About the "y" in the χ_y- characteristic of Hirzebruch

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Abstract. We indicate how Hecke algebras, Yang-Baxter equation, Hall-Littlewood polynomials, Macdonald polynomials, are related to the parameter y that Hirzebruch introduced in his study of Riemann-Roch theorem.

Geometers want to compute the dimensions of spaces of cohomology $H^i(\mathcal{M}, V \otimes \Omega^j)$, where V is a vector bundle and Ω^j the sheaf of j-differential forms over a complex manifold $\mathcal{M}$. In his study of Riemann-Roch theorem, Hirzebruch introduced a generating function

$$\sum_{i,j} (-1)^i \, y^j \, dim(H^i(\mathcal{M}, V \otimes \Omega^j))$$

that he called χ_y*-genus* or χ_y*-characteristic*.

This characteristic or its specialization for different values of y can be met in many places in [**Hi**], [**HBJ**], [**At**]. I shall only speak of the case of a relative flag variety and shall enumerate many missed opportunities of giving to the extra parameter "y" the status it deserves.

1. Relative flag variety

Let V be a vector bundle of rank n over a complex manifold $\mathcal{M}$ and $\mathcal{F}(V) \mapsto \mathcal{M}$ the associated relative flag variety, the fibers of which are the usual flag variety of complete flags of vector spaces

$$V_1 \hookrightarrow V_2 \hookrightarrow \cdots \hookrightarrow V_n = \mathbb{C}^n$$

of successive ranks $1, 2, \ldots, n$.

The variety $\mathcal{F}(V)$ comes equipped with *tautological line bundles* $L_1, \ldots, L_n$. Their Chern classes generate the cohomology ring $H(\mathcal{F}(V))$ (with coefficients in $H(\mathcal{M})$), and their classes generate the Grothendieck ring $K(\mathcal{F}(V))$ of classes of

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vector bundles (with coefficient in $K(\mathcal{M})$). We shall denote the first Chern classes, as well as the classes of the line bundles by the variables $x_1, \ldots, x_n$.

The cohomology and Grothendieck rings are thus quotient of the ring of polynomials in $x_1, \ldots, x_n$, modulo some ideals generated by symmetric functions.

When $\mathcal{M}$ is a point, symmetric functions have to be specialized to their constant term in the case of the cohomology ring, while they have to be specialized to their value in $x_1 = 1, \ldots, x_n = 1$ in the case of the Grothendieck ring.

We shall rather lift computations at the level of the ring of (analytic) functions in $x_1, \ldots, x_n$, using its structure of free module of dimension $n!$ over the ring $\mathfrak{S}ym$ of symmetric functions.

At this level, we do not distinguish between elements of $H(\mathcal{F}(V))$ and $K(\mathcal{F}(V)$. However, the two Gysin morphisms from $H(\mathcal{F}(V))$ to $H(\mathcal{M})$, and from $K(\mathcal{F}(V))$ to $K(\mathcal{M})$ are different. For the cohomology, it is

$$\partial_\omega : \quad H(\mathcal{F}(V)) \ni f(x_1, \ldots, x_n) \longrightarrow \sum_{\mu \in \mathfrak{S}(n)} \Big(\frac{f}{\prod_{i<j}(x_i - x_j)}\Big)^\mu$$

while it is

$$\pi_\omega : \quad K(\mathcal{F}(V)) \ni f(x_1, \ldots, x_n) \longrightarrow \sum_{\mu \in \mathfrak{S}(n)} \Big(\frac{f}{\prod_{i<j}(1 - x_j/x_i)}\Big)^\mu$$

for the Grothendieck ring, denoting by f^μ the image of a function $f(x_1, \ldots, x_n)$ under a permutation μ belonging to the n-th symmetric group $\mathfrak{S}(n)$ acting on the x_i.

The two rings are connected by the Chern morphism $x \mapsto e^x$. However, one summation involves dividing by the Vandermonde in the x_i, while the other uses the Vandermonde in the exponentials e^{x_i}. To get a commutative diagram where the vertical arrows are the two Gysin morphisms, one must introduce the correcting factor of proportionality between the two summations

$$Todd := \prod_{i<j} (x_i - x_j) \,/\, (1 - e^{x_j - x_i}) \ ,$$

$$\begin{tikzcd}
K(\mathcal{F}) \arrow[r, "Chern"] \arrow[d, "\pi_\omega"'] & H(\mathcal{F}) \arrow[r, "Todd"] & H(\mathcal{F}) \arrow[d, "\partial_\omega"] \\
K(\mathcal{M}) \arrow[rr, "Chern"] & & H(\mathcal{M})
\end{tikzcd}$$

Knowing that the class of the relative cotangent bundle in $K(\mathcal{F}(V))$ is $\sum_{i<j} x_j/x_i$, one recognizes into the factor of proportionality between the two above summations the *Todd class* of the relative cotangent bundle $\Omega_{\mathcal{F}/\mathcal{M}}$ and in the commutative diagram, Riemann-Roch theorem for a relative flag variety.

