. According to the preceding theorem, one can write this function

$$f(\mathbb{A}_n) = \sum\nolimits_I c_I \mathbb{Y}_I(\mathbb{A}, \mathbb{B}) \ . \tag{9.7.1}$$

Let us check that all the indices I corresponding to non zero coefficients c_I are of the type

$$I = [0 \le i_1 \le i_2 \cdots \le i_n \, 0\, 0 \dots] \ .$$

Indeed, $f(\mathbb{A}_n)$ depending only on $a_1, \dots, a_n$, is annihilated by all ∂_k, with $k > n$. Similarly, being symmetrical in $a_1, \dots, a_n$, it is also annihilated by $\partial_1, \dots, \partial_{n-1}$.

Therefore, all the Schubert polynomials appearing on the right side must be annihilated by the same divided differences, because otherwise one would have a sum of Schubert polynomials equal to 0. This is equivalent to say that I can have only the decrease $\dots i_n,\, 0 \dots$. □

These special Schubert polynomials are called *Grassmannian Schubert polynomials*, because they arise in the description of the cohomology of Grassmann manifold[4].

Let us show an example of the interpolation of a symmetric function in two variables, say $\mathbb{Y}_{25}(\mathbb{A}, \mathbb{C})$, where $\mathbb{C}$ is another infinite alphabet. Write $\mathbb{Y}_I \mathbb{Y}_J$ for the product $\mathbb{Y}_I(\mathbb{B}, \mathbb{C}) \mathbb{Y}_J(\mathbb{A}, \mathbb{B})$. Then formula (9.7.1) reads

$$\begin{aligned} \mathbb{Y}_{25}(\mathbb{A}, \mathbb{C}) = {} & \mathbb{Y}_{25}\mathbb{Y}_{00} + \mathbb{Y}_{204}\mathbb{Y}_{01} + \mathbb{Y}_{014}\mathbb{Y}_{11} + \mathbb{Y}_{2003}\mathbb{Y}_{02} + \mathbb{Y}_{0103}\mathbb{Y}_{12} \\ & + \mathbb{Y}_{20002}\mathbb{Y}_{03} + \mathbb{Y}_{0003}\mathbb{Y}_{22} + \mathbb{Y}_{01002}\mathbb{Y}_{13} + \mathbb{Y}_{200001}\mathbb{Y}_{04} + \mathbb{Y}_{00002}\mathbb{Y}_{23} \\ & + \mathbb{Y}_{010001}\mathbb{Y}_{14} + \mathbb{Y}_{200000}\mathbb{Y}_{05} + \mathbb{Y}_{000001}\mathbb{Y}_{24} + \mathbb{Y}_{010000}\mathbb{Y}_{15} + \mathbb{Y}_{000000}\mathbb{Y}_{25} . \end{aligned}$$

In the case $\mathbb{B} = 0$, the above expansion expresses the Schubert polynomial $\mathbb{Y}_{25}(\mathbb{A}, \mathbb{C})$ as a linear combination of all the Schur functions $S_I(a_1 + a_2) = \mathbb{Y}_I(\mathbb{A}, 0)$, $I \subseteq 25$. Cauchy formula (10.2.10) gives more generally the expansion of any polynomial $\mathbb{Y}_v(\mathbb{A}, \mathbb{C})$.

The Grasmannian Schubert polynomials $\mathbb{Y}_I(\mathbb{A}, \mathbb{B})$ are a natural generalization of Schur functions : the leading term of $\mathbb{Y}_I(\mathbb{A}, \mathbb{B})$ (i.e. term of higher degree in $\mathbb{A}$) is the Schur function $S_I(\mathbb{A}_n) = \mathbb{Y}_I(\mathbb{A}, 0)$. They are called *Double Schur functions* by Chen, Li and Louck [**34**], who give a combinatorial description in terms of non-intersecting paths.

They also can be written as determinants. Indeed, let $I = i_1 \le \cdots \le i_n$, and $K = I^\omega + \rho := i_n + n - 1, \dots, i_2 + 1, i_1 + 0$. Still writing $\mathbb{B}_k$ for $\{b_1, \dots, b_k\}$, the Schubert polynomial $\mathbb{Y}_K$ is dominant and equal to

$$\mathbb{Y}_K(\mathbb{A}, \mathbb{B}) = R(a_n, \mathbb{B}_{k_1}) \cdots R(a_1, \mathbb{B}_{k_n}) = S_{K^\omega}(\mathbb{A}_n - \mathbb{B}_{k_1}, \dots, \mathbb{A}_1 - \mathbb{B}_{k_n}) \ .$$

The image of $\mathbb{Y}_K$ under ∂_ω, with $\omega = n \cdots 1$, is by definition $\mathbb{Y}_I$. However, it can also be computed directly. Using the factorization

$$\partial_\omega = (\partial_{n-1})\,(\partial_{n-2}\partial_{n-1}) \cdots (\partial_1 \cdots \partial_{n-1}) \ ,$$

[4] They first occurred as natural generalizations of Chern classes of a difference of two vector bundles [**93**].

one obtains, as for (9.5.1),

$$\mathbb{Y}_I = S_I(\mathbb{A}_n - \mathbb{B}_{k_1}, \ldots, \mathbb{A}_n - \mathbb{B}_{k_n}) \ , \tag{9.7.2}$$

the leading term being evidently the Schur function $S_I(\mathbb{A}_n)$.

The image of $\mathbb{Y}_K(\mathbb{A}, \mathbb{B})$ under $\partial_\omega = \sum_{\sigma \in \mathfrak{S}_n} \pm\sigma\, \Delta^{-1}$ can also be written as the quotient

$$\mathbb{Y}_I = \left| R(a, \mathbb{B}_{k_1}), \ \ldots, \ R(a, \mathbb{B}_{k_n}) \right|_{a \in \mathbb{A}_n} / \, \Delta(\mathbb{A}_n) \ , \tag{9.7.3}$$

i.e.

$$\mathbb{Y}_I = \sum\nolimits_\mu (-1)^{\ell(\mu)} \Big(R(a_1, \mathbb{B}_{k_1}) \cdots R(a_n, \mathbb{B}_{k_n}) \Big)^\mu / \, \Delta(\mathbb{A}_n) \ . \tag{9.7.4}$$

If I has repeated components, one can reduce the preceding summation to a subset of the symmetric group, as for (7.7.3).

Let us describe in more details the simplest case, the case of "double complete functions", when I has only one non-zero component. Let $m \in \mathbb{N}$ and $r = m+n-1$; then

$$\mathbb{Y}_{0^{n-1}m} = \mathbb{Y}_{r\,00\ldots} \partial_1 \cdots \partial_{n-1} = L_{\mathbb{A}_n} \left(R(a_1, \mathbb{B}_r) \right) \ .$$

Leibnitz formula suffices in fact to compute $\mathbb{Y}_{0^{n-1}m}$, and the way that $\mathbb{Y}_{0^{n-1}m}$ is obtained from $\mathbb{Y}_r(\mathbb{A}, \mathbb{B})$ using it can be visualized as follows.

A string of boxes, some of them occupied by a $\bullet$, means a product of linear factors according to the following recipe :

" in an empty box at position j, put the factor $a_i - b_j$, where $i-1$ is equal to the number of dots on the left of the box. Finally erase the $\bullet$'s."

For example,

$$\square\square \bullet\bullet \square \bullet \ \square = (a_1 - b_1)(a_1 - b_2) \bullet\bullet (a_3 - b_5) \bullet (a_4 - b_7) \ .$$

The starting point $\mathbb{Y}_r$ is equal to a string of r boxes. Its image under ∂_1 is equal to the sum of all strings with one dot,

$$\mathbb{Y}_{0\,r-1} = \sum \square \ldots \square \bullet \square \ldots \square \ .$$

The next step, using ∂_2, puts one dot in a box right of the existing dot, and thus $\mathbb{Y}_{0\,0\,r-2}$ is the sum of all strings of length r with two dots :

$$\mathbb{Y}_{00\,r-2} = \sum \square \ldots \square \bullet \ \square \ldots \square \bullet \ \square \ldots \square \ .$$

Finally, $\mathbb{Y}_{0^{n-1}m}$ is equal to the sum of all strings of r boxes with $n-1$ dots. In other words,

$$\mathbb{Y}_{0^{n-1}m} = \sum (a_1{-}b_1) \cdots (a_1{-}b_i)\, (a_2{-}b_{i+2}) \cdots (a_2{-}b_j) \cdots (a_n{-}b_h) \cdots (a_n{-}b_r) \ . \tag{9.7.5}$$

Suppressing the void boxes, one could also write this summation as attached to a diagram of m boxes, with shifts in the indices of the b's given by the *contents* of the boxes.

For more general diagrams[5], with $\mathbb{B} = \{0, 1, 2, \ldots\}$, this construction is due to Biedenharn-Louck [**17**], who defined *factorial Schur functions*. This the case of "Stirling interpolation" at points $\mathbb{B} = \mathbb{N} := \{0, 1, 2, \ldots\}$ of symmetric functions. Factorial Schur functions are indeed specializations of Grassmannian Schubert polynomials at $\mathbb{B} = \mathbb{N}$.

[5]One further generalization is for *Key polynomials* [**102**], using crystal graphs.

In that case, $R(a,\mathbb{N}_n) = a(a-1)\cdots(a-n+1)$ may be considered as a polarization of the nth-power and may be conveniently represented by the symbol $a^{[n]}$. Identity (9.7.3) reads now

$$(9.7.6) \qquad \mathbb{Y}_I(\mathbb{A},\mathbb{N}) = \left| a^{[k_1]},\ \ldots,\ a^{[k_n]} \right|_{a\in\mathbb{A}_n} / \,\Delta(\mathbb{A}_n)\ ,$$

while (9.7.5) becomes

$$(9.7.7)\ \ \mathbb{Y}_{0^{n-1}m} = \sum (a_1)\cdots(a_1{-}i)\,(a_2{-}i{-}2)\cdots(a_2{-}j)\cdots(a_n{-}h)\cdots(a_n{-}r{+}1)\ ,$$

sum on all $-1 \le i \le j \le \cdots \le h \le r$.

Expression (9.7.6) shows that factorial Schur functions can be characterized by their vanishing properties[6], as noticed by Okounkov. Factorial Schur functions play an essential role in the study of central elements in the universal enveloping algebra $\mathfrak{U}(\mathfrak{Gl}(\mathfrak{n}))$ (cf. the work of Olshanski, Okounkov,Nazarov, Molev).

There is a less evident way of relating factorial Schur functions to Schubert polynomials. They now occur as specializations of some Schubert polynomials in $\mathbb{A} = \{1,1,1\ldots\}$ and $\mathbb{B} = \{0,0,0\ldots\}$, the variables[7] being now the components of the vector I indexing the Schubert polynomial. This leads to a theory of binomial determinants, as part of a bijective combinatorics so notoriously illustrated by our friend Xavier Viennot.

The relevant properties of factorial Schur functions are now related to dimensions of representations of the linear groups or unitary groups, as functions of the components of highest weight vectors.

Some operations on symmetric functions can be explicited using Newton's interpolation. Let us mention, for example, the uniform translation $\mathbb{A} \longrightarrow \mathbb{A}^{\dagger} := \{a_1+1, a_2+1, \ldots, a_n+1\}$. Since divided differences are invariant under this translation, (9.7.1) becomes, for any $J = j_1 \le j_2 \le \cdots \le j_n,\ 0, 0, \ldots$:

$$(9.7.8) \qquad \mathbb{Y}_J(\mathbb{A}^{\dagger}) = \sum_I \mathbb{Y}_{J/I}(\mathbf{1},\mathbf{0})\,\mathbb{Y}_J(\mathbb{A},\mathbf{0})\ ,$$

denoting $\mathbb{Y}_{J/I}$ the image of $\mathbb{Y}_J$ under ∂^I, by $\mathbf{1}$ the alphabet $\{1,1,\ldots\}$, by $\mathbf{0}$ the alphabet $\{0,0,\ldots\}$.

In that case, too, the operation $\mathbb{A} \to \mathbb{A}^{\dagger}$ is easier to understand when applied to the argument of a Schubert polynomial rather than only a Schur function.

We shall not explain further how to recover most of the properties of Schur functions from the theory of Schubert polynomials. We would need for that to introduce more combinatorial tools attached to them.

9.8. Key Polynomials

Other operators satisfying braid relations can be used to define new linear bases of the ring of polynomials in $\mathbb{A}$.

We shall now employ isobaric divided differences, instead of divided differences, with the same starting points as for Schubert polynomials.

Key polynomials will be indexed, as Schubert polynomials, by vectors in $\mathbb{N}^{\infty}$ having only a finite number of non-zero components. We identify a finite vector with the infinite one obtained by concanating to it, on its right, an infinite number

[6] More generally, Schubert polynomials can also be characterized by vanishing properties.

[7] Thus, one gets only the restriction of factorial Schur functions on positive integral points, but this is sufficient to determine them.

of 0 components. Recall that for $v \in \mathbb{N}^\infty$, we write vs_i for the exchange of two coordinates:

$$v\, s_i := [v_1, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \ldots] \ .$$

DEFINITION 9.8.1. *Key polynomials* K_v *are defined recursively as follows. If* v *is dominant, then*

$$K_v = \mathbb{Y}_v(\mathbb{A}, \mathbf{0}) = a^v \ .$$

Otherwise, if v and i are such that $v_i > v_{i+1}$, then

$$K_{vs_i} = K_v\, \pi_i \ .$$

The coherence of the definition is insured by the braid relations. Contrary to the case of Schubert polynomials, in the orbit of a polynomial indexed by a dominant v, there is no other dominant polynomial, and therefore there is no compatibility to check between dominant polynomials.

Notice that when $v_i \leq v_{i+1}$, then K_v is symmetrical in a_i, a_{i+1} and preserved by π_i.

It is easy to check that $K_v = a^v +$ terms smaller for the lexicographic order (reading from the right), and therefore K_v has the same leading term than $\mathbb{Y}_v(\mathbb{A}, \mathbb{B})$.

Given $\in \mathbb{N}$ and $\lambda = [l_1 \leq \cdots \leq \lambda_n \leq 0]$ a decreasing partition, then the polynomials $K_\lambda\, \pi_\sigma$, $\sigma \in \mathfrak{S}_n$ are linearly independent (as we see from their leading term), and therefore generate a space of dimension the number of different permutations of λ $(= n!/|\mathfrak{S}_\lambda|$, where $\mathfrak{S}_\lambda$ is the subgroup of $\mathfrak{S}_n$ stabilizing λ).

In the case where v is an increasing partition (followed by zeroes), we have seen that $Y_v(\mathbb{A}, \mathbb{B})$ is a double Schur function. The same property is true for K_v (but we defined key polynomials in only one alphabet).

LEMMA 9.8.2. *Let* $n \in \mathbb{N}$, $I \in \mathbb{N}^n$, $I = [i_1 \leq \cdots \leq i_n]$. *Then* K_I *is equal to the Schur function of index* I, *in* $a_1, \ldots, a_n$, *and thus is equal to* $\mathbb{Y}_I(\mathbb{A}, \mathbf{0})$..

Proof. Same proof as for Grassmannian Schubert polynomials, except that isobaric divided differences increase alphabets without decreasing degree. □

There is a more general family for which $K_v = \mathbb{Y}_v(\mathbb{A}, \mathbf{0})$ (this is the case where v is a *vexillary code*). This happens to be exactly the case when K_v is a multi-Schur function [**106**], [**109**], [**134**].

We shall state in (10.5.4) that, a Schubert polynomial $\mathbb{Y}_v(\mathbb{A}, \mathbf{0})$ decomposes into a positive sum of key polynomials. The sum is reduced to one element iff the code is vexillary.

Exercises

EX. 9.1. For any positive integer p, let $\mathbb{N} =: \{0, 1, \ldots, p\}$. Define the *Stirling numbers* (of the second kind) to be the complete functions in $\mathbb{N}_p$, the (positive) Stirling numbers of the first kind to be the elementary symmetric functions of $\mathbb{N}_p$.

Show that, for any $n \in \mathbb{N}$, one has

$$\begin{aligned} x^n &= S^{n-1}(\mathbb{N}_1)\, x + S^{n-2}(\mathbb{N}_2)\, x(x-1) + S^{n-3}(\mathbb{N}_3)\, x(x-1)(x-2) + \cdots, \\ &= \sum_{i,j} (-1)^i x^{j-i} S^{n-j}(\mathbb{N}_j) \Lambda^i(\mathbb{N}_{i-1}), \\ \frac{1}{z-x} &= \frac{1}{z} + \frac{x}{z(z-1)} + \frac{x(x-1)}{z(z-1)(z-2)} + \cdots + \frac{S^p(x-\mathbb{N}_{p-1})}{S^{p+1}(z-\mathbb{N}_p)} + \cdots \end{aligned}$$

Ex. 9.2. Let x be an element of binomial type, n a positive integer.
Following [**140**], write x^n as a linear combination of $\Lambda^n(x)$, $\Lambda^n(x+1), \ldots, \Lambda^n(x+n-1)$.

Ex. 9.3. Let $f(x)$ be a function of one variable, $\{x_1, x_2, \ldots\}$ be an infinite alphabet. Write $[i,j]$ for the image of $f(x_i)$ under the product of divided differences $\partial_i \cdots \partial_{j-1}$.

For any n, write the product of $\begin{vmatrix} [n,n{+}1] & \cdots & [n,2n] \\ \vdots & & \vdots \\ [1,n{+}1] & \cdots & [1,2n] \end{vmatrix}$ by the two Vandermonde determinants $\Delta(x_1, \ldots, x_n)$ and $\Delta(x_{n+1}, \ldots, x_{2n})$ as a single determinant.

Ex. 9.4. Let $\mathbb{A}$ be arbitrary, $k, n \in \mathbb{N}$, $k \geq n$. Show that the remainder in Newton's interpolation of $f(m) := \Lambda^k(\mathbb{A} + m)$ at points $\{0, 1, \ldots, n{-}1\}$ is equal to

$$\mathcal{R} = \sum_{j=0}^{m-1} \Lambda^n(m-j-1)\, \Lambda^{k-n-1}(\mathbb{A} + j) \ .$$

Ex. 9.5. Given two finite alphabets $\mathbb{B}$, $\mathbb{A} = \{a_1, \ldots, a_n\}$, write the Newton interpolation, at points $a_1, a_2, \ldots$, of any remainder in the division of $R(x, \mathbb{B})$ by $R(x, \mathbb{A})$. Write the coefficients of all Newton's polynomials $(x-a_1)\cdots(x-a_k)$ as minors of the same matrix of order n.

Ex. 9.6. Let $f(x)$ be a polynomial of degree n. After Carlitz, compute the determinant $\left| f(q^{i+j}) \right|_{0 \leq i,j \leq n}$.

Ex. 9.7. Let $n \in \mathbb{N}$, $\mathbb{B}_k = \{b_1, \ldots, b_n\}$, $0 \leq k \leq n$. Write the interpolation of polynomials in x in the basis $\{x^n = S^n(x{-}\mathbb{B}_0), x^{n-1}(x{-}b_1) = S^n(x{-}\mathbb{B}_1), \ldots, S^n(x{-}\mathbb{B}_n)\}$, knowing $f(0)$ and the values $c_i = f(b_i)$, $1 \leq i \leq n$.

Ex. 9.8. Let $m, n \in \mathbb{N}$. Help the rev. Blissard[8][**19**], p.160, finish proving that

$$\frac{1}{m} + \frac{n{+}m}{m(m{-}1)} + \cdots + \frac{(n{+}m)\cdots(n{+}2)}{m\cdots 1} = \frac{1}{n}\left(\frac{(n{+}m)\cdots(n{+}1)}{m\cdots 1} - 1\right) .$$

Ex. 9.9. In 1819, Sarrus, in the Annales de Gergonne **10**, p.220, asked for a direct proof of the identity :

$$2n = \frac{2n}{2n{+}3} + \frac{2n}{2n{+}3}\frac{2n{+}2}{2n{+}5} + \frac{2n}{2n{+}3}\frac{2n{+}2}{2n{+}5}\frac{2n{+}4}{2n{+}7} + \cdots$$

Answer him. Beware that

$$\frac{2n}{2n{+}2} + \frac{2n}{2n{+}2}\frac{2n{+}2}{2n{+}4} + \cdots$$

diverges!

Ex. 9.10. Let $\mathbb{A} = \{a_1, \ldots, a_n\}$, x, y be of rank 1. Prove that

$$\frac{S^{n-1}(x{+}y - \mathbb{A})}{(y{-}a_2)\cdots(y{-}a_n)} = 1 + \frac{x{-}a_1}{y{-}a_2} + \frac{(x{-}a_1)(x{-}a_2)}{(y{-}a_2)(y{-}a_3)} + \cdots + \frac{(x{-}a_1)\cdots(x{-}a_{n-1})}{(y{-}a_2)\cdots(y{-}a_n)} .$$

[8] *I have found this by observation merely. It is doubtless susceptible of proof.*

Ex. 9.11. Let $\mathbb{B}$ be finite, $\mathbb{A}$ be infinite, writing $\mathbb{A}_n = \{a_1, \dots, a_n\}$. Show that for any $k \in \mathbb{N}$, one has

$$S^k(\mathbb{B}) = S^k(\mathbb{A}_1) + S^{k-1}(\mathbb{A}_2)S^1(\mathbb{B}-\mathbb{A}_1) + S^{k-2}(\mathbb{A}_3)S^1(\mathbb{B}-\mathbb{A}_2) + \cdots$$

Ex. 9.12. Let $\mathbb{A}$ be of cardinality $n+1$, $\mathbb{E}$ be of cardinality n, $\mathbb{B}$ be infinite, z be of rank 1. Write $\mathbb{B}_k = \{b_1, \dots, b_k\}$, $k \geq 0$.

Show that

$$\frac{\prod_{e\in\mathbb{E}}(z-e)}{\prod_{a\in\mathbb{A}}(z-a)} = \sum_k \frac{S^k(\mathbb{A}-\mathbb{E}-\mathbb{B}_n)}{S^{k+1}(z-\mathbb{B}_{k+1})} \ .$$

Ex. 9.13. Let $f_1, \dots, f_n$ be functions of one variable, having integrals $F_1, \dots, F_n$. Let $W = W(f_1, \dots, f_n; x_1, \dots, x_n)$ be the discrete Wrońskian $W = \det(f_i \partial_1 \dots \partial_{j-1})_{1\leq i,j\leq n}$, where the divided differences are relative to $\mathbb{X} = \{x_1, \dots, x_n\}$.

Evaluate the integral

$$\int_{y_2}^{y_1} dx_1 \cdots \int_{y_{n+1}}^{y_n} dx_n \, W \, \Delta(X)$$

Ex. 9.14. Let k, n be two positive integers, $\mathbb{B} = \{b_1, b_2, \dots\}$ be an infinite alphabet with periodicity conditions $b_{i+k} = b_i$, $i \in \mathbb{N}$.

Compute the determinant

$$\Big| Y_{0^i, jk-i} \Big|_{1\leq i,j\leq n}$$

in Schubert polynomials in the two alphabets $\mathbb{A} = \{a_1, \dots, a_{n+1}\}$ and $\mathbb{B}$, putting $Y_{0^i,r} = 0$ for $r < 0$.

Ex. 9.15. (Factorization of the Vandermonde matrix).

Let $\mathbb{A}$ be of cardinality n, and $V := \left[a_i^{n-j}\right]_{1\leq i,j\leq n}$. For every $k : 1 \leq k \leq n$, let $\mathbb{A}_k := \{a_1, \dots, a_k\}$. Using Newton to interpolate each a_i^k in $\mathbb{A}_{i-1}$, show that V factorizes into

$$V = \left[R(a_{i-1}, \mathbb{A}_{j-1})\right]\left[S^{j-i}(\mathbb{A}_i)\right]$$

(with $R(x, \emptyset) = 1$, $R(\mathbb{A}, \mathbb{B}) = \prod_{a\in\mathbb{A}, b\in\mathbb{B}}(a-b)$). Decompose these two matrices into products of matrices with entries in the set $\{0, 1, a_1, \dots, a_n, (a_j - a_i) : 1 \leq i < j \leq n\}$.

Ex. 9.16. Let $\mathbb{A}$ be infinite, and, for any k, let $\mathbb{A}_k$ denote $\{a_1, \dots, a_k\}$. Expand the characteristic polynomial of the (right-justified) $n \times n$ Pascal matrix

$$M := \left[\Lambda^{j-i}(\mathbb{A}_{i-1})\right]_{1\leq i,j\leq n} \ .$$

Ex. 9.17. Recall that the *Rogers-Szegö polynomials* are $H_n(x) := S^n((1+x)(1-q)^{-1})\, S^n((1-q)^{-1})^{-1}$, with x, q of rank 1.

For $n \geq 0$, define $Y_n := (x+1)(x+q)\cdots(1+q^{n-1})$.

Find the matrices of change of bases between $\{H_n\}$ and $\{Y_n\}$.

Ex. 9.18. Write $[i] := 1 + q + \cdots + q^{i-1}$. For every positive integer n, show the identity

$$\frac{(x-[n])\cdots(x-[2n-1])}{x\cdots(x-[n-1])} = 1 - \frac{[n]^2}{x} + q^2\frac{[n]^2[n-1]^2}{[2]\,x(x-[1])} -$$
$$q^6\frac{[n]^2[n-1]^2[n-2]^2}{[2][3]\,x(x-[1])(x-[2]} + \cdots \pm q^{n(n-1}\frac{[n]^2[n-1]^2\cdots[1]^2}{[1]\cdots[n]\,x(x-[1])\cdots(x-[n-1])}$$

which is due to Lindelöf for $q = 1$.

CHAPTER 10

The Ring of Polynomials as a Module over Symmetric ones

10.1. Quadratic Form on $\mathfrak{Pol}$

Let us forget totally that we have obtained, in the preceding chapter, a linear basis of the ring of polynomials in an infinite alphabet $\mathbb{A}$. We shall recover Schubert polynomials from a different approach.

Instead of relying on interpolation methods, we now want to generalize the description of the ring of symmetric polynomials: starting with a Cauchy kernel was sufficient to define a scalar product and to obtain the basis of Schur functions.

It is easy to imagine a non-symmetric Cauchy kernel; remarkably, the ensuing theory is a natural extension of the theory of symmetric functions: at the combinatorial level for example, the fundamental objects are now indexed by permutations instead of partitions, and conversely, partitions appear as special permutations.

Let n be a fixed integer, and $\mathbb{A} = \{a_1, a_2, \ldots, a_n\}$. The space that we want to describe is the ring $\mathbb{Z}[\mathbb{A}]$, as a module over $\mathfrak{Sym}(\mathbb{A})$.

To be able to use vanishing properties, we introduce a second alphabet $\mathbb{B} = \{b_1, b_2, \ldots, b_n\}$, and denote by $\mathfrak{Pol}(\mathbb{A})$ the ring

$$\mathfrak{Pol}(\mathbb{A}) = \mathbb{Q}[\mathbb{A}, \mathbb{B}]/\big(\mathfrak{Sym}(\mathbb{A}) = \mathfrak{Sym}(\mathbb{B})\big) \ ,$$

where the ideal $(\mathfrak{Sym}(\mathbb{A}) = \mathfrak{Sym}(\mathbb{B}))$ is the ideal generated by all $f(\mathbb{A}) - f(\mathbb{B})$, f symmetrical. We shall consider $\mathfrak{Pol}(\mathbb{A})$ as a module over $\mathfrak{Sym}(\mathbb{A})$, which is now defined to be the ring of symmetric functions in $\mathbb{A}$ with coefficients in $\mathbb{B}$.

We have already stated that $\mathfrak{Pol}(\mathbb{A})$ is a $\mathfrak{Sym}(\mathbb{A})$-module, with basis the monomials a^I, $I \leq \rho := [n{-}1, \ldots, 1, 0]$. We shall now give a more precise description of this free module.

First, the simplest method of computation in a free module is to have a quadratic form, and pairs of adjoint bases, or even better, orthonormal bases.

An appropriate quadratic form[1] is defined by using the maximal divided difference of $\mathfrak{S}(\mathbb{A})$:

$$\mathfrak{Pol}(\mathbb{A}) \ni f, g \Rightarrow (f\ ,\ g) = (fg\ ,\ 1) := fg\,\partial_\omega \in \mathfrak{Sym}(\mathbb{A}) \tag{10.1.1}$$

LEMMA 10.1.1. *The divided differences ∂_i, $i = 1, \ldots, n{-}1$, are self-adjoint with respect to $(\ ,\ )$. Therefore, for $\sigma \in \mathfrak{S}(\mathbb{A})$, ∂_σ is adjoint to $\partial_{\sigma^{-1}}$.*

Moreover σ is adjoint to $(-1)^{\ell(\sigma)}\sigma^{-1}$.

[1] It extends the intersection form in the cohomology ring of the flag manifold.

Proof. Given any $i < n$, factorize $\omega = s_i\,\sigma$. Then $(f\,\partial_i, g) = \big(f\partial_i\, g\,\partial_i\big)\partial_\sigma$. Because $f\,\partial_i$ is a scalar for ∂_i, the preceding expression is equal to

$$\Big((f\partial_i)\,(g\partial_i)\Big)\,\partial_\sigma$$

and the symmetry between f and g implies that it is also equal to $(f\,,\,g\partial_i)$. By product, one gets the assertion for every ∂_σ.

Similarly

$$(f\sigma\,,\,g) = f^\sigma\,g\,\partial_\omega = f\,g^{\sigma^{-1}}\,\sigma\partial_\omega = (-1)^{\ell(\sigma)} f\,g^{\sigma^{-1}}\,\partial_\omega = (-1)^{\ell(\sigma)}(f\,,\,g\sigma^{-1})\ .$$

□

10.2. Kernel

Let

$$\mathbf{K}_n(\mathbb{A},\mathbb{B}) := \prod_{1\le i<j\le n} (b_i - a_j)\ . \tag{10.2.1}$$

This function has fundamental and evident vanishing properties:

LEMMA 10.2.1. *For any permutation $\sigma \in \mathfrak{S}_n$, the specialization $\mathbf{K}_n(\mathbb{A}^\sigma,\mathbb{B})\Big|_{\mathbb{A}=\mathbb{B}}$ vanishes, except the case $\mathbf{K}_n(\mathbb{B},\mathbb{B}) = \Delta(\mathbb{B})$.*

Proof. By induction on n, one sees that all specializations have at least a factor $b_i - b_i$, except in the case of the identity permutation. □

The following proposition shows that $\mathbf{K}_n(\mathbb{A},\mathbb{B})$ is a reproducing kernel.

PROPOSITION 10.2.2. *For any $f \in \mathfrak{Pol}(\mathbb{A})$, one has*

$$\big(f(\mathbb{A})\,,\,\mathbf{K}_n(\mathbb{A},\mathbb{B})\big) \equiv f(\mathbb{B}) \tag{10.2.2}$$

(we have written $\equiv$ instead of an equality to point out that we are identifying symmetric functions in $\mathbb{A}$ with those of $\mathbb{B}$).

Proof. Any polynomial in $\mathbb{A}$ can be written as a linear combination of a^I, $I \le \rho = [n-1,\ldots,1,0]$, with coefficients in $\mathfrak{Sym}(\mathbb{A})$. However, $a^I\,\mathbf{K}_n(\mathbb{A},\mathbb{B})$ is a linear combination of monomials a^{I+J}, $I \le \rho$, $J \le [0,1,\ldots n{-}1]$, that is, of monomials having an exponent $I+J \le (n-1)^n$. The image of such a monomial under ∂_ω is 0, except if all its exponents are different, and this is precisely the case where $I+J$ is a permutation of ρ. In that case $a^{I+J}\,\partial_\omega = \pm 1$.

Therefore, the image of each monomial a^{I+J} is independent of $\mathbb{A}$, and one can specialize $\mathbb{A}$ to $\mathbb{B}$ in the expression

$$f(\mathbb{A})\mathbf{K}_n(\mathbb{A},\mathbb{B})\,\partial_\omega = \sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)} f(\mathbb{A}^\sigma)\mathbf{K}_n(\mathbb{A}^\sigma,\mathbb{B})\frac{1}{\Delta(\mathbb{A})}$$

However $\mathbf{K}_n(\mathbb{B}^\sigma,\mathbb{B})$ is 0 if σ is not the identity permutation, and the summation reduces to a single term

$$f(\mathbb{B})\mathbf{K}_n(\mathbb{B},\mathbb{B})/\Delta(\mathbb{B}) = f(\mathbb{B})\ .$$

□

Now, we shall see that Schubert polynomials are compatible with the quadratic form (they are "almost" an orthonormal basis, apart from a twist in indices).

We first need to write them differently, using permutations rather than vectors to index them. Indeed, given $\sigma = [\sigma_1, \ldots, \sigma_n] \in \mathfrak{S}(n)$, its *code* is the vector $c[\sigma] = [c_1, \ldots, c_n]$ such that $c_i := card(j > i, \sigma_j < \sigma_i)$, $1 \le i \le n$.

It is immediate that $\sigma \mapsto c[\sigma]$ is a bijection between $\mathfrak{S}(n)$ and the vectors $c \in \mathbb{N}^n$ such that $c \le \rho$ (i.e. such that $c_i \le n - i$, $i = 1, \ldots, n$).

We shall write $\mathbb{X}_\sigma$ or $\mathbb{Y}_{c(\sigma)}$ indifferently for the same Schubert polynomial.

Given a permutation in $\mathfrak{S}(n)$, there exists at least one path in the permutohedron, from $\omega = [n, \ldots, 1]$ to it, and therefore all Schubert polynomials $\mathbb{X}_\sigma$, $\sigma \in \mathfrak{S}_n$ can be obtained from a top one $\mathbb{X}_\omega$.

In summary, Schubert polynomials for $\mathfrak{S}(n)$ are defined by

$$\left\{ \begin{array}{l} \mathbb{X}_\omega(\mathbb{A}, \mathbb{B}) = \prod_{1 \le i,j \le n,\, i+j \le n} (a_i - b_j) \\ \mathbb{X}_\sigma(\mathbb{A}, \mathbb{B}) = \mathbb{X}_\omega(\mathbb{A}, \mathbb{B})\, \partial_{\omega\sigma} \end{array} \right. \tag{10.2.3}$$

Notice that $\mathbf{K}_n(\mathbb{A}, \mathbb{B}) = (-1)^{\binom{n}{2}}\, \mathbb{X}_\omega(\mathbb{A}^\omega, \mathbb{B})$.

In fact, we have decomposed the resultant $R(\mathbb{A}, \mathbb{B})$ into the three factors $\mathbb{X}_\omega(\mathbb{A}, \mathbb{B})$, $\pm\mathbf{K}_n(\mathbb{A}, \mathbb{B})$, and the diagonal $\prod_i (a_i - b_i)$. Graphically, a box in position $[i, j]$ standing for a factor $(a_i - b_j)$, this decomposition is

$$R(\{a_1, a_2, a_3\}, \{b_1, b_2, b_3, b_4\}) = \begin{array}{cccc} \heartsuit & \blacksquare & \blacksquare & \blacksquare \\ \square & \heartsuit & \blacksquare & \blacksquare \\ \square & \square & \heartsuit & \blacksquare \\ \square & \square & \square & \heartsuit \end{array} ,$$

the three factors being represented by $\square$, $\blacksquare$ and $\heartsuit$ respectively.

The preceding definition implies the recursions

$$\mathbb{X}_\sigma(\mathbb{A}, \mathbb{B})\partial_i = \mathbb{X}_{\sigma s_i}(\mathbb{A}, \mathbb{B}) \quad \text{if } \ell(\sigma s_i) < \ell(\sigma) . \tag{10.2.4}$$

(because $\partial_i^2 = 0$, then $\mathbb{X}_\sigma(\mathbb{A}, \mathbb{B})\partial_i = 0$ if $\ell(\sigma s_i) > \ell(\sigma)$).

For $\mathfrak{S}_3$, the Schubert polynomials are

$$\begin{array}{ccccc}
 & & (a_1-b_1)(a_1-b_2)(a_2-b_1) & & \\
 & \partial_1 \;/\!/ & & \diagdown\; \partial_2 & \\
(a_1-b_1)(a_2-b_1) & & & & (a_1-b_1)(a_1-b_2) \\
\partial_2 \Big| & & & & \Big\| \partial_1 \\
(a_1-b_1) & & & & a_1+a_2-b_1-b_2 \\
 & \partial_1 \;\diagdown\!\diagdown & & / \;\partial_2 & \\
 & & 1 & &
\end{array}$$

Proposition (10.2.2) can be generalized.

THEOREM 10.2.3. *For any $f \in \mathfrak{Pol}(\mathbb{A})$, any permutation $\sigma \in \mathfrak{S}(n)$, one has*

$$\big(f(\mathbb{A}),\, (-1)^{\ell(\sigma\omega)} \mathbb{X}_{\sigma\omega}(\mathbb{A}^\omega, \mathbb{B})\big) \equiv f\, \partial_{\sigma^{-1}}(\mathbb{B}) . \tag{10.2.5}$$

Proof. One has

$$\mathbb{X}_{\sigma\omega}(\mathbb{A}^\omega, \mathbb{B}) = \mathbb{X}_\omega(\mathbb{A}, \mathbb{B})\, \partial_{\omega\sigma\omega}\, \omega = \mathbb{X}_\omega(\mathbb{A}, \mathbb{B})\omega\, \big(\omega \partial_{\omega\sigma\omega}\, \omega\big) = \mathbb{X}_\omega(\mathbb{A}, \mathbb{B})\, \omega\, \partial_\sigma ,$$

and one can shift the divided difference ∂_σ to the left of $(\,,\,)$ thanks to Lemma (10.1.1). Now the formula results from proposition 10.2.2 for the function $f\,\partial_{\sigma^{-1}}$. $\square$

The theorem shows a symmetry property of Schubert polynomials which is not evident from their definition with operators acting on $\mathbb{A}$ only. Introducing $\mathbb{C}$, a third alphabet of cardinality n, take $f(\mathbb{A}) = \mathbb{X}_\omega(\mathbb{A},\mathbb{C})$. Then (10.2.5) implies

$$\big(\mathbb{X}_\omega(\mathbb{A},\mathbb{C})\,,\ (-1)^{\ell(\sigma\omega)}\mathbb{X}_{\sigma\omega}(\mathbb{A},\mathbb{B})\omega\big) = \mathbb{X}_{\omega\sigma^{-1}}(\mathbb{B},\mathbb{C})\ .$$

On the other hand, putting ω on the left,

$$\big((-1)^{\ell(\omega)}\mathbb{X}_\omega(\mathbb{A}^\omega,\mathbb{C})\,,\ (-1)^{\ell(\sigma\omega)}\mathbb{X}_{\sigma\omega}(\mathbb{A},\mathbb{B})\omega\big) = \mathbb{X}_{\sigma\omega}(\mathbb{C},\mathbb{B})\ ,$$

because $(-1)^{\ell(\omega)}\mathbb{X}_\omega(\mathbb{A}^\omega,\mathbb{C})$ is a reproducing kernel. Therefore, one has the symmetry :

$$\mathbb{X}_\sigma(\mathbb{A},\mathbb{B}) = (-1)^{\ell(\sigma)}\mathbb{X}_{\sigma^{-1}}(\mathbb{B},\mathbb{A})\ . \tag{10.2.6}$$

COROLLARY 10.2.4. *The quadratic form $(\,,\,)$ is non-degenerate. The space $\mathfrak{Pol}(\mathbb{A})$ is a free module over $\mathfrak{Sym}$, with the two adjoint bases $\{\mathbb{X}_\sigma(\mathbb{A},\mathbb{B})\}$ and $\{(-1)^{\ell(\sigma\omega)}\mathbb{X}_{\sigma\omega}(\mathbb{A}^\omega,\mathbb{B}) : \sigma\in\mathfrak{S}(\mathbb{A})\}$.*

More precisely

$$\big(\mathbb{X}_\sigma(\mathbb{A},\mathbb{B})\,,\ (-1)^{\ell(\nu\omega)}\mathbb{X}_{\nu\omega}(\mathbb{A}^\omega,\mathbb{B})\big) = \delta_{\sigma,\nu}\ .$$

Any element of $\mathfrak{Pol}(\mathbb{A})$ expands into the basis of Schubert polynomials $\mathbb{X}_\sigma$, $\sigma\in\mathfrak{S}(\mathbb{A})$, as follows :

$$\begin{aligned} f(\mathbb{A}) &\equiv \sum_\sigma \big(f\,,\ (-1)^{\ell(\sigma\omega)}\mathbb{X}_{\sigma\omega}(\mathbb{A}^\omega,\mathbb{B})\big)\,\mathbb{X}_\sigma(\mathbb{A},\mathbb{B}) \\ &\equiv \sum_\sigma f\partial_{\sigma^{-1}}(\mathbb{B})\,\mathbb{X}_\sigma(\mathbb{A},\mathbb{B}) \end{aligned} \tag{10.2.7}$$

Newton's interpolation in several variables has given another expansion in terms of Schubert polynomials, this time with coefficients independent of $\mathbb{A}$ and indexed by permutations of arbitrary order; in expansion (10.2.7), the coefficients, on the contrary, belong to $\mathfrak{Sym}(\mathbb{A})$. For example, for $n=2$, this expansion is

$$a_1^2 \equiv b_1^2 + (a_1+a_2)(a_1-b_1)$$

while Newton states

$$a_1^2 = b_1^2 + (b_1+b_2)(a_1-b_1) + (a_1-b_1)(a_1-b_2)\ .$$

Both equations coincide modulo $\mathfrak{Sym}(\mathbb{A}) = \mathfrak{Sym}(\mathbb{B})$.

Notice that the two expansions are identical when $f(\mathbb{A})$ is a linear combination of the a^I, $I\le\rho$, with coefficients in $\mathbb{B}$, because no symmetric function of $\mathbb{A}$ (except constants) is used in that case.

Let us show now that the interpretation of divided differences as scalar products allows to express them in the basis of permutations.

PROPOSITION 10.2.5. *Let $\sigma\in\mathfrak{S}_n$. Let $\partial_\sigma = \sum\zeta\,c_\zeta^\sigma$ be the expansion of ∂_σ in the basis of permutations. Then*

$$(-1)^{\ell(\zeta)}\,c_\zeta^\sigma = (-1)^{\ell(\omega\sigma)}\,\mathbb{X}_{\sigma^{-1}\omega}(\mathbb{A}^{\omega\zeta},\mathbb{A})\frac{1}{\Delta(\mathbb{A})} = \mathbb{X}_{\omega\sigma}(\mathbb{A},\mathbb{A}^{\omega\zeta})\frac{1}{\Delta(\mathbb{A})}\ . \tag{10.2.8}$$

Proof. Let us replace, in formula (10.2.5)

$$\big(f(\mathbb{A})\,,\ (-1)^{\ell(\omega\sigma)}\mathbb{X}_{\sigma^{-1}\omega}(\mathbb{A}^{\omega}, \mathbb{B})\big) \equiv f\,\partial_{\sigma}\,(\mathbb{B})$$

the scalar product by its expression $(f,g) = \sum \pm (fg)^{\sigma}\Delta(\mathbb{A})^{-1}$. It becomes

$$\sum_{\zeta} \pm f(\mathbb{A}^{\zeta})\mathbb{X}_{\sigma^{-1}\omega}(\mathbb{A}^{\omega\zeta}, \mathbb{A})\big)\,\frac{1}{\Delta} = f(\mathbb{A})\partial_{\sigma}\ ;$$

erasing f which is irrelevant, one gets the proposition. □

We had written in (7.4.1) the expansion of any element of $\mathfrak{Diff}$ by making it act on $\mathbf{K}_n(\mathbb{A}, \mathbb{B})$. Since $\mathbf{K}_n(\mathbb{A}, \mathbb{B}) = (-1)^{\ell(\omega)}\,\mathbb{X}_{\omega}(\mathbb{A}^{\omega}, \mathbb{B})$ and since $\omega\partial_{\sigma}\omega = \partial_{\omega\sigma\omega}$, the two expressions involve specializations of Schubert polynomials, but not the same polynomials, as mentioned in [**115**]. One has in fact, for any pair of permutations,

$$(-1)^{\ell(\sigma)}\mathbb{X}_{\sigma}(\mathbb{A}, \mathbb{A}^{\zeta}) = \mathbb{X}_{\omega\sigma\omega}(\mathbb{A}^{\omega}, \mathbb{A}^{\omega\zeta})\ . \tag{10.2.9}$$

One can view Schubert polynomials as a basis of $\mathfrak{Pol}(\mathbb{A})$ depending on parameters $b_1, \ldots, b_n$, and let the parameters vary. Indeed, let $\mathbb{C} = \{c_1, \ldots, c_n\}$. Then the following theorem shows that the matrix of change of bases has Schubert polynomials as entries. The case $\zeta = \omega$ should be considered as the *non-symmetric Cauchy formula* :

THEOREM 10.2.6. *Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ be three alphabets of cardinality $\geq n$, and let $\zeta \in \mathfrak{S}(n)$. Then*

$$\mathbb{X}_{\zeta}(\mathbb{A}, \mathbb{C}) = \sum_{\eta,\sigma} \mathbb{X}_{\eta}(\mathbb{B}, \mathbb{C})\,\mathbb{X}_{\sigma}(\mathbb{A}, \mathbb{B})\ , \tag{10.2.10}$$

sum over all reduced factorizations $\zeta = \eta\sigma$ (i.e. factorizations such that $\ell(\sigma) + \ell(\eta) = \ell(\zeta)$).

Proof. One applies corollary(10.2.4) to the function $f(\mathbb{A}) = \mathbb{X}_{\zeta}(\mathbb{A}, \mathbb{C})$. Since $f(\mathbb{A})$ belongs to the span of monomials in $\mathbb{A}$ of degree $\leq \rho$, $\equiv$ is in fact an equality. Moreover, to require non nullity of $\mathbb{X}_{\zeta}\partial_{\eta^{-1}}$ is the same as to require that the product $(\zeta\eta^{-1})(\eta)$ be reduced. □

For example, for $\zeta = [2,4,1,3]$, the expansion is

$$\mathbb{X}_{2413}(\mathbb{A}, \mathbb{C}) = \begin{array}{c} \mathbb{X}_{2413}(\mathbb{B}, \mathbb{C})\mathbb{X}_{1234}(\mathbb{A}, \mathbb{B}) \\ -\mathbb{X}_{2143}(\mathbb{B}, \mathbb{C})\mathbb{X}_{1324}(\mathbb{A}, \mathbb{B}) \\ +\mathbb{X}_{1243}(\mathbb{B}, \mathbb{C})\mathbb{X}_{2314}(\mathbb{A}, \mathbb{B})\ +\ \mathbb{X}_{2134}(\mathbb{B}, \mathbb{C})\mathbb{X}_{1423}(\mathbb{A}, \mathbb{B}) \\ -\mathbb{X}_{1234}(\mathbb{B}, \mathbb{C})\mathbb{X}_{2413}(\mathbb{A}, \mathbb{B}) \end{array}\ .$$

The special case $\zeta = \omega$ is worth of interest. It states that $\prod_{i+j\leq n}(a_i - c_j)$ is the sum of all products $\mathbb{X}_{\omega\sigma^{-1}}(\mathbb{B}, \mathbb{C})\,\mathbb{X}_{\sigma}(\mathbb{A}, \mathbb{B})$, where $\mathbb{B}$ has been chosen arbitrarily. One can take $\mathbb{B} = 0$, but also $\mathbb{B} = \{0, 1, 2, \ldots\}$ or $\{q^0, q^1, q^2, \ldots\}$.

The usual Cauchy formula for the resultant follows from (10.2.10). Indeed, given $n, m \in \mathbb{N}$, let $\zeta = [m{+}1, \ldots, m{+}n, 1, \ldots, m]$. Then

$$\mathbb{X}_{\zeta}(\mathbb{A}, \mathbb{B}) = \prod_{1\leq i\leq n}\ \prod_{1\leq j\leq m} (a_i - b_j) = R(\mathbb{A}_n, \mathbb{B}_m)\ ,$$

and the expansion

$$\mathbb{X}_{\zeta}(\mathbb{A}, \mathbb{B}) = \sum_{\eta,\sigma} \mathbb{X}_{\eta}(\mathbf{0}, \mathbb{B})\,\mathbb{X}_{\sigma}(\mathbb{A}, \mathbf{0})$$

involves only Schubert polynomials which are symmetrical in $\mathbb{A}_n$, or $\mathbb{B}_m$, thus, which are Schur functions. It just remains to recognize that reduced factorizations of ζ correspond to pair of partitions complementary in m^n (taking the codes of σ and η^{-1}).

10.3. Shifts

A Schubert polynomial indexed by a permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma_n = n$ can be obtained from the dominant Schubert polynomial $\mathbb{Y}_{n-2,\ldots,1,0,0}$, instead of starting with $\mathbb{Y}_{n-1,\ldots,1,0}$. Therefore Schubert polynomials are invariant under the embedding $\mathfrak{S}_{n-1} \hookrightarrow \mathfrak{S}_{n-1}\times\mathfrak{S}_1$, which consists in adding to a permutation in $\mathfrak{S}_{n-1}$ a fixed point n; equivalently, one concatenates to the right of the code a zero:

$$\sigma \in \mathfrak{S}_{n-1} \Rightarrow \ \mathbb{X}_\sigma = \mathbb{X}_{[\sigma,n]} \quad \& \quad \mathbb{Y}_I = \mathbb{Y}_{I,0} \ .$$

We also need the embedding $\mathfrak{S}_{n-1} \hookrightarrow \mathfrak{S}_1\times\mathfrak{S}_{n-1}$. It induces the transformation

$$\mathbb{X}_\sigma(\mathbb{A},\mathbb{B}) \to \mathbb{X}_{\sigma^+}(\mathbb{A},\mathbb{B}) \quad \& \quad \mathbb{Y}_I(\mathbb{A},\mathbb{B}) \to \mathbb{Y}_{0,I}(\mathbb{A},\mathbb{B})$$

on the Schubert basis, where $\sigma^+ = [1, \sigma_1{+}1, \ldots, \sigma_{n-1}{+}1]$, and we shall now describe this embedding at the level of polynomials.

Let us also denote, for any alphabet $\mathbb{A}$, the shift of indices by $\mathbb{A}^+ = \{a_2, a_3, \ldots\}$.

PROPOSITION 10.3.1. *Let $\sigma \in \mathfrak{S}_n$, I be its code. Then*

$$\mathbb{X}_{\sigma^+}(\mathbb{A},\mathbb{B}) = \mathbb{X}_\sigma(\mathbb{A},\mathbb{B}^+)\,(a_1{-}b_1)\cdots(a_n{-}b_1)\,\partial_n\cdots\partial_1 \ , \tag{10.3.1}$$

$$= \mathbb{X}_\sigma(\mathbb{A},\mathbb{B}^+)\,\big(1+\partial_n(a_{n+1}{-}b_1)\big)\cdots\big(1+\partial_1(a_2{-}b_1)\big) \ , \tag{10.3.2}$$

$$= \mathbb{X}_\sigma(\mathbb{A},\mathbb{B}^+)\,(\pi_n - b_1\partial_n)\cdots(\pi_1 - b_1\partial_1) \ . \tag{10.3.3}$$

Equivalently,

$$\mathbb{Y}_{0,I}(\mathbb{A},\mathbb{B}) = \mathbb{Y}_I(\mathbb{A},\mathbb{B}^+)\,(a_1{-}b_1)\cdots(a_n{-}b_1)\,\partial_n\cdots\partial_1 \ . \tag{10.3.4}$$

Proof. Let us first treat the case of $\mathbb{X}_{\omega^+}$.

$$\mathbb{X}_{\omega^+}(\mathbb{A},\mathbb{B}) = \mathbb{X}_{n+1,\ldots,1}(\mathbb{A},\mathbb{B})\,\partial_n\cdots\partial_1 = \mathbb{X}_\omega(\mathbb{A},\mathbb{B}^+)(a_1{-}b_1)\cdots(a_n{-}b_1)\partial_n\cdots\partial_1 \ ,$$

the first formula is indeed true for ω.

Now, $\mathbb{X}_\sigma(\mathbb{A},\mathbb{B}^+)$ is the image of $\mathbb{X}_\omega(\mathbb{A},\mathbb{B}^+)$ under some product of divided differences ∂_i, with $1 \le i \le n{-}1$. One has to study an expression of the type

$$\mathbb{X}_\omega(\mathbb{A},\mathbb{B}^+)\partial_i\cdots\partial_j\,(a_1{-}b_1)\cdots(a_n{-}b_1)\,\partial_n\cdots\partial_1 \ .$$

But each divided difference ∂_i commutes with the factor $(a_1{-}b_1)\cdots(a_n{-}b_1)$, because it is symmetrical in a_i, a_{i+1}. Moreover, the braid relation

$$\partial_i\,(\partial_n\cdots\partial_1) = (\partial_n\cdots\partial_1)\,\partial_{i+1}$$

shows that one can put all these divided differences on the right, increasing their index by 1. Finally, the RHS of (10.3.1) is equal to

$$\mathbb{X}_\omega(\mathbb{A},\mathbb{B}^+)(a_1{-}b_1)\cdots(a_n{-}b_1)\partial_{i+1}\cdots\partial_{j+1} = \mathbb{X}_{\omega^+}(\mathbb{A},\mathbb{B})\partial_{i+1}\cdots\partial_{j+1} \ ,$$

and (10.3.1) is true in general.

By Leibnitz' formula,

$$(a_n{-}b_1)\cdots(a_1{-}b_1)\,\partial_n = \Big(1+\partial_n(a_{n+1}{-}b_1)\Big)(a_{n-1}{-}b_1)\cdots(a_1{-}b_1) \ ,$$

or

$$(a_n{-}b_1)\cdots(a_1{-}b_1)\,\partial_n = \Big(\pi_n - b_1\partial_n\Big)\Big)(a_{n-1}{-}b_1)\cdots(a_1{-}b_1) \ .$$

Iterating, on gets the second and third expressions. $\square$

In the case where σ is a dominant permutation, we shall check directly the preceding formulas. Let $\lambda \in \mathbb{N}^n$ be a decreasing partition.

The Schubert polynomial $\mathbb{Y}_{0,\lambda}(\mathbb{A},\mathbb{B})$ is the image of $\mathbb{Y}_{\lambda+1^n}(\mathbb{A},\mathbb{B})$ under $\partial_n \cdots \partial_1$. Writing

$$\mathbb{Y}_{\lambda+1^n}(\mathbb{A},\mathbb{B}) = S_{\lambda_n+1;\ldots;\lambda_1+1}(\mathbb{A}_n - \mathbb{B}_{\lambda_n+1};\ldots;\mathbb{A}_1 - \mathbb{B}_{\lambda_1+1}) ,$$

one gets

$$\mathbb{Y}_{0,\lambda}(\mathbb{A},\mathbb{B}) = S_{\lambda_n;\ldots;\lambda_1}(\mathbb{A}_{n+1} - \mathbb{B}_{\lambda_n+1};\ldots;\mathbb{A}_2 - \mathbb{B}_{\lambda_1+1}) ,$$

the effect of each divided difference ∂_i in $\partial_n \cdots \partial_1$ being to increase $\mathbb{A}_i$ to $\mathbb{A}_{i+1}$, and decrease $\lambda_i + 1$ to λ_i.

By iteration, one gets the following lemma.

LEMMA 10.3.2. *Let $\lambda \in \mathbb{N}^n$ be a decreasing partition, $k \in \mathbb{N}$. Then*

$$\mathbb{Y}_{0^k,\lambda}(\mathbb{A},\mathbb{B}) = S_{\lambda_n;\ldots;\lambda_1}(\mathbb{A}_{n+k} - \mathbb{B}_{\lambda_n+k};\ldots;\mathbb{A}_{1+k} - \mathbb{B}_{\lambda_1+k}) . \tag{10.3.5}$$

Extending the case (9.5.1), we shall call such a code $[0^k,\lambda]$ an *almost dominant code*, and correspondingly, define an *almost dominant permutation* and an *almost dominant Schubert polynomial.*

We now want to check (10.3.3). We have first to see what is the image of a complete function $S^k(\mathbb{A}_n - \mathbb{B}_j^+)$ under $(a_n - b_1)\partial_1$.

Since $S^k(\mathbb{A}_n - \mathbb{B}_j^+) = S^k(\mathbb{A}_n + b_1 - \mathbb{B}_{j+1})$, $S^k(\mathbb{A}_n - \mathbb{B}_j^+)$ is sent to

$$S^k(\mathbb{A}_n + b_1 - \mathbb{B}_{j+1}) - b_1 S^{k-1}(\mathbb{A}_n + b_1 - \mathbb{B}_{j+1}) = S^k(\mathbb{A}_n + b_1 - \mathbb{B}_{j+1} - b_1) = S^k(\mathbb{A}_n - \mathbb{B}_{j+1}) .$$

Therefore, the image of $S_{\lambda_n+1;\ldots;\lambda_1+1}(\mathbb{A}_n - \mathbb{B}^+_{\lambda_n};\ldots;\mathbb{A}_1 - \mathbb{B}^+_{\lambda_1})$ under $(a_n - b_1)\partial_1$ is

$$S_{\lambda_n;\lambda_{n-1}+1;\ldots;\lambda_1+1}(\mathbb{A}_{n+1} - \mathbb{B}_{\lambda_n+1};\mathbb{A}_{n-1} - \mathbb{B}^+_{\lambda_{n-1}};\ldots,\mathbb{A}_1 - \mathbb{B}^+_{\lambda_1}) .$$

By iteration, one obtains that

$$\mathbb{Y}_{0,\lambda}(\mathbb{A},\mathbb{B}) = S_{\lambda_n+1;\ldots;\lambda_1+1}(\mathbb{A}_n - \mathbb{B}^+_{\lambda_n};\ldots;\mathbb{A}_1 - \mathbb{B}^+_{\lambda_1})\,(a_n - b_1)\partial_n \cdots (a_1 - b_1)\partial_1 ,$$

which is Eq. 10.3.4.

10.4. Generating Function in the NilCoxeter Algebra

Several formulas given in the preceding section involve taking reduced factorizations. This restriction can be conveniently implemented by using generating functions in an extra algebra of divided differences (not acting on the variables a_i, nor b_i). We shall follow the common terminology and call it the *NilCoxeter algebra.*

DEFINITION 10.4.1. Let $n \in \mathbb{N}$. The *NilCoxeter algebra* $\mathfrak{NC}_n$ of order n is the algebra with $\mathbb{Z}$-linear basis u_σ, $\sigma \in \mathfrak{S}_n$, satisfying

$$u_\sigma u_\nu = u_{\sigma\nu} \text{ if } \ell(\sigma\nu) = \ell(\sigma) + \ell(\nu) \qquad \& \qquad u_\sigma u_\nu = 0 \text{ otherwise} . \tag{10.4.1}$$

Equivalently, $\mathfrak{NC}_n$ is generated by $u_1 = u_{s_1}, \ldots, u_{n-1} = u_{s_{n-1}}$ satisfying $u_i^2 = 0$ together with the braid relations

$$u_i u_j = u_j u_i \text{ if } |i-j| \geq 2 \quad , \quad u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} . \tag{10.4.2}$$

This algebra has been used in [**107**] to give a decomposition of Schubert polynomials according to the powers of the first variable.

Define the *NilCauchy kernel* to be

$$\mathfrak{NK}_n(\mathbb{A},\mathbb{B}) = \sum_{\sigma \in \mathfrak{S}_n} \mathbb{X}_\sigma(\mathbb{A},\mathbb{B})\, u_\sigma . \tag{10.4.3}$$

Formulas involving reduced factorizations are conveniently written in $\mathfrak{NC}_n$. Indeed, the product fg of two general elements of $\mathfrak{NC}_n$, $f = \sum f_\eta u_\eta$ and $g = \sum g_\sigma u_\sigma$ is such that

$$fg = \sum_{\zeta\in\mathfrak{S}_n} (fg)_\zeta\, u_\zeta \quad \text{with } (fg)_\zeta = \sum_{\eta,\sigma:\eta\sigma=\zeta} f_\eta g_\sigma\,,$$

where only reduced products $\eta\sigma$ appear, because otherwise $u_\eta u_\sigma = 0$.

One can now translate Eq. 10.2.10 into the following theorem :

THEOREM 10.4.2. *Let $\mathbb{A},\mathbb{B},\mathbb{C}$ be infinite alphabets, let n be a positive integer. Then*

$$\mathfrak{NK}_n(\mathbb{A},\mathbb{C}) = \mathfrak{NK}_n(\mathbb{B},\mathbb{C})\,\mathfrak{NK}_n(\mathbb{A},\mathbb{B})\ . \tag{10.4.4}$$

In particular,

$$\mathfrak{NK}_n(\mathbb{A},\mathbb{C}) = \mathfrak{NK}_n(\mathbf{0},\mathbb{C})\,\mathfrak{NK}_n(\mathbb{A},\mathbf{0})\ , \tag{10.4.5}$$

$$1 = \mathfrak{NK}_n(\mathbf{0},\mathbb{A})\,\mathfrak{NK}_n(\mathbb{A},\mathbf{0})\ . \tag{10.4.6}$$

We shall now show that the NilCauchy kernel $\mathfrak{NK}_n(\mathbb{A},\mathbf{0})$ can be factorized into simple factors of the type $(1+a_iu_j)$. But we first give a factorization which is equivalent to the expansion of Schubert polynomials stated in Prop. 1.7 of [**107**]. Let us extend the shift of permutations and alphabets to the NilCoxeter algebra and put $u_\nu^+ = u_{\nu^+}$, e.g. $u_{3412}^+ = u_{14523}$.

PROPOSITION 10.4.3. *The NilCauchy kernel factorizes into*

$$\mathfrak{NK}_n(\mathbb{A},\mathbf{0}) = (1+a_1u_{n-1})\cdots(1+a_1u_1)\sum_{\nu\in\mathfrak{S}_{n-1}} X_\nu(\mathbb{A}^+)\,u_\nu^+\ . \tag{10.4.7}$$

Proof. Specializing Eq.10.4.4 in $\mathbb{B}=\{a_1,0,0,\ldots\}$, $\mathbb{C}=\{0,0,\ldots\}$, one obtains

$$\sum_{\sigma\in\mathfrak{S}_n} X_\sigma(\mathbb{A})u_\sigma = \mathfrak{NK}_n(\mathbb{A},0) = \sum X_\zeta(\{a_1,0,\ldots\})\,u_\zeta \sum_\eta \mathbb{X}_\eta(\mathbb{A},\{a_1,0,\ldots\})\,u_\eta\ .$$

But, according to Lemma 9.5.2, $\mathbb{X}_\eta(\mathbb{A},\{a_1,0,\ldots\}) = 0$ if $\eta\notin\mathfrak{S}_1\times\mathfrak{S}_{n-1}$, and $\mathbb{X}_\eta(\mathbb{A},\{a_1,0,\ldots\}) = X_\nu((\mathbb{A}^+)$ if $\eta=\nu^+$. Moreover $\sum X_\zeta(\{a_1,0,\ldots\})\,u_\zeta = (1+a_1u_{n-1})\cdots(1+a_1u_1)$. □

Iterating, one gets the following factorization obtained by Fomin-Stanley [**45**] and Fomin-Kirillov [**44**] :

COROLLARY 10.4.4. *The NilCauchy kernel factorizes into*

$$\mathfrak{NK}_n(\mathbb{A},\mathbf{0}) = \Big((1{+}a_1u_{n-1})\cdots(1{+}a_1u_1)\Big) \Big((1{+}a_2u_{n-1})\cdots(1{+}a_2u_2)\Big)\cdots\Big((1{+}a_{n-1}u_{n-1})\Big)\,. \tag{10.4.8}$$

In fact, these authors state :

$$\mathfrak{NK}_n(\mathbb{A},\mathbb{B}) = \Big((1{+}(a_1{-}b_{n-1})u_{n-1})\cdots(1{+}(a_1{-}b_1)u_1)\Big) \Big((1{+}(a_2{-}b_{n-2})u_{n-1})\cdots(1{+}(a_2{-}b_1)u_2)\Big)\cdots\Big((1+(a_{n-1}{-}b_1)u_{n-1})\Big)\,, \tag{10.4.9}$$

but this also can easily be obtained from (10.4.4). We preferred stressing the link between the expansion of the NilCauchy kernel and the usual "Hopf decomposition" of any symmetric function: $S(a+\mathbb{B}) = \sum_i a^i D_{S^i}(S)(\mathbb{B})$. Let us point out that

$\mathfrak{NK}_n(\mathbb{A},\mathbb{B})$ *is not a Yang-Baxter element*[2], though one can use Yang-Baxter relations to prove that it is the generating function of Schubert polynomials. We shall later use Yang-Baxter elements in $\mathfrak{NC}_n$ to obtain Schubert polynomials.

The factorized form of $\mathfrak{NK}_n(\mathbb{A},\mathbb{B})$ is better displayed in two dimensions, using the convention to read the successive columns from left to right.

$$\mathfrak{NK}_4(\mathbb{A},\mathbb{B}) = \begin{matrix} (1+(a_1-b_3)u_3) & & \\ (1+(a_1-b_2)u_2) & (1+(a_2-b_2)u_3) & \\ (1+(a_1-b_1)u_1) & (1+(a_2-b_1)u_2) & (1+(a_3-b_1)u_3) \end{matrix}$$

The indices of a, b are just the coordinates, the u_i's are numbered starting from the diagonal.

The expansion of $\mathfrak{NK}_n(\mathbb{A},\mathbb{B})$ gives the Schubert polynomials as sums of products[3] of factors of the type $a_i - b_j$. For example the coefficient of u_3 in $\mathfrak{NK}_4(\mathbb{A},\mathbb{B})$ is equal to

$$(a_1-b_3) + (a_2-b_2) + (a_3-b_1) = \mathbb{X}_{1243}(\mathbb{A},\mathbb{B}) \ .$$

10.5. NilPlactic Kernel

In this section, we show that Schubert polynomials can be obtained from noncommutative computations. We shall not give proofs, but want only to illustrate what kind of combinatorics is involved. Other combinatorial approaches of Schubert polynomials are possible: the *Weintrauben* of Kohnert [**84**], the enumeration of all reduced decompositions by Billey, Jockush, Stanley [**18**], the configurations of Fomin-Kirillov, the lattice graphs of Lin and Yang [**127**] &c. With M.P. Schützenberger, I preferred using *key polynomials* [**113**][**114**], already introduced in (9.8.1), because they can be interpreted as sums of tableaux extending to *Demazure characters*[4] the classical description of Schur functions, and furnish a link with representation theory. Reiner and Shimozono [**152**] have shown how to obtain key polynomials from an enumeration of reduced decompositions.

The NilCauchy kernel can be refined by replacing Coxeter relations by the *nilplactic relations* (we use other variables y_i instead of u_i, to distinguish the two cases):

$$\begin{matrix} y_i & y_{i+1} \\ & y_i \end{matrix} \equiv \begin{matrix} y_{i+1} & \\ y_i & y_{i+1} \end{matrix}$$

and, for $i < j < k$,

$$\begin{matrix} y_j & \\ y_i & y_k \end{matrix} \equiv \begin{matrix} y_j & y_k \\ & y_i \end{matrix} \quad \& \quad \begin{matrix} y_k & \\ y_i & y_j \end{matrix} \equiv \begin{matrix} y_i & y_k \\ & y_j \end{matrix}$$

Two reduced words which are equivalent in the nilplactic monoid gives the same permutation, when evaluated in the symmetric group ($y_i \to s_i$).

[2]An element of a Yang-Baxter basis, to be defined in (10.7.1).

[3]Other expansions of the same type are obtained by using Monk's formula.

[4]Young tableaux of a given shape index a basis of an irreducible representation of $Gl(\mathbb{C}^n)$. More generally, the tableaux appearing in a key polynomial index a basis of the space of sections of an ample line bundle over a Schubert variety [**37**].

One can now define a *NilPlactic kernel* $\mathfrak{NPlK}_n$

$$\mathfrak{NPlK}_n = \Big((1+a_1y_n)\cdots(1+a_1y_1)\Big) \Big((1+a_2y_n)\cdots(1+a_2y_2)\Big)\cdots\cdots\Big((1+a_ny_n)\Big)\ . \tag{10.5.1}$$

Notice that each block expands as a sum of columns[5] in the y_i's

The expansion of $\mathfrak{NPlK}_n$ is given by the following theorem, that we shall not prove in this text, because it requires combinatorial properties of tableaux outside our present scope[6].

One needs the notion of a *bottom key* of a tableau t, which is the collection of all rows which appear when permuting the rows of t using the Nil-action of the symmetric group. For example,

$$\begin{array}{l}5\\46\\235\\1124\end{array} \mapsto \begin{array}{l}5\\46\\1235\\124\end{array} \mapsto \begin{array}{l}5\\1246\\35\\124\end{array} \mapsto \begin{array}{l}5\\1246\\135\\24\end{array} \mapsto \begin{array}{l}1256\\4\\135\\24\end{array} \mapsto \begin{array}{l}1256\\145\\3\\24\end{array} \mapsto \begin{array}{l}1256\\145\\34\\2\end{array}$$

In that case, the bottom key is the set of bottom rows (with the same multiplicities as the lengths of rows of the tableau, to fill the same shape) :

$$\{\,[2],\,[2,4],\,[1,2,4],\,[1,2,3,4]\,\}\ . \tag{10.5.2}$$

This construction provides a code, that we shall call *bottom code* and note $\mathfrak{bc}(t) = [\mathfrak{bc}_1, \mathfrak{bc}_2, \ldots]$. By definition, $\mathfrak{bc}_i$ is the number of occurrences of i in the bottom key of t.

The bottom code is a permutation of the lengths of the rows of t. In the preceding example, the bottom code[7] is $[2,4,1,3]$.

Expansions in the nilplactic algebra involve now key polynomials, instead of Schubert polynomials.

THEOREM 10.5.1. *The NilPlactic kernel expands, in the nilplactic algebra, as a sum of tableaux, with key polynomials in* $\mathbb{A}$ *as coefficients :*

$$\mathfrak{NPlK}_{n+1} \overset{NPl}{\equiv} \sum_t K(\mathfrak{bc}(t))\ t\ , \tag{10.5.3}$$

sum over all tableaux (read by columns, and which are reduced words) that appear as subwords of

$$(y_n\cdots y_1)\,(y_n\cdots y_2)\cdots(y_n)\ ,$$

$\mathfrak{bc}(t)$ *being the bottom code of* t.

For example,

$$\mathfrak{NPlK}_3 \overset{NPl}{\equiv} a_1^2a_2 \begin{array}{|c|c}\hline y_2 & \\ \hline y_1 & \multicolumn{1}{c|}{y_2} \\ \hline\end{array} + a_1^2 \begin{array}{|c|}\hline y_2 \\ \hline y_1 \\ \hline\end{array} + a_1a_2 \begin{array}{|c|c|}\hline y_1 & y_2 \\ \hline\end{array} + a_1 \boxed{y_1} + (a_1+a_2)\boxed{y_2} + 1$$

(the term y_2y_2 has disappeared, because it is not a reduced word).

[5] i.e. strictly decreasing word.

[6] The NilCauchy kernel and the NilPlactic kernel are very similar. However, properties of the first one are an easy consequence of the Cauchy formula (10.2.10), but the second requires a more refined combinatorics involving crystal graphs.

[7] We started with a tableau of shape $[1,2,3,4]$. In general, a bottom code is not a permutation.

Projecting the nilplactic algebra onto the NilCoxeter algebra, and comparing (10.5.3) and (10.4.3) one obtains that Schubert polynomials are positive sums of key polynomials.

COROLLARY 10.5.2. *Let σ be a permutation. Then*

$$\mathbb{X}_\sigma(\mathbb{A}, \mathbf{0}) = \sum_t K_{\mathfrak{bc}(t)} \ , \tag{10.5.4}$$

sum over all tableaux which are a reduced decomposition of σ.

Equations (10.4.8) and (10.5.3) look almost the same. However, the first one is merely a rewriting of Cauchy formula, while the second relies on rather subtle properties of the nilplactic monoid. The reader should not confuse the NilCoxeter world with the Nilplactic one!

From the Rothe diagram[8] of a permutation, one gets at least two tableaux (which can be the same!) by packing rows horizontally to the left, then packing vertically down, or performing these two operations in the other order.

For example, these two packings give for $\sigma = [21543]$:

$$\begin{smallmatrix}\cdot&\cdot&4&\cdot\\\cdot&\cdot&3&4\\\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot\end{smallmatrix} \to \begin{smallmatrix}4&\cdot&\cdot&\cdot\\3&4&&\\\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot\end{smallmatrix} \to \begin{smallmatrix}\cdot&\cdot&\cdot&\cdot\\4&\cdot&\cdot&\cdot\\3&\cdot&\cdot&\cdot\\1&4&\cdot&\cdot\end{smallmatrix} \qquad \& \qquad \begin{smallmatrix}\cdot&\cdot&4&\cdot\\\cdot&\cdot&3&4\\\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot\end{smallmatrix} \to \begin{smallmatrix}\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&4&\cdot\\1&\cdot&3&4\end{smallmatrix} \to \begin{smallmatrix}\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\4&\cdot&\cdot&\cdot\\1&3&4&\cdot\end{smallmatrix}$$

In the case of a vexillary permutation σ, there is only one tableau which is a reduced decomposition of σ. The smallest example of a non-vexillary permutation is $\sigma = [2, 1, 4, 3]$. In that case, $\mathfrak{NPlK}_4$ contains the terms $a_1 a_1 y_3 y_1$ and $a_1 a_2 y_1 y_3 + a_1 a_3 y_1 y_3$, the coefficients being the two key polynomials K_2 and K_{101}, $s_3 s_1$ and $s_1 s_3$ being the two tableaux which are reduced decompositions of $\sigma = [2, 1, 4, 3]$.

One recovers that the coefficient of $u_{2143} = u_1 u_3 = u_3 u_1$ in $\mathfrak{NK}_4$ is equal to

$$\mathbb{X}_{2143}(\mathbb{A}, \mathbf{0}) = K_2 + K_{101} = (a_1^2) + (a_1 a_2 + a_1 a_3) \ .$$

For $\mathfrak{S}_5$, there are 17 non-vexillary permutations, and only two have more than two tableaux as reduced decompositions. Indeed, for $\sigma = [21543]$ (the case $\sigma = [32154]$ is obtained by symmetry), one has the extra tableau :

$$\begin{smallmatrix}\cdot&\cdot&4&\cdot\\\cdot&\cdot&3&4\\\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot\end{smallmatrix} \to \begin{smallmatrix}\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&3&4\\\cdot&\cdot&\cdot&3\\1&\cdot&\cdot&\cdot\end{smallmatrix} \to \begin{smallmatrix}\cdot&\cdot&\cdot&\cdot\\3&4&\cdot&\cdot\\\cdot&3&\cdot&\cdot\\1&\cdot&\cdot&\cdot\end{smallmatrix} \to \begin{smallmatrix}\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\3&4&\cdot&\cdot\\1&3&\cdot&\cdot\end{smallmatrix}$$

Because $\begin{smallmatrix}4&\\3&\\1&4\end{smallmatrix} \equiv \begin{smallmatrix}4&\\3&4\\&1\end{smallmatrix}$, its bottom key is $\{[1, 4], [1], [1]\}$, giving the bottom code $[3, 0, 0, 1]$. Similarly, $\begin{smallmatrix}4&&\\1&3&4\end{smallmatrix} \equiv \begin{smallmatrix}1&3&4\\&&3\end{smallmatrix}$ has bottom key $\{[1, 3, 4], [3]\}$ and bottom code $[1, 0, 2, 1]$. Finally $\begin{smallmatrix}3&4\\1&3\end{smallmatrix}$ has bottom key $\{[1, 3], [1, 3]\}$ and bottom code $[2, 0, 2]$. In total

$$\mathbb{X}_{21543}(\mathbb{A}, \mathbf{0}) = K_{3001} + K_{1021} + K_{202} \ .$$

The reader can check by himself that the decomposition of the other non-vexillary Schubert polynomials for $\mathfrak{S}_5$ is given by the two horizontal and vertical packings of the Rothe diagram: $\mathbb{X}_{13254} = K_{0101} + K_{02}$; $\mathbb{X}_{21354} = K_{1001} + K_2$; $\mathbb{X}_{21435} = K_{101} + K_2$; $\mathbb{X}_{31254} = K_{2001} + K_3$; $\mathbb{X}_{21534} = K_{102} + K_3$; $\mathbb{X}_{21453} = K_{1011} + K_{2001}$; $\mathbb{X}_{23154} = K_{1101} + K_{12}$; $\mathbb{X}_{31524} = K_{202} + K_{301}$; $\mathbb{X}_{24153} = K_{1201} + K_{22}$; $\mathbb{X}_{31542} = K_{2021} + K_{3011}$; $\mathbb{X}_{25143} = K_{1301} + K_{23}$; $\mathbb{X}_{42153} = K_{3101} + K_{32}$; $\mathbb{X}_{32514} = K_{212} + K_{311}$; $\mathbb{X}_{52143} = K_{4101} + K_{42}$; $\mathbb{X}_{32541} = K_{2121} + K_{3111}$.

[8]In the Cartesian plane, not using matrix conventions.

10.6. Basis of Elementary Symmetric Functions

Any decomposition of the kernel

$$\mathbf{K}_n(\mathbb{A},\mathbb{B}) = \sum_{i=1}^{n!} U_i(\mathbb{A}) V_i(\mathbb{B})$$

gives a pair of adjoint bases. Indeed, for any $1 \le j \le n!$,

$$\sum_i \big(U_i(\mathbb{A}), V_j(\mathbb{A})\, V_j(\mathbb{B})\big) = \big(\mathbf{K}_n(\mathbb{A},\mathbb{B}), V_j(\mathbb{A})\big) = V_j(\mathbb{B}) \ ,$$

and this shows that $(U_i, V_j) = \delta_{i,j}$.

For any $J = [j_1, \ldots, j_n]$, $0^n \le J \le \rho = [n{-}1, \ldots, 0]$, any $\mathbb{B}$, writing $\mathbb{B}_k = \{b_1, \ldots, b_k\}$, define

$$P_J(\mathbb{B}) := \Lambda^{j_1}(\mathbb{B}_{n-1}) \cdots \Lambda^{j_{n-1}}(\mathbb{B}_1) \ .$$

The kernel can be written

$$\mathbf{K}_n(\mathbb{A},\mathbb{B}) = S^{n-1}(a_n - \mathbb{B}_{n-1}) S^{n-2}(a_{n-1} - \mathbb{B}_{n-2}) \cdots S^1(a_2 - \mathbb{B}_1) \ , \tag{10.6.1}$$

and therefore one has the following proposition.

PROPOSITION 10.6.1. $\{P_J(\mathbb{A}) : 0^n \le J \le \rho\}$ *and* $\{a^I : 0^n \le I \le [0, \ldots, n{-}1]\}$ *are two adjoint bases of* $\mathfrak{Pol}(\mathbb{A})$. *The pairing is*

$$\Big(P_J(\mathbb{A})\, ,\, (a^{\rho-J})^{\omega}\Big) = (-1)^{|\rho-J|} \ . \tag{10.6.2}$$

This proposition provides an analog of the Kostka matrix: the scalar products $(a^I, \mathbb{Y}_J(\mathbb{A}, \mathbf{0}))$, $I \le [0, \ldots, n{-}1]$, $J \le [n{-}1n \ldots, 0]$ are obtained when writing a monomial in the Schubert basis, and when writing Schubert polynomials in the basis P_I. Inversely, the scalar products $(P_I, \mathbb{Y}_J(\mathbb{A}^{\omega}, \mathbf{0}))$ occur in the expansion of P_I in the Schubert basis, as well as when expanding a Schubert polynomial in terms of monomials.

For example, for $\mathfrak{S}_4$, each row of the following matrix give the expansion of a Schubert polynomial Y_I in terms of monomials (listed in the same order than the codes indexing the Schubert polynomials, and identified with their exponents).

0000	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
0010	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
0100	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1000	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
0200	.	.	.	.	1	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
0110	.	.	.	.	.	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2000	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1010	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1100	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3000	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
0210	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	.	.	.	.	.	.	.	.	.
2010	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.
1200	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.
1110	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
2100	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
3010	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.
2200	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
3100	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.
1210	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.
2110	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
3200	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.
3110	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
2210	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
3210	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

The columns gives the expansion of elementary symmetric polynomials in the Schubert basis. For example, the 9-th column must be read as the expansion

$$P_{2110} = \mathbb{X}_{2431} + \mathbb{X}_{3241} + \mathbb{X}_{4132} + \mathbb{X}_{3412} \ ,$$

because the codes of the reversed permutations are

$$[0,1,1,0],\ [0,2,0,0],\ [1,1,0,0],\ [1,0,1,0] \ .$$

The basis $\{P_J(\mathbb{A})\}$ is specially interesting in the non-commutative case, since there are many algebras in which it is natural to define non-commutative elementary symmetric functions. But to recover a Schubert calculus in these algebras, one first needs to rewrite the action of divided differences purely in terms of the basis P_J ([**110**], Def. 3.1.; [**52**]) :

Lemma 10.6.2. *For any $1 \le k \le n-1$, any integers p, q such that $p \le k$, $q \le k-1$, one has*

(10.6.3)

$$\Lambda^p(\mathbb{A}_k)\Lambda^q(\mathbb{A}_{k-1})\,\partial_k = \begin{cases} \Lambda^{p-1}(\mathbb{A}_k)\Lambda^q(\mathbb{A}_{k-1}) + \cdots + \Lambda^0(\mathbb{A}_k)\Lambda^{q+p-1}(\mathbb{A}_{k-1}) & \\ \quad -\Lambda^{q+1}(\mathbb{A}_k)\Lambda^{p-2}(\mathbb{A}_{k-1}) - \cdots - \Lambda^{q+p-1}(\mathbb{A}_k)\Lambda^0(\mathbb{A}_{k-1}) & p \le q \\ \Lambda^q(\mathbb{A}_k)\Lambda^{p-1}(\mathbb{A}_{k-1}) + \cdots \Lambda^0(\mathbb{A}_k) + \Lambda^{q+p-1}(\mathbb{A}_{k-1}) & \\ -\Lambda^p(\mathbb{A}_k)\Lambda^{q-1}(\mathbb{A}_{k-1}) - \cdots - \Lambda^{q+p-1}(\mathbb{A}_k)\Lambda^0(\mathbb{A}_{k-1}) & p > q \end{cases}$$

Proof. The product

$$\Lambda^p(\mathbb{A}_k)\Lambda^q(\mathbb{A}_{k-1}) = \Lambda^{pq}(\mathbb{A}_{k-1}) + a_k\Lambda^{p-1,q}(\mathbb{A}_{k-1})$$

is sent to $\Lambda^{p-1,q}(\mathbb{A}_{k-1})$ under ∂_k. From Pieri formula, one gets a sum of $\Lambda_{ij}(\mathbb{A}_{k-1})$; each Schur function can be transformed by adding a_i to its top row, and expanding it again:

$$\Lambda_{ij}(\mathbb{A}_{k-1}) = \Lambda^i(\mathbb{A}_k)\Lambda^j(\mathbb{A}_{k-1}) - \Lambda^{j+1}(\mathbb{A}_k)\Lambda^{i-1}(\mathbb{A}_{k-1}) \ .$$

Summing up this terms, one gets the lemma. □

Since for any k, any P_J can be written $P_J = P_{J'00J''}\,\Lambda^p(\mathbb{A}_k)\Lambda^q(\mathbb{A}_{k-1})$, with $J = J'pqJ''$, the lemma describes how divided differences act on the basis $\{P_J\}$.

Using the expansion of any polynomial in $\mathbb{A}$ in the basis P_J is the easiest way to lift it to the free algebra, or to the plactic algebra [**110**]. In the case of Schubert polynomials, one obtains the same expansion than in (10.5.4). However, it is not evident from this method that the lift of a Schubert polynomial to the plactic algebra will be a positive sum of tableaux. This approach must be combined with representation theory, and a description of *Schubert functors* (cf. Kraskiewicz and Pragacz [**87**]).

10.7. Yang-Baxter Basis

Let us go back to the NilCoxeter algebra again, and introduce new linear bases, still noting ∂_i the algebraic generators of $\mathfrak{NC}_n$.

Let $x_1, \ldots, x_n$ be variables[9] which are invariant under ∂_i; write $\mathbf{x}$ for the alphabet $\{x_1, \ldots, x_n\}$.

One defines recursively, by induction on the length, a *Yang-Baxter basis*[10] $\{\mathcal{Y}_\sigma := \mathcal{Y}_\sigma(\mathbf{x}),\, \sigma \in \mathfrak{S}_n\}$, depending on $\mathbf{x}$, starting with $\mathcal{Y}_1 = 1$:

$$\mathcal{Y}_{\sigma s_i} = \mathcal{Y}_\sigma \left(1 + (x_{\sigma_{i+1}} - x_{\sigma_i})\,\partial_i\right),\ \ell(\sigma s_i) > \ell(\sigma). \tag{10.7.1}$$

The coherence of the definition is insured by the Yang-Baxter relations stated in the next proposition.

Proposition 10.7.1. *For any $\alpha, \beta \in \mathbb{C}$, for any $i \in \mathbb{N}$, the following relation is true :*

$$(\partial_i + \frac{1}{\alpha})\,(\partial_{i+1} + \frac{1}{\alpha+\beta})\,(\partial_i + \frac{1}{\beta}) = (\partial_{i+1} + \frac{1}{\beta})\,(\partial_i + \frac{1}{\alpha+\beta})\,(\partial_{i+1} + \frac{1}{\alpha}) \ . \tag{10.7.2}$$

The graphical display of these relations is, taking $i = 1$, and also figuring a trivial commutation relation :

[9] Yang says *spectral parameters*.

[10] The original case of Yang [**181**] was in the group algebra of the symmetric group. However, because one uses only reduced decompositions, Yang's construction is valid in the Hecke algebra, or any of its specialization. Jian-Yi Shi has extended the notion of Yang-Baxter basis to any Coxeter group.

$$\begin{tikzcd}
 & {[1\,2\,3]} \arrow[dl, dash, double, "\partial_1+\frac{1}{\alpha}"'] \arrow[dr, dash, "\partial_2+\frac{1}{\beta}"] & \\
{[2\,1\,3]} \arrow[d, dash, "\partial_2+\frac{1}{\alpha+\beta}"'] & & {[1\,3\,2]} \arrow[d, dash, double, "\partial_1+\frac{1}{\alpha+\beta}"] \\
{[2\,3\,1]} \arrow[dr, dash, double, "\partial_1+\frac{1}{\beta}"'] & & {[3\,1\,2]} \arrow[dl, dash, "\partial_2+\frac{1}{\alpha}"] \\
 & {[3\,2\,1]} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & {[1\,2\,3\,4]} \arrow[dl, dash, double, "\partial_1+\frac{1}{\alpha}"'] \arrow[dr, dash, "\partial_3+\frac{1}{\beta}"] & \\
{[2\,1\,3\,4]} \arrow[dr, dash, "\partial_3+\frac{1}{\beta}"'] & & {[1\,2\,4\,3]} \arrow[dl, dash, double, "\partial_1+\frac{1}{\alpha}"] \\
 & {[2\,1\,4\,3]} &
\end{tikzcd}$$

In our case, it is more convenient to take factors $(1+\alpha\partial_i)$ instead of $(\partial_i+\alpha^{-1})$.

For any choice of sufficiently generic spectral parameters, in particular when taking indeterminates[11], one gets a new Yang-Baxter basis. Notice that the Yang-Baxter basis associated to $\mathbf{x}^\omega = \{x_n, \ldots, x_1\}$ is not the same as the Yang-Baxter basis for $\mathbf{x}$. To understand how they are related, or more generally, to expand any element of $\mathfrak{NC}_n$ into the Yang-Baxter basis, one first introduces a quadratic form :

$$\mathfrak{NC}_n \ni f, g \ \Rightarrow\ (f,g) := f\bar{g} \cap \partial_\omega \ , \tag{10.7.3}$$

where $g \to \bar{g}$ is the morphism $x_i \to x_{n+1-i}$, $\partial_\sigma \to \partial_{\sigma^{-1}}$, and where the intersection sign $\cap\partial_\omega$ means *extracting the coefficient of* ∂_ω.

THEOREM 10.7.2. *The Yang-Baxter basis* $\{\mathcal{Y}_\sigma(\mathbf{x})\}$ *is self-adjoint :*

$$\big(\mathcal{Y}_\sigma, \mathcal{Y}_{\omega\nu}\big) = \begin{cases} = \pm\Delta(\mathbf{x}) & \textit{if } \sigma = \nu \\ = 0 & \textit{otherwise}\ . \end{cases} \tag{10.7.4}$$

Proof. The theorem is true when $\nu = \omega$, because only there is only one product $\mathcal{Y}_1\mathcal{Y}_{\omega\nu}$ having a term ∂_ω. Two elements $\mathcal{Y}_{\omega\nu}$ and $\mathcal{Y}_{\omega\nu s_i}$ differ by a factor of the type $(1+(x_j-x_k)\partial_i)$. However,

$$\Big(f\,,\, g\,(1+(x_j-x_k))\Big) = \Big(f\,(1+(x_{n+1-j}-x_{n+k-1}))\,,\, g\Big)\ ,$$

and the theorem becomes now a statement about the morphism " multiplication by $(1+x\partial_i)$" on the linear span of $\mathcal{Y}_\sigma$ and $\mathcal{Y}_{\sigma s_i}$, which is immediate to check. $\square$

Let us illustrate how one would compute directly the scalar product of two Yang-Baxter elements without knowing the preceding theorem. Let us take the "biggest" scalar product for $\mathfrak{S}_3$, that is, $(\mathcal{Y}_{321}, \mathcal{Y}_{321})$.

One has to extract the coefficient of ∂_ω in the product of $\mathcal{Y}_{321}$ by

$$\overline{Y_{321}} = (1{+}\partial_1(x_1{-}x_2))\,(1{+}\partial_2(x_1{-}x_3))\,(1{+}\partial_1(x_2{-}x_3))\ .$$

One introduces each factor of $\overline{Y_{321}}$ in succession;

[11]There are interesting integral specializations which gives Young's idempotents, cf. Jucys [**67**] and Cherednik [**35**].

$$Y_{321}(1+\partial_1(x_1-x_2)) = \heartsuit Y_{231} + \diamondsuit Y_{321} \ ,$$
$$\heartsuit(1+\partial_1(x_2-x_1))(1+\partial_2(x_3-x_1))(1+\partial_2(x_1-x_3))(1+\partial_1(x_2-x_3)) \cap \partial_{321} = 0 \, ,$$
$$\diamondsuit\, Y_{321}(1+\partial_2(x_1-x_3)) = \clubsuit Y_{321} + \spadesuit Y_{312} \ ,$$
$$\clubsuit(1+\partial_1(x_2-x_1))(1+\partial_2(x_3-x_1))(1+\partial_1(x_3-x_2))(1+\partial_1(x_2-x_3)) \cap \partial_{321} = 0 \, ,$$
$$\spadesuit Y_{312}(1+\partial_1(x_2-x_3)) \cap \partial_{321} = 0 \ .$$

In total, the coefficient is null, as required by Theorem 10.7.2.

Having the adjoint basis of $\{\mathcal{Y}_\sigma\}$, one can now expand any element of $\mathfrak{NC}_n$ in the Yang-Baxter basis by evaluating scalar products.

Let us however find directly the developed expression of the Yang-Baxter elements in the basis $\{\partial_\sigma\}$ without having recourse to the adjoint basis.

Let σ and i be such that $\ell(\sigma) < \ell(\sigma s_i)$. Suppose known the expansion

$$\mathcal{Y}_\sigma = \sum_\nu \partial_\nu \, \mathbb{X}_\nu(\mathbf{x}^\sigma, \mathbf{x}) + \partial_{\nu s_i} \, \mathbb{X}_{\nu s_i}(\mathbf{x}^\sigma, \mathbf{x}) \ ,$$

with $\nu : \ell(\nu) < \ell(\nu s_i)$. Then its product by $1+(x_{\sigma_{i+1}} - x_{\sigma_i})\partial_i$ is

$$\sum_\nu \partial_\nu \, \mathbb{X}_\nu(\mathbf{x}^\sigma, \mathbf{x}) + \partial_{\nu s_i} \left(\mathbb{X}_{\nu s_i}(\mathbf{x}^\sigma, \mathbf{x}) + \mathbb{X}_\nu(\mathbf{x}^\sigma, \mathbf{x})(x_{\sigma_{i+1}} - x_{\sigma_i}) \right)$$

and the identities

$$\mathbb{X}_\nu(\mathbf{x}^{\sigma s_i}, \mathbf{x}) \qquad \& \qquad \mathbb{X}_{\nu s_i}(\mathbf{x}^{\sigma s_i}, \mathbf{x}) = \mathbb{X}_{\nu s_i}(\mathbf{x}^\sigma, \mathbf{x}) + \mathbb{X}_\nu(\mathbf{x}^\sigma, \mathbf{x})(x_{\sigma_{i+1}} - x_{\sigma_i})$$

give a similar expansion for $\mathcal{Y}_{\sigma s_i}$ and prove the following theorem.

THEOREM 10.7.3. *The matrix of change of basis between $\{\mathcal{Y}_\sigma\}$ and $\{\partial_\sigma \Delta(\mathbf{x})\}$, and its inverse, have entries which are specializations of Schubert polynomials :*

$$\mathcal{Y}_\sigma = \sum_{\nu \le \sigma} \partial_\nu \, \mathbb{X}_\nu(\mathbf{x}^\sigma, \mathbf{x}) \, , \tag{10.7.5}$$

$$\partial_\nu \, \Delta(\mathbf{x}) = \sum \mathcal{Y}_\sigma \, \mathbb{X}_{\omega\nu}(\mathbf{x}, \mathbf{x}^{\omega\sigma}) \tag{10.7.6}$$

Remark that recursions on Schubert polynomials provided by divided differences are easier to understand than recursions on Yang-Baxter coefficients.

10.8. Yang-Baxter Elements as Permutations

We have obtained Schubert polynomials in the expansion of divided differences in the basis of permutations (10.2.8), or of permutations in the basis of divided differences (9.6.3).

We obtained them again in Theorem 10.7.3, this time when developing Yang-Baxter elements. In fact, one passes from the expression of a permutation σ to the expression of $\mathcal{Y}_\sigma$ by just changing formally $\mathbf{x}$ into $\mathbb{A}$ in Eq. (10.7.5).

This formal change can be interpreted as developing products like $(1+\partial_1(a_2-a_1))(1+\partial_2(a_3-a_1)) \cdots$ ignoring the fact that the ∂_i's act on the variables a_j. Physicists would call this type of computation *normal reordering*, and would write

$$: Y_\sigma(\mathbb{A}) : = \sigma \ .$$

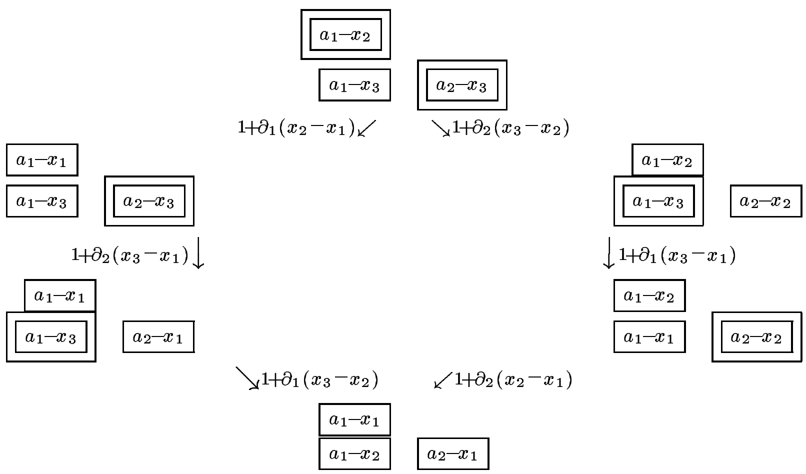
Let us show one way to see why normal reordering works in the case of Yang-Baxter elements.

Every element of $\mathfrak{Diff}$ or $\mathfrak{NC}$ is characterized by its action on the Schubert basis $\{\mathbb{X}_\sigma(\mathbb{A}, \mathbf{0})\}$. Equivalently, one can take a second alphabet $\mathbb{B}$ such that the $\mathbb{X}_\sigma(\mathbf{0}, \mathbb{B})$ be linearly independent; the maximal polynomial $\mathbb{X}_\omega(\mathbb{A}, \mathbb{B})$ is a generating function of the Schubert basis in $\mathbb{A}$, and therefore, as we already used it several times, every element of $\mathfrak{Diff}$ or $\mathfrak{NC}$ is characterized by its action on $\mathbb{X}_\omega(\mathbb{A}, \mathbb{B})$.

For what concerns Yang-Baxter elements, we choose to see how they act on

$$\mathbb{X}_\omega(\mathbb{A}, \mathbf{x}^\omega) \ .$$

Because of the recursive definition of Yang-Baxter elements, one can display the action by taking the permutohedron with the Yang-Baxter labelling of edges, putting $\mathbb{X}_\omega(\mathbb{A}, \mathbf{x}^\omega)$ at the origin and using $1 + \partial_1(x_2 - x_1)$, $1 + \partial_2(x_3 - x_1), \ldots$ instead of $1/\alpha + \partial_1$, $1/(\alpha+\beta) + \partial_2, \ldots$. The successive images are obtained by taking, for each edge, the image of the element on top of the edge under the operator $(1 + (x - x')s_i)$ written on the edge. In a word, we use the Yang-Baxter permutohedron to push down $\mathbb{X}_\omega(\mathbb{A}, \mathbf{x}^\omega)$.



We notice that for each edge of color ∂_i, the polynomial on top is the product of a polynomial symmetrical in a_i, a_{i+1} by a factor $(a_i - x)$, that we indicate by a double box. Its image under $(1+(x' - x'')\partial_i)$ is $(a_i - x'')$, because one also has that $x = x'$. Therefore, all the polynomials are products of factors $(a - x)$. More precisely, they are equal to $\mathbb{X}_\omega(\mathbb{A}, \mathbf{x}^{\omega\sigma})$, the Yang-Baxter elements (in $\mathbb{A}$) act as permutations on $\mathbf{x}$.

It is not difficult to extend the computation to any $\mathfrak{S}_n$, The combinatorial property to observe, and prove by induction on length, is the following one.

LEMMA 10.8.1. *For any σ, the diagram representing $\mathbb{X}_\omega(\mathbb{A}, \mathbf{x}^\omega)\, \mathcal{Y}_\sigma(\mathbf{x})$ is such that for any $j : 1 \le j \le n-1$, its j-th column is equal to*

$$\prod\nolimits_{k \in \{\sigma_{j+1}, \ldots, \sigma_n\}} (a_j - x_k) \ .$$

The preceding result translates into the following algebraic statement.

THEOREM 10.8.2. *The image of $\mathbb{X}_\omega(\mathbb{A}, \mathbf{x}^\omega)$ under the Yang-Baxter element $\mathcal{Y}_\sigma(\mathbf{x})$ is*

$$\mathbb{X}_\omega(\mathbb{A}, \mathbf{x}^{\sigma\omega}) \ .$$

The reader should be aware that the images that we computed are simple because we started from $\mathbb{X}_\omega(\mathbb{A},\mathbf{x}^\omega) = \mathbf{K}(\mathbf{x},\mathbb{A})$. Any other element, e.g. $\mathbb{X}_\omega(\mathbb{A},\mathbf{x})$, would have given polynomials difficult to interpret.

We end this chapter by pointing out that we have obtained the same coefficients by computations obeying different rules :

- Expanding $(1+\partial_1(x_2-x_1))(1+\partial_2(x_3-x_1))(1+\partial_1(x_3-x_2))$ in $\mathfrak{NC}_3$, using Yang-Baxter relations.
- Expanding $(1+\partial_1(a_2-a_1))(1+\partial_2(a_3-a_2))(1+\partial_1(a_2-a_1))$ in $\mathfrak{Diff}_3$, using braid relations and Leibnitz' rule.
- Expanding $(1-s_1)\frac{1}{a_1-a_2}(1-s_2)\frac{1}{a_2-a_3}(1-s_1)\frac{1}{a_1-a_2}$ in $\mathfrak{Diff}_3$, using braid relations, together with the commutations $f\,\sigma = \sigma\, f^\sigma$.
- Expanding $\mathbb{X}_\sigma(\mathbb{A},\mathbf{x}^\nu)$ in the basis $\{\mathbb{X}_\sigma(\mathbb{A},\mathbf{x})\}$.

Exercises

Ex. 10.1. Let $\mathbb{A} = \{a_1,\dots,a_n\}$. Write the Vandermonde $\Delta(\mathbb{A}) := \prod_{i<j}(a_i - a_j)$ in terms of simple Schubert polynomials $X_\sigma(\mathbb{A})$ and $X_\sigma(\mathbb{A}^\omega)$ (with ω maximal element of $\mathfrak{S}(\mathbb{A})$). Write in a similar manner the product $\prod_{i<j}(a_i + a_j)$.

Ex. 10.2. Let $\mathbb{A} = \{a_1,\dots,a_4\}$. For any $\sigma \in \mathfrak{S}(\mathbb{A})$, compute the image under ∂_σ of $(x-a_2)^{-1}(x-a_3)^{-2}(x-a_4)^{-3}$.

Ex. 10.3. Denote $[i]$ the q-integer $[i] := 1+q+\cdots+q^{i-1}$. Fix two parameters α,β and for any indeterminate x, any $k\in\mathbb{N}$, define its modified k-power $x^{<k>}$ by

$$x^{<k>} := (x-\alpha)(x-q\alpha-[1]\beta)\cdots(x-q^{k-1}\alpha-[k-1]\beta)\ .$$

Given $K = [k_1,\dots,k_n] \in \mathbb{N}^n$, and $x_1,\dots,x_n$, compute the quotient of the determinant $\left|x_i^{<k_j>}\right|$ by the Vandermonde in $x_1,\dots,x_n$.

Ex. 10.4. Denote by Y_J^{+k} a Schubert polynomial $Y_J(\{a_{1+k},a_{2+k},\dots\},\mathbb{B})$. For any $n,k\in\mathbb{N}$, factorize the triangular matrix $\left[Y^{+i-1}_{0^{j-i},\,k-j}\right]_{0\le i,j\le n-1}$ into a product of matrices, each depending only upon a single $b_i\in\mathbb{B}$ (and upon $\mathbb{A}$; one has put $Y_{0^{j-i}r} = 0$ when $j<i$).

Ex. 10.5. Let $m,n\in\mathbb{N}$. Let $\mathcal{I}$ be the family of partitions contained in n^n, $\mathcal{S}$ be set of permutations σ such that $Y_{0^m,n^n}\partial_\sigma \neq 0$. For each increasing partition $I \le n^n$, denote by $Y_{[I]}$ the Schubert polynomial in $\mathbb{A},\mathbb{B}$ of index $[0^m,i_1,\dots,i_n]$.

Compute the inverse of the matrix $\left[Y_{[I]}\partial_\sigma\right]_{I\in\mathcal{I}i,\sigma\in\mathcal{S}}$.

Ex. 10.6. Let n be a positive integer, ω be the maximum element of $\mathfrak{S}(x_1,\dots,x_n)$. Compute the image of

$$Todd = \Big((n-1)x_1+\cdots+0x_n\Big)^{\binom{n}{2}}$$

under ∂_ω.

Ex. 10.7. Develop $\mathbb{X}_\sigma(\mathbb{A},\{b,0,0,\dots\})$ in terms of powers of b. Comment the case where σ is a Grasmannian permutation.

Ex. 10.8. Given a positive integer n, and $\mathbb{A}$ of cardinality n, write the linear morphism on the ring of polynomials in $\mathbb{A}$, as a free-module over $\mathfrak{Sym}(\mathbb{A})$, induced by $X_\sigma(\mathbb{A}, \mathbf{0}) \to X_{\sigma^{-1}}(\mathbb{A}, \mathbf{0})$, $\sigma \in \mathfrak{S}_n$, without using the Schubert basis.

Adding a second alphabet $\mathbb{B}$, do the same for the linear morphism induced by $X_\sigma(\mathbb{A}, \mathbf{0}) \to \mathbb{X}_\sigma(\mathbb{A}, \mathbb{B})$.

CHAPTER 11

The plactic algebra

One way to better understand symmetric functions is to define non-commutative symmetric functions. There are several ways to do it. For example, since symmetric polynomials are polynomials in the elementary symmetric functions, then one can take formal non commutative variables Λ^i and claim that the free algebra generated by them is the ring of non-commutative symmetric polynomials. There are some grounds to support this claim, see [**56**].

We shall take here a simpler point of view, defining an action of the symmetric group on the free algebra, and picking out some appropriate invariants as generalizations of Schur functions. Indeed, considering Schur functions as sums of "tableaux" is the point of view introduced by Littlewood [**131**], which could be better understood after the work of Schensted [**158**].

To get a free version of Schubert polynomials, we shall sketch two strategies: using isobaric divided differences, but this require to first define free key polynomials, or using a non-commutative kernel.

We do not give proofs in this chapter, but just show how easily many constructions from the preceding chapters can be adapted to the free algebra, or to its quotient, the plactic algebra. Combinatorial properties of tableaux in themselves would require a book (we refer to Fulton [**51**] or Lothaire [**124**]), and more objects, related to diagrams of permutations or to reduced decompositions, are needed to prove properties of free Schubert polynomials.

11.1. Tableaux

Let us take a totally ordered alphabet $\mathbb{A} = \{a_1 < a_2 < \ldots\}$. In the examples, we usually take $\mathbb{A} = \{1, 2, \ldots, n\}$.

A nondecreasing word $v \in \mathbb{A}^*$ is called a *row*. Let $u = x_1 \cdots x_r$ and $v = y_1 \cdots y_s$ be two rows ($x_i, y_j \in A$). We say that u *dominates* v ($u \rhd v$) if $r \le s$ and for $i = 1, \ldots, r$, $x_i > y_i$. Clearly, every word w has a unique factorization $w = u_1 \cdots u_k$ as a product of rows of maximal length. A *tableau* is a word w such that $u_1 \rhd u_2 \rhd \ldots \rhd u_k$. It is customary to think of tableaux as planar objects and to represent w as the left justified superposition of its rows. For instance, taking $\mathbb{A} = \{1 < 2 < \ldots\}$,

$$t = 68\ 4556\ 223357\ 1112444$$

is a tableau whose planar representation is

6	8					
4	5	5	6			
2	2	3	3	5	7	
1	1	1	2	4	4	4

Similarly, a strictly decreasing word is called a *column*. Reading from top to bottom the lengths of the rows of a tableau t, one gets an (increasing) partition $\mathfrak{sh}(t)$ which is called the *shape* of t.

A word (in particular a tableau) is *standard* if it is a permutation of $1, 2, 3, \ldots$. Any word can be *standardized* by indexing, from left to right, the different occurrences of the same letter (and renormalizing after that, the alphabet to be $1, 2, \ldots$). For example

$$2\,1\,22\,111 \mapsto 2_1\,1_1\,2_2 2_3\,1_2 1_3 1_4 \mapsto stand(2\,122111) = 5\,1\,67\,234\ .$$

11.2. Plactic Algebra

Schensted described an algorithm to associate to any word w a tableau $P(w)$. This algorithm can in fact be traced back to Robinson. One can give an algebraic formulation to this algorithm by defining, after Knuth [**82**] the *plactic relations*

$$xzy \equiv zxy \quad (x \le y < z)\,, \tag{11.2.1}$$

$$yxz \equiv yzx \quad (x < y \le z)\,. \tag{11.2.2}$$

Two words are *congruent* iff they differ by a sequence of plactic relations. The *plactic algebra* $\mathfrak{Plac}$ is the quotient of the free algebra under the plactic relations. The plactic algebra is an intermediate quotient between the free algebra $\mathbb{Z}[\mathbb{A}^*]$ and the algebra of polynomial :

$$\mathbb{Z}[\mathbb{A}^*] \twoheadrightarrow \mathfrak{Plac} = \mathbb{Z}[\mathbb{A}^* / \equiv] \twoheadrightarrow \mathfrak{Pol}(\mathbb{A}) = \mathbb{Z}[\mathbb{A}]\ .$$

Let us note *ev* the projection map (*evaluation*) onto $\mathfrak{Pol}(\mathbb{A})$.

Given a word $w = [x_1, \ldots, x_\ell]$, not only can one find the tableau $P(w)$, but one can also record the sequence of shapes of

$$P(x_1),\ P(x_1x_2),\ \ldots,\ P(x_1x_2\cdots x_\ell)\ .$$

This sequence can be encoded into a standard tableau $Q(w)$ by labeling the boxes of the successive shapes $\mathfrak{sh}(P(x_1))$, $\mathfrak{sh}(P(x_1x_2))$, $\mathfrak{sh}(P(x_1x_2x_3))\ldots$, in the order of their creation.

We can now give Schensted's result :

THEOREM 11.2.1. *Each word w is congruent to a tableau and only one tableau, which is no other than $P(w)$.*

The morphism $w \mapsto (P(w), Q(w))$ is a bijection, and thus the elements of the plactic class of w are in bijection with standard tableaux of the same shape as $P(w)$.

The class of w is mapped, under standardization, to the class of $stand(w)$. Moreover $Q(w) = Q(stand(w))$.

Let u be a word. Define the *free Schur function $\mathcal{S}_u$ of index u* to be the sum of all words w such that $Q(w) = stand(P(u))$.

We shall see that the commutative image of $\mathcal{S}_u$ is the Schur function $S_J(\mathbb{A})$, where J is the shape of $P(u)$. But already, we notice that the image of $\mathcal{S}_u$ in $\mathfrak{Plac}$ is equal to the sum of all tableaux of shape J, sum that we shall still denote $S_J(\mathbb{A})$.