The weighted sum of all the classes of the exterior powers of $\Omega_{\mathcal{F}/\mathcal{M}}$ is

$$\prod_{i<j} (1 + y x_j/x_i) \ .$$

Thus, Hirzebruch's characteristic χ_y is the morphism

$$K(\mathcal{F}(v)) \ni f(x_1, \ldots, x_n) \to \sum_{\mu \in \mathfrak{S}(n)} \big(f \prod_{i<j} \frac{x_i + yx_j}{x_i - x_j}\big)^\mu .$$

It is $K(\mathcal{M})$-linear, and thus is determined by its values on monomials (it would be sufficient, since $K(\mathcal{F}(V))$ is a free module of rank $n!$, to give only $n!$ values, but one would have then to use interpolation technics to compute the image of a general polynomial).

2. Hall-Littlewood polynomials

The above summation, when the function f is a *dominant* monomial, taking the factor $\prod_{i<j} \frac{x_i - yx_j}{x_i - x_j}$ instead of $\prod_{i<j} \frac{x_i + yx_j}{x_i - x_j}$, has been used by Littlewood to define *Hall-Littlewood polynomials* (cf. [**Li**],[**Ma**]). Denote it by χ.

However, the $\chi(x^I)$, $I \in N^n$, are not linearly independent over $\mathbb{C}(t)$. A basis is given by the Hall-Littlewood polynomials $Q_I := \chi(x^I)$, I *dominant* (i.e. $I = [i_1, i_2, \ldots, i_n]$, $i_1 \geq i_2 \geq \cdots \geq i_n \geq 0$; one also says that I is a *partition*).

In the case when the monomial is not dominant, one must "straighten" it into a sum of dominant monomials. This amounts using the following relations, where $\star$ denotes a function not depending on x_i, x_{i+1}

$$\chi\left(\star\, (x_i^{a+b} x_{i+1}^b + x_i^a x_{i+1}^{a+b})\,(yx_i - x_{i+1})\right) = 0 \qquad (\clubsuit)$$

(which come from the fact that the image of a function symmetrical in x_i, x_{i+1}, under the morphism $\sum (-1)^{\ell(\mu)} \mu$, is zero).

Now, from the tables of these polynomials, one reads dimensions of cohomology spaces or the image of any element in the Grothendieck ring $K(\mathcal{F}(V))$ under the χ_y characteristic of Hirzebruch.

For example, to get the cohomology of the sheaf of differential forms of any order on the Grasmannian $G_{3,2}(\mathbb{C}^5)$, twisted by the fourth power of the line bundle giving the Plücker embedding, one must look at $Q_{44400}(x_1, \ldots, x_5)$. The precise decoding is given in section 12.

3. Associativity

Flags of vector spaces can be split into pieces. This induces factorisations of the χ_y characteristic . Another way to formulate this splitting is to state that there is an associative law on Hall-Littlewood polynomials indexed by integral vectors (i.e. the set of all $\chi(x^I)$, $I \in \mathbb{N}^\star$), which is just the concatenation of the indexing vectors. Any Hall-Littlewood polynomial can be decomposed (uniquely) into the basis of standard Hall-Littlewood polynomials indexed by dominant vectors by successive applications of ($\clubsuit$), that is to say, according to the following commutation rules

$$y[\cdots, a{+}1, b, \cdots] + y[\cdots, b{+}1, a, \cdots] - [\cdots, a, b{+}1, \cdots] - [\cdots, b, a{+}1, \cdots] = 0 \quad (\clubsuit)$$

The special case $y = 0$ has many applications. In fact, it is the translation, at the level of the Grothendieck ring, of Bott's theorem for the cohomology of homogeneous bundles on a relative grassmannian $G_{q,r}(V)$. This variety has a *tautological quotient* Q of rank q and a *tautological kernel* R of rank r. Two dominant weights

I, J determine two irreducible representations of the linear groups $Gl(q)$ and $Gl(r)$ and two homogenous bundles $S_I(Q)$ and $S_J(R)$, the classes of which in $K(G_{q,r})$ are two Schur functions. Bott's theorem is that the cohomology of the tensor product $S_I(Q) \otimes S_J(R)$ is either zero, or concentrated in only one degree, and in that case is equal to an irreducible representation of $Gl(q+r)$. The class of this representation is the Schur function of index the concatenation of I and J. In the case that the index is not dominant, recall that the rule for straightening Schur functions is

$$S_{\dots,a,b,\dots} = -S_{\dots,b-1,a+1,\dots} \tag{$\diamond$}$$

4. Yang-Baxter equation

Geometry however indicates that the description of the cohomology or the Grothendieck ring of $\mathcal{F}(V)$ is a pure algebraic consequence of the case of a $\mathbb{P}^1$-fibration (i.e. the case where V is of rank 2).