Now, given t, and a row $[x_1, x_2, \ldots]$, then an analysis of Schensted's construction [**82**] shows that the shape of $P(t\,x_1 x_2 \cdots)$ is obtained by adding to the shape of t an horizontal strip of boxes (the boxes being added from left to right). This implies, in fact, that Pieri's formulas are satisfied by Schur functions in the plactic algebra. Thanks to Giambelli's characterization of the ring of symmetric function, that we saw in Proposition 1.9.3, it gives the following theorem which allows us to recover symmetric functions at the level of the plactic algebra [**95**] :

THEOREM 11.2.2. *The ring $\mathfrak{Sym}(\mathbb{A})$ is canonically embedded into the plactic algebra, by sending a Schur function $S_J(\mathbb{A})$ to the sum of all tableaux of shape J.*

The Pieri formula (for any partition J, any positive integer k)

$$S_J(\mathbb{A})\, S_k(\mathbb{A}) = \sum_{I \in \{J \otimes k\}} S_I(\mathbb{A})$$

induces a bijection

$$\{(t,\, u) : \mathfrak{sh}(t) = J,\, \mathfrak{sh}(u) = [k]\} \longleftrightarrow \{t : \mathfrak{sh}(t) \in \{J \otimes k\}\}\ ,$$

and a bijection

$$\{(u,\, t) : \mathfrak{sh}(t) = J,\, \mathfrak{sh}(u) = [k]\} \longleftrightarrow \{t : \mathfrak{sh}(t) \in \{J \otimes k\}\}\ .$$

For the tableau given above, here are two possible ways to write using the left or right Pieri's bijections :

$$\boxed{2\,|\,2\,|\,4} \cdot \begin{array}{|c|c|c|c|c|c|} \hline 6 & 8 \\ \hline 5 & 5 & 6 & 7 \\ \hline 3 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 2 & 4 & 4 \\ \hline \end{array} \equiv \begin{array}{|c|c|c|c|c|c|c|} \hline 6 & 8 \\ \hline 4 & 5 & 5 & 6 \\ \hline 2 & 2 & 3 & 3 & 5 & 7 \\ \hline 1 & 1 & 1 & 2 & 4 & 4 & 4 \\ \hline \end{array} \equiv \begin{array}{|c|c|c|c|c|c|c|} \hline 6 & 8 \\ \hline 4 & 5 & 5 \\ \hline 2 & 2 & 3 & 6 \\ \hline 1 & 1 & 1 & 3 & 4 & 5 & 7 \\ \hline \end{array} \cdot \boxed{2\,|\,4\,|\,4}$$

The multiplication of a Schur function S_J by the simplest one, S_1, can be described as adding a box to the diagram of J in all possible manners furnishing a diagram. The non-commutative version, i.e. the "simplest" Pieri formula, states that given a letter x and a tableau t, there exists a unique pair t', x' such that $xt \equiv t'x'$ and such that t' has the same shape as t. With M.P. Schützenberger, I used this property to put a rank-order structure on the set of tableaux, the rank (the *charge*) being an invariant useful in the theory of Hall-Littlewood polynomials [**135**].

Cauchy formula is also an easy consequence of Robinson-Schensted's construction.

Given two totally ordered alphabets $X = \{x_1, x_2, \ldots\}$, $Y = \{y_1, y_2, \ldots\}$, a monomial w in the commutative variables $\binom{y}{x}$, $x \in X$, $y \in Y$, is called a *biword.*

Ordering lexicographically w, with priority to Y, one writes

$$w = \begin{pmatrix} y_1 \\ a_1 \end{pmatrix} \cdots \begin{pmatrix} y_1 \\ b_1 \end{pmatrix} \begin{pmatrix} y_2 \\ a_2 \end{pmatrix} \cdots \begin{pmatrix} y_2 \\ b_2 \end{pmatrix} \begin{pmatrix} y_3 \\ a_3 \end{pmatrix} \cdots \begin{pmatrix} y_3 \\ b_3 \end{pmatrix} \cdots \tag{11.2.3}$$

where $a_1 \cdots b_1$, $a_2 \cdots b_2$, $\ldots$ are rows in the monoid X^*, and a_i, b_j belong to X.

One can erase the y_j's and keep the underlying word in X^*:

$$w \mapsto w^x := [a_1 \cdots b_1,\, a_2 \cdots b_2,\, a_3 \cdots b_3, \ldots]\ .$$

Ordering lexicographically w with priority to X, one obtains similarly a word w^y in Y^*.

Schensted's bijection can now, following Bender and Knuth[**13**], be reinterpreted as combinatorial version of Cauchy formula :

THEOREM 11.2.3. *The map* $w \mapsto \big(P(w) := P(w^x)\,, P(w^y)\big)$ *is a bijection between biwords and pairs of tableaux of the same shape:*

$$\prod_{x\in X}\prod_{y\in Y}\left(1-\binom{y}{x}\right)^{-1} = \sum\ (t,t')\ , \tag{11.2.4}$$

(writing an equality instead of a bijection).

Noticet that biwords are monomials in commutative letters, but that they nevertheless give rise to non-commutative statements.

11.3. Littlewood-Richardson Rule

Commutativity of the product in $\mathfrak{Sym}(\mathbb{A})$ has interesting consequences at the level of $\mathfrak{Plac}$.

For example, combining the left and right Pieri bijections, one obtains a bijection

$$\{(u,\,t) : \mathfrak{sh}(u) = [k],\, \mathfrak{sh}(t) = J\} \longleftrightarrow \{(t',\,u') : \mathfrak{sh}(t') = J, \mathfrak{sh}(u') = [k]\}\ ,$$

which is used to provide a statistics on tableaux related to Hall-Littlewood polynomials ($k = 1$ suffices for that purpose).

The fact the module generated by the non commutative Schur functions is an algebra gives an interesting description of the decomposition coefficients of products

$$S_I(\mathbb{A})\,S_J(\mathbb{A}) = \sum_K g^K_{I,J}\,S_K(\mathbb{A})\ .$$

Say that a word w is a *Yamanouchi word* iff, for every factorization $w = w'\,w''$, then $|w''|_1 \geq |w''|_2 \geq |w''|_3 \geq \cdots$. In particular, a *Yamanouchi tableau* is of the type

$$\mathbf{y}_\lambda := \cdots 3^{\lambda_3} 2^{\lambda_2} 1^{\lambda_1}\ , \text{ with } \cdots \leq \lambda_3 \leq \lambda_2 \leq \lambda_1\ .$$

Notice that a Yamanouchi tableau is determined by its shape, as well as by its commutative evaluation.

It is easy to check :

LEMMA 11.3.1. *A word is Yamanouchi iff it belongs to the class of a Yamanouchi tableau.*

The second part of the following proposition is the celebrated *Littlewood-Richardson rule* [**133**], the first part is due to M.P. Schützenberger [**161**] :

PROPOSITION 11.3.2. *Given three partitions* I, J, K, *and a tableau* $t_3 : sh(t_3) = K$, *the number of solutions of the equation*

$$t_1 t_2 \equiv t_3\ ,\ sh(t_1) = I,\, sh(t_2) = J,\, sh(t_3) = K$$

is independent of the choice of the tableau t_3, *and equal to* $g^K_{I,J}$.

In particular, the number of solutions is equal to the number of tableaux t_1 *such that* $t_1\mathbf{y}_J$ *is a Yamanouchi word in the class of* $\mathbf{y}_K$.

We have just seen that Littlewood-Richardson rule is a consequence of the embedding of $\mathfrak{Sym}(\mathbb{A})$ in $\mathfrak{Plac}$, which in turn follows from Pieri formula at the level of Schensted construction.

There is a stronger statement in the free algebra, which is even easier to prove, and that I have used in different constructions with M.P. Schützenberger [**105**].

THEOREM 11.3.3. *Let $\mathbb{A}'$ and $\mathbb{A}''$ be two sub-alphabets such that $a' < a''$, for all $a \in \mathbb{A}'$, $a'' \in \mathbb{A}''$. Given two tableaux $t' \in \mathbb{A}'^*$ and $t'' \in \mathbb{A}''^*$ one has*

$$\left(\sum_{P(w')=t'} w'\right) \sqcup\!\sqcup \left(\sum_{P(w'')=t''} w''\right) = \sum_{t\in SH(t',t'')} \sum_{P(w)=t} w$$

where $SH(t',t'')$ is the set of all tableaux t such that $t|_{\mathbb{A}'} = t'$ and $P(t|_{\mathbb{A}''}) = t''$, that is, of all tableaux t occurring in the shuffle product of t' and a word in the plactic class of t''.

Thus the shuffle of a plactic class of $\mathbb{A}'$ and a plactic class of $\mathbb{A}''$ is a union of plactic classes of $\mathbb{A}$ (identifying a class and the sum of its elements). It is in fact a direct consequence of the following property :

LEMMA 11.3.4. *Let $\mathbb{B}$ be an interval in the alphabet $\mathbb{A}$. Then*

$$w \equiv w' \Rightarrow w|_{\mathbb{B}} \equiv w'|_{\mathbb{B}}$$

Proof. It is enough to check the lemma in the case when w' differs from w by a single plactic transformation, and this amounts to the observation that erasing x or z in (11.2.1) or (11.2.2), we are left with $xy = xy$ or $yz = yz$. □

Proof of theorem. The words occurring in the shuffle are exactly those w such that $w|_{\mathbb{A}'} \equiv t'$ and $w|_{\mathbb{A}''} \equiv t''$. By the above Lemma, this set of words is saturated with respect to the plactic congruence, hence is a union of plactic classes. □

For example, to compute S_{12}^2, one can take the shuffle of the two classes $a_2a_1a_1 + a_1a_2a_1$ and $a_4a_3a_5 + a_4a_5a_3$. The multiplicity 2 in $S_{12}^2 = 2S_{123} + \cdots$ comes from the fact that there are two tableaux $\begin{smallmatrix} \boxed{4} & & \\ \boxed{2} & \boxed{3} & \\ \boxed{1} & \boxed{1} & \boxed{5} \end{smallmatrix}$, $\begin{smallmatrix} \boxed{4} & & \\ \boxed{2} & \boxed{5} & \\ \boxed{1} & \boxed{1} & \boxed{3} \end{smallmatrix}$ of shape $[1,2,3]$ in the shuffle of the two classes.

We refer to [**124**] for a stronger statement involving free Schur functions S_u.

11.4. Action of the Symmetric Group on the Free Algebra

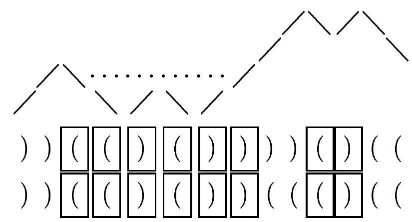
Let us now reintroduce the missing entity, the symmetric group $\mathfrak{S}(\mathbb{A})$, by defining an action on words compatible with the plactic relations.

In the case of an alphabet of cardinality 2, the action of $\S_2$ is in fact commonly used in computer science. Let the letters be ')' and '(', with the order $) \leq ($. Given a word w, pair all successive parentheses $(\ldots)$. The sub-word of un-paired letters is of the type $)^m\,(^n$. Change it, inside w, into the subword $)^n\,(^m$.

For example, the pairing for the word

$$))(()()))()(($$

can be visualized by interpreting an opening parenthesis as a North-East step, a closing parenthesis, as a South-East edge. We have put in a box the parentheses which are paired, and pointed out by dots one pairing. In this representation, paired letters are two sides facing each other at the same level.



image

More generally, given an integer i, and a word w, treat a_i as ')' and a_{i+1} as '(' to exchange the number of occurrences of a_i and a_{i+1} in the word w, and denote by $s_i(w)$ the resulting word (this is an action on values, we denote it on the left). It is clear that $w \mapsto s_i(w)$ is an involution on words, but in fact it is even a representation of the symmetric group, as shows the following theorem (cf. [**105**]).

THEOREM 11.4.1. *The operator s_i on words preserves the Q-symbol and preserves the set of tableaux :*

$$P(s_i(w)) = s_i\big(P(w)\big) \quad \& \quad Q(s_i(w)) = Q(w) \ .$$

The operators s_i satisfy the Moore-Coxeter relations

$$s_i^2 = 1\,, \tag{11.4.1}$$

$$s_i s_j = s_j s_i \quad (|i-j|>1)\,, \tag{11.4.2}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\,. \tag{11.4.3}$$

Proof. The first part is checked directly, the exchange of unpaired letters being compatible with Schensted's construction.

Equations (11.4.1,11.4.2) are immediate. The first part of the theorem restricts checking of (11.4.3) to tableaux in $1,2,3$, with no more than two rows because complete columns 321 can be ignored. We refer to ([**124**]) to finish the proof. □

Because the action of the symmetric group preserves the Q-symbol, all the free Schur functions S_u are invariants under this action, but they do not generate any more all the invariants.

The easiest way to generate words in a plactic class is to use the celebrated *jeu de taquin* due to M.P. Schützenberger. To define it, one needs the notion of *skew Young tableaux*, and their deformations through vertical and horizontal moves of boxes in such a way that weakly increasing conditions in rows, and strictly decreasing conditions in columns are satisfied at any step.

Notice that the plactic class of a tableau with two rows contains products of two rows which are determined by their lengths , as a consequence of Theorem 11.2.1:

LEMMA 11.4.2. *Let t be a tableau of shape $[\alpha,\beta]$. Then for any pair of integers (γ,δ), $\alpha\le\gamma,\delta\le\beta$, $\gamma+\delta=\alpha+\beta$, there exists a unique pair of rows (u,v): $|u|=\gamma$, $|v|=\delta$, such that $uv\equiv t$.*

The jeu de taquin is a way to get successively all these words; in fact they come as a *string* of words, the two ends being the tableau and the *contre-tableau.*

Exchanging the role of the P and Q-symbols, one gets a *right action* of the symmetric group which is the counterpart of theorem (11.4.1), and such that elementary transpositions are described by (11.4.2).

THEOREM 11.4.3. *Given a tableau t of shape $J \in \mathbb{N}^\ell$, then for any permutation $\sigma \in \mathfrak{S}_\ell$, there exist in the plactic class of t a unique word which is a product of rows, whose sequence of lengths is J^σ.*

For example, here are the words of shape $[1,2,2,3], [1,3,2,2], [3,1,2,2], [3,2,2,1]$ in the class of the tableau 43423112 :

$$\begin{array}{|c|cc}\hline 4 & & \\ \hline 3 & \multicolumn{1}{c|}{4} & \\ \hline 2 & \multicolumn{1}{c|}{3} & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \equiv \begin{array}{ccc} \boxed{4} & & \\ \boxed{1} & \boxed{3} & \boxed{4} \\ & \boxed{2} & \boxed{3} \\ & \boxed{1} & \boxed{2} \end{array} \equiv \begin{array}{cccc} \boxed{1} & \boxed{4} & \boxed{4} & \\ & & \boxed{3} & \\ & & \boxed{2} & \boxed{3} \\ & & \boxed{1} & \boxed{2} \end{array} \equiv \begin{array}{ccc} \boxed{1} & \boxed{4} & \boxed{4} \\ & \boxed{3} & \boxed{3} \\ & \boxed{2} & \boxed{2} \\ & & \boxed{1} \end{array}$$

11.5. Free Key Polynomials

We have lifted the permutation action of the symmetric group to the free algebra. More generally, one can lift the action of elementary isobaric differences, or of the generators T_i of the Hecke algebra, to the free algebra, but these new operators do not satisfy Moore-Coxeter relations any more.

It is however possible to obtain elements corresponding to permutations (independent of the choice of a reduced decomposition), by starting from special words (the analogs of *dominant weights*).

Let us first reinterpret the action of the symmetric group $\mathfrak{S}_2$, using letters $1, 2$ instead of parentheses.

Define a *string* to be a maximal set of words having same the same subword of paired letters.

Here is a string, where the paired letters are on top of each other.

$$\begin{array}{l}11122\\ \quad 11111\end{array} \to \begin{array}{l}11122\\ \quad 11112\end{array} \to \begin{array}{l}11122\\ \quad 11\,122\end{array} \to \begin{array}{l}11122\\ \quad 11\,222\end{array}$$
$$\to \begin{array}{l}11222\\ \quad 11\,222\end{array} \to \begin{array}{l}12222\\ \quad 11\,222\end{array} \to \begin{array}{l}22222\\ \quad 11\,222\end{array} .$$

The action of $\mathfrak{S}_2$ is the symmetry with respect to the middle of the string. For example, the image of $\begin{array}{l}11122\\ \quad 11112\end{array}$ is $\begin{array}{l}12222\\ \quad 11\,222\end{array}$.

But, on commuting letters, the image of $a_2^i a_1^j$, $i \le j$, under π_1 is

$$a_2^i a_1^j + a_2^{i+1} a_1^{j-1} + a_2^{i+2} a_1^{j-2} + \cdots + a_2^j a_1^i \ ,$$

i.e. is the sum of all monomials interpolating between $a_2^i a_1^j$ and $a_2^j a_1^i$. It forces us to define the image under π_1 of a a word w in a_1, a_2, with more occurences of a_1 than of a_2, to be the sum of all words in the string containing w, between w and $s_1(w)$.

We extend the action to all words by requiring that $w + s_1(w)$ be invariant.

For example, the image of $12222\,11222$ is

$$- \begin{array}{l}11122\\ \quad 11\,122\end{array} - \begin{array}{l}11122\\ \quad 11\,222\end{array} - \begin{array}{l}11222\\ \quad 11\,222\end{array} .$$

We now apply this construction to all pairs of consecutive letters a_i, a_{i+1}, by ignoring the other letters and treating a_i, a_{i+1} as we did with a_1, a_2, thus obtaining operators π_i on the free algebra.

It is convenient to also use $\widehat{\pi}_i := \pi - 1$, i.e. to subtract the identity operator to π_i.

Free key polynomials can now be defined by adapting the definition given in chapter 9. However, the property that a key polynomial does not depend on the choice of a reduced decomposition must be proved, because as we already said, the operators π_i or $\widehat{\pi}_i$ do not satisfy the braid relations anymore (cf. [**113**]).

DEFINITION 11.5.1. *Free key polynomials* $K_v^{\mathcal{F}}$, indexed by $v \in \mathbb{N}^\infty$, are defined recursively as follows. If v is dominant, then

$$K_v^{\mathcal{F}} = \cdots a_3^{v_3} a_2^{v_2} a_1^{v_1} \ .$$

Otherwise, if v and i are such that $v_i > v_{i+1}$, then

$$K_{vs_i}^{\mathcal{F}} = K_v^{\mathcal{F}} \pi_i \ .$$

For example, the different key polynomials obtained from $K_{210}^{\mathcal{F}} = \genfrac{}{}{0pt}{}{2}{11}$ are $K_{120}^{\mathcal{F}} = \genfrac{}{}{0pt}{}{2}{11} + \genfrac{}{}{0pt}{}{2}{12}$, $K_{201}^{\mathcal{F}} = \genfrac{}{}{0pt}{}{2}{11} + \genfrac{}{}{0pt}{}{3}{11}$, $K_{102}^{\mathcal{F}} = \genfrac{}{}{0pt}{}{2}{11} + \genfrac{}{}{0pt}{}{2}{12} + \genfrac{}{}{0pt}{}{3}{11} + \genfrac{}{}{0pt}{}{2}{13} + \genfrac{}{}{0pt}{}{3}{13}$, $K_{021}^{\mathcal{F}} = \genfrac{}{}{0pt}{}{2}{11} + \genfrac{}{}{0pt}{}{2}{12} + \genfrac{}{}{0pt}{}{3}{11} + \genfrac{}{}{0pt}{}{3}{12} + \genfrac{}{}{0pt}{}{3}{22}$, $K_{012}^{\mathcal{F}} = \genfrac{}{}{0pt}{}{2}{11} + \genfrac{}{}{0pt}{}{2}{12} + \genfrac{}{}{0pt}{}{3}{11} + \genfrac{}{}{0pt}{}{2}{13} + \genfrac{}{}{0pt}{}{3}{13} + \genfrac{}{}{0pt}{}{3}{12} + \genfrac{}{}{0pt}{}{3}{22} + \genfrac{}{}{0pt}{}{3}{23}, \ldots$

If v is such that $v_i \le v_{i+1}$ then $K_v^{\mathcal{F}}$ is the image of some element under π_i, and therefore, is symmetrical in a_i, a_{i+1}. It implies that if v is of the type I, $0, 0, \ldots$, with $I \in \mathbb{N}^n$ partition, then $K_v^{\mathcal{F}}$ is symmetrical in $a_1, \ldots, a_n$. In fact, because it is a positive sum of different tableaux of shape I, and because it projects onto the key polynomial $K_v = \mathbb{Y}_v(\mathbb{A}, \mathbf{0}) = S_I(a_1 + \cdots + a_n)$, then it coincides with the free Schur function S_I.

Key polynomials are characters of Demazure modules. We refer to Kashiwara [**79**], Littlemann [**129**], [**130**] for more general constructions in the framework of Kac-Moody algebras.

11.6. Plactic Schubert Polynomials

Inspired by the constructions of Young, Littlewood found a non-commutative version of Schur functions by allowing repetitions of letters in Young tableaux. However, it took a long time to define a multiplication on the module generated by these new functions. We shall easily find a free version of Schubert polynomials, but unfortunately will loose multiplication. Even at the commutative level, there is up to now no extension of the Littlewood-Richardson rule for Schubert polynomials, except in special cases. The product of a Schubert polynomial by a Schur function is described by Kohnert [**85**].

Let $\mathbb{A} = \{a_1, \ldots, a_n\}$ be an alphabet (with non-commuting letters), and $\mathbb{B} = \{b_1, \ldots, b_n\}$ be an alphabet of commutative indeterminates.

Define, in $\mathfrak{Pol}(\mathbb{B}) \otimes \mathfrak{Plac}(\mathbb{A})$ the following kernel :

$$\mathcal{K}(\mathbb{B}, \mathbb{A}) := \begin{array}{llll} (b_1 + a_{n-1}) & & & \\ (b_1 + a_{n-2}) & (b_2 + a_{n-2}) & & \\ \vdots & \vdots & & \\ (b_1 + a_1) & (b_2 + a_1) & \cdots & (b_{n-1} + a_1) \end{array}$$

One reads this expression by columns, from left to right, and evaluates the words in $\mathfrak{Plac}(\mathbb{A})$.

One defines the *plactic Schubert polynomials* $X_\sigma^{\mathcal{F}}(\mathbb{A})$ as coefficients of the Schubert polynomials in $\mathbb{B}$. In more details,

$$\mathcal{K}(\mathbb{B},\mathbb{A}) = \sum_{\sigma\in\mathfrak{S}_n} \mathbb{X}_{\sigma\omega}(\mathbb{B},\mathbf{0})X_\sigma^{\mathcal{F}}(\mathbb{A}) \ . \tag{11.6.1}$$

For example, for $n=4$, the coefficient of $\mathbb{X}_{2341}(\mathbb{B},\mathbf{0}) = b_1b_2b_3$ is

$$X_{1432}^{\mathcal{F}} = \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} + \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \ .$$

It is not straightforward to show, from this definition, that plactic Schubert polynomials are positive sums of tableaux (without multiplicities). This is why one needs to use some combinatorial properties of reduced decompositions, the nilplactic algebra being the proper set-up, as we saw in chapter 10.

We refer to a forthcoming study of Lin and Yang for a combinatorial interpretation, in terms of lattice graphs, of general Schubert polynomials[**127**] in two alphabets.

APPENDIX A

Complements

A.1. Minor Identities

Plücker relations have been obtained in the 18^e century in the process of eliminating unknowns inside systems of linear equations. The treatise of Muir [**138**] gives even a reference to Fontaine des Bertins, in 1748, for the three-terms relation, and to Bézout, in 1779, for the general one.

For finite sequences of integers A, B, write AB for the concatanation, $A \setminus a$ when erasing an element a of A. Let $n \in \mathbb{N}$, M be an $n \times \infty$ matrix. For any A of length n, write $[A]$ for the minor of M taken on the columns specified by A.

Given two sequences A, B of respective lengths $n{+}1$, $n{-}1$, then *Plücker relation* is

$$\sum_{i=1}^{n+1} (-1)^i \, [A \setminus a_i] \, [a_i B] = 0 \; . \tag{A.1.1}$$

The typical relation is for $A = \{1, 2, 3\}$, $B = \{4\}$:

$$[12]\,[34] - [13]\,[24] + [23]\,[14] = 0 \; , \tag{A.1.2}$$

and there are many ways to justify that it "generates" all Plücker relations, i.e. that one can write $n{+}1$ terms relations, $n > 2$, in terms of (A.1.2) only.

In the case

$$M = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \end{bmatrix}$$

relation (A.1.2) becomes

$$(x_1 - x_2)(x_3 - x_4) - (x_1 - x_3)(x_2 - x_4) + (x_2 - x_3)(x_1 - x_4) = 0 \tag{A.1.3}$$

and is immediate to check.

However (A.1.3) implies that Plücker relation is also true for

$$M = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & \cdots \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & \cdots \end{bmatrix}$$

and, then for

$$M = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & \cdots \\ x_{12} & x_{22} & x_{32} & x_{42} & \cdots \end{bmatrix}$$

the entries of M being now independent indeterminates, i.e. the matrix being a $2 \times \infty$ generic matrix.

More generally, a Vandermonde matrix

$$V_n(x_1, x_2, \ldots) = \begin{bmatrix} x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & x_4^{n-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & 1 & 1 & \cdots \end{bmatrix}$$

will be generic enough to check the validity of the $n+1$ terms Plücker relation (A.1.1).

In that case, it is directly related to the Lagrange interpolation. Indeed, for $\mathbb{A}, \mathbb{B}$ of cardinalities $n+1$, $n-1$, then the Lagrange remainder for $S^{n-1}(y-\mathbb{B})$ at points $\mathbb{A}$ vanishes :

$$\sum_a S^{n-1}(a-\mathbb{B})\, R(a, \mathbb{A}\setminus a)^{-1} = 0\ . \tag{A.1.4}$$

Multiplying by $\Delta(\mathbb{A})\Delta(\mathbb{B})$, it can be rewritten

$$\sum_{a\in\mathbb{A}} \pm\, \Delta(\mathbb{A}\setminus a)\, \Delta(a\mathbb{B}) = 0\ , \tag{A.1.5}$$

and this is indeed (A.1.1) for the Vandermonde matrix.

For example,

$$\begin{aligned}
[123]\,[4bc] - [124]\,[3bc] + [134]\,[2bc] - [234]\,[1bc] &= 0\\
\frac{(a_4-b)(a_4-c)}{(a_4-a_1)(a_4-a_2)(a_4-a_3)} + \frac{(a_3-b)(a_3-c)}{(a_3-a_1)(a_3-a_2)(a_3-a_4)} &\\
+\frac{(a_2-b)(a_2-c)}{(a_2-a_1)(a_2-a_3)(a_2-a_4)} + \frac{(a_1-b)(a_1-c)}{(a_1-a_2)(a_1-a_3)(a_1-a_4)} &= 0\\
\Delta(a_1,a_2,a_3)\Delta(a_4,b,c) - \Delta(a_1,a_2,a_4)\Delta(a_3,b,c) &\\
+\Delta(a_1,a_3,a_4)\Delta(a_2,b,c) - \Delta(a_2,a_3,a_4)\Delta(a_1,b,c) &= 0
\end{aligned}$$

are three ways to write the same relation.

At this stage appears a fruitful interpretation, in terms of symmetric groups representations. $\mathfrak{S}_{2n}$ acts on a matrix and its minors by permuting columns; in the case of a Vandermonde matrix, it amounts acting on the polynomial $\Delta(x_1,\ldots,x_n)\cdot \Delta(x_{n+1},\ldots,x_{2n})$ by permuting variables. It is known since Young [**182**] that the linear span of the orbit of the preceding polynomial is an irreducible representation (in char. 0) of shape 2^n.

For any permutation $\sigma\in\mathfrak{S}_{2n}$, write $[[\sigma]]$ for the product $[\sigma_1,\ldots,\sigma_n]\,[\sigma_{n+1},\ldots,\sigma_{2n}]$ of minors of $V_n(x_1,x_2,\ldots)$. Then (A.1.5) can be rewritten

$$[[\sigma]]\,(1-\sum_{i=1}^{n}\tau_{i,n+1}) = 0\ , \tag{A.1.6}$$

sum over all transpositions $\tau_{i,n+1}$.

Replacing $[\sigma]$ by the Young idempotent attached to σ, one has now an identity, which is a key ingredient in the description by Young [**182**] of irreducible representations.

For example, (A.1.2) has to be decoded as

$$(1+s_1)(1+s_3)s_2(1-s_1)(1-s_3)\,(1-\tau_{13}-\tau_{23}) = 0\ . \tag{A.1.7}$$

The important point is that

$$(1-s_1)\,(1-\tau_{13}-\tau_{23}) = \sum_{\zeta\in\mathfrak{S}_3} (-1)^{\ell(\zeta)}\,\zeta$$

and this allowed Young to conclude that the LHS of (A.1.7) vanishes, because there is no element alternating under $\mathfrak{S}_3$ in the space generated by the minors of order

2. More generally, Young uses that

$$\sum_{\nu\in\mathfrak{S}_n} \pm\nu\,(1-\sum_{i=1}^{n}\tau_{i,n+1}) = \sum_{\zeta\in\mathfrak{S}_{n+1}} (-1)^{\ell(\zeta)}\,\zeta$$

and therefore, that, for any $\sigma\in\mathfrak{S}_{2n}$, one has

$$\text{(A.1.8)}\quad [[\sigma]]\,(1-\sum_{i=1}^{n}\tau_{i,n+1}) = [[\sigma]]\,\frac{1}{n!}\sum_{\nu\in\mathfrak{S}_n}\pm\nu\left(1-\sum_{i=1}^{n}\tau_{i,n+1}\right)$$
$$= [[\sigma]]\,\frac{1}{n!}\sum_{\zeta\in\mathfrak{S}_{n+1}}(-1)^{\ell(\zeta)}\,\zeta = 0\ .$$

The modern version of the theory of representations [**143**] uses simpler relations than Young did. It states that the "tableau" [12][34] is an eigenvector for the Jucys element $\tau_{13}+\tau_{23}$, with eigenvalue 1. More generally Young's orthogonal idempotents (which are in bijection with tableaux) are eigenvectors [**67**] of the *Jucys-Murphy* elements $J_k := \sum_{i<k}\tau_{ik}$, and relations like (A.1.1) become nullities of sums of eigenvalues.

Let us describe the counterpart, in terms of minors, of the generalization of the Lagrange interpolation :

$$\text{(A.1.9)}\qquad \mathfrak{Sym}(n|n)\ni f\longrightarrow\sum\frac{f(\mathbb{A}',\mathbb{A}'')}{R(\mathbb{A}',\mathbb{A}'')} = f(a_1,\dots,a_{2n})\,\partial_{n|n}\ ,$$

where $\partial_{n|n}$ stands for $(\partial_n\cdots\partial_{2n-1})\cdots(\partial_1\cdots\partial_n)$.

Instead of having a sum over the elements of a set of cardinality $n{+}1$ like in (A.1.1), one can take sums on subsets. Let $p,n\in\mathbb{N}$, $p{+}q=n{+}1$, and let A,B,C be sequences of numbers of respective lengths $n{+}1$, $n{-}p$, $n{-}q$. Let M be an $n\times\infty$ matrix. Then

$$\text{(A.1.10)}\qquad \sum_{A'+A''=A}\epsilon(A',A'')[BA']\,[A''C] = 0\ ,$$

sum over all decompositions of A into two increasing subsequences of lengths p,q, $\epsilon(A',A'')$ being the sign of the permutation $A'A''$ of A.

When M is a Vandermonde matrix, (A.1.10) becomes the *Garnir relation* [**55**]

$$\text{(A.1.11)}\qquad \sum_{\mathbb{A}'+\mathbb{A}''=\mathbb{A}}\epsilon(\mathbb{A}',\mathbb{A}'')\Delta(\mathbb{B}\mathbb{A}')\,\Delta(\mathbb{A}''\mathbb{C}) = 0\ ,$$

$\mathbb{A},\mathbb{B},\mathbb{C}$ being now interpreted as sets of variables.

But if one divides this relation by $\Delta(\mathbb{A})\Delta(\mathbb{B})\Delta(\mathbb{C})$, then one gets

$$\text{(A.1.12)}\qquad \sum_{\mathbb{A}'+\mathbb{A}''=\mathbb{A}}\frac{R(\mathbb{B},\mathbb{A}')\,R(\mathbb{A}'',\mathbb{C})}{R(\mathbb{A}',\mathbb{A}'')} = 0$$

or

$$\text{(A.1.13)}\qquad S_{(q-1)^p}(a_1+\cdots+a_p-\mathbb{B})\,S_{(p-1)^q}(a_{p+1}+\cdots+a_{p+q}-\mathbb{C})\,\partial_{p|q} = 0\ .$$

However, this last nullity is a direct consequence of the corollary 7.7.2 of Bott's theorem :

$$\begin{aligned} S_{q^p}(\mathbb{C}-\mathbb{B}) &= \pm S_{q^p}(a_1+\cdots+a_p-\mathbb{B})S_{(p)^q}(a_{p+1}+\cdots+a_{p+q}-\mathbb{C})\partial_{p|q} \\ &= \pm(a_1\cdots a_p)S_{(q-1)^p}(a_1+\cdots+a_p-\mathbb{B})\,(a_{p+1}\cdots a_{p+q}) \\ &\qquad S_{(p-1)^q}(a_{p+1}+\cdots+a_{p+q}-\mathbb{C})\,\partial_{p|q} \\ &= \pm S_{(q-1)^p}(a_1+\cdots+a_p-\mathbb{B})S_{(p-1)^q}(a_{p+1}+\cdots+a_{p+q}-\mathbb{C})\partial_{p|q}\,a_1\cdots a_{p+q}\ , \end{aligned}$$

the vanishing of $S_{q^p}(\mathbb{C}-\mathbb{B})$ being due to the cardinalities $q-1$, $p-1$ of $\mathbb{B},\mathbb{C}$.

The reader may have already guessed that the Garnir relations can be deduced from factorizations in the group algebra of the symmetric group; unfortunately, the present text is not the proper place to comment the work of Young and the numerous relations he wrote in his *substitutional analysis*.

In characteristic 0, quadratic relations are sufficient to generate all relations between minors. However, for concrete applications to the theory of symmetric functions, or the theory of orthogonal polynomials, one needs to have recourse to other types of relations. Apart from Plücker relations, we essentially used only a polarized version of *Bazin's relation*, that we explain now.

LEMMA A.1.1. *Let $n,r\in\mathbb{N}$, $n>r$. Let $A=\{a_1,\dots,a_{2r-2}\}$, $B=\{b_1,\dots,b_r\}$, $C=\{c_1,\dots,c_{n-r}\}$ be sequences of integers, M be an $n\times\infty$ matrix. Then*

$$\Big|[a_i\dots a_{i+r-2}\,b_j\,C]\Big|_{1\le i,j\le r} = [a_1\dots a_r\,C]\cdots[a_{r-1}\dots a_{2r-2}\,C]\,[BC]\ . \tag{A.1.14}$$

For example, for $A=\{1,2,3,4\}$, $B=\{b,b',b''\}$, $C=\emptyset$ and $n=3$, one has

$$\begin{vmatrix} [12\,b] & [12\,b'] & [12\,b''] \\ [23\,b] & [23\,b'] & [23\,b''] \\ [34\,b] & [34\,b'] & [34\,b''] \end{vmatrix} = [123]\,[234]\,[bb'b'']\ .$$

For $n=6$, $r=4$, $A=\{1,2,3,6,7,8\}$, $B=\{b,c,d,e\}$ and $C=\{4,5\}$, Bazin states

$$\begin{vmatrix} [12345\,b] & [12345\,c] & [12345\,d] & [12345\,e] \\ [23456\,b] & [23456\,c] & [23456\,d] & [23456\,e] \\ [34567\,b] & [34567\,c] & [34567\,d] & [34567\,e] \\ [45678\,b] & [34567\,c] & [34567\,d] & [34567\,e] \end{vmatrix} = [123456]\,[234567]\,[345678]\,[45bcde]\ .$$

All the relations between minors that we have written up to now are easy consequences of the multi-linearity of a determinant. For more sophisticated relations, we refer to the article of Leclerc [**119**], and his analysis of Turnbull's theorem.

The reader may encounter quadratic relations looking different from the vanishing of an (alternating or not) sum of products, and he must be aware that trying to iterate Plücker relations is not always a good strategy to prove these relations.

We shall take a single example, as an illustration of why one has to supplement minor identities with representation theory. Consider Ciucu's identities about Schur functions. It states that a certain sum, indexed by all subsets of cardinality n of a set of cardinality $2n$, of products of Schur functions, is equal to 2^n a product of Schur functions. The question for us will be : writing Schur functions as determinants, is there a proof which avoids painful repetitive applications of Plücker identities[1] ?

One has first to rewrite the identity as the following minor identity (Ciucu's identity is when taking a matrix of complete functions):

[1] A combinatorial proof, in terms of non-intersecting paths, was given by Fulmek[**53**].

LEMMA A.1.2. *Given a matrix of order $n\times 2n$, for any permutation $\sigma\in\mathfrak{S}_{2n}$ let $[[\sigma]]$ code, as before, the product of the two minors $[\sigma_1,\dots,\sigma_n]$ and $[\sigma_{n+1},\dots,\sigma_{2n}]$. Then one has*

$$\sum_{\sigma\in\mathfrak{S}^{n|n}} [[\sigma]] = 2^n[1,3,\dots,2n{-}1]\,[2,4,\dots,2n]\ , \tag{A.1.15}$$

sum over all permutations such that $\sigma_1<\cdots<\sigma_n$, $\sigma_{n+1}<\cdots<\sigma_{2n}$.

One step further is to translate the statement in Young's terms, as a relation inside the irreducible representation of shape 2^n or $[n,n]$ (the correspondence between a shape and its conjugate involves signs, and the involution $s_i\to -s_i$). But instead of taking products of Vandermonde determinants as above, one can use simpler elements in the quotient ring $\mathbb{C}[\mathbb{A}]/\mathfrak{Sym}_+(\mathbb{A})$ (ring of polynomials modulo the ideal generated by symmetric polynomials without constant term), with $\mathbb{A}$ of cardinality $2n$. In that set-up, the product of two minors of order n is replaced by a *single monomial* in the orbit of $(a_1\dots a_n)^n$ under $\mathfrak{S}_{2n}$. We refer to [**27**] for this correspondence, and its application to the *straightening* of elements belonging to an irreducible representation.

More surprisingly, relation (A.1.15) boils down to the nullities $a_i^{2n}\equiv 0$, $1\le i\le 2n$. We shall not give here more details about representations, our only aim was to stress that with an appropriate trilingual dictionary *Symmetric Functions/Minors/Representations*, computations may be greatly simplified.

To get a flavour, take $n=3$. We have just claimed that the three following statements are equivalent:

- For any increasing sequence of 6 positive integers $\mathbb{A}=\{\alpha_1<\beta_1<\alpha_2<\beta_2<\alpha_3<\beta_3\}$, then
  $$\sum S_{a_1,a_2-1,a_3-2}\,S_{a_4,a_5-1,a_6-2} = 8S_{\alpha_1,\alpha_2-1,\alpha_3-2}\,S_{\beta_1,\beta_2-1,\beta_3-2}\ ,$$
  sum over all Grassmannian permutations $a_1<a_2<a_3$; $a_4<a_5<a_6$ of $\mathbb{A}$.
- For any 3×6 matrix
  $$\sum_\sigma[\sigma_1,\sigma_2,\sigma_3][\sigma_4,\sigma_5,\sigma_6] = 8[1,3,5]\,[2,4,6]\ ,$$
  sum over all Grassmannian permutations: $\sigma_1<\sigma_2<\sigma_3$, $\sigma_4<\sigma_5<\sigma_6$.
- $$\sum(-1)^{\ell(\sigma)}(a^{000333})^\sigma \equiv 8\,a^{000333}\ ,$$
  sum over all shuffles[2] of $[0,0,0]$ and $[3,3,3]$

However, the last statement is immediate, because, with $\mathbb{A}'=\{a_1,a_3,a_5\}$, $\mathbb{A}''=\{a_2,a_4,a_6\}$, it is

$$\begin{aligned}&\Psi_{000}(\mathbb{A}')\Psi_{333}(\mathbb{A}'')-\Psi_3(\mathbb{A}')\Psi_{33}(\mathbb{A}'')+\Psi_{33}(\mathbb{A}')\Psi_3(\mathbb{A}'')\\&\qquad-\Psi_{333}(\mathbb{A}')\Psi_{000}(\mathbb{A}'')\equiv\big(1-(-3)+3-(-1)\big)\,a_2^3a_4^3a_6^3=8\,a^{030303}\ ,\end{aligned}$$

and is a direct consequence of $a_i^6\equiv 0$.

[2]The sum is over all different monomials in the orbit of x^{000333}, but there is a sign, because one takes the occurrence of the representation of shape $[3,3]$ in maximal degree ($=9$), and not the Specht representation of shape $[2,2,2]$, which occurs in degree 6.

Fundamentally, one replaces a summation over the symmetric group (for the first two equations), by a summation on monomials, i.e. by a polynomial. At this stage, one can use algebraic operations on polynomials (here multiplication) which were missing at the level of permutations[3].

A.2. Exact sequences

The notion of exact sequence is widely used in algebraic geometry. It applies to categories of objects, for example, vector bundles on which acts a group, which are too sophisticated for our present needs. We shall keep from this notion only the idea of "positive enumeration", and restrict to *exact sequence of integers.*

DEFINITION A.2.1. $(0 = m_0, m_1, \ldots, m_n, m_{n+1} = 0)$ is an *exact sequence of integers* iff the partial alternating sums

$$m_j - m_{j-1} + m_{j-2} - \cdots \pm m_0 \ , \ j = 0, \ldots, n+1 \,,$$

are non-negative.

One can disguise exact sequences of integers into seemingly more elaborate sequences : exact sequences of classes of representations of the symmetric group (in char. 0), exact sequences of symmetric functions, exact sequences of elements of a free algebra, &c.

In the first case, $(0 = V_0, V_1, \ldots, V_n, V_{n+1})$ is an *exact sequence of representations* iff for every partition J, denoting by $(J : V_i)$ the multiplicity of the irreducible representation of shape J in V_i, then the sequence $(J : V_0)$, $(J : V_1)$, $\ldots$, $(J : V_{n+1})$ is exact.

Similarly, a sequence of symmetric function $(f_0, \ldots, f_{n+1})$ is exact, iff for any partition J, the sequence $\big((f_0, S_J), \ldots, (f_{n+1}, S_J)\big)$ is exact (multiplicities can be written as scalar products). Choosing another basis of symmetric functions than Schur functions would entail different exact sequences.

For a sequence of elements of the free algebra: $f_i = \sum_w c^i_w w$, one requires that for any word w, the sequence $(c^0_w, \ldots, c^{n+1}_w)$ be exact.

The Jacobi-Trudi determinantal expression of a Schur function gives rise to exact sequences that I used in my thesis [**97**] in relation with syzygies of determinantal ideals. In particular, I needed the following property.

LEMMA A.2.2. *Let $K = [k_1, \ldots, k_n]$ be a partition, let M be the Jacobi-Trudi matrix expressing S_K in terms of complete functions. Denote by M^j the minor obtained from M by suppressing its last row and its j-th column Then*

$$0, M^1 S_{k_1-n+1},\, M^2 S_{k_2-n+2},\, \ldots,\, M^n S_{k_n},\, S_K,\, 0$$

is an exact sequence of symmetric functions.

Proof. Each minor M^i is a Schur function, and its product by a complete function is given explicitly by Pieri formula. One notices that if a Schur function appears

[3]writing factorizations in the group algebra of the symmetric group is more complex than decomposing polynomials, though most identities of Young, through rewritings using the Yang-Baxter relations, can be reduced to the relation $(s_i+1)(s_i-1) = 0$, s_i being any simple transposition.

in the sequence, then it appears also at the preceding or following step, and only there. Thus, the restriction of the sequence to any J appearing in it is of the type

$$0,\, 0, \dots,\, 0,\, S_J,\, S_J,\, 0, \dots,\, 0\ ,$$

and this insures exactness. $\square$

For example, for $J = [2478]$, the determinant is $\begin{vmatrix} S_2 & S_5 & S_9 & S_{11} \\ S_1 & S_4 & S_8 & S_{10} \\ S_0 & S_3 & S_7 & S_9 \\ S_{-1} & S_2 & S_6 & S_8 \end{vmatrix}$ and gives the sequence (that we prefer to write with arrows)

$$0 \to S_{-1}\, S_{589} \to S_2\, S_{289} \to S_6\, S_{249} \to S_8\, S_{247} \to S_{2478} \to 0$$

which translates into a sequence of sets of partitions

$$\emptyset \to \emptyset \to \{2 \otimes [289]\} \to \{6 \otimes [249]\} \to \{8 \otimes [247]\} \to \{[2478]\} \to \emptyset\ .$$

Since $\mathfrak{Sym}$ is a commutative sub-algebra of the plactic algebra, the preceding lemma induces a similar statement in the plactic algebra, but now Schur functions are sums of words (one can choose to write products of Schur functions by complete functions on the right or on the left, for each term in the sequence) :

LEMMA A.2.3. *Let $K = [k_1, \dots, k_n]$ be a partition. Then*

$$0,\, S_{k_2+1,\dots,k_n+1}\, S_{k_1-n+1},\, S_{k_1,k_3+1,\dots,k_n+1}\, S_{k_2-n+2},\, \dots,\, S_{k_1,\dots,k_{n-1}}\, S_{k_n},\, S_K,\, 0$$

is an exact sequence in the plactic algebra.

The restriction of the sequence to any word w which appears in it, is of the type

$$0, \dots,\, 0,\, w,\, w,\, 0, \dots,\, 0\ .$$

Iterating Laplace expansion of Jacobi-Trudi determinant, one obtains a sequence of products of complete functions which is no less exact :

PROPOSITION A.2.4. *Let $K \in \mathbb{N}$ be a partition, $\mathbb{A}$ and alphabet. Let*

$$M_\ell := \sum_{\sigma,\, \ell(\sigma)=\ell} S^{(K+\rho)^\sigma - \rho}(\mathbb{A})\ .$$

Then

$$0 \to S_K \to M_0 \to M_1 \to \cdots \to M_{\binom{n}{2}} \to 0$$

is an exact sequence inside the plactic algebra.

For example, in the case of $K = [2, 2, 2]$, one has the exact sequence

$$0 \to S_{222} \to S^{222} \to S^{312} + S^{213} \to S^{330} + S^{411} \to S^{420} \to 0.$$

Restricting it to the class of the tableau $\begin{array}{|c|c|c|} \cline{1-1} 3 \\ \cline{1-2} 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$, one finds the sequence

$$0,\, 0,\, 23\,12\,11 + 13\,22\,11,\, 123\,2\,11 + 23\,112\,1,\, 0,\, 0,\, 0\ ,$$

which becomes exact once evaluated in the plactic algebra.

Instead of tableaux and products S^J, one could use paths, whether intersecting or not, and relate the preceding proposition to Gessel and Viennot [**58**] constructions on Jacobi-Trudi type determinants. In our case, we do not need to describe an involution to find the objects corresponding to a Schur function, pairs of "intersecting paths", that have to be eliminated, correspond to identical words in the plactic algebra.

The $\mathfrak{Sym}$-operator "multiplication by a Schur function", and the Littlewood-Richardson rule can also be formulated in terms of exact sequences.

Vexillary Schubert polynomials can be treated in a similar manner, and the constructions of Chen, Li and Louck [**34**] rewritten inside the plactic algebra.

A.3. Cauchy Residue Formula

Given a rational function $f(z)$, and $a \in \mathbb{C}$, let $f(z) = \sum_{-\infty}^{+\infty} c_k (z-a)^j$ be its expansion in a neighbourhood of $z = a$. Cauchy defined the residue of f in $z = a$ to be the coefficient c_{-1} and expressed it as the integral

$$Res(f,a) := c_{-1} = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma f(z) dz \ ,$$

where Γ is any simple curve in the complex plane enclosing the point $z = a$.

This definition extends to more general functions, but we shall restrict to the space $\mathfrak{Rat}$ of rational functions in one variable z. In that case, the map $f \mapsto Res(f,a)$ is a linear functional that one computes without integration, but with the help of Laurent expansions. It is natural to extend Cauchy residue to any finite set of points (not necessarily distinct) in $\mathbb{C}$ (however, we shall treat these points as rank-1 elements).

Let us see how to do it. More developments can be found in the book of Fuhrmann [**49**].

First, any rational function $f = P/Q$, with P, Q relatively prime, $Q = R(x, \mathbb{A})$, can be uniquely decomposed into

$$f = f_+ + f_-/Q \ ,$$

where f_- is the remainder of P modulo Q.

A rational function of the type P/Q, with $deg(P) < deg(Q)$ will be called *proper*, and the linear span of such functions will be denoted $\mathfrak{Rat}_-$. The above decomposition means that one has a direct sum decomposition

$$\mathfrak{Rat} \simeq \mathfrak{Pol} \oplus \mathfrak{Rat}_- \ ,$$

the explicit isomorphism being given by division.

Elements of $\mathfrak{Rat}_-$ have an expansion as formal power series in z^{-1}, that we shall write with the help of alphabets :

$$\beta \frac{1}{z} \sigma_{1/z}(\mathbb{A}) \ .$$

On $\mathfrak{Rat}$, one defines a bilinear symmetric form $<,>$ by

$$\text{(A.3.1)} \qquad < f \, , \, g >= 0 \quad , \quad f,g \in \mathfrak{Pol} \quad \text{or } f,g \in \mathfrak{Rat}_- \ ,$$

$$\text{(A.3.2)} \qquad < S^m(z - \mathbb{B}) \, , \, 1/z \, \sigma_{1/z}(\mathbb{A}) >= S^m(\mathbb{A} - \mathbb{B}) \ , \ m \in \mathbb{N}, \text{ any } \mathbb{B} \ .$$

As a special case, one has

$$\text{(A.3.3)} \qquad < S^m(z - \mathbb{B}) \, , \, \frac{1}{z-a} >=< S^m(z - \mathbb{B}) \, , \, \frac{1}{z} \sigma_{1/z}(a) >= S^m(a - \mathbb{B}) \ ,$$

a being a rank-1 element.

When $\mathbb{A}$ is of finite cardinality, say n, one can obtain the functional $f \mapsto < f, 1/z\, \sigma_{1/z}(\mathbb{A}) >$ by symmetrization from the case $card(\mathbb{A}) = 1$. Indeed, let ω be the maximal permutation in $\mathfrak{S}(\mathbb{A})$. Ordering totally $\mathbb{A}$, one has

$$(1 - a_1 z)^{-1}\pi_\omega = \prod_{a\in\mathbb{A}} (1 - az)^{-1} \text{ and } S^m(\mathbb{A}-\mathbb{B}) = S^m(a_1 - \mathbb{B})\pi_\omega \ ,$$

which implies that π_ω is compatible with the form $<,>$.

Because $1/S^n(z - \mathbb{A}) = z^{1-n}/z \prod(1 - az) = z^{1-n}/(z - a_1)\pi_\omega$, one has

$$< S^m(z - \mathbb{B})\, ,\ \frac{1}{S^n(z - \mathbb{A})} >= S^{m-n+1}(\mathbb{A}-\mathbb{B}) \ .$$

More generally, given $\mathbb{D}$ of cardinality k, one gets

$$< S^m(z - \mathbb{B})\, ,\ \frac{S^k(z - \mathbb{D})}{S^n(z - \mathbb{A})} >= S^{m-n+k+1}(\mathbb{A}-\mathbb{B}-\mathbb{D}) \ , \tag{A.3.4}$$

because $S^k(z - \mathbb{D})/S^n(z - \mathbb{A}) = z^{k-n}\sigma_z(\mathbb{A} - \mathbb{D})$.

It is now clear that the quadratic form $<,>$ is non-degenerate, and that the orthogonal of $\mathfrak{Pol}$ is $\mathfrak{Rat}_-$, i.e.

$$\forall f \in \mathfrak{Pol}, < f\, ,\, g >= 0 \ \Leftrightarrow\ g \in \mathfrak{Rat}_- \ . \tag{A.3.5}$$

Formula (A.3.4) easily produces pairs of adjoint bases of some natural spaces. Take for example the linear span $\mathfrak{Rat}_\mathbb{A}$ of $z^i/S^n(z - \mathbb{A})$, $i = 0, \dots, n - 1$.

LEMMA A.3.1.
- *The orthogonal of* $\mathfrak{Rat}_\mathbb{A}$ *is the space of polynomials of degree* $\le n{-}1$.
- *The adjoint basis of* $\{z^i/S^n(z{-}\mathbb{A}), 0 \le i \le n{-}1\}$ *is* $\{S^0(z{-}\mathbb{A}), \dots, S^{n-1}(z{-}\mathbb{A})\}$.
- *The adjoint basis of* $\{S^{n-1}(z + a{-}\mathbb{A})/S^{n-1}(z - \mathbb{A}), a \in \mathbb{A}\}$ *is* $\{S^{n-1}(z + a{-}\mathbb{A})/S^{n-1}(2a - \mathbb{A})\}$.

Proof. Let us write $\mathbb{D}{=_k}\,\mathbb{A}$ if $\Lambda^i(\mathbb{D}) = \Lambda^i(\mathbb{A})$, $i \le k$ (this implies that symmetric functions in $\mathbb{D}$ of degree $\le k$ coincide with those of $\mathbb{A}$).

The first two statements correspond to (A.3.4) with $\big(\mathbb{D}{=_k}\,\mathbb{A}$ and $\mathbb{B} = 0\big)$, or $\big(\mathbb{D} = 0$ and $\mathbb{B}{=_m}\,\mathbb{A}\big)$ respectively. In that case

$$< S^m(z - \mathbb{B})\, ,\ \frac{S^k(z - \mathbb{D})}{S^n(z - \mathbb{A})} >= S^{m-n+k+1}(0)$$

and this proves the two statements.

For the last one, picking out any pair $a, a' \in \mathbb{A}$, we take $\mathbb{B} = \mathbb{A}{-}a$, $\mathbb{D} = \mathbb{A}{-}a'$. The function $S^{n-1}(\mathbb{A} - \mathbb{A} + a - \mathbb{A} + a') = S^{n-1}(a + a' - \mathbb{A})$ vanishes if $a \ne a'$ (because $\mathbb{A} - a - a'$ is an alphabet of cardinality $n{-}2$), and otherwise becomes $S^{n-1}(2a - \mathbb{A})$. □

We have widely used Euclidean division on $\mathfrak{Pol}$. Let us finish this section by finding out the operation adjoint to taking a first remainder, as an operator on $\mathfrak{Rat}$.

LEMMA A.3.2. *Let* $\mathbb{X}$ *be an alphabet of cardinality* $n \ge 1$. *Then the adjoint of "First remainder modulo* $R(z, \mathbb{X})$*" is the linear morphism on* $\mathfrak{Rat}_-$ *:*

$$\frac{1}{z}\,\sigma_{1/z}(\mathbb{A}) \ \mapsto\ \frac{S^{n-1}(z + \mathbb{A} - \mathbb{X})}{S^n(z - \mathbb{X})} \ .$$

Proof. Let us take $f(z) = S^m(z-\mathbb{B})$, as usual. Then the first remainder of f modulo $S^n(z-\mathbb{X})$ is $\mathcal{R} = (-1)^{n-1}S_{1^{n-1};m-n+1}(\mathbb{X}-z;\mathbb{X}-\mathbb{B})$. Therefore

$$\text{(A.3.6)}\quad <\mathcal{R}\,,\,\frac{1}{z}\sigma_{1/z}(\mathbb{A})> \;=\; (-1)^{n-1}S_{1^{n-1};m-n+1}(\mathbb{X}-a;\mathbb{X}-\mathbb{B})\,\pi_\omega$$

$$\text{(A.3.7)}\quad = (-1)^{n-1}S_{1^{n-1};m-n+1}(\mathbb{X}-\mathbb{A};\mathbb{X}-\mathbb{B})\ .$$

On the other hand, let $\mathbb{D}$ be the alphabet of cardinality $n-1$ such that $S^{n-1}(z-\mathbb{D}) = S^{n-1}(z+\mathbb{A}-\mathbb{X})$. Then

$$<f(z)\,,\,\frac{S^{n-1}(z-\mathbb{D})}{S^n(z-\mathbb{X})} = S^m(\mathbb{X}-\mathbb{D}-\mathbb{B}) = \sum_{i=0}^{n-1} S^i(-\mathbb{D})S^{m-i}(\mathbb{X}-\mathbb{B})$$

$$= (-1)^{n-1}\sum_{i=0}^{n-1} S_{1^i}(\mathbb{X}-\mathbb{A})S^{m-i}(\mathbb{X}-\mathbb{B}) = (-1)^{n-1}S_{1^{n-1};m-n+1}(\mathbb{X}-\mathbb{A};\mathbb{X}-\mathbb{B})$$

which is the same function. □

Notice that the expansion of $\frac{S^{n-1}(z+\mathbb{A}-\mathbb{X})}{S^n(z-\mathbb{X})}$ at $z=\infty$ is

$$\sum z^{-i-1}S^i(\mathbb{A}+(\mathbb{X}-\mathbb{A}-\mathbb{D})) = z^{-1} + z^{-2}S^1(\mathbb{A}) + \cdots + z^{-n}S^{n-1}(\mathbb{A}) +$$
$$z^{-n-1}\big(S^n(\mathbb{A}) - S^n(\mathbb{A}-\mathbb{X})\big) + z^{-n-2}\big(S^{n+1}(\mathbb{A}) - S^{n+1}(\mathbb{A}-\mathbb{X}) - S^1(\mathbb{X})S^n(\mathbb{A}-\mathbb{X})\big)$$
$$+z^{-n-3}\big(S^{n+2}(\mathbb{A}) - S^{n+2}(\mathbb{A}-\mathbb{X}) - S^1(\mathbb{X})S^{n+1}(\mathbb{A}-\mathbb{X}) - S^2(\mathbb{X})S^n(\mathbb{A}-\mathbb{X})\big) + \cdots$$

A.4. Differential calculus on symmetric functions

To be able to perform computations independently of the number of variables motivated the creation of algebraic and combinatorial tools already at the start of the theory of symmetric functions.

In these notes, we have mostly used the canonical scalar product on $\mathfrak{Sym}$ to get expansions in different bases, and symmetrizing operators like divided differences to provide recursions.

However, Cayley, Sylvester, MacMahon were taking another approach, by systematizing the fact that $\mathfrak{Sym}$ can be considered in different ways as a ring of polynomials (the variables being the elementary symmetric functions, the complete functions, or the powers sums).

They were using differential calculus in these variables. The functions they wanted to study, like the resultant or the discriminant, or more elaborate invariants, were characterized as solutions of differential equations.

This point of view is now much relevant to the modern trends of the theory of symmetric functions. One can even mix differentials and discrete operators like divided differences, as in the case of Dunkl operators.

Restricting to operators on functions of one variable (and symmetrizing afterwards) already provides interesting operators on $\mathfrak{Sym}$. For example, the *Debiard-Sekiguchi-Macdonald operator* $\mathfrak{DSM}$ is of this type [**135**] :

$$\mathfrak{Sym}(\mathbb{A}) \ni f(a_1,\ldots,a_n) \mapsto f(qa_1,\ldots,a_n) \mapsto$$
$$f(qa_1,\ldots,a_n)\prod_{i\geq 2}(ta_1-a_i) \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{n-1}} \mathfrak{DSM}(f)$$

Let us just show how to use the first derivative in a single variable to build differential operators on the space of symmetric functions. Let $\mathbb{A} = \{a_1, \ldots, a_n\}$, and $D := \frac{d}{d_{a_1}}$.

LEMMA A.4.1. *Let K be any function of a_1, with coefficients in $\mathfrak{Sym}(\mathbb{A})$. Let $\nabla = \nabla_K$ be the operator*

$$\mathfrak{Sym}(\mathbb{A}) \ni f \mapsto \nabla(f) := \pi_\omega(KD(P)) \ .$$

Then ∇ is a derivation, i.e.

$$\nabla(fg) = \nabla(f) \cdot g + f \cdot \nabla(g) \ .$$

Indeed $D(fg) = D(f)\, g + f\, D(g)$, and f, g being symmetrical, commute with the total symmetrizer π_ω.

By varying $K = K(a_1)$, one can obtain as many differential operators as is needed to fill a grant application. Here are only two of them, the derivatives with respect to Λ^i or to Ψ^i :

PROPOSITION A.4.2. *Let $j \in \mathbb{N}$, $1 \le j \le n$.*

(1) *Let $K = (-a_1)^{1-j}$. Then $\nabla = \frac{d}{d_{\Lambda_j}}$.*
(2) *Let $K = (-1)^{n-1} a_1^{1-j-n}\,(a_1 \cdots a_n)$. Then $\nabla = j\, \frac{d}{d_{\Psi_j}}$.*

Proof. Because the ∇'s are derivations, we have only to check their action on algebraic bases of $\mathfrak{Sym}(\mathbb{A})$. In the first case,

$$D(\Lambda^i(\mathbb{A})) = D\Big(\Lambda^i(\mathbb{A}{-}a_1) + a_1\Lambda^{i-1}(\mathbb{A}{-}a_1)\Big) = \Lambda^{i-1}(\mathbb{A} - a_1) \ ,$$

and $\Lambda^{i-1}(\mathbb{A} - a_1)\, a_1^{1-j} = S_{1^{i-1};1-j}(\mathbb{A}; a_1)$ is sent by π_ω onto $S_{1^{i-1},1-j}(\mathbb{A})$ which is 0, except for $S_{1^{j-1};1-j}(\mathbb{A}) = (-1)^{j-1} S_{0\ldots0}(\mathbb{A})$.

In the second, $D(\Psi_i(\mathbb{A})) = i a_1^{i-1}$, and $i a_1^{i-1} a_1^{2-j-n}\, a_2 \cdots a_n$ is sent by π_ω onto $i S_{1^{n-1},i-j-(n-1)}(\mathbb{A}) = (-1)^{n-1}\, i$ if $i = j$, and sent to 0 otherwise. □

Having at our disposal the derivatives with respect to elementary symmetric functions or power sums, one can combine them to obtain more sophisticated differential operators. The articles of Cayley, Sylvester, MacMahon about invariant theory are full of such operators.

One can also obtain in this way representations of the Witt or Virasoro algebras ([**77**]). Another application is to describe the product by a conjugacy class, in the group algebra of the symmetric group, as a differential operator [**118**] . The reader will understand that we do not fix any boundary to what is possible with symmetric functions.

Let us just end this section by illustrating that even at the level of notations, symmetric functions can be useful in differential calculus.

A function $y(x)$ of x is the equation of a "line in the plane" iff $y'' = 0$. To find the differential equation satisfied by a general planar conic $y^2 + \cdots = 0$, one writes as many derivatives (the third, fourth and fifth) as is needed to eliminate the arbitrary coefficients. The solution (*Monge equation*) has been obtained by Monge. Let us see how to recover it without computations.

Let us write

$$\frac{1}{i!(i+2)!}\frac{dy}{dx} = \Lambda^i(\mathbb{A}) \ , \ i \ge 0$$

(one has to write $\Lambda^0(\mathbb{A})$ instead of 1). The general theory indicates that the looked-for expression does not depend on y and y', and, as a symmetric function of $\mathbb{A}$ written in the basis of power sums, does not depend upon $\Psi_1(\mathbb{A})$. Since it is of degree 3 in the variables $\Lambda^i(\mathbb{A})$ (three derivatives have been used), and also of degree 3 in $\mathbb{A}$ (the leading term is proportional to $\Lambda^3(\mathbb{A})$, and the expression must be homogeneous), then Monge equation must be identical to

$$\Psi^3(\mathbb{A}) = 0 \ ,$$

up to a scalar. Indeed,

$$-2\Psi^3(\mathbb{A}) = \begin{vmatrix} \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & 0 \\ \Lambda^1(\mathbb{A}) & 2\Lambda^2(\mathbb{A}) & \Lambda^0(\mathbb{A}) \\ 2\Lambda^2(\mathbb{A}) & 6\Lambda^3(\mathbb{A}) & 2\Lambda^1(\mathbb{A}) \end{vmatrix} = \begin{vmatrix} y''/2 & y'''/6 & 0 \\ y'''/6 & y^{iv}/24 & y''/2 \\ y^{iv}/24 & y^{v}/120 & y'''/3 \end{vmatrix} \ ,$$

the determinant on the right being Monge's expression, once expanded.

The first differential invariants of Halphen [**63**] could be as easily obtained. However, symmetric functions do not explain the structure of the space of differential invariants. They just help eliminating unnecessary computations, and arriving faster to the heart of the matter.

APPENDIX B

Solution of exercises

B.1

Ex. 1.1 Take $\mathbb{B}$ of cardinality k. Then

$$S^k(1-z\mathbb{B})\,\sigma_z(\mathbb{A}) = \sigma_z(\mathbb{A}-\mathbb{B}) = (-1)^k \sum_j z^j S_{1^k;\,j-k}(\mathbb{B};\mathbb{A}) \ .$$

The specialization $S^i(-\mathbb{B}) = S^i(-\mathbb{A})$, $i=0,\ldots,k$ consists in replacing $\mathbb{B}$ by $\mathbb{A}$ in the determinants $S_{1^k;j-k}(\mathbb{B};\mathbb{A})$, because $\mathbb{B}$ occurs in degree k at most in them. In summary,

$$S^k(1-z\mathbb{B})\,\sigma_z(\mathbb{A}) = (-1)^k \sum_j z^j S_{1^k,\,j-k}(\mathbb{A}) \ .$$

Ex. 1.3 Since $\mathbb{B}$ is arbitrary, one cannot directly apply the transformation Lemma (1.4.1) which requires cardinality conditions on alphabets to be subtracted. However, the Laplace expansion along the last n columns imply the result, because the list of partitions contained in I is a subset of the list of partitions obtained by taking minors in the last n columns.

There is, however, a better reasoning. Because $i_r \le n$, the Laplace expansion will involve only Schur functions $S_J(\mathbb{B})$ with $\ell(J) \le n$. Therefore, one can suppose that $\mathbb{B}$ be of cardinality n, in which case subtracting $\mathbb{B}$ in the first r rows produces the block $S_I(\mathbb{A}-\mathbb{B})$.

Ex. 1.4 One sees that the determinant, coming from Newton's recursions, expressing the power sum $\Psi^k(\mathbb{A})$ in terms of elementary symmetric functions, can be interpreted as $-\Lambda_{1^{k-1};1}(\mathbb{A};\mathbb{B})$. Expanding, one gets

$$\Psi^k(\mathbb{A}) = S^k(\mathbb{A}-\mathbb{B}) - S^k(\mathbb{A}) \ ,$$

from which follow all required decompositions.

Ex. 1.5 Writing $F(z) = \prod_a (1-za)^{-1}$, one has $\Phi(z) = \sum_a a(1-za)^{-1}$, $\Phi^{(k)}(z) = k!\,\sum a^{k+1}(1-za)^{-1}$,

$$\Phi^{(k)}(0) = k!\,\Psi_{k+1}(\mathbb{A}) \ .$$

Therefore, the required expression is

$$S^n(\mathbb{A}) = \sum_I \Psi^I(\mathbb{A})(\Psi^I,\Psi^I)^{-1} \ ,$$

sum over all partitions of weight n.

Thus

$$\begin{aligned} 24\,S^4(\mathbb{A}) &= 6\Psi^4 + 8\Psi^{13} + 3\Psi^{22} + 6\Psi^{112} + \Psi^{1111} \\ &= \Phi'''' + 4\Phi\Phi'' + 3\Phi'\Phi' + 6\Phi^2\Phi' + \Phi^4 \ . \end{aligned}$$

Ex. 1.6 Permuting rows and columns, putting signs, one recognizes in the determinant $\Lambda_{n;0^n}(\mathbb{A};b)$, with $b = -x$, a rank 1 element, and $\Lambda^i(\mathbb{A}) = y_i$. Therefore it is equal to the polynomial $\Lambda_n(\mathbb{A}-b)$. One could as well have seen that the determinant is the resultant of $\Lambda_n(\mathbb{A}-z)$ and $z-x$.

Ex. 1.7 To see that one can apply the transformation Lemma (1.4.1), let $\mathbb{A}$ be of cardinality m, and let $\mathbb{X}_i$, $\mathbb{Y}_i$, $1 \le i \le n$ be of cardinality $i-1$.

Consider the determinant

$$M(\mathbb{A},\mathbb{X},\mathbb{Y}) := \det\left(\Lambda^j(\mathbb{A}+\mathbb{X}_j+\mathbb{Y}_i)\right)_{1\le i,j\le n} \ .$$

From (1.4.1) one can replace the $\mathbb{X}_j$ by 0. Expanding by linearity in $\mathbb{Y}_i$, we similarly get that the resulting matrix is the product of $\left[\Lambda^j(\mathbb{A})\right]$ by

$$\begin{bmatrix} \Lambda^0(\mathbb{Y}_1) & 0 & 0 & \cdots \\ \Lambda^0(\mathbb{Y}_2) & \Lambda^1(\mathbb{Y}_2) & 0 & \cdots \\ \Lambda^0(\mathbb{Y}_3) & \Lambda^1(\mathbb{Y}_3) & \Lambda^2(\mathbb{Y}_3) & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

Therefore, $M(\mathbb{A},\mathbb{X},\mathbb{Y}) = \Lambda_{1^n}(\mathbb{A})\,\Lambda^1(\mathbb{Y}_2)\cdots\Lambda^{n-1}(\mathbb{Y}_n)$, and one gets the value of the original determinant by specializing all the letters in the different alphabets to 1, that is one gets

$$\Lambda_{1^n}(m) = S_n(m) = \binom{n+m-1}{n} \ ,$$

writing m for the specialization of $\mathbb{A}$. Notice that the essential fact which allowed us to simplify the determinant was that it had regularly increasing alphabets in rows and columns. The same computation would have led to the evaluation, given $k \in \mathbb{N}$, of $\det\left(\Lambda^{j+k}(\mathbb{A}+\mathbb{X}_j+\mathbb{Y}_i)\right)_{1\le i,j\le n}$.

Ex. 1.9 The involution $\mathbb{A} \mapsto -\mathbb{A}$ in the argument of a Schur function corresponds to conjugation of partitions. It is more complex to follow it in a multi-Schur function. The above identity is a case where there is only an extra term apart from the one indexed by $(1^k,2)^\sim = (1,k+1)$.

Expanding both sides of the required identity with respect to $\mathbb{B}$, we transform them into

$$(-1)^{k+2}\left(S_{1^k2}(\mathbb{A}) + S_{1^k1}(\mathbb{A})S_1(-\mathbb{B}) + \cdots + S_{1^k,-k}(\mathbb{A})S_{k+2}(-\mathbb{B})\right) = \\ S_{1,k+1}(-\mathbb{A}) + S_{k+1}(-\mathbb{A})S_1(-\mathbb{B}) + S_{k+2}(-\mathbb{B}) \ ,$$

the terms not written being null (=Schur functions with two identical columns). □

Ex. 1.10 Let $\mathbb{A}$, $\mathbb{B}$ be of respective cardinality α,β, and let $\square = \beta^\alpha$. Then, from lemma (1.4.1), one gets

$$S_I(\mathbb{A}+\mathbb{B})\prod(a-b) = S_{I^\sim;\square}(-\mathbb{B};\mathbb{A}-\mathbb{B}) = S_{I^\sim;\square^\sim}(-\mathbb{A};\mathbb{B}-\mathbb{A}) \ .$$

Ex. 1.11 Multiplying the successive rows by the Vandermonde determinants $\Delta(\mathbb{A}_r)$, $\Delta(T\mathbb{A}_r)$, $\Delta(T^2\mathbb{A}_r),\ldots$, one recognizes a Bazin-type determinant. For example, for $r=4$, $K=[\alpha,\beta,\gamma]$, one gets the determinant

$$\begin{vmatrix} [a_5a_6\,012\,\alpha] & [a_5a_6\,012\,\beta] & [a_5a_6\,012\,\gamma] \\ [a_5a_1\,012\,\alpha] & [a_5a_1\,012\,\beta] & [a_5a_1\,012\,\gamma] \\ [a_1a_2\,012\,\alpha] & [a_1a_2\,012\,\beta] & [a_1a_2\,012\,\gamma] \end{vmatrix}$$

of minors of the 6×12 matrix

$$\begin{array}{c|ccccccccc|}
\textit{indices} & a_1 & \cdots & a_6 & 0 & 1 & 2 & \alpha & \beta & \gamma \\
\hline
 & 1 & & 0 & a_1^0 & a_1^1 & a_1^2 & a_1^\alpha & a_1^\beta & a_1^\gamma \\
 & 0 & \ddots & 0 & \vdots & & \vdots & \vdots & \vdots & \vdots \\
 & 0 & & 1 & a_6^0 & a_6^1 & a_6^2 & a_6^\alpha & a_6^\beta & a_6^\gamma
\end{array},$$

the left part being the identity matrix of order 6.

According to Bazin, the determinant is equal to

$$[a_5a_6a_1012][a_6a_1a_2012][012\alpha\beta\gamma] = \Delta(a_2, a_3a_4)\cdot\Delta(a_3a_4a_5)\cdot\Delta(\mathbb{A}_6)S_{\alpha-3,\beta-4,\gamma-5}(\mathbb{A}_6)\,.$$

The general case, with $\rho = [0, 1, \ldots, n{-}1]$, will be

$$D(K,r)\,\Delta(\mathbb{A}_r)\cdots\Delta(T^{n-1}\mathbb{A}_r) \\ = \Delta(T\mathbb{A}_{r-1})\cdots\Delta(T^{n-1}\mathbb{A}_{r-1})\,\Delta(\mathbb{A}_{n+r-1})\,S_{K-\rho}(\mathbb{A}_{n+r-1})\,.$$

Notice that the same computation would work for a multi-Schur function $S_{K-\rho}(\mathbb{A}-\mathbb{B}_1;\ldots;\mathbb{A}-\mathbb{B}_n)$, starting with $\left|S^{k_j}(T^{i-1}\mathbb{A}_r - \mathbb{B}_j)\right|$. The reader can give another proof, writing inside the initial determinant $T^i(\mathbb{A}_r) = \mathbb{A}_{n+r-1} - (\mathbb{A}_{n+r-1} - T^i\mathbb{A}_r)$.

Ex. 1.12 Let us take four letters, and $A = x_1 + x_2$, $B = y_1 + y_2$. Specializing some of them allow to recover the smaller cardinalities.

We have nullities $S_I(\mathbb{A}-\mathbb{B}) = 0$ when $I \supseteq [3,3,3]$. In the case where $I \supseteq [2,2]$, $I \not\supseteq [3,3,3]$, on has a factorization of $S_I(\mathbb{A}-\mathbb{B})$ in three factors given by (1.4.3). There remains to explicit the case where I is a hook.

Ex. 1.15 The entries of the adjoint matrix are minors of the original matrix. In our case, they are minors of the Toeplitz matrix $\left[S^{j-i}\right]$, since the Jacobi-Trudi matrix itself is so.

The required entries are therefore skew Schur functions, up to sign. The example should be written

$$\begin{bmatrix} S_{56/11} & -S_{56/1} & S_{56} \\ S_{16/11} & -S_{16/1} & S_{16} \\ S_{14/11} & -S_{14/1} & S_{14} \end{bmatrix}$$

and this gives the general pattern.

Ex. 1.16 For each $k, r \in \mathbb{N}$, the identity

$$S_{12\ldots kr} = S_{12\ldots k}S^r - S_{12\ldots k/1}S^{r+1} + S_{12\ldots k/11}S^{r+2} - \cdots$$

given by the expansion along the last column, shows that

$$\begin{bmatrix} S_0 & \cdot & \cdot & \cdot \\ -S_0 & S_1 & \cdot & \cdot \\ S_{12/11} & -S_{12/1} & S_{12} & \cdot \\ -S_{123/111} & S_{123/11} & -S_{123/1} & S_{123} \end{bmatrix} \mathbb{S}_{1234} = \begin{bmatrix} S_1 & S_3 & S_5 & S_7 \\ \cdot & S_{12} & S_{14} & S_{16} \\ \cdot & \cdot & S_{123} & S_{125} \\ \cdot & \cdot & \cdot & S_{1234} \end{bmatrix}.$$

It remains to normalize by left multiplication by a diagonal matrix.

Ex. 1.17 Using $S_{k-i}(\mathbb{A}_i - \mathbb{A}_{k-1}) = \sum_j S_{j-i}(\mathbb{A}_i)S_{k-j}(-\mathbb{A}_{k-1})$, one obtains that the product of the given matrix by $\left[S_{j-i}(-\mathbb{A}_{j-1})\right]_{1\le i,j\le n}$, is the identity matrix, the terms above the diagonal being null because, for $k > i$, $\mathbb{A}_{k-1} - \mathbb{A}_i$ is an alphabet of cardinality $< k - i$.

Ex. 1.18 If $[I] = [i_1, \ldots, i_n] \in \mathbb{N}^n$ is (weakly) increasing, and denote a shifted Wrońskian, then derivating it by rows, one gets

$$\frac{d}{dx}([I]) = \sum\nolimits_{\epsilon} [I+\epsilon] \ ,$$

sum over all vectors which are a permutation of $[1, 0^{n-1}]$. However, the terms such that $[I+\epsilon]$ is not a partition vanish (coding determinants having two identical rows), and therefore the sum is

$$\frac{d}{dx}([I]) = \sum\nolimits_{J \in \{I \otimes 1\}} [J] \ ,$$

i.e. is given by Pieri formula. One recognizes another occurrence of the space of symmetric functions. Finally,

$$\frac{d^k}{dx^k} W = \sum_{I \in \mathbb{N}^n, |I| = k} d_I \, [I] \ ,$$

where the sum is over partitions of weight k, and where each constant $d_I = \big(S_I, (S_1)^k\big)$ is the dimension of the representation of shape I of the symmetric group (in char. 0).

This should be considered as a concise answer, as required by Muir (IV, p.246) commenting the work of Konigsberger.

Ex. 1.20 Take $\mathbb{A}$ of cardinality $\geq n$, let y, z be rank-1 elements, $x = 1 - y/z$. Let $f(z) = \sigma_z(-\mathbb{A})$. From Taylor formula around $x = 0$ one gets

$$f(y) = f(z) - xzf'(z) + x^2 z^2/2!\, f''(z) - \cdots$$

But $f'(z)f(z)^{-1} = \sum_{a \in \mathbb{A}} a(1{-}za)^{-1}$, so that

$$f^r(z)f(z)^{-1} = \sum_{\mathbb{B} \subset \mathbb{A}} \prod_{b \in \mathbb{B}} b(1{-}zb)^{-1} \ ,$$

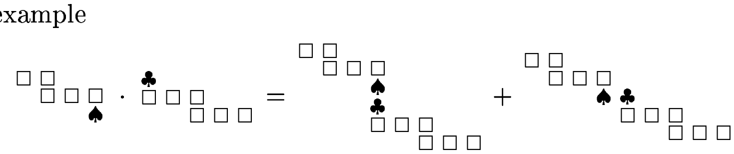
sum over all subsets $\mathbb{B}$ of cardinality r. Therefore

$$f^r(z)f(z)^{-1} = \sum_{I : \ell(I) = r} z^{|I|} \Psi_I(\mathbb{A}) \ .$$

The conclusion follows by identifying coefficients in $f(z{-}xz)$ and $f(z) \sum (-xz)^i f^{(i)}(z) (i! f(z))^{-1}$.

Ex. 1.21 Expanding the determinant expressing a ribbon along any row, or column, one gets sums of products of smaller ribbons, and this allows in particular to check that the product of two ribbons is obtained by gluing, in two different manners, the last box of the first one to the first box of the second one, using diagrams of skew partitions to represent a ribbon.

For example



Expanding a determinant of ribbons along a row, or a column, one can try, by induction on the order of the determinant, to identify the symmetric function it represents. However, even for the order 2, it is not straightforward, because two products of ribbons can be equal without being identical.

One must proceed by a step-by-step transformation of the successive rows of the determinant. We refer to [**103**] for the details. Another method, still explained in [**103**], is to recognize that, more generally, Bazin theorem allows to express any Schur function as a determinant of ribbons.

Ex. 1.22 From the definition of tan(), and from $\exp\left(\sum_{i\geq 1}(-z)^i\Psi^i/i\right) = \sum z^i\Lambda^i$, one gets the first equality. Using the determinantal expression of the coefficients of the quotient $\lambda_{-y}(\mathbb{B})/\lambda_{-y}(\mathbb{A})$ of two series, for $y = z^2$, and replacing $\Lambda^i(\mathbb{A})$ by Λ^{2i}, $\Lambda^i(\mathbb{B})$ by Λ^{2i+1}/Λ^1 , one recognizes in them the specified Schur functions. For example,

$$S_{1234/12} = \Lambda_{1234/12} = \begin{vmatrix} \Lambda^1 & \Lambda^3 & \Lambda^5 & \Lambda^7 \\ \Lambda^0 & \Lambda^2 & \Lambda^4 & \Lambda^6 \\ 0 & \Lambda^0 & \Lambda^2 & \Lambda^4 \\ 0 & 0 & \Lambda^0 & \Lambda^2 \end{vmatrix} = \Lambda^1 \begin{vmatrix} \Lambda^0(\mathbb{B}) & \Lambda^1(\mathbb{B}) & \Lambda^2(\mathbb{B}) & \Lambda^3(\mathbb{B}) \\ \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) & \Lambda^3(\mathbb{A}) \\ 0 & \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) \\ 0 & 0 & \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) \end{vmatrix} .$$

Given a formal series f and two other series g, h such that $f = g/h$, then g, h are of course not uniquely determined by f. However, one has here the constraint that the identity is valid for $\mathbb{A}$ of any cardinality. In other words, each truncation $(z\Lambda^1 - \cdots \pm z^{2n+1}\Lambda^{2n+1})/(1 - \cdots \pm z^{2n}\Lambda^{2n})$ or $(z\Lambda^1 + \cdots \pm z^{2n+1}\Lambda^{2n+1})/(1 - \cdots \pm z^{2n+2}\Lambda^{2n+2})$ is a rational approximation of $\tan(\mathcal{Z})$, and now the coefficients are uniquely determined if the approximation exists (normalizing the approximation as $(z + \cdots)/(1 + \cdots)$).

Since $\mathcal{Z}$ involves only the odd power sums, the elementary symmetric functions are rational functions of the odd power sums. If $\mathbb{A}$ is of finite cardinality n, then the right member is a rational function, which is determined by the Taylor expansion of $\tan(\mathcal{Z})$ to the order involving the same number of parameters. Thus $\Lambda^1, \ldots, \Lambda^n$ are rational functions of $\Psi^1, \Psi^3, \ldots, \Psi^{2n+1}$, and so is any symmetric function of $\mathbb{A}$.

Ex. 1.23 This is another way of obtaining the symmetric analog of the tangent or cotangent function seen in the preceding exercise. With $x_i = \cot(a_i)$, $y_i = 1/x_i = \tan(a_i)$, $\mathbb{A} = \{a_1, \ldots, a_n\}$ $\mathbb{X} = \{x_1, \ldots, x_n\}$, $\mathbb{Y} = \{y_1, \ldots, y_n\}$, one has

$$\begin{aligned} \tan(\mathbb{A}) &= \frac{\Lambda^1(\mathbb{Y}) - \Lambda^3(\mathbb{Y}) + \Lambda^5(\mathbb{Y}) - \cdots}{1 - \Lambda^2(\mathbb{Y}) + \Lambda^4(\mathbb{Y}) + \cdots} = \sigma_1\big(\lambda_{-1}(\mathbb{Y})\big) \\ \cot(\mathbb{A}) &= \frac{1}{\tan(\mathbb{A})}\frac{\Lambda^n(\mathbb{X})}{\Lambda^n(\mathbb{X})} = \frac{\Lambda^n(\mathbb{X}) - \Lambda^{n-2}(\mathbb{X}) + \Lambda^{n-4}(\mathbb{X}) - \cdots}{\Lambda^{n-1}(\mathbb{X}) - \Lambda^{n-3}(\mathbb{X}) + \Lambda^{n-5}(\mathbb{X}) + \cdots} \end{aligned}$$

Ex. 1.24 This is almost the same computation as in Ex.1.22. We have to explicit the Taylor expansion of

$$\left(1 - z^2\Lambda^2 + z^4\Lambda^4 + \cdots\right)\left(z\Lambda^1 - z^3\Lambda^3 + z^5\Lambda^5 - \cdots\right)^{-1} .$$

But we know how to write the coefficients of the quotient of any two series as determinants in the coefficients of the two series. In our case, with $\theta = \Lambda^1(\mathbb{A})$, $y = z^2/\theta$, the expansion is

$$\frac{1}{\theta}\big(1 - yS_{12}(\mathbb{A}) - y^2 S_{123}(\mathbb{A}) - y^2 S_{1234/1}(\mathbb{A}) - \cdots y^n S_{1\ldots n+1/1\ldots n-2}(\mathbb{A}) - \cdots\big)$$

Ex. 1.25 We are in the case of a Schur function of a difference of two alphabets, which factorizes according to Proposition 1.4.3:

$$S_{j,n^n}\left(\mathbb{A} - \overline{\mathbb{A}}\right) = S_j\left(-\overline{\mathbb{A}}\right) S_{n^n}\left(\mathbb{A} - \overline{\mathbb{A}}\right) = \Lambda^j(\mathbb{A})\, S_{n^n}\left(\mathbb{A} - \overline{\mathbb{A}}\right) .$$

The extra factor, which is the resultant of $\mathbb{A}$ and $\overline{\mathbb{A}}$, is equal to

$$S_{n^n}\left(\mathbb{A}-\overline{\mathbb{A}}\right)=\prod_{1\le i,j\le n}(a_i+a_j)=S_{1,\ldots,n-1}(\mathbb{A})\,S_{1,\ldots,n}(\mathbb{A})\,.$$

On the other hand, $S_{j,n^n}\left(\mathbb{A}-\overline{\mathbb{A}}\right)$ is a polynomial in $\Psi_i\left(\mathbb{A}-\overline{\mathbb{A}}\right)=(1-(-1)^i)\Psi_i(\mathbb{A})$, $1\le i\le 2n$. In all, only the odd powers sums $\Psi_1(\mathbb{A}),\Psi_3(\mathbb{A}),\ldots,\Psi_{2n-1}(\mathbb{A})$ appear in the expression of $S_{j,n^n}\left(\mathbb{A}-\overline{\mathbb{A}}\right)$, $S_{1,\ldots,n-1}(\mathbb{A})$, $S_{1,\ldots,n}(\mathbb{A})$.

Ex. 1.26 Take the determinantal expression of $S_{1\ldots n/1^k}=\Lambda_{1\ldots n/k}$ in terms of the Λ^i. It factorizes, a factor being equal to Λ^{n-k}. One moreover knows that $S_{1\ldots n}$ depends only on the odd power sums $\Psi^1,\Psi^3,\ldots,\Psi^{2,n-1}$, because one cannot remove of ribbon of even length from the diagram of $[1,\ldots,n]$ (cf. (1.8.9)). One can obtain the skew functions $S_{1\ldots n/I}$, any I, by derivation with respect to power sums, and therefore they also involve only odd power sums.

Ex. 1.27 Minors of $S_{m^n}(\mathbb{A})$ are skew Schur functions. The expansion of $Q(x,y)$ also produces skew Schur functions that one has to check to be the same.

For example, the adjoint to $\mathbb{S}_{6666}(\mathbb{A})$ is

$$\begin{bmatrix} S_{666} & -S_{667} & S_{677} & -S_{777}\\ S_{666/001} & -S_{667/001} & S_{677/001} & -S_{777/001}\\ S_{666/011} & -S_{667/011} & S_{677/011} & -S_{777/011}\\ S_{666/111} & -S_{667/111} & S_{677/111} & -S_{777/111}\end{bmatrix}=\begin{bmatrix} S_{666} & -S_{667} & S_{677} & -S_{777}\\ -S_{566} & S_{567}+S_{666} & -S_{667}-S_{577} & S_{677}\\ S_{556} & -S_{566}-S_{557} & S_{567}+S_{666} & -S_{667}\\ -S_{555} & S_{556} & -S_{566} & S_{666}\end{bmatrix}$$

Ex. 1.28 Identify $\mathfrak{Sym}$ with $\mathfrak{Sym}(\mathbb{A})$. The right factor of ∇ is the operator $f\mapsto f(\mathbb{A}-1/z)$, as is checked on the basis $\Psi^K(\mathbb{A})$. The second factor is "Multiplication by $\sigma_z(\mathbb{A})$". Assuming the property to be true for length $<\ell$, we have, for any $I\in\mathbb{Z}^{\ell-1}$, any $k\in\mathbb{Z}$

$$\nabla_k\big(S_I(\mathbb{A})\big)=S_i(\mathbb{A}-1/z)\sigma_z(\mathbb{A})\cap z^k=\sum_{i,j}(-z)^{-j}S_{I/1^j}(\mathbb{A})\,z^iS^i(\mathbb{A})\cap z^k$$

and this is the expansion of the determinant $S_{Ik}(\mathbb{A})$ along its last column.

Ex. 1.29 The function $\sum_{-n}^{\infty}$ is simpler. Indeed, the Laplace expansion of each $S_{I,j}(\mathbb{A})$ produces

$$\sum_{-n}^{\infty}\sum_{k=0}^{n}S_{I/1^k}(\mathbb{A})S^{j+k}(\mathbb{A})x^j=\sum_k S_{I/1^k}(\mathbb{A})(-x)^{-k}\,\sigma_x(\mathbb{A})=S_I\left(\mathbb{A}-x^{-1}\right)\,\sigma_x(\mathbb{A})\ ,$$

and this is clearly a rational function of x with denominator $\lambda_{-x}(\mathbb{A})$.

Ex. 1.30 Expanding the determinant along the last two rows, one would get Schur functions of $\mathbb{A}$ indexed by partitions of length ≤ 2. One thus loses no information by supposing that $\mathbb{A}$ is of cardinality 2, and similarly, that $\mathbb{B}$ is of cardinality 3.

Subtract now $\mathbb{A}$ in rows $1,\ldots,5$ and $\mathbb{B}$ in columns $4,\ldots,8$. The new determinant factorizes into

$$S_{0^3}(\mathbb{B})\,S_{(k+3)^5}(\mathbb{C}-\mathbb{A}-\mathbb{B})\,S_{0^2}(\mathbb{A})\ .$$

Ex. 1.31 The Vandermonde $\Delta(\mathbb{A})$ is a Schur function $S_{n-1,n-2,\ldots,0}(a_1,\ldots,a_n)$, but the transformation is not given by lemma (1.4.1) because one modifies each row. However, it also is given by multiplication by a triangular matrix with unit diagonal. One could have even added or subtracted different alphabets, $\mathbb{A}_1,\mathbb{A}_2,\ldots$ in successive columns.

In the special case where $B = -\mathbb{A}$, getting rid of signs which are uniform, one obtains a determinant of elementary symmetric functions in the subalphabets $\mathbb{A} - a_i$.

Ex. 1.32 The matrix is the product of the Vandermonde matrix in the x_i^2, and of the matrix $\left[\Lambda^{n-i+j-1}(\mathbb{A})\right])_{1\le i,j\le n}$. Therefore, its determinant is equal to

$$\pm \prod (x_i^2 - x_j^2)\Lambda_{12\ldots n-1}(\mathbb{A}) = \pm \prod (x_i - x_j)\Lambda_{12\ldots n-1}(x_1 + \cdots + x_n)\,\Lambda_{12\ldots n-1}(\mathbb{A}) \ .$$

Ex. 1.33 The determinant is equal to $\Lambda^{n-1}(\mathbb{A})\,\Delta(\mathbb{A})$, and thus proportional to the first elementary symmetric function of the alphabet $\{1/1, \ldots, 1/n\}$.

Ex. 1.34 By induction on n, one has to prove that

$$\Lambda^{n-1}/\Lambda^{n} + S_{2^n}/(\Lambda^n \Lambda^{n+1}) = \Lambda^n/\Lambda^{n+1} \ ,$$

which is evident, once noticed that $S_{2^n} = \Lambda_{nn}$.

This expression is used by Whittaker (1918; MuirV, p.272) as an approximation of the smallest root of $\sigma_x(\mathbb{A}) = 0$.

Ex. 1.35 Muir's rule states that $\Psi_{1^\alpha 2^\beta} = \sum S_v$, sum over all permutations v of $[0^b, 1^a, 2^b]$. All the vectors such that there is no null component at distance 2, on its right, of a component equal to 2, give a non-zero contribution. Such a vector can be considered as a word in $0, 1, (20), 2$ and is of the type $[0^\delta, u, 2^\delta]$, with $\delta \le \beta$, u being an arbitrary permutation of $1^\alpha, (20)^\gamma$, with $\gamma = \beta - \delta$. Therefore,

$$\Psi_{1^\alpha 2^\beta} = \sum_{\gamma\le\beta} (-1)^\gamma \binom{\alpha+\gamma}{\alpha} S_{1^{\alpha+2\gamma}, 2^{\beta-\gamma}} \ .$$

Ex. 1.36 Let $N \ge n(k-1)$. Muir's rule states that $\Psi_{k^n} = \sum S_v$, sum over all permutations of $[0^N, k^n]$. The Schur functions which are null in this sum are those indexed by a v such that there exists $i : v_i = k$, $v_{i+k} = 0$ (because then S_v has two identical columns). It is easy to see that all the other vectors give different Schur functions.

Ex. 1.37 For the mutiplication of S_I by Λ^k, one first concat 0^k to I. Now ${}^rT_k^+(S_{0^k I})$ reduces to a single non-zero determinant, the one where indices have been increased in the first k rows. This determinant factorizes into $\Lambda^k S_I$, and therefore ${}^cT_k^+$ gives the multiplication by Λ^k.

For the multiplication by S^2 or S^3, one writes $S_2 = \Psi_2 + \Psi_{11}$, $S_3 = \Psi_3 + \Psi_{12} + \Psi_{111}$. One has to check that the terms S_{I+H} , with H a permutation of $[0\cdots 02]$, $[0\cdots 011]$; $[0\cdots 03]$, $[0\cdots 012]$, $[0\cdots 0111]$ are zero functions, or cancel two by two, and that the terms which survive are exactly those such that J/I is an horizontal strip.

Ex. 1.38 Write $[a..b][c..d]\cdots$ for the family of partitions J such that $a \le j_1 \le b$, $c \le j_2 \le d], \ldots$ Then $\{[0038]\otimes 5\} \subset [0..0][0..3][3..8][8..]$, $\{[0035]\otimes 8\} = [0..0][0..3][3..5][5..]$, $\{[0068]\otimes 2\} = [0..0][0..2][6..8][8..]$. Therefore, partitions belonging to $\{[0038]\otimes 5\}$ are distributed in two subsets, according to the value of their third part. It implies that, in the expansion of the determinant along its last row, the vanishing $S_{35}S^8 - S_{38}S^5 + S_{68}S^2$ is realized by eliminating each middle term with one term on its right or on its left.

But, if one takes instead the expansion of the Schur function S_{357}, then $\{[0068]\otimes 1\} \subset \{[0038]\otimes 4\}$, the difference $\{[0038]\otimes 4\} \setminus \{[0068]\otimes 2\}$ is a strict subset of $\{[0035]\otimes 7\}$, the complement being S_{357}.

More generally, the ordered expansion, along its last row, of the Jacobi-Trudi determinant expressing S_J, J partition $\in \mathbb{N}^n$, will involve a sequence of sets of partitions $\mathcal{E}_0 = \emptyset, \mathcal{E}_1, \dots, \mathcal{E}_n$ such that each $\mathcal{E}_i$ decomposes into $\mathcal{E}'_i \cup \mathcal{E}''_i$, with $\mathcal{E}'_i = \mathcal{E}''_{i-1}$, $\mathcal{E}''_i = \mathcal{E}'_{i+1}$, defining $\mathcal{E}'_{n+1} := \{J\}$, and such that $\mathcal{E}_{i-1} \cap \mathcal{E}_{i+1} = \emptyset$. .

Ex. 1.39 (MacMahon [**136**], I, p.39). Q_k is characterized by the equations $D_{S^1}(Q_n) = 0$, $D_{S^r}(Q_n) = Q_{n-r}$, $r > 1$. Let us put $\Phi_{-1} = 0 = \Phi_{-2}$. Since $D_{S^i}(\Phi_j) = 0$ for $i > 2$, $D_{S^1}(\Phi_j) = \Phi_{j-1}$, and $D_{S^2}(\Phi_j) = \Lambda^j + \Phi_{j-2}$, one gets the equalities $P_k = Q_k$, by checking the equations

$$D_{S^1}(P_k) = \sum (-1)^i \Phi_i S^{k-i-1} + \sum (-1)^i \Phi_{i-1} S^{k-i} = 0 \,.$$

$$D_{S^r}(P_k) = \sum (-1)^i \Phi_i S^{k-i-r} + \sum (-1)^i \Phi_{i-1} S^{k-i-r+1} + \sum (-1)^i \big(\Phi_{i-2} + \Lambda^i\big) S^{k-i-r+2}$$

and this last expression is equal to P_{k-r}, because the terms in Λ^i vanish: for any j, $\sum_i \pm \Lambda^i S^{j-i} = 0$.

Ex. 1.40 The identity comes from the fact that the adjoint D_{Ψ^k} of multiplication by Ψ^k is a derivation.

Ex. 1.41 Recall that $D_{S^i}(\Psi_{i,J}) = \Psi_J$, and $D_{S^i}(\Psi_J) = 0$ if $i \notin J$. The operators $\theta_i = \xi_i\big(D_{S^i}()\big)$ commute, and allow to filter the space of monomial functions according to the smallest part in their index. Therefore,

$$\begin{aligned} 1 &= \theta_1 + \theta_2\big(1-\theta_1\big) + \theta_3\big((1-\theta_2)(1-\theta_1)\big) + \cdots \\ &= \xi_1\Big(D_{S^1}\Big) + \xi_2\Big(D_{S^2}(1-\theta_1)\Big) + \xi_3\Big(D_{S^3}\big((1-\theta_2)(1-\theta_1)\big)\Big) + \cdots \end{aligned}$$

Ex. 1.42 The entries of the inverse are the minors of order n of the original matrix, up to sign. Identifying $c_i = S^i(\mathbb{A})$ (or to $S^{i+1}(\mathbb{A})$, if one does not want c_0 to be equal to 1), then they are minors of the matrix $\mathbb{S}(\mathbb{A})$, that is they are skew Schur functions of $\mathbb{A}$.

The general entry of the inverse matrix is

$$\pm\, S_{(n^n/1^i)/1^j}(\mathbb{A}) = \pm\, S_{[(n-1)^i, n^{n-i}]/1^j}(\mathbb{A}) \ .$$

This matrix can be considered as a discrete Wrońskian of the functions

$$S_{n^n}(\mathbb{A}),\ S_{n-1,n^{n-1}}(\mathbb{A}), \dots,\ S_{(n-1)^n}(\mathbb{A}) \ .$$

Ex. 1.43 For this type of properties of matrices, one can suppose by continuity the matrix to be diagonal, in which case the statements are just rewritings of the definitions of elementary symmetric functions, complete functions and power sums.

Ex. 1.44 Suppose M to be diagonal.

Ex. 1.45 There is, of course, no special property of the sequence $2, 4, \dots, 2n$ of powers of M. One could take any sequence of powers. The required nullity is a variation about Cayley-Hamilton theorem :

$$M^n - M^{n-1}\Lambda^1(\mathbb{A}) + M^{n-2}\Lambda^1(\mathbb{A}) - \cdots \pm M^0 \Lambda^n(\mathbb{A}) \ ,$$

and results from the fact that the sequence of powers of M is a recurrent sequence with characteristic polynomial $S^n(x - \mathbb{A})$. Thus Cayley-Hamilton's nullity should be written $S^n(M - \mathbb{A}) = 0$, and the required relation comes from the Laplace expansion, along the bottom row, of

$$\begin{vmatrix} S^2(\mathbb{A}) & S^4(\mathbb{A}) & S^6(\mathbb{A}) & \cdots & S^{2n}(\mathbb{A}) \\ \vdots & \vdots & \vdots & & \vdots \\ S^{3-n}(\mathbb{A}) & S^{5-n}(\mathbb{A}) & S^{7-n}(\mathbb{A}) & \cdots & S^{n+1}(\mathbb{A}) \\ M^2 & M^4 & M^6 & \cdots & M^{2n} \end{vmatrix}$$

Ex. 1.46 By continuity, it is sufficient to treat the case of a diagonal matrix, with $\mathbb{A}$ as a set of eigenvalues. In that case,

$$det_2(1+yM) = \prod_{a\in\mathbb{A}}(1+ya)\exp(-ya) = \exp(yt_1 - y^2t^2/2 + y^3t_3/3 - \cdots)\exp(-yt_1)$$
$$= \exp(0 - y^2t^2/2 + y^3t_3/3 - \cdots)\,,$$

and the equivalence between the different expressions that we gave of $\det_2(1+yM)$ is clear.

Now det_2 appears as the image of the generating series of elementary symmetric functions under the specialization $\Psi_1 = 0$. Thus the determinants figuring in the RHS of $(*)$ are the determinants expressing elementary functions in the basis of power sums, with a diagonal of 0's instead of a diagonal of t_1.

More generally, one can define k-regularized determinants by the formulas

$$det_k(1+yM) = det(1+yM)\exp\left(-t_1 + t_2/2 - \cdots \pm t_{k-1}/(k-1)\right)\ ,$$

and their expansions involves the specializations $t_1 = 0 = \cdots = t_{k-1}$, of the usual determinants expressing Newton's relations between power sums and elementary symmetric functions.

B.2

Ex. 2.1 The sum is the logarithmic derivative of $S^n(y - \mathbb{X})$ with respect to y (with $\mathbb{X} = x_1 + \cdots + x_n$). Therefore $\sum_i 1/(y - x_i) = S^{n-1}(2y - \mathbb{X})/S^n(y - \mathbb{X})$.

Ex. 2.2 Both sides are linear in $S^j(\mathbb{A})$; one can take $\mathbb{A} = a$ of rank 1, in which case the identity is $a^n - \binom{k}{1}a^{n-1}(a-x) + \binom{k}{2}a^{n-2}(a-x)^2 + \cdots = a^{n-k}(a-(a-x))^k$

Ex. 2.3 Subtracting $0, x, 2x, \ldots$ in rows upwards, one gets the determinant $\left|S_{jn+j-i}\big((i{+}1)x - j\mathbb{A}\big)\right|_{1\le i,j\le k-1}$. Writing $S^{n+1}(x{-}A) = f(x)$, one has $f^j = S^{(n+1)j}(x{-}jA)$, and one recognizes in the determinant the Wrońskian of $f^0, \ldots, f^{k-1}$ (minus its first row and first column), up to the factors $0!, 1!, \ldots, (k{-}1)!$. Therefore, according to Wroński himself, the determinant is equal to

$$\big(f'(x)\big)^{\binom{k}{2}} = \big(S^n(2x - \mathbb{A})\big)^{\binom{k}{2}}\ .$$

For example, for $k = 3$ one has

$$S_{n;2n}(3x - \mathbb{A}; 3x - 2\mathbb{A}) = \big(S^n(2x - \mathbb{A})\big)^3\ .$$

Ex. 2.4 Let us give an example of how to use elements of rank 1 to obtain identities about elements of binomial type.

One has

$$\frac{q}{q-1}S^n\left((q-1)\mathbb{A}\right) = \sum_{\alpha+\beta=n}(-1)^\beta q^\alpha S_{1^\beta\alpha}(\mathbb{A}) = \frac{q}{q-1}\sum S^I(q-1)\Psi_I(\mathbb{A})$$
$$= \sum q^{1+|I|-\ell(I)}(q-1)^{\ell(I)-1}\Psi_I(\mathbb{A}) = \sum S^{|I|}(q+1-\ell(I))\Psi_I(\mathbb{A}) \ .$$

Taking the image of this identity under $q^i = S^i(q) \to S^i(r)$, one gets

$$\sum(-1)^\beta S^\alpha(r)S_{1^\beta\alpha}(\mathbb{A}) = \sum S^{|I|}\big(r+1-\ell[I]\big)\,\Psi_I(\mathbb{A}) \ .$$

For example, in weight 4, one has

$$S^4(r)\Psi_4(\mathbb{A}) + S^4(r-1)\big(\Psi_{13}(\mathbb{A}) + \Psi_{22}(\mathbb{A})\big) + S^4(r-2)\Psi_{112}(\mathbb{A}) + S^4(r-3)\Psi_{1111}(\mathbb{A})$$
$$= S^4(r)S_4(\mathbb{A}) - S^3(r)S_{13}(\mathbb{A}) + S^2(r)S_{112}(\mathbb{A}) - S^1(r)S_{1111}(\mathbb{A}) \ .$$

Ex. 2.5 We do not need α, q to be rank-1 elements, but just formal parameters. We recognize that the Jacobi-Trudi determinant expressing S_I is a specialization of Cauchy determinant $\left|(1-xy)^{-1}\right|$, for $\mathbb{X} = \{\alpha, \alpha q, \ldots, \alpha q^{n-1}\}$, $\mathbb{Y} = \{q^{i_1-n+1}, \ldots, q^{i_n}\}$, and therefore $S_I(\mathbb{A})$ is equal to the product of all the entries of the determinant times $\Delta(\mathbb{X})\,\Delta(\mathbb{Y})$.

When $\alpha = 1$, one has the limit case $S_n(\mathbb{A}) = 1/n$ for $q \to 1$.

Ex. 2.6 cf. Riordan, [**153**], p.169. By recursion, one shows that

$$\Lambda^n(\mathbb{A}) = \alpha(\alpha-n\beta)^{-1}\, S^n(\alpha-n\beta) \quad , \quad \Psi^n(\mathbb{A}) = \alpha(n\beta-1)\cdots(n\beta-n+1)/(n-1)!\ .$$

Ex. 2.7 Subtract x in all the rows, except the last one, and subtract y in all columns, but the first one. Then the determinant factorizes, one block being a Schur function of $\mathbb{A}-x-y$, the other being $\begin{vmatrix} S^0(0) & \star \\ 0 & S^0(0)\end{vmatrix} = 1$.

Ex. 2.8 Decomposing each column as a sum of columns involving only one letter, acccording to $\Psi^k = \sum m_a a^k$, one gets a sum of determinants, but the only ones which are non zero are those for which no two columns contains the same letter. This proves nullity for order $> n$. For order n, the remaining determinants are in bijection with permutations of the letters, and their sum is just $\prod_a m_a$ times the sum obtained when all m_a are equal to 1. Therefore, a minor of index I, J, $I, J \in \mathbb{N}^n$ is equal to

$$\prod_a m_a\, S_I(\mathbb{A})\, S_J(\mathbb{A})\, \Delta(\mathbb{A})^2 \ .$$

Bellavitis (1857), Muir II, p.353, gives the case $I = 0^n,\ J = k^n$.

Ex. 2.9 (Glaisher(1879), Muir III p.243). The first determinant is the specialization of the expression of Ψ^n in terms of complete functions, for $S^n(\mathbb{A}) = n$. Therefore, $f(x) = \sigma_x(\mathbb{A}) = (1+x^3)(1-x)^{-1}(1-x^2)^{-1}$, and $\{\Psi^i(\mathbb{A})\}$, which is our first determinant, is periodic of period 6, with values, starting from $i = 1$, equal to $1, 3, 4, 3, 1, 0,\ 1, 3, 4, \ldots$. The second case is $S^n(\mathbb{A}) = (n+1)^2$, with generating function $f(x) = \sigma_x(\mathbb{A}) = (1+x)(1-x)^3$ (we should have a first row $[1.2^2, 2.3^2, 3.4^2, \ldots]$, but adding to it the second row, one transforms it into the sequence of cube $[1^3, 2^3, \ldots]$). Therefore $\Psi^n(\mathbb{A}) = 3 - (-\zeta)^n$, with ζ specializing to -1; in other words, $\Psi^{2n} = 2$ and $\Psi^{2n+1} = 4$.

Ex. 2.10 First notice that, for the properties that we examine, the set of integers decomposes according to each prime.

The statement involving convolution comes from the identity

$$S^k(\mathbb{A}_p+\mathbb{B}_p)=\sum S^i(\mathbb{A}_p)\,S^{k-i}(\mathbb{B}_p)\ .$$

Products of $S_k(-1)$ are 0 if one of the k is bigger than 1, and the sign is the parity of the number of prime divisors.

For all n, $\mu_{1,\ldots,1,\ldots}(n)=1$, and the convolution square of this function is τ.

The sequence $\mathcal{A}=(\mathbb{A}_2,\ldots,\mathbb{A}_p,\ldots)$ gives the function $\mu_{\mathcal{A}}$ such that $\mu_{\mathcal{A}}(n)=n$, i.e. gives the identity function.

One checks that φ is the convolution of μ and $\mu_{\mathcal{A}}$, and that σ is the convolution of $\mu_{\mathcal{A}}$ and μ_1 (with $\mu_1(n)=1$ for all n).

For more properties of the Moebius function on the integers, and of multiplicative functions, we refer to Dixon [**39**] and Apostol [**6**].

Ex. 2.12

$$f(y)=\sum_{p\geq 0}S^p(\mathbb{A})\,\left(\sum_{n\geq 1}x^nS^n(\mathbb{B})\right)^p\ .$$

Cauchy's formula $\left(\sum_{n\geq 0}x^nS^n(\mathbb{B})\right)^p=\sum\Psi_B(p)\,S^I(\mathbb{B})$ gives by homogeneity

$$\left(\sum\nolimits_{n\geq 1}x^nS^n(\mathbb{B})\right)^p=\sum\nolimits_{I:\ell(I)=p}\Psi_B(p)S^I(\mathbb{B})\ ,$$

and therefore

$$f(y)=\sum_I x^{|I|}S^{\ell(I)}(\mathbb{A})\,\Psi_I(p)\,S^I(\mathbb{B})\ .$$

The required derivative being the coefficient of x^n, the conclusion follows from the fact that for every $j\geq 0$, $S^j(\mathbb{B})=D^j(y)/j!$.

Ex. 2.13 The compatibility of $\gamma^i(\mathbb{A})$ with the involution $\mathbb{A}\to-\mathbb{A}$ is seen on the expression of $S^i(\mathbb{A}^\diamond)$. The compatibility with addition comes from the fact that generating functions multiply: $\gamma_z(\mathbb{A}+\mathbb{B})=\gamma_z(\mathbb{A})\gamma_z(\mathbb{B})$.

For what concerns multiplication, one has for example the inequality

$$\Psi^2(\mathbb{A}^\diamond\mathbb{B}^\diamond)=\Psi^2(\mathbb{A}\mathbb{B})-2\Psi^1(\mathbb{A}\mathbb{B})\neq\left(\Psi^2(\mathbb{A})-2\Psi^1(\mathbb{A})\right)\left(\Psi^2(\mathbb{B})-2\Psi^1(\mathbb{B})\right)\ .$$

Ex. 2.14 For any partition J, one recognizes in $\Lambda_J(\mathbb{A}^\diamond)$ a multi-Schur function. For example, for $J=[1,2,4]$, one has

$$\Lambda_{124}(\mathbb{A}^\diamond)=\begin{vmatrix}\Lambda^1(\mathbb{A}) & \Lambda^3(\mathbb{A}+2q) & \Lambda^6(\mathbb{A}+5q)\\ \Lambda^0(\mathbb{A}-q) & \Lambda^2(\mathbb{A}+q) & \Lambda^5(\mathbb{A}+4q)\\ \Lambda^{-1}(\mathbb{A}-2q) & \Lambda^1(\mathbb{A}) & \Lambda^4(\mathbb{A}+3q)\end{vmatrix}=\Lambda_{1;2;4}(\mathbb{A}-2q;\mathbb{A};\mathbb{A}+3q)\ ,$$

the last equality being obtained by adding $0,q,2q,\ldots$ to the alphabets in the successive rows, upwards, of the determinantal expression of $\Lambda_{1;2;4}(\mathbb{A}-2q;\mathbb{A};\mathbb{A}+3q)$.

In general, the shift will be $[j_1-n,\ldots,j_n-1]$.

If J is a rectangular partition $J=k^r$, then one can also transform by columns the alphabets, and obtain

$$\Lambda_{k^r}(\mathbb{A}^\diamond)=\Lambda_{k^r}(\mathbb{A}+q(k-r))\ .$$

The two cases needed for a continued fraction expansion are

$$\Lambda_{n^n}(\mathbb{A}^\diamond)=\Lambda_{n^n}(\mathbb{A})=S_{n^n}(\mathbb{A})$$

and

$$\Lambda_{n^{n-1}}(\mathbb{A}^\diamond)=\Lambda_{n^{n-1}}(\mathbb{A}+q)=S_{(n-1)^n}(\mathbb{A}+q)\ .$$

Ex. 2.15 The formulas of change of basis are linear in $S^n(\xi)$, and one can take any element $\mathbb{A}$ instead of ξ, as long as the $S^n(\mathbb{A})$ are linearly independent. For example, with z of rank 1, then one has $S^n(z-ny)=(z-y)^n$ and

$$S^n(z) = \big((z-y)+y\big)^n = \sum_j \binom{n}{n-j} y^{n-j} S^j(z-jy) \ .$$

Replacing z by $x\xi - y$, one gets

$$\frac{x^n}{n!} - \frac{yx^{n-1}}{(n-1)!} = S^n(x\xi - y) = \sum_j \binom{n}{n-j} y^{n-j} S^j(z-(j+1)y) \ ,$$

and therefore

$$\frac{x^n}{n!} = \sum_{j=0}^{n} \binom{n+1}{n-j} y^{n-j} S^j(z-(j+1)y) \ .$$

Ex. 2.16 With q, z of rank 1, the q-exponential is $\sigma_1(z/(1-q))$, and therefore, for every $n \geq 1$, $n^{-1}\Psi_n(z/(1-q)) = z^n(n(1-q))^{-1}$.

Ex. 2.17 Since

$$\sum_{i,j} \Lambda^i(x)\Lambda^j(x) u^i v^j = (1+u)^x(1+v)^x = \big(1+(u+v+uv)\big)^x = \sum_n \Lambda^n(x)((u+v+uv)^n \ ,$$

one has

$$\Lambda^i(x)\Lambda^j(x) = \sum_n \Lambda^n(x) \binom{n}{n-i,\, n-j,\, i+j-n} \ .$$

To decompose more complicated expressions in the $\Lambda^i(x)$, the best way is to use Newton's interpolation at points $0, -1, -2, \ldots$.

Ex. 2.18

$$\begin{aligned}\Lambda^n(x-y) &= \Lambda^n\left(x - \frac{y(1-q)}{1-q}\right) = \sum_{i+j=n} (-1)^i S^i\left(\frac{y(1-q)}{1-q}\right)\Lambda^j(x) \\ &= \sum (-1)^i \Lambda^j(x) \frac{y^i(1-q)^i}{(1-q)\cdots(1-q^i)} \ ,\end{aligned}$$

and this proves that the generating function factorizes as stated.

The limit case $q=1$ gives the *Charlier polynomials* treated by Touchard [**172**].

Ex. 2.19 The first equation results from the expansion of $S^n(1+q\,\frac{1-q^{m+1}}{1-q})$, the second one, from the expansion of $q^{-k(k-1)/2}\,\Lambda^k\Big((1+\cdots+q^{n-1})+(q^n+\cdots+q^{n+m-1})\Big)$.

Ex. 2.20 Leaving only 1 in the RHS, writing $[n] = 1+q+\cdots+q^{n-1}$, this is

$$q^{m(m-1)/2}\,\Lambda^m\Big(([m]+q^m[n]) - q^m[n]\Big) = \Lambda^m([m])\, q^{m(m-1)/2} \ .$$

Ex. 2.21 It suffices to write $(1-q)^{-1} = 1 + q(1-q)^{-1}$ to get the recursion. Writing $(a+x)(1-q)^{-1} = x(1-q)^{-1} + a(1-q)^{-1}$, one can expand r_n in terms of Gauss polynomials:

$$r_n = \sum \begin{bmatrix} n \\ k \end{bmatrix} a^{n-k} x^k \ .$$

Finally, the generating function

$$\sum z^n \frac{r_n(x,a)}{(1-q)\cdots(1-q^n)} = \sigma_z\left(\frac{a}{1-q}\right)\sigma_z\left(\frac{x}{1-q}\right) = \prod_n \frac{1}{(1-zaq^n)(1-zxq^n)}$$

gives Gauss' identity for $a = 1 = -x$.

Ex. 2.22 The sum is

$$(1-q^2)\cdots(1-q^{2n})S^n\left(\frac{1+q}{1-q^2}\right) = (1-q^2)\cdots(1-q^{2n})S^n\left(\frac{1}{1-q}\right) ,$$

and therefore the result is $(1+q)(1+q^2)\cdots(1+q^n)$.

This gives the value of a Rogers-Szegö polynomial at $x = \sqrt{q}$ (and $a = 1$).

Ex. 2.23 The identity to show is

$$\sigma_1\left(\frac{q^{2k+1}}{1-q^2}\right) = \sigma_1\left(-\frac{q-q^{2k+1}}{1-q^2}\right)\sigma_1\left(\frac{q}{1-q^2}\right)$$

and thus amounts to $q^{2k+1} = -(q-q^{2k+1}) + q$.

But not all q-identities reduce to $a = -(b-a) + b$.

Ex. 2.24 go to next.

Ex. 2.25 Recognizing that $C_n^\alpha = S^n(\mathbb{AB})$, with $\mathbb{B} = (1-\alpha)(1-q)^{-1}$, $\mathbb{A} = \{a_1, a_2\}$ of rank 2 such that $\Lambda^1(\mathbb{A}) = 2\cos(\theta)$, $\Lambda^2(\mathbb{A}) = 1$, one expands $S^n(a_1\mathbb{B} + a_2\mathbb{B})$.

Ex. 2.26 Putting in factor $(1-q)\cdots(1-q^n)q^{n(n+1)/2}$, letting a be of rank 1, one rewrites the functional

$$f \to \int' f = \sum_k S^k\left(\frac{1-a}{1-q}\right)S^{n-k}\left(\frac{1-aq^{2n}}{1-1/q}\right)q^{kn}f(k) .$$

By linearity in f, it suffices to write the case $f(k) = y^k$. One can suppose y to be of rank 1, in which case the sum is equal to

$$S^n\left(\frac{1-a}{1-q}q^n y + \frac{1-aq^{2n}}{1-1/q}\right) := F_n(y) .$$

One sees that the terms in the expansion of $F_n(0)$, according to powers of a, are equal to those of $aF_n(q^2)$. Therefore, the functional vanishes on $f : f(k) = 1 - aq^{2k}$, and this gives a proof of the preceding exercise. One can add the vanishing of $f : f(k) = q^{-k} - a^2q^{3k}$, by similarly comparing the expansions of $F_n(q^{-1})$ and $F_n(q^3)$, when $n \geq 2$.

The more general vanishing properties, obtained by direct expansions, are that

$$F_n(q^{-r+1} - a^r q^{r+1})$$

vanishes for $n \geq r$, that is

$$\sum \pm \frac{\left[{n \atop k}\right](q^{k(1-r)} - a^r q^{k(r+1)})}{(aq^k;q)_{n+1}} = 0 .$$

Stated differently, they are the identities :

$$S^n\left((1-q^{n-r}) + a\frac{q^{n-r}-q^{2n}}{1-1/q}\right) = a^r\, S^n\left(\frac{(1-q^{n+r}) + a(q^{n+r}-q^{2n})}{1-1/q}\right) \quad , \quad n \geq r .$$

Ex. 2.27 Introduce two elements x, y of rank 1. Recognize that the above expression is the coefficient of x^n in $\frac{d}{dy}f(x,y)\Big|_{y=1}$, where

$f(x,y) := \sum_i (xy)^i S^i(p\mathbb{A}) \sum_j (-xy^p)\Lambda^j(\mathbb{A})$. Moreover, one can take $\mathbb{A}$ to be a sum $\sum a$ of rank 1 elements; in that case $f(x,y) = \prod_a (1-axy^p)/(1-axy)^p$. The logarithmic derivative $\frac{d}{dy}\log f(x,y)$ is equal to $\sum_a \frac{-p(-ax)}{1-axy} + \frac{p(-axy^{p-1})}{1-axy^p}$, from which one sees the required nullity in $y = 1$.

Ex. 2.28 The $[k,j]$-entry, $k,\ell = 0,1,\ldots$, is equal to

$$\sum\nolimits_{\ell(I)=k;\,|I|=j} \Psi_I(k)\, S^I(\mathbb{A}) \ .$$

For example,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & S^1 & S^2 & S^3 \\ 1 & 2S^1 & 2S^2+S^{11} & 2S^3+2S^{12} \\ & 3S^1 & 3S^2+3S^{11} & 3S^3+6S^{12}+S^{111} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & S^1 & S^2 & S^3 \\ 0 & 0 & S^{11} & 2S^{12} \\ 0 & 0 & 0 & S^{111} \end{bmatrix}$$

Ex. 2.29 By definition,

$$\prod_{e\in\mathbb{E}}(z+e) = \sum z^{n-i}\Lambda^i(\mathbb{E}) = \prod_{a\in\mathbb{A}}\big((z{+}x)+a\big)$$
$$= (z{+}x)^n + (z{+}x)^{n-1}\Lambda^1(\mathbb{B}-x) + (z{+}x)^{n-2}\Lambda^2(\mathbb{B}-x) + \cdots + \Lambda^n(\mathbb{B}-x)\,.$$

Expanding each $\Lambda^j(\mathbb{B}-x) = \sum(-x)^i\Lambda^{j-i}(\mathbb{B})$, one gets the required expressions, whose main interest was to illustrate that the '+' sign, as well a the '-' sign, can have different meanings.

Ex. 2.30 $S^{n-1}(2x-\mathbb{A})/S^n(x-\mathbb{A})$ is the logarithmic derivative of $S^n(x{-}\mathbb{A})$, and this settles the first point.

One recognizes in $S_{n^{n-1}}(\mathbb{A}{-}a{-}(\mathbb{A}^{der}{+}a))$ a resultant $R(\mathbb{A}{-}a, \mathbb{A}^{der}{+}a)$. However, for any $a' \in \mathbb{A}$, one has $R(\mathbb{A}^{der}, a') = R(\mathbb{A}{-}a', a')$, and therefore

$$R(\mathbb{A}{-}a, \mathbb{A}^{der}{+}a) = \pm R(\mathbb{A}{-}a, a) \prod_{a'\in\mathbb{A}\setminus a} R(\mathbb{A}{-}a', a') = \pm\Delta(\mathbb{A})^2 \ .$$

Ex. 2.31 Writing $\Psi_i(\mathbb{A}) = nS^i(\mathbb{A}-\mathbb{A}^{der})$, then $\sum\Psi_i(\mathbb{A}) = n\sigma_1(\mathbb{A}-\mathbb{A}^{der})$, and, indeed, $\sigma_1(-\mathbb{A}^{der}) = 1-(1{-}1/n)\Lambda^1(\mathbb{A}) + (1{-}2/n)\Lambda^2(\mathbb{A}) - \cdots$

Ex. 2.32 Write $-\mathbb{A}^{der} = (\mathbb{A}-\mathbb{A}^{der})-\mathbb{A}$. Therefore $S^k(x-\mathbb{A}^{der}) = \sum S^{k-i}(x-\mathbb{A})S^i(\mathbb{A}-\mathbb{A}^{der}) = \sum S^{k-i}(x-\mathbb{A})\,\Psi_i(\mathbb{A})/\Psi_0(\mathbb{A})$.

Ex. 2.33 Since $P_k(x) := S_{k^k,0}(\mathbb{A}-\mathbb{A}^{der}, x)$, using the preceding exercise, it can be written as

$$\begin{vmatrix} \frac{1}{n}\Psi_k(\mathbb{A}) & \cdots & \frac{1}{n}\Psi_{2k-1}(\mathbb{A}) & x^{k+j} \\ \vdots & & \vdots & \vdots \\ \frac{1}{n}\Psi_0(\mathbb{A}) & \cdots & \frac{1}{n}\Psi_{k-1}(\mathbb{A}) & x^j \end{vmatrix} .$$

Summation on $x = a \in \mathbb{A}$ transforms the last column into a multiple of a previous one, hence the sum vanishes.

Using Plücker relations, one can write

$$P_k(x)^2 = S_{k^{k-1}}(\mathbb{B}-2x)S_{k^{k+1}}(\mathbb{B}) + S_{k+1^{k-1}}(\mathbb{B}-2x)S_{k-1^{k+1}}(\mathbb{B}) \ ,$$

with $\mathbb{B} = \mathbb{A}-\mathbb{A}^{der}$.

Take now $n=4$, to simplify indices. The candidate to nullity is

$$\Big| 1,\ S_{11}(\mathbb{B}) + S_2(\mathbb{B}-2a),\ S_2(\mathbb{B}-2a)S_{222}(\mathbb{B}) + S_{33}(\mathbb{B}-2a)S_{11}(\mathbb{B}),$$
$$S_{33}(\mathbb{B}-2a)S_{3333}(\mathbb{B}) + S_{444}(\mathbb{B}-2a)S_{222}(\mathbb{B}) \Big|_{a\in\mathbb{A}} .$$

By successive subtractions, it simplifies into

$$\Big| 1,\ S_2(\mathbb{B}-2a),\ S_{33}(\mathbb{B}-2a)S_{11}(\mathbb{B}),\ S_{444}(\mathbb{B}-2a)S_{222}(\mathbb{B}) \Big| .$$

However, the last exercise shows that the elements $S_{444}(\mathbb{B}-2a)$ are constant, and thus the last column is a multiple of the first one and the determinant vanishes. □

Ex. 2.34 Since $\sum z^i\Psi_i(\mathbb{A})/n = \sigma_z(\mathbb{A} - \mathbb{A}^{der})$, then

$$K_j/(j-1)! = \Psi_j(\mathbb{A} - \mathbb{A}^{der}) = \Psi_j(\mathbb{A}) - \Psi_j(\mathbb{A}^{der}) ,$$

and one expresses $\Psi_j(\mathbb{A})$ and $\Psi_j(\mathbb{A}^{der})$ with the help of Waring's formula.

The first values are $nK_1 = \Lambda^1$, $n^2K_2 = -4n\Lambda^2 + (2n-1)\Lambda^{11}$, $n^3K_3 = (12n-18n^2)\Lambda^{12} + 18n^2\Lambda^3 + (6n^2-6n+2)\Lambda^{111}$.

Ex. 2.35 Put $z := 1-x$. Then z is a rank 1 element, and thus $S_J(1+x) = S_J(2-z) = \sum(-z)^i S_{J/1^i}(2)$. Apart from one part-partitions, the only J for which $S_J(1+x)$ can be non zero are of the type $J = [1^m, h, k]$, $h \geq 1$, $m \geq 0$. The corresponding summation reduces to $\Big((-z)^m\Lambda^2(2) + (-z)^{m+1}\Lambda^1(2) + (-z)^{m+2}\Big)\, S_{h-1,k-1}(2)$. Finally, for such J, one has

$$S_{1^m,h,k}(1+x) = (k-h+1)\,(x-1)^m x^2 .$$

Ex. 2.36 Taking generating series, one has to show that

$$\frac{u}{\beta+1}\sum n u^{n-1} S_n\big((\beta{+}1)\mathbb{A}\big) = \frac{d}{dz}\left(u^n S_n\big((\beta{+}z)\mathbb{A}\big)\right)\Big|_{z=1} ,$$

but both sides are equal to $\prod_{a\in\mathbb{A}}(1-ua)^{-\beta-1}\sum_a ua\,(1-ua)^{-1}$.

Ex. 2.37 The Tchebychef polynomials are given by the generating function $(1-xz)(1-2xz+z^2)^{-1} = \sum z^n T_n(x)$. In other words, $T_n(x) = \Psi_n(\mathbb{B})/2$, with $\mathbb{B}$ of cardinality 2 such that $\Lambda^1(\mathbb{B}) = 2x$, $\Lambda^2(\mathbb{B}) = 1$. The alphabet $\mathbb{A} = (1-\zeta)\mathbb{B}$, with ζ of rank 1, is such that $\Psi_n = 2(1-\zeta^n)\,T_n(x)$. Therefore, the identity to be shown is just the specialization $\zeta = -1$ of

$$\log\big(\sigma_z((1-\zeta)\mathbb{B})\big) = \log\left(\frac{1-2zx\zeta+z^2\zeta^2}{1-2zx+z^2}\right) = \sum \Psi_n(\mathbb{B})/n .$$

Ex. 2.38 The left hand side is $r!\,S_r(x)$, with x of binomial type, the right side being its expansion in the basis Ψ^J. This gives the first equality. To take into account the restriction on parity, one introduces a rank 1 element ξ which will be specialized to -1. Then $\Psi_i\big((1-\xi)\,x/2\big) = x$ if i is odd, and $= 0$ otherwise. The left hand side becomes $r!S_r\big((1-\xi)\,x/2\big)$, and one finds its values by taking the generating series

$$\sigma_z\big((1-\xi)\,x/2\big) = \left(\frac{1-z\xi}{1-z}\right)^{x/2} = \left(\frac{1+z}{1-z}\right)^{x/2} .$$

The same method would allow to explicit the summation $\sum_J x^{\ell(J)} 1/(\Psi^J,\Psi^J)$ restricted to partitions with all parts multiple of a fixed integer p. One has to take ξ to be a primitive p-th root of unity instead of being -1.

Ex. 2.39 Since a monomial function $\Psi_J(x)$ is proportional to $N_{\ell(J)}$, according to Formula (2.2.2), one has just to expand sf in the basis of monomial functions.

$$sf(x) = \sum_J \frac{(sf\,,\,S^J)}{m_1!\,m_2!\cdots} N_{\ell(J)}\ ,$$

sum over all partitions $J = 1^{m_1}2^{m_2}\cdots$.

Ex. 2.40 Instead of derivating, one can take the Taylor expansion of $S_I(x+y)$. However, the values of Schur functions in an element of binomial type are not linearly independent, and the usual expansion as a sum over all partitions contained in I is too big. Let us see that one can give a summation on hook-partitions only, and prove that

$$(\star)\qquad \frac{d}{dx}S_I(x) = \sum_{\alpha,\beta:\,\alpha+\beta\geq 1} \frac{(-1)^\beta}{\alpha+\beta} S_{I/[1^\beta,\alpha]}(x) = \sum_{k=1}^{\infty} \frac{d}{d_{\Psi_k}}\big(S_I\big)(x)\ .$$

The RHS can be interpreted as the limit, for $q=1$, q a rank-1 element, of

$$\int_0^1 \frac{S_I(x+z(1-q)) - S_I(x)}{z(1-q)} dz\ ,$$

z being also treated as a rank 1 element.

But taking the power sum basis Ψ^J instead, one sees that

$$\int_0^1 \frac{\prod_{j\in J}\big(x+z^j(1-q^j)\big) - x^{\ell(J)}}{z(1-q)}\, dz = \sum_{j\in J} \frac{z^j}{j}\,\frac{1-q^j}{1-q}\,\frac{\Psi^J(x)}{x}$$

has the limit $\ell(J)x^{\ell(J)-1}$, and therefore is indeed the value of the derivative of $\Psi^J(x) = x^{\ell(J)}$. This proves the first equality of $(\star)$. The second expression results from

$$k\frac{d}{d_{\Psi_k}} = D_{\Psi^k} = \sum\nolimits_H \pm D_{S_H}\ ,$$

sum over all hook-partitions of weight k. In fact, it is immediate to check directly that d/d_x, acting on symmetric functions in x expressed in terms of power sums, coincides with the sum of derivatives with respect to the power sums. □

Notice that, for a complete function,

$$\frac{d}{dx}S_k(x) = S_{k-1}(x) + \frac{1}{2}S_{k-2}(x) + \frac{1}{3}S_{k-3}(x) + \cdots$$

and this gives the derivative of a binomial coefficient $\binom{x+k-1}{k}$.

Ex. 2.41 The power sums $\Psi^i(\mathbb{A}) = \Psi^i(x)\Psi^i(1+\zeta) = x(1+\zeta^i)$ specialize into $2x$ if i is even, 0 if i is odd. The expression of monomial functions in terms of power sums is compatible with doubling the parts of a partition I. Denote this operation by $I\to 2\star I$. Then

$$\Psi_J(\mathbb{A})\Big|_{\zeta=-1} = \Psi_I(2x) \quad \text{if } J = 2\star I \quad \text{or 0 if } J \text{ has an odd part}\ .$$

The determination of Schur functions is a little more subtle. One needs to use the factorization, due to Littlewood [**132**], of $S_I(\Omega\,\mathbb{B})$, for Ω is the "alphabet of p-th roots of unity" (here p=2), and $\mathbb{B}$ is arbitrary. The outcome in our case is that the specialization of a Schur function S_I is equal to $\pm S_{I'}(x)\,S_{I''}(x)$ if I is a partition without 2-core, and with 2-quotient $[I', I'']$, and is null if I has a core. In particular, if $I = [2n]$, then $I' = [n]$, $I'' = 0$, and this gives Gillis'identity

$$\sum_{i=0}^{2n} (-1)^i \binom{x}{i} \binom{x}{2n-i} = \binom{x}{n} .$$

Ex. 2.42 Because $f(z, \mathbb{A})$ is linear in the $S^n(\mathbb{A})$, one can restrict to $\mathbb{A} = \alpha$, with α binomial. In that case

$$f(z,\alpha) = \frac{1}{z-\alpha} = 1 + \frac{1}{z} + \frac{\alpha}{z(z+1)} + \frac{\alpha(\alpha+1)}{z(z+1)(z+2)} + \frac{\alpha(\alpha+1)(\alpha+2)}{z(z+1)(z+2)(z+3)} + \cdots$$

identity that is a special case of Newton's interpolation formula. Now everybody agrees that $z - \alpha = (z+y) - (\alpha + y)$ and this settles the first point.

As for the product $\left(\frac{1}{z-\alpha}, \frac{1}{z-\beta}\right) \mapsto \frac{1}{(z-\alpha)(z-\beta)} = \left(\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right)\frac{1}{\alpha-\beta}$, one is reduced to evaluate all the $\big(S^n(\alpha) - S^n(\beta)\big)/(\alpha - \beta)$, $n \in \mathbb{N}$. The expansion of $S^n(\alpha)$ involves Stirling numbers of the first kind, that is, the elementary symmetric functions of an alphabet specialized in $\{0, 1, \ldots, n{-}1\}$. It remains to express the $(\alpha^k - \beta^k)/(\alpha - \beta)$ in terms of products of $S^i(\alpha)$, $S^j(\beta)$.

Ex. 2.43 Taking advantage that the interval of summation has not been specified, one chooses $-r \le n < \infty$. The Laplace expansion of the determinantal expression of $S_{Jn}(x)$ along its last column gives :

$$\begin{aligned}
\sum_n z^n S_{Jn}(x) &= \sum_n \sum_{i=0}^{r} (-1)^i z^n S_{J/1^i}(x) S_{n+i}(x) \\
&= \sum_i (-z)^{-i} S_{J/1^i}(x) \sum_n z^{n+i} S_{n+i}(x) \\
&= \sum_i (-z)^{-i} S_{J/1^i}(x)\, (1-z)^{-x} .
\end{aligned}$$

Notice that the summation in i is restricted to $0 \le i \le \ell(J)$. For example,

$$(1-z)^x \sum_{n \ge -1} z^n S_{1n}(x) = S_1(x) - z^{-1} S_{1/1}(x) = x - z^{-1} ,$$

$$(1-z)^x \sum_{n \ge -2} z^n S_{23n}(x) = S_{23}(x) - z^{-1} S_{23/1}(x) + z^{-2} S_{23/11}(x)$$

$$= \frac{1}{24} x^2 (x{+}1)(x{+}2)(x{-}1) - \frac{1}{24} x(x{-}1)(5x{+}6)(x{+}1) z^{-1} + \frac{1}{3} x(x{+}1)(x{-}1) z^{-2}$$

Ex. 2.44 For all positive integers n, k, one has $D_{\Psi_k} S_{n+k} = S_n$. On another hand, expressing S_{n+k} in the basis Ψ^I, one gets

$$D_{\Psi_k} S_{n+k} = k \frac{d}{d_{\Psi^k}} \sum \frac{\Psi^I}{(\Psi^I, \Psi^I)} = k \sum m_k \frac{\Psi^I}{(\Psi^I, \Psi^I) \Psi^k} .$$

The unknown sum is therefore equal to $\sum \Psi^k S_n / k$. Using multiplication of Schur function by monomial functions, one gets a sum on Schur functions indexed by vectors of the type $[k, 0^i] + [0^i, n]$, $i = 0, \ldots, k{-}1$, which, by reordering, finally gives

$$S_{n+k} + S_{k,n} - S_{1,\,k-1,\,n} + S_{1,\,1,\,k-2,\,n} - \cdots$$

Ex. 2.45 Derivations of polynomials $S^n(x-\mathbb{A})$ rather than of series $\sigma_x(\mathbb{A})$ are easier to write. Truncating the series to an arbitrary big order, one can write it $f(x) = S^n(x-\mathbb{A})$. Now, Ω becomes

$$\begin{aligned}\Omega &= S^n(x-\mathbb{A}) + \binom{m+1}{1} yxS^{n-1}(x-\mathbb{A}) + \binom{m+2}{2} y^2x^2S^{n-2}(x-\mathbb{A}) + \cdots \\ &= S^n(x - \mathbb{A} + (m+1)xy) \ .\end{aligned}$$

On the other hand, a direct computation shows that

$$\frac{xf(x) - xyf(xy)}{x - xy} = S^n(x + xy - \mathbb{A}) \text{ and } \frac{1}{m!}\frac{d}{dz^m} z^m S^n(z+\mathbb{B}) = S^n((m+1)z + \mathbb{B}) \ ,$$

for all $\mathbb{B}$ and all rank 1 element z. One concludes by putting $z = xy$, $\mathbb{B} = x - \mathbb{A}$.

Ex. 2.46 First, the explicit expression of Hermite polynomials is equivalent to the generating function $\sum z^n H_n/n! = \exp(-2xz - z^2)$. Therefore, $\mathbb{A}$ is such that $\Psi^1(\mathbb{A}) = -2x$, $\Psi^2(\mathbb{A}) = -2$, $\Psi^i(\mathbb{A}) = 0$ for $i > 2$. The forgotten functions for $\mathbb{A}$ are equal, by definition, to the monomial functions for $\mathbb{A}'$, such that $\Psi^1(\mathbb{A}') = -2x$, $\Psi^2(\mathbb{A}') = 2$, $\Psi^i(\mathbb{A}') = 0$ for $i > 2$. In particular, the only surviving functions are those indexed by partitions of the type $1^n 2^k$. Because

$$2\Psi_{1^n2^k}(\mathbb{A}') = \Psi^2(\mathbb{A}')\Psi_{1^n2^k}(\mathbb{A}') = \Psi_{1^n2^{k+1}}(\mathbb{A}') \ ,$$

one can ignore the parts equal to 2, and therefore, one has only to compute for the indices 1^n, in which case one obtains $H_n/n!$.

Ex. 2.47 This determinant is a special case of the first determinant in Eq. (2.4.6). Therefore, the determinant is to $(-1)^n n!$ times a Gegenbauer of order n, but for the parameter $-k\alpha$. In particular, Gegenbauer polynomials can be expressed as determinants in the Tchebychef polynomials $G(n, 1, x)$, but more simply, one can choose to express them in terms of the coefficients of $(1-2xz+z^2)$ (taking $\alpha = -1$).

In that case, the determinant is tridiagonal, with main diagonal $[2\alpha x, 2(\alpha+1)x, \ldots, 2(\alpha+n-1)x]$, above diagonal $[2\alpha, 2\alpha+1, \ldots, 2\alpha+n-2]$, subdiagonal $[1, 2, \ldots, n-1]$. For example, $4!\, G((4, \alpha, x)$ is equal to the following determinant :

$$\begin{vmatrix} 2\alpha x & 2\alpha & 0 & 0 \\ 1 & 2(1+\alpha)x & 1+2\alpha & 0 \\ 0 & 2 & 2(2+\alpha)x & 2+2\alpha \\ 0 & 0 & 3 & 2(3+\alpha)x \end{vmatrix}$$

$$= 96x^4\alpha + 176x^4\alpha^2 + 96\alpha^3x^4 - 96x^2\alpha - 144x^2\alpha^2 + 16\alpha^4x^4 - 48x^2\alpha^3 + 12\alpha + 12\alpha^2 \ .$$

Ex. 2.48 The generating function given in the preceding exercise shows that the Gegenbauer polynomials are the complete functions of $\alpha\mathbb{A}$, with $\mathbb{A}$ of rank 2 such that $\Lambda^1(\mathbb{A}) = 2x$, $\Lambda^2(\mathbb{A}) = 1$, and α of binomial type. The summation to study is $\sum \Psi_J(-1/\alpha)S^J(\alpha\mathbb{A})$, and the Cauchy formula gives its value $S^n((-1/\alpha)\,\alpha\mathbb{A}) = (-1)^n\Lambda^n(\mathbb{A})$. Therefore the sum is $-2x$ for $n = 1$, 1 for $n = 2$ and 0 otherwise.

Ex. 2.49 The hypotheses are $\sigma_z(2x) = (1 - z\,2x + z^2)^{-1}$. Therefore

$$\sigma_z(x) = (1 - z\,2x + z^2)^{-1/2}$$

and the $S^n(x)$ are the Legendre polynomials.

Ex. 2.50 Since $1/y$ specializes to $x - \sqrt{x^2-1}$, the first statement is equivalent to the generating function

$$\sum z^n P_n = (1 - z(y + 1/y) + z^2)^{-1/2} = (1 - z\,2x + z^2)^{-1/2} \ .$$

One deduces from it that the polynomials are characterized by the recursions

$$P_n = \left(2-\frac{1}{n}\right) x\, P_{n-1} - \left(1-\frac{1}{n}\right)\, P_{n-2}\ ,\ P_0 = 1,\ P_{-1} = 0\ .$$

One then checks that the other expressions also satisfy the same recursions.

It is interesting to spraid out these compact expressions, to recognize that λ-rings are not too far from the XIX century analysis. For example, by definition, $\sum z^i \Lambda^i(n(\zeta{+}1)) = \big((1+z)(1-z)\big)^n$, and therefore ([**153**], p.66) :

$$S^n\big((n{+}1)x - n(\zeta{+}1)\big) = \sum S^{n-2k}\big((n{+}1)x\big)\,\Lambda^{2k}\big(n((\zeta{+}1))\big) = \\ \sum \pm \frac{(n{+}1)\cdots(n{+}n{-}2k)}{(n-2k)!}\binom{n}{k} x^{n-2k}$$

The next expression is due to Murphy ([**153**], p.66) :

$$S^n\big((n{+}1)x - n\zeta\big) = \sum x^k S^k(n{+}1)(-\zeta)^{n-k}\Lambda^{n-k}(n) = \sum_0^n x^k \frac{(n{+}k)!}{k!\,k!\,(n{-}k)!}\ .$$

$$\Lambda^n\big(n-(n{+}1)c\big) = \sum (-1)^k \binom{n}{n{-}k}\binom{n{+}k}{k}\left(\frac{1{-}x}{2}\right)^k$$

is also a classical expansion, and the last one is due to Dirichlet :

$$\Lambda^n(na+nb) = \sum a^k\Lambda^k(n)\, b^{n-k}\Lambda^{n-k}(n) = \sum \binom{n}{k}^2\left(\frac{x{+}1}{2}\right)^k\left(\frac{x{-}1}{2}\right)^{n-k}\ .$$

Ex. 2.51 Use the expression $P_n(x) = \Lambda^n(na+nb)$, with $a = \frac{x+1}{2}$, $b = \frac{x-1}{2}$, as in the preceding exercise. Then, according to Lagrange-Bürman formula (2.4.3), $\sum z^n P_n(x)/n$ is the logarithm of $g(z)/z$, where $g(z)$ is the inverse of

$$z\sigma_{-z}(a+b) = z\left(1 - z\frac{x{+}1}{2}\right)^{-1}\left(1 - z\frac{x{-}1}{2}\right)^{-1}$$

Ex. 2.52 (cf. Schur [**160**] III, p.361, Prosper [**150**]. The equations defining $F(x)$ are

$$(-z)^n\Psi^n(\mathbb{A}) + \sum \frac{n}{i}\Lambda^{n-i}(-iz\mathbb{A})\lambda_z(i\mathbb{A})\ \cap\ \{z,\ldots,z^n\} = \{0,\ldots,0\}\ ,$$

writing $f\cap z^i$ for the coefficient of z^i in f.

These equations are very similar to those involved in Lagrange inversion, and admits the solution

$$F(x) = (-1)^n\Psi^n(\mathbb{A}) + \sum_{i=1}^n \frac{n}{i} z^i\,\Lambda^{n-i}(-i\mathbb{A})\ .$$

Ex. 2.53 T_x is the operator $\mathbb{A}\mapsto\mathbb{A}+x$, and T_{-x}, the operator $\mathbb{A}\mapsto\mathbb{A}-x$, the expansions $(\star)$, $(\star\star)$ being special cases of the Taylor formula describing the morphism $\mathbb{A}\mapsto\mathbb{A}+X$ on the ring $\mathfrak{Sym}(\mathbb{A})$ (with coefficients in $\mathfrak{Sym}(X)$). With this interpretation, the four formulas are immediate.

More general operators appear in the theory of Hall-Littlewood polynomials [**75**].

Ex. 2.54 By induction on n, one has to compute the image of

$$\sigma_1\big((1-t)\mathbb{A}(\mathbb{Z}-z_1)\big)\, f\big(\mathbb{A}-\mathbb{Z}^\vee + z_1^{-1}\big)$$

under ∇_{z_1}. But $\exp\big((-z_i)^i D_{\Psi^i}/i\big)$ is the operator $f \to f(\mathbb{A} - z^{-1})$, and $\exp\big((1-t^i)z^i\Psi^i/i\big)$ is the operator "multiplication by $\sigma_z\big((1-t)\mathbb{A}\big)$".

Therefore, $\sigma_1\big((1-t)\mathbb{A}(\mathbb{Z}-z_1)\big)\, f\big(\mathbb{A}-Z^\vee + z_1^{-1}\big)$ becomes, under ∇_{z_1},

$$\begin{aligned}
&\sigma_1\big((1-t)(\mathbb{A}-z_1^{-1})(\mathbb{Z}-z_1)\big)\, f(\mathbb{A}-\mathbb{Z}^\vee)\\
&\qquad = \sigma_1\big((1-t)\mathbb{A}\mathbb{Z}\big)\,\sigma_1\big((1-t)(-z_1\mathbb{A} - z_1^{-1}(\mathbb{Z}-z_1)\big)\, f(\mathbb{A}-\mathbb{Z}^\vee)\\
&\qquad\text{and finally} \qquad = \sigma_1\big((1-t)\mathbb{A}\mathbb{Z}\big)\,\sigma_{1/z_1}\big((t-1)(\mathbb{Z}-z_1)\big)\, f(\mathbb{A}-\mathbb{Z}^\vee)\ .
\end{aligned}$$

Ex. 2.55 The reader, undoubtly, will accept that the example be generic enough. Once more the required identity comes from Plücker relations. Take the matrix

$$\begin{array}{r} \\ \\ \\ \textit{alphabets} \\ \textit{indexing} \end{array}\begin{bmatrix} 2 & 5 & 5 & 6 & 7\\ 1 & 4 & 4 & 5 & 6\\ 0 & 3 & 3 & 4 & 5\\ 0 & \mathbb{A} & \mathbb{A}-x & \mathbb{A}-x & \mathbb{A}-x\\ {\scriptstyle 0} & {\scriptstyle 5} & {\scriptstyle 5'} & {\scriptstyle 6} & {\scriptstyle 7}\end{bmatrix},$$

and recognize that

$$[05'6][567] - [056][5'67] = [55'6][067]$$

gives what is wanted, since

$$[55'6] = S_{544}(\mathbb{A};\mathbb{A}-x;\mathbb{A}-x) = S_{54;4}(\mathbb{A};\mathbb{A}-x) - xS_{53;4}(\mathbb{A};\mathbb{A}-x) = xS_{444}(\mathbb{A})\ .$$

Ex. 2.56 The ring $\mathfrak{Sym}(\mathbb{A}) \otimes \mathfrak{Sym}(\mathbb{B})$, with coefficients in Q, decomposes into a product $\bigotimes \mathbb{Q}\big[\Psi^k(\mathbb{A}), \Psi^k(\mathbb{B})\big]$, this decomposition being compatible with the scalar product. Fixing k and putting $x = \Psi^k(\mathbb{A})$, $y = \Psi^k(\mathbb{B})$, one is reduced to study the space $Q[x, y]$ with scalar product

$$(x^i y^j\ ,\ x^m y^n) = i!\, k^i\, j!\, k^j\, \delta_{i,m}\delta_{j,n}\ .$$

But one can write the scalar product with a polynomial $f(x, y) = \sum f_{ij}x^i y^j$ as

$$(f, g) = \sum f_{i,j} \frac{d^i}{k^i dx^i}\frac{d^j}{k^j dy^j}(g)\Big|_{x=0=y}\ .$$

Therefore the operator $g \to \big((x+y)^i(x-y)^j\ ,\ g\big)$ coincides with

$$g \to k^{i+j}(d_{x+y})^i\,(d_{x-y})^j(g)\Big|_{x=0=y}$$

which annihilates all monomials in $(x+y)$, $(x-y)$ of total degree $i+j$, except $(x+y)^i(x-y)^j$. This proves the orthogonality of $\Psi^I(\mathbb{A}-\mathbb{B})\,\Psi^J(\mathbb{A}+\mathbb{B})$ for partitions with all parts equal to k, and therefore for all partitions without restriction.

The property just obtained can be used to describe, at the level of characters, the branching rules of the hyperoctahedral group B_n into $\mathfrak{S}_{2n}$.

Ex. 2.57 The scalar product corresponds to the Cauchy kernel $\sigma_1(\mathbb{A}_0\mathbb{B}_0 + \cdots + \mathbb{A}_{n-1}\mathbb{B}_{n-1})$. One has first to rewrite the argument. Since it is bilinear in $\mathbb{A}_i$, $\mathbb{B}_j$, one can use $a^0,\ldots,a^{n-1}$, $b^0,\ldots,b^{n-1}$, a,b of rank 1, instead of the alphabets. Write now

$$1+ab+\cdots+a^{n-1}b^{n-1} = S^{n-1}(1+ab) = S^{n-1}\big((1+b[k])+b(a-[k])\big)$$
$$= \sum_{i+j=n-1} S^j(1+b[k])\, b^i S^i(a-[k])$$
$$= \sum_{i+j=n-1} \Big(b^i+S^1([k])b^{i+1}+\cdots+S^j([k])b^{i+j}\Big)\Big(a^i-\Lambda^1([k])a^{i-1}+\cdots\pm\Lambda^k([k])a^{i-k}\Big)$$

(suppressing the eventual negative powers of a in the last sum).

Therefore $\mathbb{A}_0\,\mathbb{B}_0+\cdots+\mathbb{A}_{n-1}\,\mathbb{B}_{n-1})$ can be decomposed into a sum of terms

$$\Big(\mathbb{B}_i+S^1([k])\mathbb{B}_{i+1}+\cdots+S^j([k])\mathbb{B}_{i+j}\Big)\Big(\mathbb{A}_i-\Lambda^1([k])\mathbb{A}_{i-1}+\cdots\pm\Lambda^k([k])\mathbb{A}_{i-k}\Big)$$

and this decomposition induces a factorisation of the Cauchy kernel into

$$\prod_{i+j=n-1} \sigma_1\Big(\big(\mathbb{B}_i+\cdots+S^j([k])\mathbb{B}_{i+j}\big)\,\big(\mathbb{A}_i\pm\Lambda^k([k])\mathbb{A}_{i-k}\big)\Big)\ .$$

B.3

Ex. 3.1 (cf. Giudice, Muir V p.183). The expression to consider is a linear operator from $\mathfrak{Pol}$ to $\mathfrak{Pol}_n$, which interpolates $f(x)$ in all $a\in\mathbb{A}$. It is therefore the identity on polynomials of degree $\le n-1$ and the null operator on the space of polynomials divisible by $S^n(x-\mathbb{A})$. It therefore coincides with the first remainder.

Ex. 3.2 $n=3$ will be generic enough. The matrix, together with indexing of columns, is

$$\begin{array}{r} \\ \\ \\ \\ \textit{alphabets} \\ \textit{indexing} \end{array}\begin{bmatrix} 0 & 1 & 2 & 3 & m & 3\\ \cdot & 0 & 1 & 2 & m-1 & 2\\ \cdot & \cdot & 0 & 1 & m-2 & 1\\ \cdot & \cdot & \cdot & 0 & m-3 & 0\\ \mathbb{A} & \mathbb{A} & \mathbb{A} & \mathbb{A} & \mathbb{A}-\mathbb{B}+x & x\\ 0 & 1 & 2 & 3 & m & x \end{bmatrix}$$

Plücker gives the following relation, on minors of order 4 sharing columns 1,2 :

$$[012x]\,[123m] = [012m]\,[123x] - [0123]\,[12mx]\ ,$$

that is, more explicitly, $S_{0000}\,S_{111,m-3}(\mathbb{A},\mathbb{A}-\mathbb{B}+x) = S_{m-3}(\mathbb{A}-\mathbb{B}+x)\,S_{111,0}(\mathbb{A},x) - S_{0000}\,S_{11,m-2,0}(\mathbb{A},\mathbb{A}-\mathbb{B}+x,x)$. Subtracting $\mathbb{A}$ in the top row, or x in the first three rows, one gets

$$-S^m(-\mathbb{B}+x) = S_{m-3}(\mathbb{A}-\mathbb{B}+x)\,S_{111}(\mathbb{A}-x) - S_{11,m-2}(\mathbb{A}-x,\mathbb{A}-\mathbb{B})$$

and therefore, $S^m(x-\mathbb{B})$ is congruent to $S_{11,m-2}(\mathbb{A}-x,\mathbb{A}-\mathbb{B})$ modulo $S_{111}(\mathbb{A}-x) = -S^3(x-\mathbb{A})$.

Ex. 3.3 The only difficulty is to produce an appropriate matrix having the required Schur functions as minors.

For example, let us look at the third remainder of $S^{n+1}(x-\mathbb{B})$ by $S^n(x-\mathbb{A})$. The input consists of the two successive second remainders $S_{1^{n-2},33}(\mathbb{A}-x,\mathbb{A}-\mathbb{B})$

and $S_{1^{n-2},44}(\mathbb{A}-x, \mathbb{A}-\mathbb{B})$, that one can place inside the following matrix

$$\begin{matrix} \textit{indexing} \\ \\ \\ \\ \\ \\ \\ \textit{alphabets} \end{matrix}\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & \alpha & \beta & \gamma \\ 0 & 1 & 2 & 3 & 4 & 7 & 8 & 9 \\ \cdot & 0 & 1 & 2 & 3 & 6 & 7 & 8 \\ \cdot & \cdot & 0 & 1 & 2 & 5 & 6 & 7 \\ \cdot & \cdot & \cdot & 0 & 1 & 4 & 5 & 6 \\ \cdot & \cdot & \cdot & \cdot & 0 & 3 & 4 & 5 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 \\ \mathbb{A}-x & \mathbb{A}-x & \mathbb{A}-x & \mathbb{A}-x & \mathbb{A}-x & \mathbb{A}-\mathbb{B} & \mathbb{A}-\mathbb{B} & \mathbb{A}-\mathbb{B} \end{bmatrix}$$

The term on the right in Plücker equation

$$[1234\alpha\beta]\,[0123\beta\gamma] - [1234\beta\gamma]\,[0123\alpha\beta] = [01234\beta]\,[123\alpha\beta\gamma]$$

is, indeed,

$$S^3(\mathbb{A}-\mathbb{B})\, S_{111,444}(\mathbb{A}-x, \mathbb{A}-\mathbb{B}) \ .$$

Ex. 3.4 Modulo $S^{n+1}(x-\mathbb{B})$, the k-th remainder can be replaced by $S_{k^k}(\mathbb{B}-\mathbb{A}-x)S^n(x-\mathbb{A})$, according to (3.1.5).

Expanding $S_{j^j}(\mathbb{B}-\mathbb{A}-b) = \sum_{i=0}^{j}(-b)^i S_{j^j/1^i}(\mathbb{B}-\mathbb{A})$, writing

$$b^i S^n(b-A)S_{k^k}(\mathbb{B}-\mathbb{A}-b) = S_{k^k;n+i}(\mathbb{B}-\mathbb{A};b) \ ,$$

and using that $\sum_b S^r(b-\mathbb{A})/R(b, \mathbb{B}\setminus b) = S^{r-n}(\mathbb{B}-\mathbb{A})$, for any $r \in \mathbb{N}$, one finally writes Kronecker's sum as :

$$\sum_{i=0}^{j}(-1)^i S_{k^k,i}(\mathbb{B}-\mathbb{A})S_{j^j/1^i}(\mathbb{B}-\mathbb{A}) \ .$$

One concludes, each $S_{k^k,i}(\mathbb{B}-\mathbb{A})$ being null from having two identical columns.

Kronecker notices that the above vanishing properties determine the remainders, up to normalization. One can also interpret his result in terms of orthogonal polynomials. Indeed, let $\int_{\mathbb{A}-\mathbb{B}}$ be the linear functional on polynomials in x such that

$$\int_{\mathbb{A}-\mathbb{B}} P(x) := \sum_b P(b)S^n(b-\mathbb{A})/R(b, \mathbb{B}\setminus b) \ .$$

Then $\int_{\mathbb{A}-\mathbb{B}} x^k = S^k(\mathbb{B}-\mathbb{A})$, and the remainders appear as the "first orthogonal polynomials" with respect to $\int_{\mathbb{A}-\mathbb{B}}$, the other $S_{k^k}(\mathbb{B}-\mathbb{A}-x)$ being null for $k > n+1$.

Ex. 3.5 Let $\mathbb{A}^\vee := \{1/a : a \in \mathbb{A}\}$. Then

$$S^n(1-x\mathbb{A}) = S^n(x-\mathbb{A}^\vee)S^n(-\mathbb{A}) \ .$$

Therefore, up to non-zero constants, the successive remainders are

$$S_{1^{n-1},2}(\mathbb{A}^\vee - x, \mathbb{A}^\vee) \ , \ S_{1^{n-2},33}(\mathbb{A}^\vee - x, \mathbb{A}^\vee) \ , \ldots, \ S_{(n+1)^n}(\mathbb{A}^\vee) \ .$$

Now, expanding

$$S_{1^{n-r},(1+r)^r}(\mathbb{A}^\vee - x, \mathbb{A}^\vee) = \sum(-x)^j S_{0^j 1^{n-r-j}(1+r)^r}(\mathbb{A}^\vee)$$

and using that Schur functions of $\mathbb{A}^\vee$ can be expressed as Schur functions of $\mathbb{A}$ indexed by complementary diagrams, one finally gets

$$\sum(-x)^j Sr^{n-r-j(1+r)^j}(\mathbb{A}) = (-x)^{n-r}(\mathbb{A}-1/x)$$

for the r-th remainder.

For example, for $n=5$, the second remainder $S_{111,33}(\mathbb{A}^\vee - x, \mathbb{A}^\vee)$ is equal to

$$S_{11133}(\mathbb{A}^\vee) - xS_{01133}(\mathbb{A}^\vee) + x^2 S_{00133}(\mathbb{A}^\vee) - x^3 S_{00033}(\mathbb{A}^\vee) \ .$$

Multiplied by $S_{33333}(\mathbb{A})$, it becomes equal to

$$S_{222}(\mathbb{A}) - xS_{223}(\mathbb{A}) + x^2 S_{233}(\mathbb{A}) - x^3 S_{333}(\mathbb{A}) = -x^3 S_{333}(\mathbb{A} - 1/x) \ .$$

Ex. 3.6 From the definition of the companion matrix $\mathcal{C}(\mathbb{A})$, one has that for any $a \in \mathbb{A}$,

$$[1,\, a,\, \ldots,\, a^{n-1}]\, \mathcal{C}(\mathbb{A}) = [1,\, a,\, a^2,\, \ldots] \ .$$

This implies that $\mathbb{V}_{0^n}(\mathbb{A})\, \mathcal{C}(\mathbb{A}) = \mathbb{V}(\mathbb{A})$, where $\mathbb{V}(A)$ is the Vandermonde matrix. Therefore, for any $J \in N^n$, one has $\Delta(\mathbb{A})\, \mathcal{C}(\mathbb{A})_J = \mathbb{V}(\mathbb{A})_J$, that is $\det(\mathcal{C}(\mathbb{A})_J) = S_J(\mathbb{A})$.

For example, for $n=3$, $J=[1,4,6]$, one has

$$\det(\mathbb{C}(\mathbb{A})_J) = \begin{vmatrix} 0 & S_{500} & S_{800} \\ 1 & S_{40} & S_{70} \\ 0 & S_3 & S_6 \end{vmatrix} = - \begin{vmatrix} S_{500} & S_{800} \\ S_3 & S_6 \end{vmatrix} = \begin{vmatrix} S_3 & S_6 \\ S_{113} & S_{116} \end{vmatrix}$$

One can find the hook functions in the determinant expressing $S_J(\mathbb{A})$ by decomposing the diagram of J into diagonal hooks [**103**].

Notice that because the left part of the companion matrix is the identity matrix, a minor of any order of the companion matrix is equal to a maximal minor.

Ex. 3.7 For any x, one has the factorization

$$[1,\, x,\, \ldots,\, x^{n-1}]\, \mathcal{C}(\mathbb{A}) = [1,\, x,\, x^2,\, \ldots]$$

modulo $S^n(x - \mathbb{A})$. Therefore, the infinite Vandermonde matrix in the x_i's factorizes, modulo the polynomials $S^n(x_i - \mathbb{A})$, into :

$$\mathbb{V}(\mathbb{X})_{0^n}\, \mathcal{C}(A) \equiv \mathbb{V}(\mathbb{X}) \ .$$

Now, thanks to Binet-Cauchy, for any $J \in \mathbb{N}^k$, one has

$$\det(\mathbb{V}(\mathbb{X})_J) = \sum_I \det(\mathbb{V}(\mathbb{X})_I)\, \det(\mathcal{C}(A)_{J/I}) \ ,$$

sum over all partitions $I \in \mathbb{N}^k$, $I \le (n-k)^k$. However, as seen in the preceding exercise, all minors of $\mathcal{C}(A)$ are Schur functions of $\mathbb{A}$, up to a sign. ☐

For example, with $k=2$,

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} S_{400}(\mathbb{A}) & S_{700}(\mathbb{A}) \\ S_{30}(\mathbb{A}) & S_{60}(\mathbb{A}) \\ S_2(\mathbb{A}) & S_5(\mathbb{A}) \end{bmatrix} \equiv \begin{bmatrix} x_1^4 & x_1^7 \\ x_2^4 & x_2^7 \end{bmatrix}$$

Extracting the factor $x_1 - x_2$, one gets the expansion

$$(x_1 - x_2)\Big(S_{235}(\mathbb{A}) - (x_1 + x_2)\, S_{135}(\mathbb{A}) + x_1 x_2\, S_{35}(\mathbb{A})\Big) \equiv x_1^4 x_2^7 - x_1^7 x_2^4$$

which can be written more compactly $(x_1 - x_2) S_{2,\,35}(\mathbb{A}{-}\mathbb{X},\, \mathbb{A})$. In fact, the general remainder can always be written as a multi-Schur function in $\mathbb{A} - \mathbb{X}$, $\mathbb{A}$.

Ex. 3.8 Expanding $S_{1^i,\,n}(\mathbb{A}, (x{-}\mathbb{A}) + (\mathbb{A}{-}\mathbb{B})) \equiv \sum_{j=0}^{n-1} S_{1^i,\,n-j}(\mathbb{A}, \mathbb{A}{-}\mathbb{B})\, S^j(x{-}\mathbb{A})$, one finds that the desired matrix is

$$\left[(-1)^j S_{1^j,n-i}(\mathbb{A}, \mathbb{A} - \mathbb{B})\right]_{0 \le i,j \le n-1} \ .$$

Since $(-1)^j S_{1^j,n-i}(\mathbb{A}, \mathbb{A} - \mathbb{B}) = \sum_k S^{j-k}(-\mathbb{A})\, S^{n-i+k}(\mathbb{A} - \mathbb{B})$, one finally gets that the matrix factorizes into $\mathbb{S}_{n^n}(\mathbb{A} - \mathbb{B})\ \mathbb{S}_{0^n}(-\mathbb{A})$. ☐

Ex. 3.9

$$\begin{aligned} S^{n-1}(x-\mathbb{A}+a)\, S^n(x-\mathbb{B}) &= S^{2n-1}(x-\mathbb{A}+a-\mathbb{B}) \\ &\equiv S^{n-1}(x-\mathbb{A})S^n(a-\mathbb{B}) + S^{n-2}(x-\mathbb{A})S^{n+1}(a-\mathbb{B}) + \cdots = \end{aligned}$$
$$S^n(a-\mathbb{B})\Big(S^{n-1}(x-\mathbb{A})a^0 + S^{n-2}(x-\mathbb{A})a^1 + \cdots\Big) = S^n(a-\mathbb{B})\, S^{n-1}(x-\mathbb{A}+a)\,.$$

This gives another proof that the determinant of $Bez(\mathbb{A},\mathbb{B})$ is $\prod_a S^n(a-\mathbb{B}) = R(\mathbb{A},\mathbb{B})$.

Ex. 3.10 Using the notation of (3.6.2), one has for the successive remainders the expression :

$$f_r = \sum_{\mathbb{B}} \Delta(\mathbb{B})^2\, S^{n-r}(x+\mathbb{B}-\mathbb{A}) = \sum_{\mathbb{B}}\sum_j \Delta(\mathbb{B})^2\, S^{n-r-j}(\mathbb{B})\, S^j(x-\mathbb{A})\ .$$

Now, we have already expressed $\sum_{\mathbb{B}} S^{n-r-j}(\mathbb{B})$ as a determinant of power sums

$$\begin{vmatrix} \psi_0(\mathbb{A}) & \cdots & \psi_{r-2}(\mathbb{A}) & \psi_{n-j-1}(\mathbb{A}) \\ \vdots & & \vdots & \vdots \\ \psi_{r-1}(\mathbb{A}) & \cdots & \psi_{2r-3}(\mathbb{A}) & \psi_{n-j+r-2}(\mathbb{A}) \end{vmatrix}\ .$$

Taking into account the expansion of $S^j(x-\mathbb{A})$, one sees that the required coefficients can be expressed as determinants of powers sums multiplied by elementary symmetric functions.

For example, for $n=4$, writing S^i for $S^i(x-\mathbb{A})$, ψ_i for $\psi_i(\mathbb{A})$, one has $f_0 = S^4(x-\mathbb{A})$;

$$f_1 = S^3(x-\mathbb{A}^{der}) = S^3(x-\mathbb{A}+\mathbb{A}-\mathbb{A}^{der}) = \frac{1}{4}\Big(\psi_3 + \psi_2\, S^1 + \psi_1\, S^2 + \psi_0\, S^3\Big)$$

$$f_2 = \sum_{j=0}^{2} \begin{vmatrix} \psi_0 & \psi_{3-j} \\ \psi_1 & \psi_{4-j} \end{vmatrix}\ S^j(x-\mathbb{A}) = \begin{vmatrix} \psi_0 & \psi_1 S^2 + \psi_2 S^1 + \psi_3 \\ \psi_1 & \psi_2 S^2 + \psi_3 S^1 + \psi_4 \end{vmatrix}$$

$$f_3 = \sum_{j=0}^{1} \begin{vmatrix} \psi_0 & \psi_1 & \psi_{3-j} \\ \psi_1 & \psi_2 & \psi_{4-j} \\ \psi_2 & \psi_3 & \psi_{5-j} \end{vmatrix}\ S^j(x-\mathbb{A}) = \begin{vmatrix} \psi_0 & \psi_1 & \psi_2 S^1 + \psi_3 \\ \psi_1 & \psi_2 & \psi_3 S^1 + \psi_4 \\ \psi_2 & \psi_3 & \psi_4 S^1 + \psi_5 \end{vmatrix}$$

$$f_4 = \begin{vmatrix} \psi_0 & \psi_1 & \psi_2 & \psi_3 \\ \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \psi_2 & \psi_3 & \psi_4 & \psi_5 \\ \psi_3 & \psi_4 & \psi_5 & \psi_6 \end{vmatrix}$$

Ex. 3.11 Some caution is needed, because of the symmetry of the role played by $\mathbb{A}$, $\mathbb{B}$ in the Euclidean division. One has to start from $S_{1^{n-r},\,(r-1)^r}(\mathbb{A}-x, \mathbb{A}-\mathbb{A}^{der})$, instead of $S_{1^{n-1-r},\,(r+1)^r}(\mathbb{A}^{der}-x, \mathbb{A}^{der}-\mathbb{A})$. Now, just replace in the determinant expressing $S_{1^{n-r},\,(r-1)^r}(\mathbb{A}-x,\mathbb{A}-\mathbb{A}^{der})$ the complete functions $S^j(\mathbb{A}-\mathbb{A}^{der})$ by $\psi_j(\mathbb{A})/n$. Laplace expansion, according to the last r columns, of such a determinant will give a sum of products of determinants of power sums times polynomials $\pm S_{1^j}(\mathbb{A}-x) = S^j(x-\mathbb{A})$, in which one recognizes the sums obtained in the preceding exercise.

For example, for $n=4$, one has

$$4^{-1} f_1 = S_{111,0}(\mathbb{A}-x, \mathbb{A}-\mathbb{A}^{der}) = 4^{-1} \sum (-1)^i \Lambda^{3-i}(\mathbb{A}-x)\, \psi_i(\mathbb{A})$$

$$4^{-2} f_2 = S_{11,11}(\mathbb{A}-x, \mathbb{A}-\mathbb{A}^{der}) = 4^{-2} \begin{vmatrix} S^1 & S^2 & \psi_3 & \psi_4 \\ S^0 & S^1 & \psi_2 & \psi_3 \\ \cdot & S^0 & \psi_1 & \psi_2 \\ \cdot & \cdot & \psi_0 & \psi_1 \end{vmatrix}$$

$$4^{-3} f_3 = S_{1,222}(\mathbb{A}-x, \mathbb{A}-\mathbb{A}^{der}) = 4^{-3} \begin{vmatrix} S^1 & \psi_3 & \psi_4 & \psi_5 \\ S^0 & \psi_2 & \psi_3 & \psi_4 \\ \cdot & \psi_1 & \psi_2 & \psi_3 \\ \cdot & \psi_0 & \psi_1 & \psi_2 \end{vmatrix}$$

$$4^{-4} f_4 = S_{3333}(\mathbb{A}-\mathbb{A}^{der}) = 4^{-4} \begin{vmatrix} \psi_0 & \psi_1 & \psi_2 & \psi_3 \\ \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \psi_2 & \psi_3 & \psi_4 & \psi_5 \\ \psi_3 & \psi_4 & \psi_5 & \psi_6 \end{vmatrix}$$

Ex. 3.12 The developed expression of $\phi(n,a,b)$ is

$$\phi(n,a,b) = \sum (-1)^i x^{n-i} \binom{n}{i} \frac{\Lambda^i(a)}{\Lambda^i(b)} .$$

From it one gets with care or computer, but without difficulty, that the respective remainders of $\phi(n,a,b)$ are:

$$\begin{aligned}
\text{by} \quad \phi(n-1,a,b) &= (n-1)\frac{a(a-b)}{b^2(b-1)}\phi(n-2,a-1,b-2) \\
\text{by} \quad \phi(n-1,a+1,b) &= (n-1)\frac{(a-n+1)(a-b+1)}{b^2(b-1)}\phi(n-2,a,b-2) \\
\text{by} \quad \phi(n-1,a,b-1) &= (n-1)\frac{a(b-n+1)(a-b+1)}{b(b-1)^2(b-2)}\phi(n-2,a-1,b-3) \\
\text{by} \quad \phi(n-1,a-1,b) &= (n-1)\frac{(a-1)(a-b+5)}{b^2(b-1)}\phi(n-2,a-2,b-2) \\
\text{by} \quad \phi(n-1,a-1,b-1) &= (n-1)\frac{(a-1)(b-n)(a-b)}{b(b-1)^2(b-2)}\phi(n-2,a-2,b-3) \\
\text{by} \quad \phi(n-1,a-1,b-2) &= (n-1)\frac{(a-1)(a+1-b)(b-n-1)}{(b-1)(b-2)^2(b-3)}\phi(n-2,a-2,b-4)
\end{aligned}$$

Ex. 3.13 One recognizes that $(n-1)S^{n-1}(x-\mathbb{B})$ is the derivative of $S^n(x-\mathbb{A})$. Therefore $S_{(n-1)^n}(\mathbb{A}-\mathbb{B})$ is the discriminant of $S^n(x-\mathbb{A})$, and the other Schur functions to compute are its multiple by $\left(\Lambda^{n-1}(\mathbb{A})\right)^{k-n+1}$.

B.4

Ex. 4.2 Let us write $I := [i_1, i_2, .., i_k]$ for a monomial $b_1^{i_1} b_2^{i_2} \cdots b_k^{i_k}$ in the expansion. By recursion, one sees that its coefficient is

$$\binom{i_1+i_2-1}{i_2}\binom{i_2+i_3-1}{i_3}\cdots\binom{i_{k-1}+i_k-1}{i_k}$$

(in particular, if b_k appears in a monomial, all other b_j, $j \le k$ appear in the same monomial) □

For example, $b_1^2 b_2^6 b_3^4$ appears as coefficient of z^{12} with coefficient $\binom{7}{6}\binom{9}{3} = 588$. Notice that taking the Stieltjes parameters $b_1 = S^1(\mathbb{A}), b_2 = -S^2(\mathbb{A})/S^1(\mathbb{A}),\ b_3 =$

$S_{22}(\mathbb{A})/S^{12}(\mathbb{A}),\ldots$, gives the expansion of $\lambda_z(\mathbb{A})$ in the rational basis of Schur functions indexed by partitions $[k^k]$, and $[(k\pm 1)^k]$. Thus

$$\Lambda^4 = S^{1111} - 3S^{112} + 3S^{22} - 2S_{2,2} - \frac{S^{222}}{S^{11}} + 2\frac{S^2 S_{22}}{S^{11}} - \frac{S_{22}^2}{S^{112}} + \frac{S_{33}}{S^2} \ .$$

Touchard [**171**] gives a development without multiplicities, in terms of words in $b_1, b_2, \ldots$, the letters not commuting any more.

For example, the third Taylor coefficient is written by Touchard as

$$b_1b_1b_1 + b_1b_1b_2 + b_1b_2b_1 + b_1b_2b_2 + b_1b_2b_3 \ .$$

Ex. 4.3 We have given the expression of the parameters in a continued fraction expansion of the type $1 + \frac{b_2 z}{1+b_3 z..}$ of a series $\sigma_z(\mathbb{A})$. None of the three required fractions is exactly of this type and we leave it to the reader to find the minor adaptations necessary.

All required parameters can still be expressed in terms of Schur functions indexed by rectangular partitions. In the case of a power $(1+z)^\alpha$, we know the explicit values of all Schur functions $S_J(\alpha)$ and this gives the first expansion. The coefficients in an expansion of the type $1 + \frac{b_2 z}{1+b_3 z/\cdots}$ are $b_2 = \alpha, b_3 = (1-\alpha)/2, b_4 = (1+\alpha)/6, b_5 = (2-\alpha)/6, \ldots, b_{2n} = (n-1+\alpha)/(4n-2), b_{2n+1} = (n-\alpha)/(4n-2), \ldots$. Another classical expansion is

$$(1+z)^\alpha = \cfrac{1}{1 + \cfrac{(0-\alpha)z}{1 + \cfrac{1(1+\alpha)z}{2 + \cfrac{1(1-\alpha)z}{3 + \cfrac{2(2+\alpha)z}{4 + \cfrac{2(2-\alpha)z}{\ddots}}}}}}$$

On the other hand, $e^z = \sigma_z(\mathbb{A})$, with $\mathbb{A}$ such that $S^n(\mathbb{A}) = 1/n!$. This is the specialization of alphabets which corresponds to dimensions of representations of the symmetric group: one has $S_{k^n}(\mathbb{A}) = (kn)!/(1\cdots k)(2\cdots(k{+}1))/(n\cdots(k{+}n{-}1))$.

Finally, $-\log(1-z) = z\sigma_z(\mathbb{A})$, with $S^k(\mathbb{A}) = 1/(k{+}1)$. The Hankel determinants that one has to evaluate are in that case specializations of Cauchy determinant $\left|1/(x_i+y_j)\right|$, the x_i's being consecutive integers, as well as the y_j's.

Ex. 4.4 More general computation in next exercise.

Ex. 4.5 Since $S^i(\mathbb{A}) = (1{-}q)\cdots(1{-}q^{i-1})S^i\big((a{-}b)(1{-}q)^{-1}\big)$, with a, b, q of rank 1, one can use the known values of $S_I\big((a{-}b)(1{-}q)^{-1}\big)$. In final,

$$\sigma_z(\mathbb{A}) = 1 + \cfrac{z(a{-}b)}{1 + \cfrac{z(bq{-}a)}{1 + \cfrac{zbq(q{-}1)}{1 + \cfrac{zq(bq^2{-}a)}{1 + \cfrac{zbq^2(q^2{-}1)}{1 + \ddots}}}}} \ ,$$

the parameters being

$$\ldots,\ q^{i-1}(bq^i - a),\ bq^i(q^i - 1),\ \ldots$$

Ex. 4.6 Expanding the determinants according to their last column, one gets the recursion between successive convergents and a_n, b_n. For example, the quotient of the two determinants

$$\begin{vmatrix} a_1 & b_2 & 0 & 0 \\ -1 & a_2 & b_3 & 0 \\ 0 & -1 & a_3 & b_4 \\ 0 & 0 & -1 & a_4 \end{vmatrix} \quad \& \quad \begin{vmatrix} a_2 & b_3 & 0 \\ -1 & a_3 & b_4 \\ 0 & -1 & a_4 \end{vmatrix}$$

is equal to the fourth convergent

$$\frac{P_4}{Q_4} = \frac{(a_1a_2a_3a_4 + a_1a_2b_4 + a_1b_3a_4 + b_2a_3a_4 + b_2b_4)}{a_2a_3a_4 + a_2b_4 + b_3a_4}$$

Instead of expanding determinants, one can have recourse to the now familiar Plücker relations and read $\begin{bmatrix} P_4 \\ Q_4 \end{bmatrix} = \begin{bmatrix} P_3 & P_2 \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} a_4 \\ b_4 \end{bmatrix}$ from the maximal minors of

$$\begin{bmatrix} 1 & 0 & 0 & a_1 & b_2 & 0 & 0 \\ 0 & 0 & 0 & -1 & a_2 & b_3 & 0 \\ 0 & 1 & 0 & 0 & -1 & a_3 & b_4 \\ 0 & 0 & 1 & 0 & 0 & -1 & a_4 \end{bmatrix} .$$

Ex. 4.8 Sylvester determinant given in the preceding exercise is in that case a minor of index $[1, 1, \ldots]$ of the matrix $\mathbb{S}(-\mathbb{A})$, up to signs which cause no problem.

Ex. 4.9 The terms $a_{2n-1}a_{2n}$ coincide with Wroński's parameters β_n. On the other hand, one needs to check that $\zeta_n = a_{2n} + a_{2n+1}$. In fact, both sides can be rewritten

$$\zeta_n = a_{2n} + a_{2n+1} = \frac{S_{(n-1)^{n-3}}S_{n^n} - S_{(n-2)^{n-2}}S_{(n+1)^{n-1}}}{S_{(n-1)^{n-2}}S_{n^{n-1}}} .$$

For example,

$$S_{12}/S_2 - S_1 = \zeta_1 = a_2 + a_3 = -S_2/S_1 + +S_{22}/S_1S_2 ,$$

$$S_{133}/S_{33} - S_{12}/S_2 = \zeta_2 = a_4 + a_5 = -S_1S_{33}/S_2S_{22} + S_2S_{333}/S_{33}S_{22} .$$

Ex. 4.12 Taking the interpretation given by theorem (8.8.1), one has to specialize $\zeta_0 = 1 = \zeta_1 = \cdots$, $\beta_1 = -1 = \beta_2 = \cdots$. One could more generally specialize all β_i's to the same value $-b$. In that case, the generating function is given by the continued fraction

$$F = \cfrac{1}{1 - zx - \cfrac{bx^2}{1 - zx - \cfrac{bx^2}{1 - zx - \cdots}}} .$$

It satisfies the functional equation

$$1/F = 1 - zx - bx^2F$$

and therefore, is equal to

$$F = \left(1 - zx - \sqrt{1 - 2zx + (z^2 - 4b)x^2}\right)/2bx^2 = 1 + xz + (b + z^2)x^2 + (3bz + z^3)x^3$$
$$+ (2b^2 + 6bz^2 + z^4)x^4 + (10b^2z + 10bz^3 + z^5)x^5 + \cdots ,$$

the case $b = 1, z = 1$ giving the generating function of the number of Motzkin paths $1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 51x^6 + 127x^7 + \cdots$.

B.5

Ex. 5.1 Adapting Formula (5.1.7), one considers the determinant

$$\begin{vmatrix} S^m(\mathbb{A}) & \cdots & S^{m+n-1}(\mathbb{A}) & x^0\sigma_x(\mathbb{A}) \\ \vdots & & \vdots & \vdots \\ S^{m-n}(\mathbb{A}) & \cdots & S^{m-1}(\mathbb{A}) & x^n\sigma_x(\mathbb{A}) \end{vmatrix} .$$

Subtracting to each row x^{-1} times the next one makes it factorize into $x^n S_{m^n}(A - x^{-1})\,\sigma_x(\mathbb{A})$.

On the other hand, expanding $\sigma_x(\mathbb{A})$ in the last column of the determinant, one gets

$$\sum_{i=-n}^{m-n-1} S_{m^n,i}(\mathbb{A})x^{n+i} + \sum_{i=m-n}^{m-1} S_{m^n,i}(\mathbb{A})x^{n+i} + \sum_{i\geq 0} S_{m^n,m+i}(\mathbb{A})x^{m+n+i} .$$

Each determinant in the middle sum is null, and therefore

$$P(x) = x^n S_{m^n}(A - x^{-1}) \qquad \& \qquad Q(x) = x^n \sum_{i=-n}^{m-n-1} S_{m^n,i}(\mathbb{A})x^{n+i}$$

is the answer, the difference being also expressed as a sum of Schur functions than one can put into a single determinant.

Frobenius' problem was in fact to find the Padé approximant of type $[m{-}1/n]$ of $\sigma_x(\mathbb{A})$ and already received many solutions much earlier (from Cauchy, Jacobi, Rosenhain in particular).

Ex. 5.2 One has to compute the $S_{(n-1)^n}(\mathbb{B})$ and $S_{n^n}(\mathbb{B})$, $n \geq 0$, with $\mathbb{B}$ such that

$$S^n(\mathbb{B}) = S^n(\mathbb{A} + ny) = \sum_i \binom{n}{i} x^i S^{n-i}(\mathbb{A}) .$$

From Lemma 1.4.1, one sees that

$$S_{(n-1)^n}(\mathbb{B}) = S_{(n-1)^n}(\mathbb{A}) \quad \& \quad S_{n^n}(\mathbb{B}) = S_{n^n}(\mathbb{A} + y) = \sum_{i=0}^{n} y^i S_{(n-1)^i,n^{n-i}}(\mathbb{A}) ,$$

and therefore, in terms of functions of $\mathbb{A}$,

$$\sigma_z(\mathbb{B}) = \cfrac{1}{1 - \cfrac{z(y+S^1)}{1 + \cfrac{zS_{11}/(y+S^1)}{1 - \cfrac{z(y^2 + yS_{12}/S_{11} + S_{22}/S_{11})/(y+S^1)}{\ddots}}}}$$

Notice that, more generally, for $I \in \mathbb{N}^n$, then

$$S_{(n-1)^n+I}(\mathbb{B}) = S_{(n-1)^n+I}(\mathbb{A} + i_1 y, \mathbb{A} + (i_2{+}1)y, \ldots, \mathbb{A} + (i_n{+}n{-}1)y) .$$

Ex. 5.3 Since $P = \prod_{a\in\mathbb{A}}\prod_{\zeta\in\Omega}(1 - a\zeta) = \prod\prod(\zeta - a)$, it is the resultant (in x) of the polynomials $\prod(x + a)$ and $\prod(x - \zeta) = x^{n+1} - 1$.

On the other hand, since $\sum x^i S^i(\Omega) = (1 - x^{n+1})^{-1}$, one has, for $0 \leq k \leq n$,

$$\begin{cases} S^{n-k}(\Omega - \mathbb{A}) & = S^{n-k}(-\mathbb{A}) \\ S^{n+k}(\Omega - \mathbb{A}) & = S^{k-1}(-\mathbb{A}) \end{cases} ,$$

and one recognizes in the given determinant $S_{n^{n+1}}(\Omega - \mathbb{A}) = R(\Omega, \mathbb{A})$.

One can give other expressions of this special resultant. For example, $R(\Omega, \mathbb{A}) = \prod_{a\in\mathbb{A}}(1 - a^{n+1})$, and therefore it is equal to

$$\Psi^{n+1}\left(\prod(1-a)\right) = \Psi^{n+1}\left(\sum \pm\Lambda^i(\mathbb{A})\right) = \sum \pm\Psi_{(n+1)^i}(\mathbb{A}) \ .$$

One can also obtain this expression from Cauchy formula (1.5.3).

Ex. 5.4 One has to compute the functions $S_{(n-1)^n}(\mathbb{A})$ and $S_{n^n}(\mathbb{A})$, for $\mathbb{A}$ such that $S^n(\mathbb{A}) = H_n(x)$. The values of the first functions can be found, among a wealth of evaluations of determinants, in the article of Krattenthaler [**88**].

To compute the second functions, one introduces the positive Hermite polynomials $H_n^+ := H_n(x\sqrt{-1})(\sqrt{-1})^{-n}$. The law required from the reader is

$$\gamma_1 = H_1^+ \ , \ \gamma_2 = -2H_0^+/H_1^+ \ , \ \gamma_3 = H_2^+/H_1^+ \ , \ \gamma_4 = -4H_1^+/H_2^+ \ ,$$
$$\gamma_{2n+1} = H_{n+1}^+/H_n^+ \qquad \& \qquad \gamma_{2n} = -2nH_{n-1}^+/H_n^+ \ ,$$

and one gets in total

$$\sum z^n H_n(x) = \cfrac{1}{1 - \cfrac{2zx}{1 + \cfrac{z/x}{1 - \cfrac{z(2x+1/x)}{1 - \cfrac{4zx/(2x^2+1)}{\ddots}}}}}$$

B. Leclerc [**121**](formula 33) shows more generally that Hankel determinants of Hermite polynomials can be expressed as Wrońskians of positive Hermite polynomials.

Ex. 5.5 The determinants S_{n^n} and $S_{(n-1)^n}$ are of Cauchy type $\left|(a-b)^{-1}\right|$, for $\mathbb{A} = \{3, 5, 7, \ldots\}$, or $\mathbb{A} = \{1, 3, 5, \ldots\}$, and $\mathbb{B} = \{0, -2, -4, \ldots\}$. They are in consequence immediate to evaluate.

One finds

$$1 + z/3 + z^2/5 + \cdots = \cfrac{1}{1 - \cfrac{1^2z/1.3}{1 - \cfrac{2^2z/3.5}{1 - \cfrac{3^2z/5.7}{1 - \cfrac{4^2z/7.9}{\ddots}}}}}$$

the parameters being equal to $n^2(2n-1)^{-1}(2n+1)^{-1}$, $n = 1, 2, \ldots$.

Ex. 5.6 Writing $f_\alpha(z) = \sigma_z(\mathbb{A})$, one has $\Lambda^i(\mathbb{A}) = (-1)^i\alpha/i$. The determinants $\Lambda_{n^n}(\mathbb{A})$, $\Lambda_{(n+1)^n}(\mathbb{A})$ are again of Cauchy type. The continued fraction expansion of the type (5.3.4) is

$$f_\alpha(z) = \cfrac{1}{1 + \cfrac{z\alpha}{1 - \cfrac{z/2}{1 - \cfrac{z/6}{1 - \cfrac{z/3}{\ddots}}}}} \,,$$

where the general parameters are

$$\gamma_{2n} = n/(4n-2) \qquad \& \qquad \gamma_{2n+1} = n/(4n+2) \ .$$

Ex. 5.7 This type of alphabet is encountered in relation with the function arcsin (cf. Sylvester [**166**], art.35). Using that $\arcsin(x)$ is the integral of $(1-x^2)^{-1/2}$, one finds, putting $x = z^2$, that

$$\sigma_z(\mathbb{A}+1/2) = \frac{\arcsin(x)}{x\sqrt{1-x^2}}$$
$$= 1 + \frac{2}{1\cdot 3}z^1 + \frac{2\cdot 4}{1\cdot 3\cdot 5}z^2 + \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7}z^3 + \frac{2\cdot 4\cdot 6\cdot 8}{1\cdot 3\cdot 5\cdot 7\cdot 9}z^4 + \cdots$$

The required Hankel determinants for this series are easy to evaluate, contrary to those for $\sigma_z(\mathbb{A})$, and one finds

$$\sigma_z(\mathbb{A}+1/2) = \cfrac{1}{1 - \cfrac{2z/3}{1 - \cfrac{2z/15}{1 - \cfrac{12z/35}{1 - \cfrac{4z/21}{1 - \cfrac{10z/33}{\ddots}}}}}}$$

the general term being $2\lceil i/2\rceil\,(2\lfloor (i+1)/2\rfloor - 1)\,(2i-1)^{-1}(2i+1)^{-1}$.

Ex. 5.8 The first expansion is obtained from the continued fraction for $\sigma_z(A)$ by writing that $1/\sigma_z(\mathbb{A}) = \lambda_{-z}(\mathbb{A}))$:

$$\Lambda^1 - z\Lambda^2 + z^2\Lambda^3 - \cdots = (1-\lambda_{-z}(\mathbb{A}))/z \ .$$

The second continued fraction is obtained from $1/(1-\lambda_{-z\gamma_1}(\mathbb{A}))$. Instead of finding the complete functions $S^i(\mathbb{A})$ as Taylor coefficients, one has their images under the shift $\Lambda^i \to \Lambda^{i+1}$. More precisely, let $h_n^+ = \left|e^+_{j-i+1}\right|_{1\le i,j\le n}$, $e_i^+ = \Lambda^{i+1}$, $i \ge 0$, $e_i^+ = 0$, $i < 0$. For example, $h_1^+ = \Lambda^2$, $h_2^+ = \Lambda^{22} - \Lambda^{13} = S_{22}$, $h_3^+ = S_{33} - S_{222}$, $h_4^+ = S_{44} - 2S_{233} + S_{2222} \ldots$. Then the Taylor expansion for the second continued fraction is

$$h_1^+ + h_2^+ z + h_3^+ z^2 + \cdots$$

Ex. 5.10 All the Schur functions of $\mathbb{A} = (1-q)^{-1}$ are known. On the other hand, for $\mathbb{B}$ such that $S^i(\mathbb{B}) = (1-q)/(1-q^{i+1})$, one sees that a Schur function $S_I(\mathbb{B})$, when I such that $I \ge (\ell(I)-1)^{\ell(I)}$, is a specialization of Cauchy's determinant $\left|(1-q)/(1-x_iy_j)\right|$, the variables x_i, y_j being specialized to some powers of q.

In final, one gets

$$1+\frac{z}{1-q}+\frac{z^2}{1-q^2}+\cdots=\cfrac{1}{1-\cfrac{z/(1-q)}{1+\cfrac{qz/(1-q^2)}{1-\cfrac{qz(1-q)/(1-q^2)(1-q^3)}{1+\ddots}}}}\,,$$

with parameters

$$\gamma_{2i+1}=q^i\frac{1-q^i}{(1-q^{2i})(1-q^{2i+1})}\qquad\&\qquad \gamma_{2i}=q^{2i-1}\frac{1-q^i}{(1-q^{2i-1})(1-q^{2i})}\,,$$

and the q-logarithm leads to

$$z+\frac{z^2}{1+q}+\frac{z^3}{1+q+q^2}+\cdots=\cfrac{z}{1-\cfrac{z/(1+q)}{1-\cfrac{zq/(1+q)((1+q+q^2)}{1-\ddots}}}\,,$$

with parameters

$$\gamma_{2i-1}=q^{i-1}\frac{(1-q^i)^2}{(1-q^{2i-1})(1-q^{2i})}\qquad\&\qquad \gamma_{2i}=q^i\frac{(1-q^i)^2}{(1-q^{2i})(1-q^{2i+1})}\,.$$

More determinants related to the q-exponential and the q-logarithm will be found in the article of Bornwein [**21**].

B.6

Ex. 6.1 Of course, one also needs z in the determinants! One has to separate the two alphabets in the expressions (6.1.2) or (6.1.3), or take the appropriate determinantal expressions of the two resultants in (6.1.1).

To Hadamard [**62**] are ascribed the two following determinants of order $\alpha+\beta+2$:

$$Q:=\begin{vmatrix}\Lambda^0(\mathbb{B}) & \cdots & \Lambda^{-\alpha}(\mathbb{B}) & \Lambda^{-\beta}(\mathbb{A}) & \cdots & \Lambda^0(\mathbb{A})\\ \vdots & & \vdots & \vdots & & \vdots\\ \Lambda^{\alpha+\beta}(\mathbb{B}) & \cdots & \Lambda^{\beta}(\mathbb{B}) & \Lambda^{\alpha}(\mathbb{A}) & \cdots & \Lambda^{\alpha+\beta}(\mathbb{A})\\ 1 & \cdots & (-z)^\alpha & 0 & \cdots & 0\end{vmatrix}\;;$$

$$P=\begin{vmatrix}\Lambda^0(\mathbb{B}) & \cdots & \Lambda^{-\alpha}(\mathbb{B}) & \Lambda^{-\beta}(\mathbb{A}) & \cdots & \Lambda^0(\mathbb{A})\\ \vdots & & \vdots & \vdots & & \vdots\\ \Lambda^{\alpha+\beta}(\mathbb{B}) & \cdots & \Lambda^{\beta}(\mathbb{B}) & \Lambda^{\alpha}(\mathbb{A}) & \cdots & \Lambda^{\alpha+\beta}(\mathbb{A})\\ 0 & \cdots & 0 & (-z)^\beta & \cdots & 1\end{vmatrix}\,.$$

Expanding them according to their first $\alpha+1$ columns, one indeed recognizes the expression given in (6.1.2):

$$Q=\pm z^\alpha\Lambda_{(\beta+1)^\alpha}(\mathbb{B}-\mathbb{A}+\frac{1}{z})\quad\&\quad P=\pm z^\beta\Lambda_{(\alpha+1)^\beta}(\mathbb{A}-\mathbb{B}+\frac{1}{z})\,.$$

For example, for the approximant [2/3] (i.e. $\alpha = 3, \beta = 2$), one finds

$$\pm z^3 \Lambda_{333}(\mathbb{B}-\mathbb{A}+\frac{1}{z}) = Q = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \Lambda^1(\mathbb{B}) & 1 & 0 & 0 & 0 & 1 & \Lambda^1(\mathbb{A}) \\ \Lambda^2(\mathbb{B}) & \Lambda^1(\mathbb{B}) & 1 & 0 & 1 & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) \\ 0 & \Lambda^2(\mathbb{B}) & \Lambda^1(\mathbb{B}) & 1 & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) & \Lambda^3(\mathbb{A}) \\ 0 & 0 & \Lambda^2(\mathbb{B}) & \Lambda^1(\mathbb{B}) & \Lambda^2(\mathbb{A}) & \Lambda^3(\mathbb{A}) & 0 \\ 0 & 0 & 0 & \Lambda^2(\mathbb{B}) & \Lambda^3(\mathbb{A}) & 0 & 0 \\ 1 & -z & z^2 & -z^3 & 0 & 0 & 0 \end{vmatrix},$$

$$\pm z^2 \Lambda_{44}(\mathbb{A}-\mathbb{B}+\frac{1}{z}) = P = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \Lambda^1(\mathbb{B}) & 1 & 0 & 0 & 0 & 1 & \Lambda^1(\mathbb{A}) \\ \Lambda^2(\mathbb{B}) & \Lambda^1(\mathbb{B}) & 1 & 0 & 1 & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) \\ 0 & \Lambda^2(\mathbb{B}) & \Lambda^1(\mathbb{B}) & 1 & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) & \Lambda^3(\mathbb{A}) \\ 0 & 0 & \Lambda^2(\mathbb{B}) & \Lambda^1(\mathbb{B}) & \Lambda^2(\mathbb{A}) & \Lambda^3(\mathbb{A}) & 0 \\ 0 & 0 & 0 & \Lambda^2(\mathbb{B}) & \Lambda^3(\mathbb{A}) & 0 & 0 \\ 0 & 0 & 0 & 0 & z^2 & -z & 1 \end{vmatrix}.$$

Ex. 6.2 (Littlewood, [**131**], p.102). A function of the type $Q(z) = z^n S_J(\mathbb{A} - 1/z)$, with J a partition in $\mathbb{N}^n$ will do. Indeed, let a be the first letter of $\mathbb{A}$ and π_ω the symmetrizing operator $a^i \mapsto S^i(\mathbb{A})$, $\forall i \geq 0$. Then multiplying the last column of $S_{J,0}(\mathbb{A}, 1/z) = S_J(\mathbb{A} - 1/z)$ by $z^n/(1 - az)$, and taking the image under π_ω, one gets

$$Q(z)\sigma_z(\mathbb{A}) = \sum_{i=0}^{\infty} x^i S_{J,i-n}(\mathbb{A}) \ .$$

Now, $S_{J,i-n}$ is null for

$$i \in \{j_1,\ j_2{+}1,\ j_n{+}n{-}1\}$$

and therefore one has just to take J such that the above set is K.

For example,

$$z^3 S_{236}(\mathbb{A} - 1/z)\,\sigma_z(\mathbb{A}) = z^3 S_{236}(\mathbb{A} - 1/z)\frac{1}{1 - za}\,\pi_\omega =$$

$$\begin{vmatrix} S^2(\mathbb{A}) & S^4(\mathbb{A}) & S^7(\mathbb{A}) & 1 + az + a^2z^2 + a^3z^3 + a^4z^4 + \cdots \\ S^1(\mathbb{A}) & S^3(\mathbb{A}) & S^6(\mathbb{A}) & z + az^2 + a^2z^3 + a^3z^4 + \cdots \\ S^0(\mathbb{A}) & S^2(\mathbb{A}) & S^5(\mathbb{A}) & + z^2 + az^3 + a^2z^4 + \cdots \end{vmatrix} \pi_\omega =$$

$$S_{235,-3}(\mathbb{A}) + zS_{235,-2}(\mathbb{A}) + z^2 S_{235,-1}(\mathbb{A}) + z^3 S_{235,0}(\mathbb{A}) + z^4 S_{235,1}(\mathbb{A}) + \cdots$$

and $S_{235,-1} = 0 = S_{2351} = S_{2354}$.

Ex. 6.3 Write $[\beta/\alpha] = \sigma_z(\mathbb{A} - \mathbb{B})$, $card(\mathbb{A}) = \alpha$, $card(\mathbb{B}) = \beta$. Then, by definition, $S^k(\mathbb{A}-\mathbb{B}) = S^k(\mathbb{D})$, $k \leq \alpha + \beta$. But for any k,

$$S^k(\mathbb{A}-\mathbb{B}) = S_{k;0^k}(-\mathbb{B}, -\mathbb{A}) \ ,$$

and all the entries of this determinant are Schur functions of $\mathbb{D}$, being the coefficients of the numerator or denominator of $[\beta/\alpha]$. This provides an answer, though not optimal.

Let us look more closely at the first non trivial coefficient. One has

$$S_{\alpha+\beta+1,\beta^\alpha}(\mathbb{A}-\mathbb{B}) = \pm S_{(\beta+1)^{\alpha+1}}(\mathbb{A}-\mathbb{B}) = 0 \ ,$$

because it is the resultant of $\mathbb{A} + 0$ and $\mathbb{B} + 0$ (which have in common the element 0). The last determinant differ from $S_{(\beta+1)^{\alpha+1}}(\mathbb{D})$ only in the first entry $S_{\alpha+\beta+1}$, and therefore, one gets

$$S_{\alpha+\beta+1}(\mathbb{A}-\mathbb{B}) = S_{\alpha+\beta+1}(\mathbb{D}) - (-1)^{\alpha} S_{(\beta+1)^{\alpha+1}}(\mathbb{D}) / S_{\beta^{\alpha}}(\mathbb{D}) \ .$$

More generally, one would use the nullity of $S_{\alpha+\beta+k,\beta^{\alpha}}(\mathbb{A}-\mathbb{B})$, $k \geq 1$ (which just says that $\left(S^i(\mathbb{A}-\mathbb{B})\right)_{i\geq 0}$ is a recursive sequence of characteristic polynomial $S^{\alpha}(1-z\mathbb{A})\big)$, because the $S^i(-\mathbb{A}) S_{\beta^{\alpha}}(\mathbb{A}-\mathbb{B}) = S_{(\beta+1)^{\alpha}/1^{\alpha-i}}(\mathbb{A}-\mathbb{B})$ are the cofactors of the elements of the first column of $S_{\alpha+\beta+k,\beta^{\alpha}}(\mathbb{A}-\mathbb{B})$.

Ex. 6.4 Images of $S^N(y_i - \mathbb{D})$ under products of divided differences are still complete functions, and therefore the entries of the two determinants under examination are complete functions. The determinant expressing $Q(z)$ is the image of $S_{(N-\beta)^{\alpha},0}(\mathcal{Y}-\mathbb{D},0)$, that we have obtained in (6.4.2), after subtracting $\mathcal{Y}_{\alpha-1},\ldots,\mathcal{Y}_{-1}$ in the alphabets of the successive rows, and $(\mathcal{Y}-\mathcal{Y}_{\beta+2}), (\mathcal{Y}-\mathcal{Y}_{\beta+3}), \ldots$ in columns $1,2,\ldots,\alpha-1$.

As for the second determinant, it is the image, under the same transformation, of $S_{N-\beta-1,(N-\beta-1)^{\alpha}}(\mathcal{Y}+z-\mathbb{D}, \mathcal{Y}-\mathbb{D}) = S_{(N-\beta-1)^{\alpha+1}}(\mathcal{Y}+z-\mathbb{D})$. Equality $(\star)$ is now recognized to be one possible definition of the remainder of degree β.

Ex. 6.5 For degree reasons, up to powers of $x = 1/z$, each of the differences is a rational function with numerator of degree 1 in x. To determine them, one needs only to compute two (appropriate) terms in the expansion of these rational functions.

The result is, up to powers of x,

$$[\beta/\alpha] - [\beta{-}2/\alpha] = \frac{S_{\beta^{\alpha+1}}(\mathbb{A}) S_{(\beta-1)^{\alpha}}(\mathbb{A}) + x S_{(\beta-1)^{\alpha+1}}(\mathbb{A}) S_{\beta^{\alpha}}(\mathbb{A})}{S_{(\beta-1)^{\alpha}}(\mathbb{A}-x) S_{(\beta+1)^{\alpha}}(\mathbb{A}-x)} \ ,$$

$$[\beta/\alpha] - [\beta/\alpha{-}2] = \frac{S_{\beta^{\alpha-1}}(\mathbb{A}) S_{(\beta+1)^{\alpha}}(\mathbb{A}) - x S_{(\beta+1)^{\alpha-1}}(\mathbb{A}) S_{\beta^{\alpha}}(\mathbb{A})}{S_{(\beta+1)^{\alpha}}(\mathbb{A}-x) S_{(\beta+1)^{\alpha-2}}(\mathbb{A}-x)} \ .$$

Ex. 6.6 For a change,we shall use Sylvester's relation, and check the statement only for $n=3, k=4$. First, because of homogeneity, one can restrict to $x=1$. Consider the following matrix, writing S_i for $S_i(\mathbb{A})$,

$$\begin{bmatrix} S_4 & S_5 & S_6 & S_7 & 1 \\ S_3 & S_4 & S_5 & S_6 & 1 \\ S_2 & S_3 & S_4 & S_5 & 1 \\ S_1 & S_2 & S_3 & S_4 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

Taking $M = \begin{bmatrix} S_5 & S_6 & S_7 \\ S_4 & S_5 & S_6 \\ S_3 & S_4 & S_5 \end{bmatrix}$ as a pivot, and recognizing, after subtracting 1 in the alphabets of the first, second and third rows, that the full determinant is equal to $-S_{444}(\mathbb{A}-1)$, one obtains the identity

$$\begin{aligned} -\det(M) S_{444}(\mathbb{A}-1) &= \begin{vmatrix} M & \\ & 1 \end{vmatrix} \begin{vmatrix} & M \\ 0 & \end{vmatrix} - \begin{vmatrix} & M \\ S_1 & \end{vmatrix} \begin{vmatrix} M & \\ & 0 \end{vmatrix} , \\ S_{555}(\mathbb{A}) S_{444}(\mathbb{A}-1) &= S_{555}(\mathbb{A}-1) S_{444}(\mathbb{A}) - S_{4444}(\mathbb{A}) S_{55}(\mathbb{A}-1) \ . \end{aligned}$$

Ex. 6.7 For $m \geq card(\mathbb{B}) = \beta$, the approximant is

$$[m/n] = (-1)^n x^{n-m} S_{m^{n+1}}(\mathbb{A}-\mathbb{B}+x) / S_{(m+1)^n}(\mathbb{A}-\mathbb{B}-x) \ , \ x = 1/z \ .$$

Both Schur functions factorize, according to Proposition 1.4.3 :

$$[m/n] = (-1)^n x^{n-m} \frac{R(\mathbb{A},\mathbb{B})R(x,\mathbb{B})(x\Lambda\mathbb{A})^{m-\beta}}{R(\mathbb{A},\mathbb{B})R(\mathbb{A},x)(\Lambda\mathbb{A})^{m+1-\beta-1}} = \frac{\lambda_{-z}(\mathbb{B})}{\lambda_{-z}(\mathbb{A})} ,$$

and thus the Padé approximant is constant, and equal to $f(z)$, for $m \geq \beta$. We refer to [**11**], p.282, for the general case of an holomorphic function.

B.7

Ex. 7.2 One needs only to test on the basis $f_i \in \{1, -y\}$. Then $1\pi = 1$, $1\partial = 0$, $-y\pi = 0$, $-y\partial = 1$, and the required formula follows immediately in that case.

For example, for $r = 3$, one has 8 words. Because $S_{10} = 0$, the words with one occurrence of ∂ give a zero contribution, and there remains only 5 terms (rewriting $S_{1,-1} = -1$) :

$$f_1f_2f_3\pi = f_1\pi f_2\pi f_3\pi - \big(f_1\pi f_2\partial f_3\partial + f_1\partial f_2\pi f_3\partial + f_1\partial f_2\partial f_3\pi\big)\, S_{11}(x+y) + f_1\partial f_2\partial f_3\partial S_{12} .$$

Ex. 7.3 Taking the alphabet $\mathbb{B} = \{a_1, qa_2, \ldots, q^{n-1}a_n\}$, one sees that δ_i is a divided difference relative to $\mathbb{B}$, and there is nothing new to prove. Since ∂_ω sends any monomial to a Schur function, then δ_ω sends any monomial in $\mathbb{B}$ (or $\mathbb{A}$, up to a power of q) to a Schur function in $\mathbb{B}$.

Ex. 7.5 $\delta_\omega = \partial_\omega (a_1 \cdots a_n)^{n-1}$.

Ex. 7.6 Rewriting

$$\delta_i = -(qa_{i+1} + p)\partial_i + \frac{q(a_i{+}a_{i+1}) + 2p}{a_i - a_{i+1}} ,$$

one can check $\delta_1\delta_2\delta_1 = \delta_2\delta_1\delta_2$.

Beware that the operators do not preserve polynomials.

A better proof is to conjugate by the Vandermonde $\Delta(\mathbb{A})$. One has

$$\Delta(\mathbb{A})^{-1}\, \delta_i\, \Delta(\mathbb{A}) = (a_i{-}a_{i+1})^{-1}\delta_i(a_i{-}a_{i+1}) = \partial_i(qa_i + p) .$$

Now, both $\Delta(\mathbb{A})^{-1}\,\delta_1\delta_2\delta_1\Delta(\mathbb{A})$ and $\Delta(\mathbb{A})^{-1}\,\delta_2\delta_1\delta_2\Delta(\mathbb{A})$ are equal to $\partial_{321}\,(qa_1{+}p)^2(qa_2{+}p)$.

More generally, with $n = |\mathbb{A}|$, $\omega = [n, \ldots, 1]$, one has

$$\Delta(\mathbb{A})^{-1}\,\delta_\omega\Delta(\mathbb{A}) = \partial_\omega \prod_{1\leq i<j\leq n} (qa_i + p)^{n-i} .$$

For example, the image of the monomial $a_1^2a_2$ under δ_{321} is

$$\Psi_{12}(\mathbb{A})\, \frac{(qa_1 + p)^2)(qa_2 + p)}{(a_1{-}a_2)(a_1{-}a_3)(a_2{-}a_3)} ,$$

where Ψ_{12} is the monomial function of index $[1, 2]$.

Ex. 7.7 Since f is a function of one variable only, $f\pi_\omega = f\pi_1\pi_2\cdots\pi_n$. By induction, one supposes that

$$f\pi_1\pi_2\cdots\pi_{n-1} = f(a_1) + f\widehat{\pi}_1 + f\widehat{\pi}_1\widehat{\pi}_2 + \ldots + f\widehat{\pi}_1\cdots\widehat{\pi}_{n-1} .$$

The image of this identity under $\pi_n = 1{+}\widehat{\pi}_n$ contains only the extra term $f\widehat{\pi}_1\cdots\widehat{\pi}_n$, since $\widehat{\pi}_n$ kills all terms not containing a_n, that is, all the terms of the RHS, except the last one.

In the case of $f = 1/(1-x)$, with $\mathbb{A}$ of cardinality n, one gets the identity

$$\frac{1}{(1{-}a_1)\cdots(1{-}a_n)} = \frac{1}{1{-}a_1}\pi_\omega = \frac{1}{1{-}a_1} + \frac{a_2}{(1{-}a_1)(1{-}a_2)} + \frac{a_3}{(1{-}a_1)(1{-}a_2)(1{-}a_3)} + \cdots$$

which just says that monomials can be filtered according to the last letter appearing in them.

Ex. 7.9 Let $g(x)$ be a polynomial. Then the remainder of $g(x)R(x,\mathbb{A}')$ modulo $R(x,\mathbb{A})$ is equal to $\sum_{a\in\mathbb{A}} g(a)R(a,\mathbb{A}')R(x,\mathbb{A}-a)R(a,\mathbb{A}-a)^{-1}$, and the sum reduces to the single term

$$g(\xi)R(\xi,\mathbb{A}')R(x,\mathbb{A}')R(\xi,\mathbb{A}')^{-1} = g(\xi)R(x,\mathbb{A}') \ .$$

Therefore, $g(\xi)$ is the coefficient of x^{n-1}, with $n=|\mathbb{A}|$, in the remainder.

However, one does not know $R(x,\mathbb{A}')$, because it would be the same as knowing $x-\xi = R(x,\mathbb{A})/R(x,\mathbb{A}')$. One has to replace in the preceding computation $R(x,\mathbb{A}')$ by

$$R(x+\mathbb{B}',\mathbb{A}') = S_{(n-1)^m}(x+\mathbb{B}'-\mathbb{A}') = S_{(n-1)^m}(x+\mathbb{B}-\mathbb{A}) \ ,$$

with $m=|\mathbb{B}|$. Thus $g(\xi)$ is the quotient of the coefficient of x^{n-1} in the remainder of $g(x)S_{(n-1)^m}(x+\mathbb{B}-\mathbb{A})$ by the extra factor $S_{(n-1)^{m-1}}(\mathbb{B}-\mathbb{A})$.

Finally, the value in ξ of a rational function $f=g/h$ will be the quotient of the coefficients of x^{n-1} in the two remainders of $g(x)S_{(n-1)^m}(x+\mathbb{B}-\mathbb{A})$ and $h(x)S_{(n-1)^m}(x+\mathbb{B}-\mathbb{A})$.

Ex. 7.10 The product is equal to $f(x) := S_{(r-1)^{m-n+r},m}(\mathbb{A}-\mathbb{B}, x-\mathbb{A})$. The remainder is equal to the image of $f(a_1)\,S^{n-1}(x+a_1-\mathbb{A})$ under the product of divided differences $\partial_1\cdots\partial_{n-1}$. Expanding $S^{n-1}(x+a_1-\mathbb{A})$ according to a_1, one has

$$f(a_1)\,S^{n-1}(x+a_1-\mathbb{A}) = \sum S_{\square,m+i}(\mathbb{A}-\mathbb{B},a_1-\mathbb{B})\,S^{n-1-i}(x-\mathbb{A}) \ ,$$

with $\square=(r-1)^{m-n+r}$. Now, the image of $S(a_1-\mathbb{B})$ being $S^{j-n+1}(\mathbb{A}-\mathbb{B})$, the image of the sum is

$$\sum S_{\square,m+i-n+1}(\mathbb{A}-\mathbb{B},\mathbb{A}-\mathbb{B})\,S^{n-1-i}(x-\mathbb{A})$$

in which one recognizes the expansion of the r-th remainder of $S^m(x-\mathbb{B})$ modulo $S^n(x-\mathbb{A})$.

In fact, we have already seen in (3.1.8) that the r-th remainder can be interpreted as a first remainder.

Ex. 7.11 The answer is

$$\begin{cases}(x-a_1)\cdots(x-a_n) & n \text{ odd}\\ -2\sum x^{2n-2j-1}\Lambda^{2j+1}(\mathbb{A}) & n \text{ even}\end{cases} ,$$

and this is a special case of computation of *Euler-Poincaré characteristic.*

Let χ be the Euler-Poincaré morphism :

$$\mathfrak{Sym}(a_1\,|\,a_2,\ldots,a_n) \ni f \xrightarrow{\ \chi\ } (a_1+a_2)\cdots(a_1+a_n)\,f\,\partial_1\cdots\partial_{n-1} \in \mathfrak{Sym}(\mathbb{A}) \ .$$

One finds that $\chi(1)=1$ when n is odd, and $=0$ otherwise, that

$$\chi\big(a_1(1-a_1)^{-1}\big) = \sum_{j\geq 1}\frac{1}{1-z}S^j\big((1-z)\mathbb{A}\big)\Big|_{z=-1} = \sum_H S_H(\mathbb{A})$$

sum over all hook partitions H, and that

$$\chi\Big((1+a_2)\cdots(1+a_n)\Big) = (1+a_1)(1+a_2)\cdots(1+a_n)\,\chi\Big((1+a_1)^{-1}\Big)$$
$$= \begin{cases} \sum \Lambda^{2j}(\mathbb{A}) & \text{if } n \text{ odd} \\ -\sum \Lambda^{2j+1}(\mathbb{A}) & \text{if } n \text{ even} \end{cases} .$$

Writing

$$(x-a_1)(x+a_2)\cdots(x+a_n) = (x+a_1)(x+a_2)\cdots(x+a_n)\big(2x(x+a_1)^{-1}-1\big) ,$$

one concludes.

Ex. 7.12 (Polya II p.82, Ex. 69) One has to compute the image of $(1-a_1)$ $(1-a_1a_2)\cdots(1-a_1a_n)$ under $L_{\mathbb{A}}$. One finds that only two terms in the expansion of this function give a non-zero contribution. For example, for $n=6$, one obtains

$$\begin{aligned} (-1)^3 a_3a_2a_1^3 &\quad\to\quad (-1)^3 \\ (-1)^6 a_6\cdots a_2a_1^6 = (a_6\cdots a_1)a_1^5 &\quad\to\quad (a_6\cdots a_1)\,, \end{aligned}$$

and this gives the required formula.

Ex. 7.13 To recognize the Lagrange interpolation, one replaces the pair $\mathbb{A},\mathbb{B}$ by the pair $\mathbb{X}=\{a_1b_1,\ldots,a_nb_n\}$, $\mathbb{B}$. The sum to compute is

$$\sum_j b_j^{-1}\, R(b_j,\mathbb{X})\, R(b_j,\mathbb{B}-b_j)^{-1} .$$

In other words, one has to compute $L_{\mathbb{B}}(x^i)$, for $-1\le i\le n-1$. Only the two extreme terms contribute, $(-1)^{n-1}(b_1\cdots b_n)^{-1}$ for the first one, 1 for the second.

Ex. 7.14 Writing $\partial_{\mathbb{B}}=\partial_1\cdots\partial_{n-1}$, one decomposes the right hand side into

$$(b_1-b_1a_1)\cdots(b_1-b_{n-1}a_{n-1})\partial_{\mathbb{B}} + -1/b_1(b_1-b_1a_1)\cdots(b_1-b_{n-1}a_{n-1})(b_na_n)\partial_{\mathbb{B}} .$$

The contribution of the first part is $(b_1^{n-1}-b_1^{n-1}a_1)\partial_{\mathbb{B}} = 1-a_1$. For what concerns the second, define $\mathbb{B}'=\{b_1,\ldots,b_{n-1}\}$. Using $\partial_{\mathbb{B}}=\partial_{\mathbb{B}'}\partial_{n-1}$, and showing by induction that

$$1/b_1(b_1-b_1a_1)\cdots(b_1-b_{n-1}a_{n-1})\partial_{\mathbb{B}'}$$

is independent of $\mathbb{B}$, one gets in total

$$(1-a_1)+a_n(1-a_1)(1+a_{n-1}+\ldots+a_{n-1}\cdots a_2) ,$$

which is the required identity.

Ex. 7.15 For $n=3$, one writes

$$(1-a_1^2)^{-1}(1-a_2^2)^{-1} = (1-a_3^2)\sigma_1(a_1^2+a_2^2+a_3^2)\,.$$

Now, $(1-a_3^2)$ is sent to 1 under π_ω. This implies the general case, because $\pi_\omega = \pi_{n-1}\pi_{n-2}\pi_{n-1}\,\pi_\omega$ sends $\sigma_1(a_1^2+\cdots+a_{n-1}^2)=(1-a_n^2)\sigma_1(\mathbb{A})$ onto $\sigma_1(\mathbb{A})$. Noticing that

$$\sigma_1(a_1^2+a_2^2) = (1+a_1a_2)^{-1}\sum_k S^{2k}(a_1+a_2) = \sum_{i\le j:\, i+j\ odd} (-1)^i S_{ij}(a_1+a_2)$$

one gets the second statement.

Ex. 7.16 Proof by induction on n, writing

$$\left(1-(a_1{+}a_2)x\right)^{-1}\cdots\left(1-(a_1{+}\cdots{+}a_{n-1})x\right)^{-1}$$
$$= (1{-}a_1x)^{n-2}\left(1-a_2\frac{x}{1{-}a_1x}\right)\cdots\left(1-(a_2{+}\cdots{+}a_{n-1})\frac{x}{1{-}a_1x}\right) .$$

Ex. 7.17 By linearity, it is sufficient to prove the statement when $f_1,\ldots,f_m$ are powers, but in that case the quotient of the determinant by $\Delta(\mathbb{A}')$ is a Schur function. Thus one can ignore determinants, and decide to prove the equality

$$F(\mathbb{X}) = \sum_{\mathbb{A}'+\mathbb{A}''=\mathbb{A}} F(\mathbb{A}')\frac{R(\mathbb{X},\mathbb{A}'')}{R(\mathbb{A}',\mathbb{A}'')} ,$$

for any symmetric function F which is a linear combination of Schur functions $S_I : I \le (n{-}m)^m$.

At this stage, to compute the image of S_I, one writes $R(\mathbb{X},\mathbb{A}'') = S_{(n-m)^m}(\mathbb{X}-\mathbb{A}'')$, and one recognizes in Borchardt's theorem the version (7.7.5) of Bott's theorem.

Ex. 7.18 From Lagrange interpolation, one sees that the function is equal to

$$S_{I^\sim;m}(\mathbb{A};a_1-\mathbb{B})\partial_1\cdots\partial_{n-1} = S_{I^\sim;m-n+1}(\mathbb{A};\mathbb{A}-\mathbb{B})$$

Since $m \le n$, there are most two non-zero terms in the expansion $\sum S_{I^\sim,m-n+1-j}(\mathbb{A})S^j(-\mathbb{B})$. Explicitly, with $I=[i,j]$, which gives $I^\sim = 1^{j-i}2^i$, one gets

$$\Lambda^i(\mathbb{A})\Lambda^{j+m-n+1}(\mathbb{B}) - \Lambda^{j+1}(\mathbb{A})\Lambda^{i+m-n}(\mathbb{B}) .$$

More generally, a sum $\sum_{a\in\mathbb{A}} R(a,\mathbb{B})R(a,\mathbb{A}{-}a)^{-1}\Lambda_I(\mathbb{A}-a)$, with $\ell(I)=r$, and $|\mathbb{A}|-|\mathbb{B}| \ge r{-}2$, would give a sum of r terms of the type $\Lambda_K(\mathbb{A})\Lambda^j(\mathbb{B})$.

Ex. 7.19 In the case where $\mathbb{A}$ has only one letter, there is no doubt that $(x{-}a)(y{-}a)^{-1} = (1-\frac{y-x}{a-x})^{-1}$.

Suppose that the identity is true for $\mathbb{A}_n = \{a_1,\ldots,a_n\}$, and let us prove it for $\mathbb{A}_{n+1} = \{a_1,\ldots,a_{n+1}\}$. Notice first that the divided difference relative to a pair of letters $(a{-}x)^{-1}$, $(b{-}x)^{-1}$ is equal to $\partial^\diamond_{ab} = -\partial_{ab}(a{-}x)(b{-}x)$.

The image of $R(x,\mathbb{A}_n)R(y,\mathbb{A}_n)^{-1}$ under $\partial^\diamond_n$ (relative to the pair a_n, a_{n+1}) is

$$-\frac{R(x,\mathbb{A}_{n+1})}{R(y,\mathbb{A}_{n+1})}\frac{(x{-}a_n)(y-a_{n+1})}{(x{-}a_n)(x{-}a_{n+1})}\partial_n(a_n-x)(a_{n+1}-x)$$
$$= -(x{-}a_n)(y-a_{n+1})\partial_n\frac{R(x,\mathbb{A}_{n+1})}{R(y,\mathbb{A}_{n+1})} = (y{-}x)\frac{R(x,\mathbb{A}_{n+1})}{R(y,\mathbb{A}_{n+1})} .$$

Therefore, the image of $\sigma_{y-x}(\mathbb{A}^\diamond_n)$ is

$$\sigma_{y-x}\left(\mathbb{A}^\diamond_{n+1}\right)\left(1-\frac{y{-}x}{a_{n+1}-x}\right)\partial^\diamond_n = \frac{-1}{a_{n+1}{-}x}\partial^\diamond_n\sigma_{y-x}\left(\mathbb{A}^\diamond_{n+1}\right)(y{-}x) ,$$

and one gets the identity for $\mathbb{A}_{n+1}$.

This property can be found in [**173**].

Ex. 7.20 The solution of this problem is one possible interpretation of Lagrange interpolation when the interpolating set is cut into two pieces $\mathbb{A},\mathbb{B}$. Indeed, let

$f(x) = S^{\gamma}(x - \mathbb{E})$. Then, Equations (7.8.9, 7.8.10) read

$$f(x) - \frac{R(x,\mathbb{B})}{R(\mathbb{A},\mathbb{B})} S_{(\beta+1)^{\alpha-1};\gamma-\alpha+1}(A-x-\mathbb{B};\mathbb{A}-\mathbb{E}) - \frac{R(x,\mathbb{A})}{R(\mathbb{B},\mathbb{A})} S_{(\alpha+1)^{\beta-1};\gamma-\beta+1}(B-x-\mathbb{A};\mathbb{B}-\mathbb{E}) = R(x,\mathbb{A}+\mathbb{B})\, S^{\gamma-\alpha-\beta}(x+\mathbb{A}+\mathbb{B}-\mathbb{E}) \ .$$

Hill [**64**] gave a determinantal solution to this problem.

Ex. 7.21 In front of a sum of $n{+}1$ rational functions with denominator of degree n, one appeals to Lagrange. Thus, the interpolation points must be

$$0,\, a_1{+}a_2,\, a_1{+}a_2{+}a_3{+}a_4,\, a_1{+}a_2{+}a_3{+}a_4{+}a_5{+}a_6, \ldots$$

This forces the function to interpolate to be

$$f(x) = \big(x - a_1\big)\big(x - (a_1{+}a_2{+}a_3)\big)\cdots\big(x - (a_1{+}a_2{+}\cdots{+}a_{2n-1})\big)$$

and even signs agree.

Since $f = x^n + \cdots$, its image under the Lagrange operator is 1.

Ex. 7.22 The required formula is given by the Lagrange interpolation for $f(k) = S^n(k\mathbb{A})$. Write B for the interpolation set $\{0,1,\ldots,n\}$. Then, because $S^n(k\mathbb{A})$ is a polynomial of degree n in the variable k, one has

$$\frac{S^n(k\mathbb{A})}{R(k,\mathbb{B})} = \sum_{i\in\mathbb{B}} \frac{S^n(iA)}{(k-i)R(i,\mathbb{B}-i)} = \sum_{i\in\mathbb{B}} \pm \frac{S^n(iA)}{(k-i)i!(n-i)!}$$

Ex. 7.23 Consider Jacobi's summation as a functional

$$\varphi : \mathfrak{Pol}(x) \to \mathfrak{Sym}(\mathbb{A}) \otimes \mathfrak{Rat}(x)$$

Since two polynomials have the same image under φ when they are congruent modulo $R(x,\mathbb{A})$ (because the functional involves only the values in $a\in\mathbb{A}$), it is sufficient to compute $\varphi(x^0),\ldots,\varphi(x^{n-1})$.

I claim that

$$\varphi(x^j) = \begin{cases} x^j\, R(x,\mathbb{A})^{-1}\,, & 0 \le j \le n{-}k{-}1 \\ k\, x^j\, R(x,\mathbb{A})^{-1}\,, & j = n{-}k \\ x^{n-k}\, \Psi^{j-n+k}(\mathbb{A})\, R(x,\mathbb{A})^{-1}\,, & n{-}k < j \le n{-}1 \end{cases}$$

Indeed, the subsum on $a\in\mathbb{A}'$ can be written $\sum_\epsilon S_{(n-k)^k+\epsilon}(\mathbb{A}')$, sum over the k vectors $\epsilon = [0^i, j{+}k{-}n, 0^{k-1-i}]$, $i = 0,\ldots,k-1$. Writing now

$$R(x,\mathbb{A}')^{-1} = R(x,\mathbb{A})^{-1} R(x,\mathbb{A}'') = R(x,\mathbb{A})^{-1}(-1)^{n-k} S_{1^{n-k}}(\mathbb{A}''-x)\,,$$

remembering that, for any $I\in\mathbb{N}^k$,

$$S_{1^{n-k}}(\mathbb{A}''-x)\, S_I(\mathbb{A}') = S_{1^{n-k};I}(\mathbb{A}-x;\mathbb{A}')$$

is sent to $S_{1^{n-k};I-(n-k)^k}(\mathbb{A}-x;\mathbb{A})$ under $\sum_{\mathbb{A}=\mathbb{A}'+\mathbb{A}''}$, one finally gets that, for any $j\in\mathbb{N}$,

$$R(x,\mathbb{A})\,\varphi(x^j) = (-1)^{n-k} \sum_\epsilon S_{1^{n-k};\epsilon}(\mathbb{A}-x;\mathbb{A}) \ .$$

Decomposing in terms of x, one obtains a summation involving Schur functions, most of them having repeated columns.

In more details[1], for $j \le n-k-1$, there remains only the contribution of $\epsilon = [j+k-n, 0^{k-1}]$, but $S_{1^{n-k};\epsilon}(\mathbb{A}-x;\mathbb{A}) = (-x)^{n-k} S_{0^n}(\mathbb{A})$ in that case.

For $j \ge n-k$, each of the k terms $S_{1^{n-k};\epsilon}(\mathbb{A}-x;\mathbb{A})$ gives a contribution $(-x)^{n-k} S_{0^{n-k},\epsilon}(\mathbb{A})$. When $j = n-k$, it is $k(-x)^{n-k} S_{0^n}(\mathbb{A})$. When $n-k < j \le n-1$, the sum of Schur functions is equal to a power sum. □

For example, for $n=3, k=2$,

$$\begin{aligned}
\text{for } x^0 &: S_{1;-1,0}(\mathbb{A}-x;\mathbb{A}) = S_{1;-1,0}(\mathbb{A}) \ ; \ S_{1;0,-1}(\mathbb{A}-x;\mathbb{A}) = 0\,,\\
\text{for } x^1 &: S_{1;0,0}(\mathbb{A}-x;\mathbb{A}) + S_{1;0,0}(\mathbb{A}-x;\mathbb{A}) = -2x S_{0;0,0}(\mathbb{A})\,,\\
\text{for } x^2 &: S_{1;0,1}(\mathbb{A}-x;\mathbb{A}) = -x S_{0;0,1}(\mathbb{A}) \ ; \ S_{1;1,0}(\mathbb{A}-x;\mathbb{A}) = 0\,.
\end{aligned}$$

Ex. 7.24 One can put $\psi(j) = S^j(\mathbb{A}-\mathbb{E})$, with $|\mathbb{E}| = r-1$, supposing $\psi(0) = 1$. Thus, the determinant becomes $S_{j^r}(\mathbb{A}-\mathbb{E})$ since $\prod R(b, A-b) = R(B, A-B)\Delta^2(B)$ Therefore $f(a) = a^{n-r} R(a,\mathbb{E})/R(a, A-a)$ is equal to

$$\sum_{\mathbb{B}} S_{(j+n-r)^r}(\mathbb{B}-\mathbb{E})/R(B, A-B) = S_{j^r}(A-E)$$

Aitken gives another proof, by numbering the letters of $\mathbb{A}$, factorizing the matrix into the product

$$\Big[a_k^{j+1-i}\Big]_{1\le i\le r; 1\le k\le n} \ Diag\Big(f(a_1),\ldots,f(a_n)\Big) \ \Big[a_i^{r-k}\Big]_{1\le i\le n; 1\le k\le r}\ ,$$

and using Binet-Cauchy.

Ex. 7.25 Let us take $f(z) = S^k(z-\mathbb{D})$ for some k and $\mathbb{D}$. Then

$$S^k(z-\mathbb{D})/R(z,\mathbb{A}) = \sum\nolimits_{i,j\ge 0} z^{i-n-j} S^{k-i}(-\mathbb{D}) S^j(\mathbb{A})\ ,$$

and the coefficient of z^{-1} is indeed $S^{k-n+1}(\mathbb{A}-\mathbb{D})$.

Taking $g(\mathbb{B}) = R(\mathbb{B},\mathbb{D})$ for some finite alphabet $\mathbb{D}$ of cardinality $\ge n$, and using $\Delta(\mathbb{B}+\mathbb{D}) = \Delta(\mathbb{B})R(\mathbb{B},\mathbb{D})/R(\mathbb{D})$, one is reduced to the preceding case.

Ex. 7.26 This is another way of asking for the limit of the Lagrange interpolation, when some points coincide. We gave Jacobi's formula (7.8.12). It translates concretely into

$$\begin{aligned}
\frac{1}{(x-a)^n(x-b)^n} &= \frac{1}{(x-a)^n(a-b)^n}\\
&\quad - \frac{n}{(x-a)^{n-1}(a-b)^{n+1}} + \frac{\binom{n+1}{2}}{(x-a)^{n-1}(a-b)^{n+1}} - \cdots \pm \frac{\binom{2n-2}{n-1}}{(x-a)(a-b)^{2n-1}} + \cdots\ ,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{(x-a)^3(x-b)^3(x-c)^3} &= \frac{1}{(x-a)^3 R(3a, b+c)}\\
&\quad - \frac{6a-3b-3c}{(x-a)^2 R(4a, b+c)} + \frac{21a^2-21ab-21ac+6b^2+9bc+6c^2}{(x-a)R(5a, b+c)} + \cdots\ ,
\end{aligned}$$

the expression written needing to be completed by symmetry in $a, b, \ldots$.

Except for the case of two distinct poles which requires only simple coefficients, one needs to have a convenient way of writing the expansion of the derivatives, with respect to a, of a function of the type $1/(x-a)R(a,B)$.

[1] However, [**71**], p.27, *Non est meum, per calculos prolixos terrorem incutere lectori, eodemque repetito negotio plurimas paginas implere.*

Ex. 7.27 It suffices to test both sides of the equation on the basis of polynomials

$$\{S^n(x+a-\mathbb{A}),\ a\in\mathbb{A}\}\cup\{S^{n+k+1}(x-\mathbb{A}),\ k\geq 0\}$$

For the first polynomials, both members are equal to 1 (we assume known the residue of $1/(x-a)$). For the second family, one has indeed

$$S^{k+1}(\mathbb{A}-\mathbb{A})=0=\text{ residue of } x^k \quad .$$

Hermite's expression can be used for Lagrange's or Newton's interpolation formula.

Ex. 7.28 The determinant obtained by replacing $\mathbb{A}$ by $\mathbb{A}_n=\{a_1,\ldots,a_n\}$ factorizes into $S_I(\mathbb{A}_n)\times S_J(\mathbb{A}_n)$. Its image under π_ω, where ω is the maximum element of $\mathfrak{S}(\mathbb{A})$ is the original determinant. On the other hand,

$$S_J(\mathbb{A}_n)=S_J(\mathbb{A}-(\mathbb{A}-\mathbb{A}_n))=\sum \pm S_{J/L^\sim}(\mathbb{A})S_L(\mathbb{A}-\mathbb{A}_n)\ .$$

The image of this sum multiplied by $S_I(\mathbb{A}_n)$ under π_ω is now the left-hand side, according to Bott's theorem.

Ex. 7.29 The determinant is the limit case, $x_1\to x,\ldots,x_n\to x$, of the determinant with i+1-th row equal to $f(x_i),\ldots,f^{n+1}(x_i)$. Therefore, it is the limit of

$$f(x_1)\cdots f(x_n)\,\Delta(x,f(x_1),\ldots,f(x_n))/\Delta(x_1,\ldots,x_n)\ ,$$

which is

$$f(x)^n\ (f(x)-x)^n\ f'(x)^{n(n-1)/2}\ .$$

Ex. 7.30 If M was satisfying the equation $(M-a_1)\cdots(M-a_5)=0$, with five different roots, then any function $f(M)$ could be obtained by Lagrange interpolation :

$$f(M)=\sum f(a_i)S^4(M+a_i-\mathbb{A})/S^4(2a_i-\mathbb{A})\ ,$$

because the difference $f(x)-\sum f(a_i)S^4(x+a_i-\mathbb{A})/S^4(2a_i-\mathbb{A})$ is divisible by $S^5(x-\mathbb{A})$.

In the present case, we have to use one of the limit case of Lagrange interpolation, for example (7.8.11) for $m=2, n=3$. The evaluation of $S^k(M-\mathbb{E})$ is equal to

$$S^k(M-\mathbb{E})=\frac{(M-b)^2}{(a-b)^6}S_{33;k-2}(3a-2b-M;3a-\mathbb{E})+\frac{(M-a)^3}{(b-a)^6}S_{4;k-1}(2b-3a-M;2b-\mathbb{E})\ .$$

Expanding the Schur functions, one gets a linear combination of $S^{k-2}(3a-\mathbb{E})$, $S^{k-1}(3a-\mathbb{E}), S^k(3a-\mathbb{E}), S^{k-1}(2b-\mathbb{E}), S^k(2b-\mathbb{E})$.

The function that we wanted to evaluate, however, was not $S^k(M-\mathbb{E})$, but $\exp(M)$. The last step, to erase the phantom alphabet $\mathbb{E}$ of cardinality k, is to recognize that $S^{k-1}(2x-\mathbb{E}), S^k(2x-\mathbb{E})$ are the derivatives of $S^k(x-\mathbb{E})$, $xS^k(x-\mathbb{E})$, and that $2S^{k-2}(3x-\mathbb{E}), 2S^{k-1}(3x-\mathbb{E}), 2S^k(3x-\mathbb{E})$ are the second derivatives of $S^k(x-\mathbb{E})$, $xS^k(x-\mathbb{E})$, $x^2S^k(x-\mathbb{E})$, and thus must be replaced by $\exp(x)$, $(1+x)\exp(x)$ and $\exp(x)$, $(2+x)\exp(x)$, $(2+4x+x^2)\exp(x)$ respectively.

Let us note that we only used that $(x-a)^3(x-b)^2$ was vanishing in M, not that this polynomial was of minimum degree.

In the case of a more complicated polynomial, one would use expressions involving only the derivatives of $f(x)$ and not of $x^j f(x)$. For example, instead of using $S_{33;k-2}(3a-2b-M;3a-\mathbb{E})$, one would subtract $2a$ in the first row, a in the second row. Now, the last column is $S^k(a-\mathbb{E}), S^{k-1}(2a-\mathbb{E}), S^{k-2}(3a-\mathbb{E})$, that is $f(a), f'(a), f''(a)/2$.

B.8

Ex. 8.1 Write $S_{n^n}(\mathbb{A}-x) = S_{n^n;0}(\mathbb{A};\, x)$. Its image under $\partial_{t,x}$ is $S_{n^n;-1}(\mathbb{A};\, x+t)$ and the linear functional has just the effect of replacing t by $\mathbb{A}$.

Ex. 8.2 Compared to the usual functional used in this chapter, we have changed $\mathbb{A}$ into $-\mathbb{A}$. Therefore, up to normalization, the orthogonal polynomials are $S_{n^n}(\mathbb{A}+x)$, and the adjoint ones are $Q_n = S_{(n-1)^{n+1}}(-A+x) = \pm S_{(n+1)^{n-1}}(\mathbb{A}-x)$. On this last expression, one sees that $Q_1, Q_2, \ldots$ are the orthogonal polynomials for the shifted functional $x^n \to S^{n+2}(\mathbb{A})/S^2(\mathbb{A})$, $n \geq 0$.

Ex. 8.3 Write the polynomial to decompose as $S^k(x-\mathbb{B})$. According to the preceding exercise, $Q_1, Q_2 \ldots$ are the orthogonal polynomials associated to the functional $x^n \to S^{n+2}(-\mathbb{A})/S^2(-\mathbb{A})$. Writing

$$Q_n(x)S^k(x-\mathbb{B}) = (-1)^{n-1}S_{(n+1)^{n-1};0}(-\mathbb{A};x)S^k(x-\mathbb{B}) = (-1)^{n-1}S_{(n+1)^{n-1};k}(-\mathbb{A};x-\mathbb{B})$$

one obtains the coefficient of decomposition

$$\frac{(Q_n, S^k(x-\mathbb{B}))}{(Q_n,Q_n)} = \frac{S_{(n+1)^{n-1};k}(-\mathbb{A};-\mathbb{A}-\mathbb{B})}{S_{(n+1)^n}(-\mathbb{A})S_{(n-1)^n}(\mathbb{A})}$$

Ex. 8.4 According to Theorem 8.3.1, $(-1)^{k-1}S_{(k-2)^k}(\mathbb{A}+x)S_{(k-1)^{(k-1)}}(\mathbb{A}-x)^{-1}$ and $(-1)^{n+k}S_{(n+k-1)^{n+k+1}}(\mathbb{A}+x)S_{(n+k)^{(n+k)}}(\mathbb{A}-x)^{-1}$ have the Taylor expansion

$$x^{-1} + x^{-2}S_1(\mathbb{A}) + \cdots + x^{-2k+2}S_{2k-3}(\mathbb{A}) + \mathcal{O}(x^{-2k+1}) \ .$$

Therefore $P_n^{(k)}S_{(k-1)^{(k-1)}}(\mathbb{A}-x)^{-1}S_{(n+k)^{(n+k)}}(\mathbb{A}-x)^{-1} = \mathcal{O}(x^{-2k+1})$ and $P_n^{(k)}$ is of degree $\leq n$ in x. Since $\{\widetilde{P}_n^{(0)}\}$ and $\{\widetilde{P}_n^{(1)}\}$ satisfy the same three terms recursion, $\{\widetilde{P}_n^{(k)}\}$ also does (but with a shift of indices). Notice that $\widetilde{P}_{-1}^{(k)} = 0$, $\widetilde{P}_0^{(k)} = S_{(k-1)^k}(\mathbb{A})^2$, and therefore $\widetilde{P}_n^{(k)}$ has degree n.

Ismail, Prodinger, Stanton [**65**] use the associated orthogonal polynomials to Al-Salam/Ismail polynomials to generalize Rogers-Ramanujan identities.

Ex. 8.5 Since the quadratic form can be defined by using the Darboux-Christoffel kernels, one has $K_\infty(x,y) = \sum y^n T_n(x)$, and therefore

$$T_n(x) = \sum_{j\geq 0}(-1)^{j-n}\frac{S_{j^j/1^n}(\mathbb{A})}{S_{(j-1)^j}(\mathbb{A})S_{j^{j+1}}(\mathbb{A})}\, S_{j^j}(\mathbb{A}-x) \ .$$

For example, writing S_I for $S_I(\mathbb{A})$,

$$\begin{aligned} T_0 &= 1 - \frac{S_1}{S_{11}}S_1(\mathbb{A}-x) + \frac{S_{22}}{S_{11}S_{222}}S_{22}(\mathbb{A}-x) - \frac{S_{333}}{S_{222}S_{3333}}S_{333}(\mathbb{A}-x) + \cdots \\ T_1 &= \frac{S_0}{S_{11}}S_1(\mathbb{A}-x) - \frac{S_{12}}{S_{11}S_{222}}S_{22}(\mathbb{A}-x) + \frac{S_{233}}{S_{222}S_{3333}}S_{333}(\mathbb{A}-x) + \cdots \end{aligned}$$

For any $k, n, k \leq n$, the projection of $T_k(x)$ in the space generated by $P_0, \ldots, P_n$ is, up to a scalar, the only polynomial of degree $\leq n$ which is orthogonal to $x^0, \ldots, x^{k-1}, x^{k+1}, \ldots, x^n$. Clearly $S_{(n+1)^n/1^k}(\mathbb{A}-x) = S_{n^k,(n+1)^{(n-k)};0}(\mathbb{A};x)$ satisfies the same property (it is the coefficient of $(-y)^k$ in $S_{(n+1)^n}(\mathbb{A}-x-y)$). The factor of proportionality is found to be $\pm S_{n^{n+1}}(\mathbb{A})$. For example, for $n=3$, one has

$$T_0 \equiv -S_{444}(\mathbb{A}-x)/S_{3^4}(\mathbb{A}) \ , \ T_1 \equiv S_{344}(\mathbb{A}-x)/S_{3^4}(\mathbb{A}) \ ,$$
$$T_2 \equiv S_{334}(\mathbb{A}-x)/S_{3^4}(\mathbb{A}) \ , \ T_3 \equiv S_{333}(\mathbb{A}-x)/S_{3^4}(\mathbb{A}) \ .$$

Ex. 8.6 The identity is the expansion of x^n in the basis of orthogonal polynomials. Therefore, the coefficients are the normalized scalar products

$$c_k = -(x^n, \widetilde{P}_k)/(\widetilde{P}_k, \widetilde{P}_k) = (-1)^{k+1} S_{k^k,n}(\mathbb{A})/S_{k^{k+1}}(\mathbb{A}) \ .$$

Ex. 8.7 Instead of the series $\sum x^{-i-1} S_i(\mathbb{A})$, we have the series $\sum x^{-i-1}(i+1)S_i(\mathbb{A})$. Therefore, to get the new convergents, we have to change $S_i(\mathbb{A})$ into $(i+1)S_i(\mathbb{A})$, $i = 1, 2, \ldots$ inside the determinantal expressions

$$P_n(x) = S_{n^n;0}(\mathbb{A};\, x) \quad \& \quad Q_n(x) = \pm S_{(n+1)^{n-1};0}(-\mathbb{A};\, x) \ .$$

Ex. 8.8 This is the specialization of the Darboux-Christoffel kernel :

$$K_{n+1}(0,0) = S_{(n+1)^n}(\mathbb{A})/S_{n^{n+1}}(\mathbb{A}) \ .$$

Ex. 8.9 The generating function of moments is $\sum_a \nu_a/(1-za)$ and, as a rational function, can be written

$$\sum_a \nu_a/(1-za) = \prod_{b\in\mathbb{B}}(1-zb)/\prod_{a\in\mathbb{A}}(1-za) \ ,$$

where $\mathbb{B}$ is of cardinality $\leq n-1$ (suppose $= n-1$, to be generic). All formulas given for $\int_{\mathbb{A}}$ for orthogonal polynomials, kernels, expansions, *involving no moment of order* $> 2n-1$, remain valid when replacing $\mathbb{A}$ by $\mathbb{A}-\mathbb{B}$, as we used for the case $\mathbb{B}-\mathbb{B}^{der}$.

For example,

$$K_n(x,y) = (-1)^{n-1} S_{n^{n-1}}(\mathbb{A}-\mathbb{B}-x-y)/S_{(n-1)^n}(\mathbb{A}-\mathbb{B})$$

and, for $f(x)$ of degree $\leq n-1$, one has

$$\oint K(x,y)f(x) = \sum_a K(a,y)\frac{R(a,\mathbb{B})}{R(a,\mathbb{A}-a)}\, f(a) = \sum_a \frac{R(\mathbb{A}-a,\, y)}{R(\mathbb{A}-a,\, a)}\, f(a) = f(y) \ .$$

Ex. 8.10 The rational function $\partial_{xy}(\widetilde{P}_{n-1}(x)/\widetilde{P}_n(x)$ can be written in terms of the Darboux-Christoffel kernel $K_n(x,y) = (-1)^{n-1} S_{n^{n-1}}(\mathbb{A}-x-y)/S_{(n-1)^n}(\mathbb{A})$. It is equal to

$$(-1)^n \frac{S_{(n-1)^n}(\mathbb{A})}{S_{(n-2)^{n-1}}(\mathbb{A})}\frac{K_n(x,y)}{\widetilde{P}_n(x)\widetilde{P}_n(y)} \ .$$

Subtracting $-1/\beta_n$ and reducing to the same denominator, one gets a multiple of

$$S_{n^{n+1}}(\mathbb{A})S_{n^{n-1}}(\mathbb{A}-x-y) - S_{n^n}(\mathbb{A}-x)S_{n^n}(\mathbb{A}-y)$$

which is equal to $S_{(n+1)^n}(\mathbb{A}-x-y)\, S_{(n-1)^n}(\mathbb{A})$, as we have already seen many times. This gives us the left side of (8.9.9).

Notice that $S_{(n+1)^n}(\mathbb{A}-y-x)$ is a "shifted" orthogonal polynomial, as used in Proposition (8.7.1).

Ex. 8.11 Let us test the statement on the basis

$$\{x^j : j = 0, \ldots, k-1\} \cup \{S_j(x-\mathbb{B}) : j = k, \ldots, k+j-1\} \ .$$

Since $P_{n,k}(x;\, \mathbb{B})x^j = S_{(n+k)^n;\, j}(\mathbb{A}-\mathbb{B};\, x)$, the left hand side, when $f(x) = x^j$, is equal to $S_{(n+k)^n;\, j}(\mathbb{A}-\mathbb{B};\, \mathbb{A})$.

On the other hand, for any h, j, one has

$$b_1^j S_h(\mathbb{A}-\mathbb{B}+b_1) = S_{h+j}(\mathbb{A}-\mathbb{B}+b_1) - b_1^{j-1}S_{h+1}(\mathbb{A}-\mathbb{B}) - b_1^{j-2}S_{h+2}(\mathbb{A}-\mathbb{B}) - \cdots$$

Therefore, with $m := n{+}k{-}1$, $b_1^j S_{m^{n+1}}(\mathbb{A}-b_2-\cdots-b_k) = b_1^j S_{m;\,m^n}(\mathbb{A}-\mathbb{B}+b_1;\ \mathbb{A}-\mathbb{B})$ is sent, when $j<k$, onto

$$S_{m+j-k+1;\,m^n}(\mathbb{A}-\mathbb{B}+\mathbb{B};\ \mathbb{A}-\mathbb{B}) = (-1)^n S_{(m-1)^n;\,j}(\mathbb{A}-\mathbb{B};\ \mathbb{A})$$

under $\partial_1\cdots\partial_{k-1}$, and this proves the statement for the first family.

For the second family, the left hand side is equal to $S_{(n+k)^n;\,j}(\mathbb{A}-\mathbb{B};\ \mathbb{A}-\mathbb{B})$, which is null when $k\le j\le k{+}n{-}1$. The right hand side is null for any $j\ge k$, because $S_j(b_1-\mathbb{B})$ is so.

Ex. 8.12 One recognizes a Lagrange summation in the RHS. Writing

$$(b-x)^{2n+1}S^n(b{+}\mathbb{A}-\mathbb{B}) = (-1)^n S_{1^n;2n+1}(\mathbb{B}-\mathbb{A}; b-(2n{+}1)x)\ ,$$

one gets that it is equal to $(-1)^n S_{1^n;n+1}(\mathbb{B}-\mathbb{A};\mathbb{B}-(2n{+}1)x)$. Since one wants a function of $\mathbb{A}-(2n{+}1)x)$, one rather writes

$$(-1)^n S_{1^n;n+1}(\mathbb{B}-\mathbb{A};\mathbb{B}-\mathbb{A}+\mathbb{A}-(2n{+}1)x)\ .$$

Fortunately, according to (8.6.16), $S^{n+1}(\mathbb{A}-\mathbb{B}) = 0 = S^{n+2}(\mathbb{A}-\mathbb{B}) = \cdots = S^{2n+}(\mathbb{A}-\mathbb{B})$, and this is equivalent to the nullities

$$S_{1^{n+1}}(\mathbb{B}-\mathbb{A}) = 0 = S_{1^n 2}(\mathbb{B}-\mathbb{A}) = \cdots = S_{1^n,n+1}(\mathbb{B}-\mathbb{A})\ .$$

Therefore the sum $(-1)^n\sum S_{1^n,n+1-i}(\mathbb{B}-\mathbb{A})S^i(\mathbb{A}-(2n{+}1)x)$ reduces to its last term

$$(-1)^n S_{1^n,-n}(\mathbb{B}-\mathbb{A})\,S^{2n+1}(\mathbb{A}-(2n{+}1)x) = S^{2n+1}(\mathbb{A}-(2n{+}1)x)\ .$$

Ex. 8.13 These polynomials are called *continuous q-Hermite polynomials* by Askey and Ismail [**8**], who take the normalization $2x = b+1/b$. Expanding

$$q^nS^n\left(\tfrac{x}{1-q}\right) = S^n\left(\tfrac{qx}{1-q}\right) = S^n\left(\tfrac{x}{1-q}-x\right)$$
$$= S^n\left(\frac{x}{1-q}\right) - xS^{n-1}\left(\frac{x}{1-q}\right) + S^{n-2}\left(\frac{x}{1-q}\right)\ ,$$

one needs only to multiply by $(1-q)\cdots(1-q^{n-1})$ to recognize the required recursions.

The expansion of the Darboux-Christoffel kernels (8.2.4) can be written in terms of the coefficients of the three-term recursion and gives

$$\frac{H_n(x)H_{n-1}(y)-H_n(y)H_{n-1}(x)}{(x-y)(1-q)\cdots(1-q^{n-1})} = \sum_{i=0}^{n-1}\frac{H_i(x)H_i(y)}{(1-q)\cdots(1-q^i)}\ .$$

Notice that the function $\sum_{i\ge 0} z^i\frac{H_i(x)H_i(y)}{(1-q)\cdots(1-q^i)}$ factorizes. It is an extension of *Mehler's formula* for the usual Hermite polynomials [**8**].

Ex. 8.14 One finds that the moments are $S_{2i+1}(\mathbb{A}) = 0$, $S_{2i} = 1/(2i+1)$, and they are, indeed, the coefficients of the Taylor expansion of $\log((x+1)/(x-1))/2$ near $x=\infty$. Note that $P_n(x)$ can be written $S^n(x)$, with $x = (y+y^{-1})/2$, y of rank 1.

Ex. 8.15 According to Euler,

$$\int_{-1}^{1}(1+x)^\alpha(1-x)^\beta = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha{+}1)\Gamma(\beta{+}1)}{\Gamma(\alpha{+}\beta{+}2)}\ .$$

Using Dirichlet's decomposition of Legendre polynomials seen in Ex.2.50,

$$P_n = \Lambda^n(na+nb)\ ,\ a = \frac{x{+}1}{2},\, b = \frac{x{-}1}{2}\ ,$$

one gets

$$\int_{-1}^{1}(1+x)^{\alpha}P_n(x)\,dx = 2^{-n}\sum(-1)^{n-k}\binom{n}{k}^2 2^{n+\alpha+1}\frac{\Gamma(\alpha+k+1)\Gamma(n-k+1)}{\Gamma(\alpha+n+2)}$$

$$= \frac{2^{\alpha+1}}{(n+1)S_{n+1}(\alpha+1)}\sum\pm\binom{n}{k}S^k(\alpha+1) = \frac{2^{\alpha+1}S^n(\alpha+1-n)}{(n+1)S_{n+1}(\alpha+1)} = 2^{\alpha+1}\frac{\alpha\cdots(\alpha-n+1)}{(\alpha+1)\cdots(\alpha+n+1)}$$

This computation is due to Hermite ([**69**] IV, p.500).

Notice that the expression vanishes for $\alpha = 0,\ldots,n-1$. It proves that that $P_n(x)$ is orthogonal to $x^0,\ldots,x^{n-1}$. Moreover, for $\alpha = n$, one finds that

$$\int P_nP_n = 2^{n+1}(2-1)\cdots(2-\frac{1}{n})\,\frac{n!^2}{(2n+1)!} = \frac{2}{2n+1}\ .$$

Ex. 8.16 One proves by induction that $S_k(\mathbb{A}) = k!$. Indeed, take the determinantal expression of $\Psi_n(\mathbb{A})$ in terms of the complete functions, and add the last column to the first one:

$$\begin{vmatrix} nS_n & S_1 & \cdots & S_n \\ (n-1)S_{n-1} & S_1 & \cdots & S_{n-1} \\ \vdots & \vdots & & \vdots \\ 0S_0 & S_{-n+1} & \cdots & S_0 \end{vmatrix} \mapsto \begin{vmatrix} (n+1)S_n & S_1 & \cdots & S_n \\ nS_{n-1} & S_1 & \cdots & S_{n-1} \\ \vdots & \vdots & & \vdots \\ 1S_0 & S_{-n+1} & \cdots & S_0 \end{vmatrix}$$

It becomes equal to $(-1)^n$ times the determinantal expression of $S_{1^{n+1}}(\mathbb{A}) = \Lambda_{n+1}(\mathbb{A})$ if $S_k(\mathbb{A}) = kS_{k-1}(\mathbb{A})$.

Ex. 8.17 One finds the same alphabet as in Ex.8.16, i.e. $S_k(\mathbb{A}) = k!$, $k \geq 0$.

Ex. 8.18

$$L_n^*(x)/L_n(x) = 2\big(x^{-1} + \frac{1}{3}x^{-3} + \frac{1}{5}x^{-5} + \cdots + \frac{1}{2m-1}x^{-2m+1} + \cdots\big)$$

the expansion being exact up to $m = n/2$. Thus the limit is

$$\log((x+1)/(x-1))\ ,$$

which is the generating function of moments of Legendre polynomials $P_n(x)$, seen in Ex.(8.14)! For example, on has

$$\frac{L_3^*}{2L_3} = \frac{x^2-9x+55/3}{x^3-9x^2+18x-6}\ \&\ \frac{P_3^*}{2P_3} = \frac{x^2-4/15}{x^3-3/5x} = \frac{1}{x}+\frac{1}{3x^3}+\cdots\ ,$$

$$\frac{L_5^*}{2L_5} = \frac{x^4-25x^3+601/3x^2-1825/3x+10003/15}{x^5-25x^4+200x^3-600x^2+600x-120}$$

$$\&\ \frac{P_5^*}{2P_5} = \frac{x^4-7/9x^2+64/945}{x^5-10/9x^3+5/21x} = \frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\cdots$$

and

$$\lim_{n\to\infty}\frac{\int_{-1}^{+1}\partial_{xy}(L_n(x))dy}{L_n(x)} = \lim_{n\to\infty}\frac{\int_{-1}^{+1}\partial_{xy}(P_n(x))dy}{P_n(x)} = \log(\frac{x+1}{x-1})\ .$$

Taking instead the "true adjoint" of Laguerre polynomials (i.e. with respect to $f(x) \mapsto \int_0^\infty e^{-x}f(x)dx$) would have given the generating series of moments: $x^{-1}+1!x^{-2}+2!x^{-3}+\cdots$.

Ex. 8.19 Go to next exercice.

Ex. 8.20 The exercice has nothing to do with orthogonal polynomials. Let $P_n(x) = S_n(x - \mathbb{B})$. Then

$$\partial_{xy}(P_n(x)) = S_{n-1}(x + y - \mathbb{B}) = x^{n-1} \sum (y/x)^i S_{n-i-1}(1 - x^{-1}\mathbb{B}) \ .$$

Therefore

$$\frac{\int S_{n-1}(x + y - \mathbb{B})dy}{S_n(x - \mathbb{B})} = x^{-1} \int 1dy + x^{-2} \int ydy + \cdots + x^{-n} \int y^{n-1}dy + x^{-n-1}(\cdots) \ ,$$

and the required limit, for whatever reasonable meaning of the term, is $\int 1/(x-y)$.

In particular, $P_n = x^n$ and $\int = \int_{-1}^{+1}$ gives at the limit $\log((x+1)/(x-1))$. One does not need to take Legendre polynomials.

Ex. 8.21 I claim that the moments are

$$S^n(\mathbb{A}) = \Big(\alpha(\alpha+1)\cdots(\alpha+n-1)\Big)^{-1} \ .$$

Indeed, let us look at $S_{n^n;0}(\mathbb{A}; x)$ for such $\mathbb{A}$. Multiplying each column by the biggest denominator it contains, and multiplying the successive rows by $1/0!$, $1/1!$, $1/2!, \ldots$, one gets the determinant

$$\begin{vmatrix} \Lambda^0(\alpha{+}n{-}1) & \cdots & \Lambda^0(\alpha{+}2n{-}2) & x^n/0! \\ \vdots & & \vdots & \vdots \\ \Lambda^n(\alpha{+}n{-}1) & \cdots & \Lambda^n(\alpha{+}2n{-}2) & x^0/n! \end{vmatrix}$$

which transforms into

$$\begin{vmatrix} \Lambda^0(\alpha{+}n{-}1) & \cdots & \Lambda^{-n+1}(\alpha{+}n{-}1) & x^n/0! \\ \vdots & & \vdots & \vdots \\ \Lambda^n(\alpha{+}n{-}1) & \cdots & \Lambda^1(\alpha{+}n{-}1) & x^0/n! \end{vmatrix} \ .$$

Introducing for convenience ξ such that $S^n(\xi) = (-1)^n/n!$, $n \geq 0$, one recognizes

$$\pm x^n S_{1^n;0}(\alpha{+}n{-}1; \xi x^{-1}) = \pm S^n\big((\alpha{+}n{-}1)x + \xi\big) \ ,$$

whose expansion is P_n.

Notice that more generally, any Schur functions $S_I(\mathbb{A})$, for $I \in \mathbb{N}^n$ such that $I \geq (n{-}1)^n$, factorizes into factors of the type $1/(\alpha + k)$. In particular, one can compute the Schur functions $S_{(n-1)^n}(\mathbb{A})$ and $S_{n^n}(\mathbb{A})$, and therefore, write continued fraction expansions of $\sigma_z(\mathbb{A})$.

In the case of the classical Bessel polynomials, the moments[2] are $S^n(\mathbb{A}) = 1/(n{+}1)!$.

Let $f(z) := \sigma_z(\mathbb{A})$, and f' be its derivative with respect to z. Then one finds

$$zf' = (1 + z - \alpha)f + a - 1 \,,$$

i.e., dividing by f,

$$\sum_{i>0} z^{i+1}\Psi^i(\mathbb{A}) = z\,f'/f = 1{+}z{-}\alpha + (\alpha - 1)\sum_{i\geq 0}(-z)^i\Lambda^i(\mathbb{A}) \ .$$

This implies

$$\Psi^i(\mathbb{A}) = (-1)^i(\alpha - 1)\,\Lambda^i(\mathbb{A}) \ , \ i \geq 0 \ .$$

[2] The specialization $S^n(\mathbb{A}) = 1/n!$ occurs in the computation of dimensions of irreducible representations of the symmetric group

Notice that the above equations characterize the Bessel polynomials, the value $\Psi^1(\mathbb{A}) = \Lambda^1(\mathbb{A})$ can be kept as an extra parameter (that one can specialize to 1 or α by homogeneity).

Finally, by recursion, using $\widetilde{P}_n := n!P_n$, one finds the expansion

$$x^n = \frac{1}{\alpha\cdots(\alpha+n-1)}\,\widetilde{P}_0 - \binom{n}{1}\frac{(\alpha+1)}{\alpha\cdots(\alpha+n)}\,\widetilde{P}_1 + \binom{n}{2}\frac{(\alpha+3)}{(\alpha+1)\cdots(\alpha+n+1)}\,\widetilde{P}_2 - \cdots$$
$$+ (-1)^k\binom{n}{k}\frac{(\alpha+2k-1)}{(\alpha+k-1)\cdots(\alpha+n+k-1)}\widetilde{P}_k + \cdots$$

For example, for $n = 4$, one has :

$$x^4 = \frac{\widetilde{P}_0}{\alpha(\alpha+3)(2+\alpha)(1+\alpha)} - \frac{4\,\widetilde{P}_1}{\alpha(\alpha+3)(2+\alpha)(\alpha+4)} + \frac{6\,\widetilde{P}_2}{(2+\alpha)(1+\alpha)(\alpha+4)(\alpha+5)}$$
$$- \frac{4\,\widetilde{P}_3}{(\alpha+3)\,(2+\alpha)\,(\alpha+4)\,(\alpha+6)} + \frac{\widetilde{P}_4}{(\alpha+3)\,(\alpha+4)\,(\alpha+5)\,(\alpha+6)}\ .$$

Ex. 8.22 The inverse of the generating function is $\sum z^i/(i+1)!$, i.e. the generating function of the moments of the classical Bessel polynomials (with parameter $a = 2$). If $\mathbb{A}$ is such that $S^i(\mathbb{A}) = 1/(i+1)!$, then $\Lambda^i(\mathbb{A}) = (-1)^i b_i$ and the associated orthogonal polynomials are

$$S_{n^n}(\mathbb{A}-x) = \pm\Lambda_{n^n}(\mathbb{A}-x) = \Lambda_{n;n^{n-1}}(\mathbb{A}-x;\mathbb{A})\ .$$

Therefore, with $n = 6$ for example, taking into account the vanishing of b_{2i+1}, $i \geq 1$, one has

$$\Lambda_{6;6^5}(\mathbb{A}-x;\mathbb{A}) = \begin{vmatrix} \Lambda^6(\mathbb{A}-x) & \cdot & b_8 & \cdot & b_{10} & \cdot \\ \Lambda^5(\mathbb{A}-x) & b_6 & \cdot & b_8 & \cdot & b_{10} \\ \Lambda^4(\mathbb{A}-x) & \cdot & b_6 & \cdot & b_8 & \cdot \\ \Lambda^3(\mathbb{A}-x) & b_4 & \cdot & b_6 & \cdot & b_8 \\ \Lambda^2(\mathbb{A}-x) & \cdot & b_4 & \cdot & b_6 & \cdot \\ \Lambda^1(\mathbb{A}-x) & b_2 & \cdot & b_4 & \cdot & b_6 \end{vmatrix}$$
$$= C\cdot\begin{vmatrix} \Lambda^6(\mathbb{A}-x) & b_8 & b_{10} \\ \Lambda^4(\mathbb{A}-x) & b_6 & b_8 \\ \Lambda^2(\mathbb{A}-x) & b_4 & b_6 \end{vmatrix} = C\cdot\begin{vmatrix} x^6+x^5b_1+x^4b_2+b_6 & b_8 & b_{10} \\ x^4+x^3b_1+x^2b_2+b_4 & b_6 & b_8 \\ x^2+xb_1+b_2 & b_4 & b_6 \end{vmatrix}\ ,$$

where C is the extra factor independent of x : $C = \begin{vmatrix} b_6 & b_8 & b_{10} \\ b_4 & b_6 & b_8 \\ b_2 & b_4 & b_6 \end{vmatrix}$.

It is convenient to write $\Lambda_{6;6^5}(\mathbb{A}-x;\mathbb{A})C^{-1}$ as

$$\Big(b_1 + (x+b_2x^{-1}+b_4x^{-3}+\cdots)\Big)\begin{vmatrix} x^5 & b_8 & b_{10} \\ x^3 & b_6 & b_8 \\ x^1 & b_4 & b_6 \end{vmatrix}\ \cap\ \mathfrak{Pol}(x)\,,$$

where $\cap\mathfrak{Pol}(x)$ means the restriction to non-negative powers of x.

Now this expression can be identified with a Padé approximation. Indeed, $\begin{vmatrix} x^5 & b_8 & b_{10} \\ x^3 & b_6 & b_8 \\ x^1 & b_4 & b_6 \end{vmatrix}$ is the numerator of an approximant of the hyperbolic tangent $\tanh(1/2x)$, and the pair of determinants to study is the numerator and denominator, up to the same factor, of an approximant of $\tanh(1/2x)$. But these approximants are obtained by decomposing Bessel polynomials into their odd and even parts. Therefore, our two determinants are, up to a common factor[3], equal to the odd and even part of a Bessel polynomial.

For $n = 6$, the Padé approximant is

$$\frac{42(x + 240x^3 + 7920x^5)}{1 + 840x^2 + 75600x^4 + 665280x^6} ,$$

the Bessel polynomial of degree 6 is the sum of the numerator and denominator, and one has

$$\begin{vmatrix} x^5 & b_8 & b_{10} \\ x^3 & b_6 & b_8 \\ x^1 & b_4 & b_6 \end{vmatrix} = -\frac{1}{11!\,7!\,6!}(x + 240x^3 + 7920x^5) .$$

Introducing a rank-1 element y such that y^2 is to be specialized to $-x^2$, one can write our determinants in a compact and unambiguous manner.

Indeed

$$\begin{vmatrix} x^4 & b_8 & b_{10} \\ x^2 & b_6 & b_8 \\ 1 & b_4 & b_6 \end{vmatrix} = \begin{vmatrix} 0 & b_8 - x^2 b_6 & b_{10} - x^2 b_8 \\ 0 & b_6 - x^2 b_4 & b_8 - x^2 b_6 \\ 1 & b_4 & b_6 \end{vmatrix} = \begin{vmatrix} \Lambda^8(\mathbb{A}+2y) & \Lambda^{10}(\mathbb{A}+2y) \\ \Lambda^6(\mathbb{A}+2y) & \Lambda^8(\mathbb{A}+2y) \end{vmatrix} = \Lambda_{89/1}(\mathbb{A}+2y) .$$

More generally, the numerators of the approximants of $\tanh(1/2x)$ are proportional to

$$\Lambda_4(\mathbb{A}+2y) , \Lambda_6(\mathbb{A}+2y) , \Lambda_{45/1}(\mathbb{A}+2y) , \Lambda_{89/1}(\mathbb{A}+2y) , \\ \Lambda_{8,9,10/12}(\mathbb{A}+2y) , \Lambda_{10,11,12/12}(\mathbb{A}+2y) \ldots$$

Ex. 8.23 One can expand $\mathcal{P}_n$ in two ways

$$\begin{aligned} \mathcal{P}_n &= \Lambda^n(\mathbb{E})S^0((\mathbb{E}) + \Lambda^{n-2}(\mathbb{E})S^2((\mathbb{E}) + \Lambda^{n-4}(\mathbb{E})S^4((\mathbb{E}) + \cdots \\ &= \Lambda^{n-1}(\mathbb{E})S^1((\mathbb{E}) + \Lambda^{n-3}(\mathbb{E})S^3((\mathbb{E}) + \Lambda^{n-5}(\mathbb{E})S^5((\mathbb{E}) + \cdots . \end{aligned}$$

With the help of these expansions, one checks that $\int x^k S_{1\ldots n}(\mathbb{E} - x) = \int S_{1\ldots n,k}(\mathbb{E} - x)$ is null for $0 \le k < n$, and that

$$\int S_{1\ldots n,n}(\mathbb{E} - x) = S_{1,\ldots,n+1}(\mathbb{E}) .$$

Thus, one can write a continued fraction expansion involving only staircase Schur functions of $\mathbb{E}$, instead of our usual developments with rectangular partitions :

$$\sum_1^\infty z^{-i}\mathcal{P}_i = \cfrac{S_1}{z - S_1 + \cfrac{S_{12}/S_1}{z + \cfrac{S_{123}/S_1 S_{12}}{z + \cfrac{S_{1234}S_1/S_{123}S_{12}}{\ddots}}}} .$$

It generalizes a continued fraction expansion of the hyperbolic tangent. The polynomials $S_{1\ldots n}(\mathbb{E}-x)$ can be considered as symmetric analogs of the Bessel polynomials, that one recovers for $\mathbb{E}$ such that $S^i(\mathbb{E}) = 1/i!$ (the *exponential alphabet*).

[3] to get this factor, and evaluate more general Hankel determinants, we refer to the survey [**88**].

Ex. 8.24 After Cauchy, one defines

$$\tan(z;\mathbb{A}) := \exp\bigl(z\Psi^1(\mathbb{A}) - z^3/3\Psi^3(\mathbb{A}) + z^5/5\Psi^5(\mathbb{A}) - \cdots\bigr) ,$$

and one obtains from him that it is equal to

$$\bigl(zS^1(\mathbb{A}) - z^3S^3(\mathbb{A}) + z^5S^5(\mathbb{A}) - \cdots\bigr)\bigl(1 - z^2S^2(\mathbb{A}) + z^4S^4(\mathbb{A}) - \cdots\bigr)^{-1} .$$

It is transformed into

$$\frac{z\mathcal{P}_2 - z^3\mathcal{P}_4 + \cdots}{P_1 - z^2\mathcal{P}_3 + \cdots} = \frac{zQ_2 - z^3Q_4 + \cdots}{Q_1 - z^2Q_3 + \cdots} ,$$

with $Q_i = 2\mathcal{P}_i$. But now, Q_i is the specialization $\zeta = -1$ of $S^i\bigl((1-\zeta)\mathbb{A}\bigr)$, and we have to study a quotient of two sums of complete functions.

Given any $\mathbb{B}$, let us write $X = \sigma_{z\sqrt{-1}}(\mathbb{B}/2)$. If $\mathbb{B}$ is such that $\Psi_{2j}(\mathbb{B}) = 0$, $j \geq 1$, then $\sigma_{-z\sqrt{-1}}(\mathbb{B}/2) = X^{-1}$. The function $\bigl(zS^2(\mathbb{B}) - z^3S^4(\mathbb{B}) + \cdots\bigr)\bigl(S^1(\mathbb{B}) - z^2S^3(\mathbb{B}) + \cdots\bigr)^{-1}$ is equal to

$$\frac{2 - X^2 - X^{-2}}{(X^2 - X^{-2})/\sqrt{-1}} = \frac{1}{\sqrt{-1}}\frac{X - X^{-1}}{X + X^{-1}} = \tan(z;\mathbb{B}/2) .$$

Therefore, when $\mathbb{B}$ is specialized to $(1-\zeta)\mathbb{A}$, and ζ is specialized to -1, one obtains that the image of $\tan(z;\mathbb{A})$ is still $\tan(z;\mathbb{A})$, and consequently, that the power sums of odd degree are invariant under the transformations $S^i \to \mathcal{P}_{i+1}/\mathcal{P}_1$.

Ex. 8.25 We have to apply once more the polarized version of Bazin relation, writing the $P_{n+2j-2}(x_i) = S_{1\ldots n;0}(\mathbb{E};x_i)$ as minors of order $n + 2k - 1$ of the same matrix.

For example, for $k = 4$, the matrix is

$$\begin{array}{c|cccccc}
\textit{indices} & 0 & \cdots & 2n{+}11 & x_1 & \cdots & x_4 \\
\hline
 & S^0(\mathbb{E}) & \cdots & S^{2n+11} & x_1^{n+6} & \cdots & x_4^{n+6} \\
 & \vdots & & \vdots & \vdots & & \vdots \\
 & S^{-n-6}(\mathbb{E}) & \cdots & S^{n+5} & x_1^0 & \cdots & x_4^0
\end{array}$$

The determinant of orthogonal polynomials is the following determinant of minors, not writing the columns in common :

$$\begin{vmatrix}
[0,2,4\,x_1] & [0,2,2n{+}7,\,x_1] & [0,2n{+}7,2n{+}9,\,x_1] & [2n{+}7,2n{+}9,2n{+}11,\,x_1] \\
[0,2,4\,x_2] & [0,2,2n{+}7,\,x_2] & [0,2n{+}7,2n{+}9,\,x_2] & [2n{+}7,2n{+}9,2n{+}11,\,x_1] \\
[0,2,4\,x_3] & [0,2,2n{+}7,\,x_3] & [0,2n{+}7,2n{+}9,\,x_3] & [2n{+}7,2n{+}9,2n{+}11,\,x_3] \\
[0,2,4\,x_4] & [0,2,2n{+}7,\,x_4] & [0,2n{+}7,2n{+}9,\,x_4] & [2n{+}7,2n{+}9,2n{+}11,\,x_4]
\end{vmatrix}$$

It factorizes into

$$[x_1x_2x_3x_4]\,[[0,2,4,2n{+}7]\,[0,2,2n{+}7,2n{+}9]\,[0,2n{+}7,2n{+}9,2n{+}11] =$$
$$\Delta(x_1,\ldots,x_4)S_{1\ldots n+3}(\mathbb{E} - x_1 - x_2 - x_3 - x_4)\,S_{1\ldots n+1}(\mathbb{E})S_{1\ldots n+3}(\mathbb{E})S_{1\ldots n+5}(\mathbb{E}) .$$

In the general case, the answer is the product

$$\Delta(x_1,\ldots,x_k)\,S_{1\ldots n+k1}(\mathbb{E} - x_1 - \cdots - x_k)\;P_{n+1}(0)P_{n+3}(0)\cdots P_{n+2k-3}(0) .$$

Ex. 8.26 We know the explicit value of Schur functions at $A = 1/(1-q)$, in terms of hook lengths. Putting these values in (8.3.2), one gets $\zeta_0 = 1/(1-q)$, $\beta_1 = q/(1-q)(1-q^2)$,

$$\zeta_n = \frac{q^{2n}(q^n-1) - q^n(q^{n-1}-1)}{(q^{2n-1}-1)(q^{2n+1}-1)} \quad \& \quad \beta_n = q^{3n-2}\frac{(q^{n-1}-1)(q^n-1)}{(q^{2n-2}-1)(q^{2n-1}-1)^2(q^{2n}-1)} ,$$

$$E(x) = \cfrac{1}{x - \cfrac{1}{1-q} + \cfrac{q/(1-q)(1-q^2)}{x - \cfrac{q^2}{q^3-1} + \cfrac{q^4((1-q)/(1-q^3)(1-q^4)}{x - \cfrac{q^5+q^4-q^2}{(q^3-1)(q^5-1)} + \ddots}}}$$

The n-th convergent, denoting $\left[{n \atop i}\right]$ a Gauss polynomial, is

$$(q^n-1)\frac{\sum_{j\le n-1}(-q^n)^{n-1-j}x^j \left[{n-1 \atop j}\right](q^{n+1}-1)\cdots(q^{n+j}-1)}{\sum_{j\le n} q^{\binom{n-j}{2}}x^j \left[{n-1 \atop j}\right](q^n-1)\cdots(q^{n-1+j}-1)} .$$

The limit, for $q=1$, is the fraction with $\zeta_0 = 1$, $\beta_1 = 1/2$,

$$\zeta_n = -(2n-1)^{-1}(2n+1)^{-1} \quad \& \quad \beta_n = (2n-1)^{-2}/4 ,$$

$$\sum x^{-1-i}/i! = \cfrac{1}{x - 1 + \cfrac{1/2}{x + 1/3 + \cfrac{1/36}{x + 1/15 + \cfrac{1/100}{x + 1/35 + \ddots}}}}$$

Ex. 8.27 The alphabet $\mathbb{B}$ such that $S_{ki}(\mathbb{B}) = S_i(\mathbb{A})$, the other $S_j(\mathbb{B})$ being 0, is not generic, and it is not clear whether there exists a family of orthogonal polynomials. However, if it exists, then it is given by the formulas that we wrote using $\mathbb{B}$ instead of $\mathbb{A}$.

Reordering rows and columns in $S_{n^n;0}(\mathbb{B};x)$, one can obtain a matrix triangular by blocks, with only one block containing x (the blocks being best explained by taking the "k-quotient" of partitions, but we shall not need it).

To be more specific, let us take $k=3$. Then, writing i for $S_i(\mathbb{A})$ and $\cdot$ for 0, one has, for example,

$$\begin{vmatrix} . & 2 & . & . & 3 & x^5 \\ . & . & 2 & . & . & x^4 \\ 1 & . & . & 2 & . & x^3 \\ . & 1 & . & . & 2 & x^2 \\ . & . & 1 & . & . & x^1 \\ 0 & . & . & 1 & . & x^0 \end{vmatrix} = \pm \begin{vmatrix} . & . & . & . & 2 & x^4 \\ . & . & . & . & 1 & x^1 \\ . & . & 2 & 3 & . & x^5 \\ . & . & 1 & 2 & . & x^2 \\ 1 & 2 & . & . & . & x^3 \\ 0 & 1 & . & . & . & x^0 \end{vmatrix} = xS_2(\mathbb{A}-x^3)\, S_{11}(\mathbb{A})\, S_{22}(\mathbb{A})$$

More generally, one finds

$$\begin{aligned} S_{(3n)^{3n}}(\mathbb{B}-x) &= S_{n^n}(\mathbb{A}-x^3)\, S_{n^n}(\mathbb{A})^2 \\ S_{(3n+1)^{3n+1}}(\mathbb{B}-x) &= xS_{(n+1)^n}(\mathbb{A}-x^3)\, S_{n^n}(\mathbb{A})\, S_{n^{n+1}}(\mathbb{A}) \\ S_{(3n+2)^{3n+2}}(\mathbb{B}-x) &= xS_{(n+1)^n}(\mathbb{A}-x^3)\, S_{n^{n+1}}(\mathbb{A})\, S_{(n+1)^{n+1}}(\mathbb{A}) . \end{aligned}$$

The last two polynomials are proportional, and therefore the $S_{n^n}(\mathbb{B}-x)$ are not a basis of polynomials. However, $S_{(3n)^{3n}}(\mathbb{B}-x)$ is a polynomial of degree n in

$y = x^3$. This indicates that, for a general k, one has to restrict to the family of $S_{(kn)^{kn}}(\mathbb{B} - x)$, in which case the triangularity illustrated above gives

$$S_{(kkn)^{kn}}(\mathbb{B} - x) = S_{n^n}(\mathbb{A} - x^k)\, S_{n^n}(\mathbb{A})^{k-1} \ ,$$

and we have recovered our usual polynomials, but this time in $y = x^k$.

Ex. 8.28 This is a special case of Christoffel determinant, with $\mathbb{B} = \{\zeta x, \ldots, \zeta^{k-1}x\}$. Write $\mathcal{Z} = \{1, \zeta, \ldots, \zeta^{k-1}\}$ (therefore $\Lambda^1(\mathcal{Z}) = 0 = \cdots = \Lambda^{k-1}(\mathcal{Z})$, $\Lambda^k(\mathcal{Z}) = (-1)^{k-1}$). According to proposition (8.4.1), $Q_n(x)$ is proportional to $S_{(n+k-1)^n}(\mathbb{A} - x\mathcal{Z})$. Expanding it in the basis $S^I(-x\mathcal{Z})$, one gets, with $\square = (n+k-1)^n$,

$$\sum_I D_{\Psi_I}\big(S_\square(\mathbb{A})\big)\, x^I S^I(-\mathcal{Z}) \ ,$$

sum over all partitions I. However, all $S^j(-\mathcal{Z})$, $j > 0$, are null, except $S^k(-\mathcal{Z}) = -1$. Using Muir's rule for the multiplication by monomial functions (or the adjoint operation), one finally gets that $Q_n(x)$ is proportional to

$$\sum_{u \in \{0,k\}^n} S_{\square - u}(\mathbb{A})\, (-x^k)^{|u|/k} \ .$$

For example, Q_2 is proportional to

$$S_{33}(\mathbb{A}) - x^2 S_{13}(\mathbb{A}) - x^2 S_{31}(\mathbb{A}) + x^4 S_{11}(\mathbb{A}) = S_{33}(\mathbb{A}) - x^2 S_{13}(\mathbb{A}) + x^2 S_{22}(\mathbb{A}) + x^4 S_{11}(\mathbb{A}) \ .$$

Ex. 8.29

$$H_n^c = H_n - 2c\binom{n-1}{1} H_{n-2} + 2^2 c(c+1)\binom{n-2}{2} H_{n-4} - 2^3 c(c+1)(c+2)\binom{n-3}{3} H_{n-6} + \cdots$$

is checked by recursion and double checked by computer.

In the case $c = -n$, the recurrence becomes

$$\begin{aligned} H_{n+1}^{-n} &= 2xH_n^{-n} + 0 \\ H_{n+2}^{-n} &= 2xH_{n+1}^{-n} - 2H_n^{-n} \ , \ldots \end{aligned}$$

and it implies that

$$H_{n+k}^{-n}(x) = H_k(x)\, H_n^{-n}(x) \ .$$

To describe $H_n^{-n}(x)$, one needs the *positive Hermite polynomials* defined by the recursions

$$H_{n+1}^+(x) = 2xH_n^+(x) - 2(n+c)H_{n-1}^+(x) \quad , \quad H_{-1}^+ = 0, H_0^+ = 1 \ .$$

One finds

$$H_n^+ = H_n + 2\frac{n!}{1!\, n-2!} H_{n-2} + +2^2 \frac{n!}{2!\, n-4!} H_{n-4} + +2^3 \frac{n!}{3!\, n-6!} H_{n-6} + \cdots$$

and this shows that $H_n^{-n}(x) = H_n^+(x)$.

Ex. 8.30 The sum $\sum (a_1 - a_i)^{-1}$ is the value in $x = a_1$ of the logarithmic derivative of $S_{n-1}(x - (\mathbb{A} - a_1))$, thus is equal to

$$\frac{S_{n-2}(2a_1 - (\mathbb{A} - a_1))}{S_{n-1}(a_1 - (\mathbb{A} - a_1)} = \frac{S_{n-2}(3a_1 - \mathbb{A})}{S_{n-1}(2a_1 - \mathbb{A})} \ .$$

Thus, we have to prove, in the case of a zero of an Hermite polynomial, that $a_1 = S_{n-2}(3a_1 - \mathbb{A})/S_{n-1}(2a_1 - \mathbb{A})$, i.e. $S_n(2a_1 - \mathbb{A}) = S_{n-2}(3a_1 - \mathbb{A})$, which can be written

$$S_n(2x - \mathbb{A}) \equiv S_{n-2}(3x - \mathbb{A}) \qquad \mathrm{mod} \quad H_n(x) \ ,$$
$$D\big(x\, S_n(x - \mathbb{A})\big) \equiv \frac{1}{2} D^2\big(S_n(x - \mathbb{A})\big)$$

and that is no other than the differential equation satisfied by Hermite polynomials. □

For more general relations see [**70**].

Notice that $2\sum_{i>1} 1/(a_1 - a_i)$ is the logarithmic derivative of the discriminant $\prod_{i<j}(a_i - a_j)^2$ with respect to a_1. Schur [**159**] determines for which $x_1, \ldots, x_n$ (belonging to some real domains) the discriminant reach its maximum. He found that the maximum is reached, when $x_1^2 + \cdots + x_n^2 \leq 1$, for the zeros of Hermite polynomials (conveniently normalized).

Ex. 8.31 The differential equation for Jacobi polynomials implies that the function $S_{n-2}(3x - \mathbb{A})/S_{n-1}(2x - A) = D^2(f)/2D(f)$ specializes, for $x = a_1$, into $-\big(p - (p+q)a_1\big)/a_1(1 - a_1) = -p/a_1 + q/(1 - a_1)$.

In the second case, $D^2(f)/2D(f)$ specializes into $1/2 - (\alpha + 1)/2a_1$.

Ex. 8.32 We have seen that when $\mathbb{A}$ is of finite cardinality n that $\Psi_i(\mathbb{A}) = nS^i(\mathbb{A} - \mathbb{A}^{der})$, with $n\, S^{n-1}(x - \mathbb{A}^{der})$ the derivative of $S^n(x - \mathbb{A})$. One can pass to infinity by just formally writing $\Psi_i(\mathbb{A}) = \Psi_0(\mathbb{A}) S^i(\mathbb{A} - \mathbb{A}^{der})$, and one can check that expressions for orthogonal polynomials remain valid when changing $S^i(\mathbb{A})$ into $S^i(\mathbb{A} - \mathbb{A}^{der}) = \Psi_i(\mathbb{A})/\Psi_0$.

For example, $\begin{vmatrix} \Psi_0 & \Psi_1(\mathbb{A}) & \Psi_2(\mathbb{A}) \\ \Psi_1(\mathbb{A}) & \Psi_2(\mathbb{A}) & \Psi_3(\mathbb{A}) \\ 1 & x & x^2 \end{vmatrix}$ is, up to normalization, an orthogonal polynomial for $\int$.

Going back to finite cardinality, say $card(\mathbb{A}) = 4$, this last polynomial can be rewritten, using the factorization

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 & x \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & x^2 \end{bmatrix} \begin{bmatrix} 1 & a_1 & 0 \\ 1 & a_2 & 0 \\ 1 & a_3 & 0 \\ 1 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \Psi_0 & \Psi_1(\mathbb{A}) & \Psi_2(\mathbb{A}) \\ \Psi_1(\mathbb{A}) & \Psi_2(\mathbb{A}) & \Psi_3(\mathbb{A}) \\ 1 & x & x^2 \end{bmatrix}$$

into

$$\sum\nolimits_{\mathbb{B}\subset\mathbb{A},\ card(\mathbb{B})=2} \Delta(\mathbb{B} + x)\Delta(\mathbb{B}) = \sum \Delta(\mathbb{B})^2\, R(\mathbb{B}, x) \ .$$

Expressions of this type have been intensively used by Sylvester [**165**] in his study of Sturm sequences.

Ex. 8.33 Write $S_J(\mathbb{X})\Delta(\mathbb{X})$ as the minor V_J on columns J of the infinite Vandermonde matrix in the x_i's. Multiplying V_J by a monomial corresponds to increasing exponents uniformly in each row. To multiply by $\Delta(\mathbb{X})$, we thus have to sum over all the monomials in the expansion of $\Delta(\mathbb{X})$. To see what happens, take $n = 3$ to be more specific.

V_J can be written $\sum_{\sigma\in\mathfrak{S}_3} (-1)^{\ell(\sigma)} \big(x^{J+[012]}\big)^\sigma$. Its product by $(-1)^{\ell(\nu)}\big(x^{[012]}\big)^\nu$, $\nu \in \mathfrak{S}_3$, is

$$(-1)^{\ell(\nu)} \sum\nolimits_\sigma \big(x^J\big)^\sigma \big(x^{[012]}\big)^\sigma \big(x^{[012]}\big)^\nu \ .$$

Therefore

$$\begin{aligned}\iint S_J(\mathbb{X})\,\Delta(\mathbb{X})^2 &= \sum\nolimits_{\sigma,\nu} (-1)^{\ell(\nu)+\ell(\sigma)} S^{J^\sigma+[012]^\sigma+[012]^\nu}(\mathbb{A}) \\ &= \sum\nolimits_{\sigma} \big(\sum\nolimits_{\eta} (-1)^{\ell(\eta)} S^{J+[222]-[210]^\eta}(\mathbb{A})\big)^\sigma \\ &= n!\,\sum\nolimits_{\eta} (-1)^{\ell(\eta)} S^{J+[222]-[210]^\eta}(\mathbb{A}) \\ &= n!\,(-1)^{n(n-1)/2}\, S_{J+[222]}(\mathbb{A})\end{aligned} \quad ,$$

putting $\eta = \nu\sigma^{-1}$. The reader may be surprised that a summation $[012]^\sigma + [012]^\eta$ results in the addition of $[222]$ $(=[012]+[210])$.

Ex. 8.34 Write $S_{(k+1)^k}(\mathbb{A} - 2x) = S_{(k+1)^k,0}(\mathbb{A} - x, x)$ and expand into:

$$\sum\nolimits_{i=0\ldots k} \pm x^i S_{k^i(k+1)^{k-i},0}(\mathbb{A}, x) = \sum \pm S_{k^i(k+1)^{k-i},i}(\mathbb{A}, x) \ .$$

All terms give the same contribution under $\int_{\mathbb{A}}$. □

To get the corollary, one has to write

$$S_{k^k}(\mathbb{A} - x)^2 = S_{(k+1)^k}(\mathbb{A} - 2x) S_{(k-1)^k}(\mathbb{A}) + S_{k^{k-1}}(\mathbb{A} - 2x) S_{k^{k+1}}(\mathbb{A}) \ .$$

Let us notice that the corollary computes the scalar product $(P_k(x)\,,\,P_k(x))$.

Ex. 8.35 Since the quadratic form is positive, its determinant is positive. But it is $S_{(n-1)^n}(\mathbb{A})$, up to reordering of columns, hence a sign.

More generally, given any partition $J = [j_1, \ldots, j_n]$, one could as well has taken the quadratic form

$$\int_0^1 g(t) \left(\sum x_i t^{j_i+i-1}\right)^2 dt \ .$$

In that case, one obtains that the minor $M_{J,J}$ is positive.

With $g(t) = \sigma_t(\mathbb{B})$, the integrals become sums of complete functions :

$$\int_0^1 t^k\, g(t) dt = \sum\nolimits_i S^i(\mathbb{B})/(i + k + 1) \ .$$

The matrix M factorizes into

$$\mathbb{S}(\mathbb{B}) \cdot N := \begin{bmatrix} S^0(\mathbb{B}) & S^1(\mathbb{B}) & S^2(\mathbb{B}) & \cdots \\ \cdot & S^0(\mathbb{B}) & S^1(\mathbb{B}) & \cdots \\ \cdot & \cdot & S^0(\mathbb{B}) & \cdots \\ \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1/1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \ .$$

The minors of N are special cases of the Cauchy determinant $\left|(x_i + y_j)^{-1}\right|$. Thus, using Binet-Cauchy theorem, one obtains (skew)-Schur functions of $\mathbb{A}$ as a sum of product of a (skew)-Schur functions of $\mathbb{B}$ by a minor of N. But these last minors are special cases of the Cauchy determinant $\left|(x_i + y_j)^{-1}\right|$ and immediate to evaluate.

For example, the initial minor of order n

$$\begin{vmatrix} \sum S^i(\mathbb{B})/(i+1) & \cdots & S^i(\mathbb{B})/(i+n) \\ \vdots & & \vdots \\ S^i(\mathbb{B})/(i+n) & \cdots & S^i(\mathbb{B})/(i+2n-1) \end{vmatrix}$$

is equal to the following sum over all partitions J

$$\sum_{J\in N^n} S_J(\mathbb{B}) N_{J,0^n} = \sum S_J(\mathbb{B})\,(n-1)!\cdots 1! \frac{\prod_{1\le\alpha<\beta\le n}\big((j_\beta+\beta) - (j_\alpha+\alpha)\big)}{\prod_{\alpha=1}^n \prod_{\beta=0}^{n-1} (j_\alpha + \alpha + \beta)} \ .$$

As for the multiple integral, one can notice that, for any r, the integral of

$$x_1^r x_2^{r+1}\cdots \begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix} = \begin{vmatrix} x_1^r & x_1^{r+1} & \cdots & x_1^{r+n-1} \\ \vdots & & & \vdots \\ x_n^{r+n-1} & x_n^{r+n} & \cdots & x_n^{r+2n-2} \end{vmatrix}$$

is already equal to the Hankel determinant, for $r = k-n+1$. Taking the multiple integral of $(n!)^{-1}\sum_{\sigma\in\mathfrak{S}_n}(-1)^{\ell(\sigma)}\left(x_1^r x_2^{r+1}\cdots x_n^{r+n-1}\right)^\sigma$ still gives the same Hankel determinant.

This last computation is due to D.Bessis [**15**].

Ex. 8.36 This is

$$(-1)^k S^{n+k}(\mathbb{A}-k) = \int_0^1 \Lambda^k(k-t)t^n g(t)dt = \int_0^1 (1-t)^k t^n g(t)dt\,.$$

Ex. 8.37

$$R_n(z) = z^{-1}\sum_j \int S_{n^n;0}(\mathbb{A};\,x)\,x^j z^{-j} = \sum z^{-n-j} S_{n^n,n+j}(\mathbb{A})\ .$$

Define the alphabet $\mathbb{A}^n$ (already encountered in the Euclidean division) by

$$S_j(\mathbb{A}^n) = S_{n^n,n+j}(\mathbb{A})/S_{n^{n+1}}(\mathbb{A})\ ,\ j=0,1,\ldots.$$

In other words,

$$R_n(z) = z^{-n}S_{n^{n+1}}(\mathbb{A})\,\sigma_{1/z}(\mathbb{A}^n)\ \&\ \frac{1}{R_n(z)} = \frac{z^n}{S_{n^{n+1}}(\mathbb{A})}\,\lambda_{-1/z}(\mathbb{A}^n)\ .$$

Therefore

$$St_{n+1}(x) = S_{n+1}(x-\mathbb{A}^n)/S_{n^{n+1}}(\mathbb{A})\ .$$

Let $\mathbb{B}$ be the alphabet of roots of $St_{n+1}(x)$. All symmetric functions of $\mathbb{A}^n$ and $\mathbb{B}$ coincide modulo $\Lambda^{n+2}(\mathbb{A}^n)$, $\Lambda^{n+3}(\mathbb{A}^n),\ldots$.

Now

$$\int_{\mathbb{A}} \widetilde{P}_n(x)\,St_{n+1}(x)x^i = \int_{\mathbb{A}} S_{n^n;\,0}(\mathbb{A};\,x)\,S_{n+1+i}(x-\mathbb{B}) = S_{n^n;\,n+1+i}(\mathbb{A};\mathbb{A}-\mathbb{B})$$
$$= \sum_{j=0}^{n+1} S_{n^n,\,n+1+i-j}(\mathbb{A})S_j(-\mathbb{B}) = S_{n^{n+1}}(\mathbb{A})\sum_{j=0}^{n+1} S_{1+i-j}(\mathbb{A}^n)S_j(-\mathbb{B})\ .$$

If $i\le n$, the sum is equal to $S_{1+i}(\mathbb{A}^n-\mathbb{B})$, therefore is 0 as wanted. As a matter of fact, the sum, for any $i\ge 0$, is equal to $(-1)^{n+1}S_{1^{n+1},\,i-n}(\mathbb{A}^n)$.

Ex. 8.38 Take the expression of a Schur function as a determinant of powers of the variables, divided by the Vandermonde. Then

$$\frac{S_{1^\beta;\,n^\alpha}(\mathbb{A}+\mathbb{B}-x;\,\mathbb{A}+\mathbb{B})}{S_{n^\alpha}(\mathbb{A}+\mathbb{B})} = \frac{\begin{vmatrix} A & C \\ B & D\end{vmatrix}}{\begin{vmatrix} A' & C \\ B' & D\end{vmatrix}}\,,$$

with $A = \left|S^1(a-x),\ldots,\,S^\beta(a-x)\right|_{a\in\mathbb{A}}$, $A' = \left|S^0(a-x),\ldots,\,S^{\beta-1}(a-x)\right|_{a\in\mathbb{A}}$, $B = \left|S^1(b-x),\ldots,\,S^\beta(b-x)\right|_{b\in\mathbb{B}}$, $B' = \left|S^0(b-x),\ldots,\,S^{\beta-1}(b-x)\right|_{b\in\mathbb{B}}$, $C = \left|a^{\beta+n},\ldots,\,a^{\alpha+\beta+n-1}\right|_{a\in\mathbb{A}}$, $D = \left|b^{\beta+n},\ldots,\,b^{\alpha+\beta+n-1}\right|_{b\in\mathbb{B}}$.

The determinant of D tends towards 0, the fraction tends towards $\det(B)\det(C)/\det(B')\det(C)$, i.e. towards $R(\mathbb{B},x)$, which is, up to a sign, the required factor of $R(x,\mathbb{A}+\mathbb{B})$. □

Supposing no root equal to 0, one can exchange the role of $\mathbb{A}$ and $\mathbb{B}$ by considering the reversed polynomial $\prod(x-1/a)\prod(x-1/b)$ and find the factor corresponding to $\mathbb{A}$.

Notice that, when $\alpha=1$, it is easy to exhibit functions having dominant term a^n, the quotient of two successive such functions having limit a. For example, one can approximate a by $S^{n+1}(a+\mathbb{B})/S^n(a+\mathbb{B})$, or $\Psi^{n+1}(a+\mathbb{B})/\Psi^n(a+\mathbb{B})$. This is Bernoulli's method to produce the root of biggest modulus, when it is unique.

Ex. 8.39 By recursion on n, one shows that the truncation of the series is equal to $S_{n^k}/S_{(n+1)^k}$, i.e. that one has

$$\frac{S_{(n-1)^k}}{S_{n^k}}+\frac{S_{n^k-1}S_{n^k+1}}{S_{n^k}S_{(n+1)^k}}=\frac{S_{(n)^k}}{S_{(n+1)^k}}\ .$$

On the other hand it is clear (and this is a special case of Wroński's result (Ex.8.38), that $S_{n^k}/S_{(n+1)^k}$ tends towards $1/(a_1\cdots a_k)$, since S_{n^k} has dominant term $(a_1\cdots a_k)^n$.

The result for $k=1$ is attributed to many authors, of various nationalities.

Considering the function $1/\prod(1-az)$, the preceding formula gives the product of its k-th smallest poles.

Ex. 8.40 The limit of

$$\frac{S_k(\mathbb{A}-a_1)}{S_{k+1}(\mathbb{A}-a_1)}=\frac{S_k(\mathbb{A})-a_1S_{k-1}(\mathbb{A})}{S_{k+1}(\mathbb{A})-a_1S_k(\mathbb{A})}$$

is $1/a_2$. Replace a_1 by $S_{k+2}(\mathbb{A})/S_{k+1}(\mathbb{A})-\epsilon_k$. Then

$$\frac{S_{k,k+1}(\mathbb{A})+\epsilon_kS_{k-1}(\mathbb{A})S_{k+1}(\mathbb{A})}{S_{k+1,k+1}(\mathbb{A})+\epsilon_kS_k(\mathbb{A})S_{k+1}(\mathbb{A})}\quad\text{tends towards}\quad\frac{1}{a_2}\ ,$$

but one cannot neglect the terms in ϵ_k! For example, when $\mathbb{A}=\{2,1\}$, $S_k(\mathbb{A})=2^{k+1}-1$, $\epsilon_k=1/(2^{k+2}-1)$, $S_{k,k+1}(\mathbb{A})=2^k(1+2)$, $S_{k+1,k+1}(\mathbb{A})=2^{k+1}$, and

$$S_{k,k+1}(\mathbb{A})/S_{k+1,k+1}(\mathbb{A})=3/2\neq1\ .$$

Ex. 8.41 Put $f(x)=(x-a)g(x)$. Then

$$f^i(x)/i!=(x-a)g^i(x)/i!+g^{i-1}(x)/(i-1)!\ .$$

Define $\mathbb{A}$ by $\Lambda^i(\mathbb{A})=g^i(x)/(i!\,g(x))$, and consider ϵ to be a rank-1 element.

Then $f^i(x)/(i!f(x))=\Lambda^i(\mathbb{A}+\epsilon^{-1})$, and therefore

$$D_n(f)=S_n(\mathbb{A}+\epsilon^{-1})\ .$$

Thus

$$x-\frac{D_{n-1}(f)}{D_n(f)}=a+\epsilon-\frac{S_{n-1}(\mathbb{A}+\epsilon^{-1})}{S_n(\mathbb{A}+\epsilon^{-1})}=a+\epsilon^{n+1}\frac{S_n(\mathbb{A})}{S_n(1+\epsilon\mathbb{A})}\ .$$

In the special case of $f(x)=1-2\sin(x)$, the determinants $D_n(f)$ have to be specialized under $f^{2i}(x)=0$, $f^{2i+1}(x)/f(x)=(-1)^i2/(2i+1)!$.

Another way to write the result is

$$\frac{1}{1-2sin(z)}=\sum z^nD_n\quad\&\quad D_{n-1}/D_n\sim\pi/6\ ,$$

but in fact the "approximation" of $1/(1-2sin(z))$ by $1/(1-z\pi/6)$ results from Bernoulli's method for approximating the biggest root of a polynomial $R(x,\mathbb{B})$ by $S^n(\mathbb{B})/S^{n-1}(\mathbb{B})$ or $\Psi_n(\mathbb{B})/\Psi_{n-1}(\mathbb{B})$.

Ex. 8.42 Take the determinant expressing a Schur function in terms of complete functions. Dividing each row by the power of q present in the first column, and each column by a power of q in such a way that the bottom row becomes q^0, $q^{2\nu_2-2\nu_1+2}$, $q^{2\nu_3-2\nu_1+4}$, ... one recognizes in the transformed determinant a Vandermonde in the powers of q figuring in the new bottom row. The hypothesis is needed to have no S_i of negative index in the determinant.

Finally, one gets

$$S_\nu = q^{\nu_1^2+\nu_2^2+\cdots} \prod_{1\le i<j\le \ell} (1-q^{2\nu_j-2\nu_i+2j-2i}) \ .$$

This implies, for the parameters in the continued fraction expansion,

$$\zeta_n = q^{2n-1}(q^{2n+2}+q^{2n}-1) \quad \& \quad \beta_n = q^{6n-4}(1-q^{2n}) \ .$$

For example,

$$x^{-1}+qx^{-2}+q^4x^{-3}+q^9x^{-4}+q^{16}x^{-5}+\cdots =$$
$$\cfrac{1}{x-q+\cfrac{q^2-q^4}{x-q^5-q^3+q+\cfrac{q^8-q^{12}}{x-q^9-q^7+q^3+\cfrac{q^{14}-q^{20}}{x-q^{13}-q^{11}+q^5+\cdots}}}}$$

Stieltjes gives an equivalent, but a little less simple, continued fraction.

Notice that the associated orthogonal polynomials have simple coefficients, because they are Schur functions indexed by partitions ν satisfying the hypothesis $\nu \supseteq (\ell{-}1)^\ell$.

For example, truncating the fraction as displayed above, one gets

$$P_4(x) = q^{28} - q^{21}\frac{1-q^8}{1-q^2}\,x + q^{14}\frac{(1-q^6)(1-q^8)}{(1-q^2)(1-q^4)}\,x^2 - q^7\frac{1-q^8}{1-q^2}\,x^3 + x^4 \ .$$

B.9

Ex. 9.1 This is Newton/Stirling interpolation at integral points.

Ex. 9.2 The required expansion is not given by Newton interpolation at integral points (called *Stirling interpolation*), because the polynomials that we are given have the same degree in x.

One notices that the determinant

$$\begin{vmatrix} x^n & \Lambda^n(x) & \cdots & \Lambda^n(x+n{-}1) \\ n^n & \Lambda^n(n) & \cdots & \Lambda^n(2n{-}1) \\ \vdots & \vdots & & \vdots \\ 1^n & \Lambda^n(1) & \cdots & \Lambda^n(n) \end{vmatrix}$$

is null, because the space of polynomials in x, divisible by x and of degree $\le n$, is of dimension n. Now, one rewrites the binomial coefficients in the determinant, as complete functions of $n+1$. For example, for $n=4$, the determinant becomes

$$\begin{vmatrix} x^4 & \Lambda^4(x) & \Lambda^4(x{+}1) & \Lambda^4(x{+}2) & \Lambda^4(x{+}3) \\ 4^4 & S^0(5) & S^1(5) & S^2(5) & S^3(5) \\ 3^4 & \cdot & S^0(5) & S^1(5) & S^2(5) \\ 2^4 & \cdot & \cdot & S^0(5) & S^1(5) \\ 1^4 & \cdot & \cdot & \cdot & S^0(5) \end{vmatrix}$$

The expansion along the first column and the first row produces the required relation

$$\sum_{0\le i,j\le n-1} \pm\Lambda^n(x+j)\,(n{-}i)^n\Lambda^{i-j}(n{+}1) = 0\ .$$

Ex. 9.3 Writing for clarity $\mathbb{X}=\{x_1,\ldots,x_n\}$, $\mathbb{Y}=\{x_{n+1},\ldots,x_{2n}\}$, and using the Newton interpolation formula for the x_i's and the y_j's, one recognizes that the determinant is equal to

$$\frac{1}{\Delta(\mathbb{X})\Delta(\mathbb{Y})}\left|\frac{f(x_i)-f(y_j)}{x_i-y_j}\right|\ .$$

Ex. 9.4 The expansion

$$\begin{aligned}&\Lambda^k(\mathbb{A}+m)\\ &=\Big(\Lambda^k(\mathbb{A})+\cdots+\Lambda^{k-n}(\mathbb{A})\Lambda^n(m)\Big)+\Big(\Lambda^{k-n-1}(\mathbb{A})\Lambda^{n+1}(m)+\cdots+\Lambda^{k-m}(\mathbb{A})\Lambda^m(m)\Big)\end{aligned}$$

shows that the second term of the RHS is the remainder. We are asked to prove that

$$\sum_{j=0}^{m-1}\Lambda^n(m-j-1)\,\Lambda^{k-n-1}(\mathbb{A}+j)=\Lambda^{k-n-1}(\mathbb{A})\Lambda^{n+1}(m)+\cdots+\Lambda^{k-m}(\mathbb{A})\Lambda^m(m)\ .$$

The equation being linear in $\Lambda^i(\mathbb{A})$, it is sufficient to prove it for $\mathbb{A}=-x$, for x of rank 1. Now, $\mathcal{R}$ can be interpreted as the remainder of $\Lambda^k(m-x)$ modulo x^{k-n}.

Ex. 9.5 We have to compute the image of the remainder under divided differences. The most appropriate expression is given by (3.1.6), the r-th remainder being congruent to $S_{\square;m}(\mathbb{A}{-}\mathbb{B}; x{-}\mathbb{B})$ modulo $R(x,\mathbb{A})$, with $m=card(\mathbb{B})$, $\square=(m{-}n{+}r)^{r-1}$. We are back to the interpolation of a complete function $S^k(x{-}\mathbb{B})$, and therefore the r-th remainder is equal to

$$\begin{aligned}&S_{\square;m}(\mathbb{A}{-}\mathbb{B}; a_1{-}\mathbb{B}) + S_{\square;m-1}(\mathbb{A}{-}\mathbb{B}; \mathbb{A}_2{-}\mathbb{B})(x{-}a_1)+\\ &\qquad\qquad + S_{\square;m-2}(\mathbb{A}{-}\mathbb{B}; \mathbb{A}_3{-}\mathbb{B})(x{-}a_1)(x{-}a_2)+\cdots\ ,\end{aligned}$$

writing, as usual, $\mathbb{A}_j=\{a_1,\ldots,a_j\}$, $j\in\mathbb{N}$.

The last remainder is the resultant $S_{m^n}(\mathbb{A}{-}\mathbb{B})$, and already of order n. But it can be written as $S_{m;\ldots;m;m}(\mathbb{A}_n{-}\mathbb{B};\ldots;\mathbb{A}_2{-}\mathbb{B};\mathbb{A}_1{-}\mathbb{B})$, and the coefficients in Newton interpolation can be realized as minors of this last matrix, using transformations of the type

$$S_{k;k;k;\star}(\mathbb{A}{-}\mathbb{B};\mathbb{A}{-}\mathbb{B};\mathbb{A}{-}\mathbb{B};\star)\to S_{k;k;k;\star}(\mathbb{A}_n{-}\mathbb{B};\mathbb{A}_{n-1}{-}\mathbb{B};\mathbb{A}_{n-2}{-}\mathbb{B};\star)\ .$$

For example, for $m = 5$, $n = 4$, the succesive remainders are equal, modulo $S^4(x-\mathbb{A})$, to

$$\begin{aligned}
S_{2^0;5}(\mathbb{A}-\mathbb{B};x-\mathbb{B}) &\equiv S_5(\mathbb{A}_1-\mathbb{B}) + S_4(\mathbb{A}_2-\mathbb{B})S^1(x-\mathbb{A}_1) + \\
&\quad +S_3(\mathbb{A}_3-\mathbb{B})S^2(x-\mathbb{A}_2) + S_2(\mathbb{A}_4-\mathbb{B})S^3(x-\mathbb{A}_3)\ , \\
S_{3;5}(\mathbb{A}-\mathbb{B};x-\mathbb{B}) &\equiv S_{3;5}(\mathbb{A}-\mathbb{B};\mathbb{A}_1-\mathbb{B}) + S_{3;4}(\mathbb{A}-\mathbb{B};\mathbb{A}_2-\mathbb{B})S^1(x-\mathbb{A}_1) + \\
&\quad +S_{3;3}(\mathbb{A}-\mathbb{B};\mathbb{A}_3-\mathbb{B})S^2(x-\mathbb{A}_2)\ , \\
S_{44;5}(\mathbb{A}-\mathbb{B};x-\mathbb{B}) &\equiv S_{44;5}(\mathbb{A}-\mathbb{B};\mathbb{A}_1-\mathbb{B}) + S_{44;4}(\mathbb{A}-\mathbb{B};\mathbb{A}_2-\mathbb{B})S^1(x-\mathbb{A}_1)\ , \\
S_{555;5}(\mathbb{A}-\mathbb{B};x-\mathbb{B}) &\equiv S_{555;5}(\mathbb{A}-\mathbb{B};\mathbb{A}_1-\mathbb{B}) = S_{5555}(\mathbb{A}-\mathbb{B})\ ,
\end{aligned}$$

and their coefficients are all minors of $S_{5;5;5;5}(\mathbb{A}_4-\mathbb{B};\mathbb{A}_3-\mathbb{B};\mathbb{A}_2-\mathbb{B};\mathbb{A}_1-\mathbb{B})$.

Ex. 9.6 Write $f(x) = S^n(x-\mathbb{B})$ (supposing the polynomial of exact degree n, we can normalize it). Using in each row of the determinant Newton's interpolation for $f(q^i x)$ at points $q^0, q^1, q^2, \ldots$, one transforms it into

$$\left| q^{ij} S^{n-j}(q^i + \cdots + q^{i+j} - \mathbb{B}) \right|_{0 \le i,j \le n}\ ,$$

up to the factor $\Delta(q^0, \ldots, q^n)$.

With q-integers $[i] := 1 + q + \cdots + q^{i-1}$, the determinant can be rewritten $\left| q^{ni} S^{n-j}([j+1] - \mathbb{B}/q^i) \right|$. Multiplying it by the inverse of the q-Pascal matrix $\left[S^{n-j-i}([j+1]) \right]$, which is no other than $\left[S^{n-i-j}(-[n-i]) \right]$, one transforms the determinant into

$$\left| q^{ni} S^{n-j}(-\mathbb{B}/q^i) \right| = \left| q^i S^{n-j}(-\mathbb{B}) \right|\ .$$

In all, one gets

$$\pm S^n(-\mathbb{B}) \cdots S^0(-\mathbb{B}) \Big(\Delta(q^0, \ldots, q^n) \Big)^2\ .$$

Ex. 9.7 This is Newton interpolation for the function $f(x)x^{-n}$ of the variable $z = 1/x$ at the points $1/b_i$, completing with the value $f(0)$. For example, for $n = 3$, and $g(x)$ of degree ≤ 2, one has

$$f(x)x^{-3} = g(x) = g(1/b_1) + g\partial_1(x^{-1} - b_1^{-1}) + g\partial_1\partial_2(x^{-1} - b_1^{-1})(x^{-1} - b_2^{-1})\ ,$$

and therefore, for f of degree ≤ 3,

$$\begin{aligned}
f(x) = g(b_1)x^3 + g\partial_1\, b_1^{-1}x^2(b_1-x) + g\partial_1\partial_2\, b_1^{-1}b_2^{-1}x(b_1-x)(b_2-x) \\
+ f(0)\, b_1^{-1}b_2^{-1}b_3^{-1}(b_1-x)(b_2-x)(b_3-x)\ ,
\end{aligned}$$

the last term being obtained by identification in $x = 0$.

Ex. 9.8 This is the Newton interpolation of the function $f(x) = (m-x)^{-1}$, at points $0, 1, \ldots, m-1$, specialized into $x = m+n$. The remainder

$$\frac{(x-0)\cdots(x-(m-1))}{(m-0)\cdots(m-(m-1))(m-x)} \quad \text{specializes in} \quad \frac{(n+m)\cdots(n+1)}{m!\,(-n)}\ ,$$

$f(x)$ specializes into $-1/n$.

Of course, the identity does not need to be restricted to positive integers m, n, provided that no 0 appears in denominator.

Ex. 9.9 Newton interpolation, for the function $f(x) = 2n/(z+x)$, in the interpolation points $\{0, 2, 4, \ldots\}$, with z specialized in $2n+3$ and x in $-2n-2$.

Ex. 9.10 Take Newton formula for $f(x) = S^{n-1}(x+y - \mathbb{A})$, in the interpolation points $a_1, \dots, a_n$:

$$f(x) = S^{n-1}(y + a_1 - \mathbb{A}) + (x-a_1)S^{n-2}(y + a_1+a_2-\mathbb{A}) + \cdots$$

Each of the complete functions factorizes, and one recognizes the required formula. Notice that the left-hand side vanishes for $x = a_n$, $y = a_1$.

Ex. 9.11 Ordering $\mathbb{B}$, take the image of Newton's interpolation of b_1^k at points $\mathbb{A}$ under the isobaric divided difference π_ω acting on $\mathbb{B}$, ω being the maximal permutation in $\mathfrak{S}(\mathbb{B})$.

Ex. 9.12 Start with Newton's interpolation of $(z-a_1)^{-1}$ at points $\mathbb{B}$, which states $(z - a_1)^{-1} = \sum S^k(a_1 - \mathbb{B}_k)/R(z, \mathbb{B}_{k+1})$. Multiplying by $S^n(a_1 - \mathbb{E})$, one takes the image of both sides under $\partial_1 \cdots \partial_n$. On the left, $S^n(a_1-\mathbb{E})(z-a_1)^{-1} = (-1)^{n-1}S_{1^n;n}(\mathbb{A}-z; a_1-\mathbb{E})/R(z, \mathbb{A})$ gives

$$(-1)^{n-1}S_{1^n;0}(\mathbb{A} - z; \mathbb{A} - \mathbb{E})/R(z, \mathbb{A}) = R(z, \mathbb{E})/R(z, \mathbb{A}) \ .$$

On the other hand, $S^n(a_1 - \mathbb{E})S^k(a_1 - \mathbb{B}_k) = S^{n+k}(a_1 - \mathbb{E} - \mathbb{B}_k)$ is sent onto $S^k(\mathbb{A} - \mathbb{E} - \mathbb{B}_k)$.

Ex. 9.13 Proof. Recall that the discrete Wrońskian is equal to $\det(f_i(x_j))/\Delta(\mathbb{X})$. The integral

$$\int_{y_2}^{y_1} f_1(x_1)dx_1 \cdots \int_{y_{n+1}}^{y_n} f_n(x_n)dx_n$$

is equal to $\big(F_1(y_2) - F_1(y_1)\big) \cdots \big(F_n(y_{n+1}) - F_n(y_n)\big)$. By summation on the symmetric group $\mathfrak{S}(\mathbb{X})$, one gets that the integral to evaluate is equal to

$$W(1, F_1, \dots, F_n; y_1, \dots, y_{n+1}) \ .$$

In particular, for $f_i(x) = x^i$, $i = 1, \dots, n$, one gets that the integral of $\Delta(x_1, \dots, x_n)$ is equal to $\Delta(y_1, \dots, y_{n+1})/(n+1)!$.

Ex. 9.14 This determinant is obtained from the discrete Wrońskian of $(1, Y_k, Y_{(2k)}, \dots, Y_{(nk)})$ by suppressing its first row and first column. The periodicity conditions are such that $Y_{(jk)} = (Y_k)^j = (a_1 - b_1)^j \cdots (a_1 - b_k)^j$, $j \geq 0$.

Let $f(x) = (x-b_1)\cdots(x-b_k)$. The Wrońskian is equal to $\prod_{i<j} \left(f(a_i) - f(a_j)\right)/(a_i - a_j)$, and therefore the original determinant is equal to the product

$$\prod_{1 \leq i < j \leq n+1} Y_{0,k-1}\left(\{a_i, a_j\}, \mathbb{B}\right) = \prod_{i<j} S^{k-1}(a_i + a_j - \mathbb{B}) \ .$$

In the case where $a_1, \dots, a_{n+1}$ tends towards x, then the determinant tends (up to the factor $1!\,2! \cdots n!$) towards the Wrońskian of derivatives f', $(f^2)'$, ..., $(f^n)'$, and each $S_{k-1}(a_i + a_j - \mathbb{B})$ tends towards f'. Wroński did obtain that the Wrońskian of $f', \dots, (f^n)'$ be equal to $1! \cdots n!\,(f')^{n(n-1)/2}$.

Segar (Muir IV, p.185) generalizes Wroński's identity by taking a Wrońskian of arbitrary powers of f instead of the powers $0, 1, 2 \ldots$.

One can replace the sequence $j = 1, \dots, n$ in the determinant under scrutinity by any sequence of positive numbers.

For example, take $n = 2$, $k = 3$, $\{j\} = \{4, 6\}$. Then the determinant $\begin{vmatrix} Y_{0,11} & Y_{0,17} \\ Y_{0,0,10} & Y_{0,0,16} \end{vmatrix}$ is the discrete Wrońskian of Y_0, Y_{12}, Y_{18} and factorizes into $S_{3,14}(\{f(a_1), f(a_2), f(a_3)\})$ times the Wrońskian of Y_0, Y_3, Y_6, where $f(x) := (x - b_1)(x - b_2)(x - b_3)$.

Ex. 9.15 An example will suffice. For $n = 4$, the right factor of V factorizes into

$$\begin{bmatrix} S^0(\mathbb{A}_1) & S^1(\mathbb{A}_1) & S^2(\mathbb{A}_1) & S^3(\mathbb{A}_1) \\ . & S^0(\mathbb{A}_2) & S^1(\mathbb{A}_2) & S^2(\mathbb{A}_2) \\ . & . & S^0(\mathbb{A}_3) & S^1(\mathbb{A}_3) \\ . & . & . & S^0(\mathbb{A}_4) \end{bmatrix} = \begin{bmatrix} 1 & a_1 & . & . \\ . & 1 & a_2 & . \\ . & . & 1 & a_3 \\ . & . & . & 1 \end{bmatrix} \begin{bmatrix} 1 & . & . & . \\ . & 1 & a_1 & . \\ . & . & 1 & a_2 \\ . & . & . & 1 \end{bmatrix} \begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & a_1 \\ . & . & . & 1 \end{bmatrix}$$

while the left one is

$$\begin{bmatrix} 1 & . & . & . \\ 1 & a_2 - a_1 & . & . \\ . & 1 & a_3 - a_1 & . \\ . & . & 1 & a_4 - a_1 \end{bmatrix} \begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & 1 & a_3 - a_2 & . \\ . & . & 1 & a_4 - a_2 \end{bmatrix} \begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & 1 & a_4 - a_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & . & . & . \\ 1 & R(a_2, \mathbb{A}_1) & . & . \\ 1 & R(a_3, \mathbb{A}_1) & R(a_3, \mathbb{A}_2) & . \\ 1 & R(a_4, \mathbb{A}_1) & R(a_4, \mathbb{A}_2) & R(a_4, \mathbb{A}_3) \end{bmatrix}$$

Ex. 9.16 Use that the coefficient of a Schubert polynomial X_σ in a sum $f = \sum c_\nu X_\nu$ is equal to $f\partial_{\sigma^{-1}}(\mathbb{A} = 0)$. Thus, one has to compute the image of $det(M - z)$ under sequences of divided differences. However, one has that $\Lambda^k(\mathbb{A}_i)\partial_i = \Lambda^{k-1}(\mathbb{A}_{i+1})$. One sees that at each step, one has a determinant with different alphabets in each row (except in rows where no $\mathbb{A}$ appears). In the case where $\mathbb{A}_i$ and $\mathbb{A}_{i-1}$ both appear in a determinant, then ∂_i transforms row $\mathbb{A}_i$ into a row identical to row $\mathbb{A}_{i-1}$, apart from the occurrence of a 'z'. Thus the determinant reduces to the cofactor of this z, and one can iterate.

In all, one finds a sum of $\binom{n}{3}$ terms, with coefficients of the type $\pm z^i(z^2-1)^j(z-1)^\epsilon$, $i, j \in \mathbb{N}$, $\epsilon = 0, 1$. A more precise description would require introducing an appropriate representation of products of divided differences.

For example, for $n = 4$, one has

$$\begin{vmatrix} -z & . & . & 1 \\ . & -z & 1 & Y_1 \\ . & 1 & Y_{01} - z & Y_{11} \\ 1 & Y_{001} & Y_{011} & Y_{111} - z \end{vmatrix} =$$

$$(z-1)^2(z+1)^2 Y_0 - z^3 Y_{111} - z^2 Y_{1010} - z^2 Y_{22} + z Y_3 - z(z-1)(z+1)Y_{01} \ .$$

The next case, $n = 5$, gives $-(z+1)^2(z-1)^3 Y_0 + z^3 Y_{222} + z^2(z-1)Y_{1001} + z^3 Y_{1201} + z^4 Y_{1111} - z Y_4 + z(z-1)(z+1)Y_{02} - z^2 Y_{33} + z^2(z-1)(z+1)Y_{011} - z^2 Y_{202}$.

Ex. 9.17 The two families of polynomials are compatible with the q-derivative $f \to D_q(f) := (f(x) - f(qx))\,(x - qx)^{-1}$. Indeed, one easily checks that

$$D_q(H_n) = \frac{1-q^n}{1-q} H_{n-1} \quad \& \quad D_q(Y_n) = \frac{1-q^n}{1-q} Y_{n-1} \ .$$

Adding the values in $x=-1$ fixes then the relations between the two bases. One finds

$$\begin{aligned} H_n &= Y_n + \frac{(1-q^n)(1-q^{n-1})}{1-q^2} Y_{n-2} + \frac{(1-q^n)\cdots(1-q^{n-3})}{(1-q^2)(1-q^4)} Y_{n-4} + \cdots \\ Y_n &= H_n - \frac{(1-q^n)(1-q^{n-1})}{1-q^2} H_{n-2} + q^2 \frac{(1-q^n)\cdots(1-q^{n-3})}{(1-q^2)(1-q^4)} H_{n-4} - \\ & \qquad q^6 \frac{(1-q^n)\cdots(1-q^{n-5})}{(1-q^2)(1-q^4)(1-q^6)} H_{n-6} + \cdots \end{aligned}$$

(there are signs and powers $q^{i(i-1)}$ in the second formula).

For more considerations about such expansions, see Kupershmidt [**91**].

Ex. 9.18 Multiplying both members by $x(x-[1])\cdots(x-[n-1])$, I claim that the identity to show becomes the Newton interpolation of $f(x)=(x-[n])\cdots(x-[2n-1])$ at the points $[n-1]$, $[n-2], \ldots$, $[0]$. Thus one has only to check the specialization of the Schubert polynomial $Y_{0^i,n-i}$, $i \le n$, in $\mathbb{A}=\{[n-1], [n-2], \ldots\}$, $\mathbb{B}=\{[n],[n+1],\ldots\}$. Multiplying each a_i, b_j by $q-1$ just introduces powers of $q-1$. Therefore, one can as well compute the specialization for the alphabets $\{(q^n-1),(q^{n-1}-1,\ldots\}$ and $\{(q^{n+1}-1),(q^{n+2}-1,\ldots\}$. Schubert polynomial being invariant under a uniform shift $a_i \to a_i+1$, $b_j \to b_j+1$, one is finally reduced to evaluate some $Y_{0^ik}(q^n, q^{n-1}, \ldots ; q^{n+1}, q^{n+2}, \ldots)$. This a q-specialization of a q-factorial Schur function, that it is immediate to evaluate on its determinantal expression.

B.10

Ex. 10.1 $\Delta(\mathbb{A}) = \mathbb{X}_\omega(\mathbb{A}, \mathbb{A}^\omega)$, and therefore, according to (10.2.10), it expands into

$$\Delta(\mathbb{A}) = \sum_{\sigma \in \mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)} X_{\sigma\omega}(\mathbb{A}) X_\sigma(\mathbb{A}^\omega) \ .$$

Similarly,

$$S_{1\ldots n-1}(\mathbb{A}) = \prod_{i<j}(a_i+a_j) = \sum_{\sigma \in \mathfrak{S}(\mathbb{A})} X_{\sigma\omega}(\mathbb{A}) X_\sigma(\mathbb{A}^\omega) \ .$$

Thus, the sum $\prod_{i<j}(a_i-a_j)+\prod_{i<j}(a_i+a_j)$ expands in terms of permutations of even length, the difference, in terms of permutations of odd length. These expressions can be used to describe invariants of the alternating group.

Ex. 10.2 Multiplying by the symmetrical factor $\prod_{a\in\mathbb{A}}(x-a)^3$, one has to compute the image of

$$(x-a_1)^3(x-a_2)^2(x-a_3) = \mathbb{X}_{4321}(\mathbb{A}, \mathbb{B}) \ ,$$

with $\mathbb{B}=\{x,x,x,x\}$, and therefore one gets $\mathbb{X}_{\omega\sigma}(\mathbb{A},\mathbb{B})R(x,\mathbb{A})^{-3}$.

Ex. 10.3 Each factor $x-q^k\alpha-[k]\beta$ can be rewritten $(x-\gamma)-q^k(\alpha-\gamma)$, with $\gamma=\beta(1-q)^{-1}$. Therefore, the required quotient is obtained from a q-factorial Schur function under an easy modification.

Ex. 10.4 The matrix is the commutative product of

$$\begin{bmatrix} a_1-b & 1 & 0 & & \\ 0 & \ddots & \ddots & & \\ & & & & 1 \\ & & & & a_n-b \end{bmatrix} \ , \ b \in \{b_1, \ldots, b_k\} \ ,$$

as can be checked by induction on k, using the relations

$$Y_{0^{m-1}j} = Y_{0^{m-2}j} + (a_m - b_{j+m-1})\, Y_{0^{m-1}j-1}\ .$$

Ex. 10.5 The required inverse is the matrix obtained by replacing each Schubert polynomial Y_ν by $\pm Y_{\nu^{-1}}$, each entry of the product of the two matrices being obtained by Cauchy formula (10.2.10).

For example, for m= 0, $n=2$, the following two matrices are inverse from each other :

$$\begin{bmatrix} 1 & Y_{01} & Y_{02} & Y_{11} & Y_{12} & Y_{22} \\ 0 & 1 & Y_{001} & Y_1 & Y_{101} & Y_{201} \\ 0 & 0 & 1 & 0 & Y_1 & Y_2 \\ 0 & 0 & 0 & 1 & Y_{001} & Y_{011} \\ 0 & 0 & 0 & 0 & 1 & Y_{01} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -Y_{01} & Y_{011} & Y_{200} & -Y_{201} & Y_{22} \\ 0 & 1 & -Y_{001} & -Y_1 & Y_{101} & -Y_{12} \\ 0 & 0 & 1 & 0 & -Y_1 & Y_{11} \\ 0 & 0 & 0 & 1 & -Y_{001} & Y_{02} \\ 0 & 0 & 0 & 0 & 1 & -Y_{01} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Of course, the same method gives the inverse of a matrix where, instead of using the family $Y_{[I]}$, $I \in \mathcal{I}$, one takes a permutation ζ and all the Schubert polynomials X_η, where η runs over all permutations lower than ζ in the permutohedron.

Notice also that the above pair of matrices looks more mysterious when specializing $\mathbb{B}$ to $\{1, q, q^2, \ldots\}$ (factorial skew Schur functions), or even when $\mathbb{B} = \mathbf{0}$. This exercise generalizes Ex. 1.17, where we used only complete functions.

Ex. 10.6 The only monomials in the expansion of $Todd$ which give a non-zero contribution are $(x^\rho)^\sigma$, $\rho = [n{-}1, \ldots, 0]$, $\sigma \in \mathfrak{S}_n$. Since the image of x^ρ under ∂_ω is 1, the image of $Todd$ is the same as the image of

$$\sum_{(I,\sigma):\sigma\in\mathfrak{S}_n,\, I=\sigma\rho} (n-1)^{i_1} \cdots 0^{i_n} \frac{\binom{n}{2}!}{i_1! \cdots i_n!} x^I\ .$$

On the other hand,

$$\sum (-1)^{\ell(\sigma)} (n{-}1)^{i_1} \cdots 0^{i_n} = \Delta(n{-}1, \ldots, 0) = (n{-}1)! \cdots 1!\ .$$

Therefore the image of $Todd$ is $\binom{n}{2}!$.

Ex. 10.7 For general $\mathbb{A}, \mathbb{B}$, we have $\mathbb{X}_\sigma(\mathbb{A}, \mathbb{B}) = \sum (-1)^{\ell(\nu)} X_\nu(\mathbb{B}) X_\eta(\mathbb{A})$, sum over reduced factorizations $\nu^{-1}\eta$ of σ. But

$$\sum_{\nu\in\mathfrak{S}_n} (-1)^{\ell(\nu)} X_\nu(\{b, 0, 0, \ldots\}) u_\nu = (1 - bu_{n-1}) \cdots (1 - bu_1)\ .$$

Taking $\zeta = \sigma^{-1}$, the two facts above can be written in terms of divided differences :

$$(-1)^{\ell(\sigma)} = \mathbb{X}_\zeta(\{b, 0, 0, \ldots\}, \mathbb{A}) = \mathbb{X}_\zeta(1 - \partial_1 b) \cdots (1 - \partial_{n-1} b)$$

the right side being evaluated in $(0, \mathbb{A})$.

In the Grassmannian case, J being the code of σ, one has $\mathbb{X}_\sigma(\mathbb{A}, \{b, 0, 0, \ldots\}) = S_J(\mathbb{A}{-}b)$, and the expansion of $S_J(\mathbb{A}{-}b)$ produces a sum over all partitions obtained from J by removal of a vertical strip. This property is more generally true for a *vexillary permutation*, though defining vertical and horizontal strips require some work, compared to the case of a partition.

For example, $\mathbb{X}_{1475236}(\mathbb{A},\mathbb{B}) = \mathbb{Y}_{0242}(\mathbb{A},\mathbb{B}) = \mathbb{Y}_{033011}(\mathbb{B},\mathbb{A})$, and

$$\mathbb{Y}_{033011}(1-\partial_1 b)\cdots(1-\partial_6 b) = \begin{array}{cc} \multicolumn{2}{c}{\mathbb{Y}_{033011}} \\ -b\mathbb{Y}_{030211} & -b\mathbb{Y}_{033010} \\ +b^2\mathbb{Y}_{030111} & +b^2\mathbb{Y}_{030210} \\ \multicolumn{2}{c}{-b^3\mathbb{Y}_{030110}} \end{array}$$

which by inversion of permutation, gives

$$\begin{array}{cc} \multicolumn{2}{c}{\mathbb{Y}_{0242}} \\ -b\mathbb{Y}_{0142} & -b\mathbb{Y}_{0232} \\ +b^2\mathbb{Y}_{0141} & +b^2\mathbb{Y}_{0132} \\ \multicolumn{2}{c}{-b^3\mathbb{Y}_{0131}} \end{array} .$$

Ex. 10.8 It is better to introduce anotherl alphabet $\{x_1,\ldots,x_n\}$, apart from $\mathbb{A} = \{a_1,\ldots,a_n\}$, $\mathbb{B} = \{b_1,\ldots,b_n\}$. Let us assume that divided differences act on the x_i's only, and write Θ for the specialization of $x_1,\ldots,x_n$ to 0.

Then, a direct consequence of the Cauchy kernel is that

$$\mathbb{X}_\sigma(x_1,\ldots,x_4;\mathbf{0}) \begin{array}{lll} 1+\partial_3 b_1 & & \\ 1+\partial_2 b_1 & 1+\partial_3 b_2 & \\ 1+\partial_1 b_1 & 1+\partial_2 b_2 & 1+\partial_3 b_3 \end{array} \quad \Theta = \mathbb{X}_{\sigma^{-1}}(\mathbb{B},\mathbf{0}) \ ,$$

where the array is a product of divided differences acting on its left, which is read by successive rows, from top to bottom (it could also be read by columns downwards, from left to right). The reader will agree that $n=4$ is generic enough.

Similarly, and more generally,

$$\mathbb{X}_\sigma(x_1,\ldots,x_4;\mathbf{0}) \begin{array}{lll} 1+\partial_3(a_3-b_1) & & \\ 1+\partial_2(a_2-b_1) & 1+\partial_3(a_2-b_2) & \\ 1+\partial_1(a_1-b_1) & 1+\partial_2(a_1-b_2) & 1+\partial_3(a_1-b_3) \end{array} \quad \Theta \ = \mathbb{X}_\sigma(\mathbb{A},\mathbb{B}) \ .$$

Therefore, the two triangles of operators, followed by the specialization Θ, constitute an answer.

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