Indeed, calling x_i and x_{i+1} the fisrt Chern classes, or the classes of the two tautological line bundles of $\mathcal{F}(V)$, the Gysin morphism for the cohomology is *Newton's divided difference*

$$\partial_i : \ f \mapsto \frac{f(x_i, x_{i+1}) - f(x_{i+1}, x_i)}{x_i - x_{i+1}}$$

and the Gysin morphism for the Grothendieck ring is

$$\pi_i : \ f \mapsto \frac{x_i f(x_i, x_{i+1}) - x_{i+1} f(x_{i+1}, x_i)}{x_i - x_{i+1}} \ .$$

Now these elementary operators satisfy the so-called *braid relations* (due to Moore, and Coxeter for more general groups):

$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \quad \text{and} \quad \partial_i \partial_j = \partial_j \partial_i \ , \ |i-j| > 1$$

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} \quad \text{and} \quad \pi_i \pi_j = \pi_j \pi_i \ , \ |i-j| > 1$$

This allows to define by products a divided difference ∂_μ and an isobaric divided difference π_μ for any $\mu \in \mathfrak{S}(n)$. In particular, the two Gysin morphisms for a relative flag variety correspond to the maximal permutation $\omega := [n, \dots, 1]$. For any *reduced* decomposition of ω one gets a factorisation of ∂_ω and of π_ω. Thus, for $n = 4$

$$\partial_{[4,3,2,1]} = \partial_1 \partial_2 \partial_1 \partial_3 \partial_2 \partial_1 = \cdots = \partial_3 \partial_2 \partial_1 \partial_3 \partial_2 \partial_3$$
$$\pi_{[4,3,2,1]} = \pi_1 \pi_2 \pi_1 \pi_3 \pi_2 \pi_1 = \cdots = \pi_3 \pi_2 \pi_1 \pi_3 \pi_2 \pi_3$$

Having reduced Gysin morphisms to the case of $\mathbb{P}^1$-fibration, one expects a similar success in defining the i-th elementary characteristic to be

$$\chi_i : \ f \mapsto \frac{(tx_i - x_{i+1}) f(x_i, x_{i+1}) - (tx_{i+1} - x_i) f(x_{i+1}, x_i)}{x_i - x_{i+1}}$$

However

$$\chi_1 \chi_2 \chi_1 \neq \chi_2 \chi_1 \chi_2 \ .$$

It can be corrected by introducing some shifts

$$\chi_1 \, (\chi_2 - \frac{t}{1+t}) \, \chi_1 = \chi_2 \, (\chi_1 - \frac{t}{1+t}) \, \chi_2 \ ,$$

and this product happens to be a factorisation of the characteristic for $n = 3$. We have now replaced the factor $\prod_{i<j}(x_i - yx_j)$ by $\prod_{i<j}(tx_i - x_j)$ but keep the same notation χ, in spite of the change of parameter.

More generally, looking at the action of the operators on a specific basis of the ring of polynomials (as a free module over $\mathfrak{S}ym$) one can check that χ factorizes into simple factors $\chi_i - (t + \cdots + t^k)/(1 + t + \cdots + t^k)$ which involve shifts by the t-integers $[k] := 1 + t + \cdots + t^{k-1}$.

For example, for $n = 4$, one has the following decomposition of χ :

$$\chi = \chi_1\,(\chi_2 - \frac{t}{1+t})\,\chi_1\,(\chi_3 - \frac{t+t^2}{1+t+t^2})\,(\chi_2 - \frac{t}{1+t})\,\chi_1\ ,$$

the general law being not difficult to infer for the special reduced decompositions of the same type. But we must have recourse to the more general Yang-Baxter equation [**Ya**] to explain the preceeding decomposition (a, b, c being any distinct constants)

$$(\chi_1 - \frac{ta-b}{a-b})(\chi_2 - \frac{ta-c}{a-c})(\chi_1 - \frac{tb-c}{b-c}) = (\chi_2 - \frac{tb-c}{b-c})(\chi_1 - \frac{ta-c}{a-c})(\chi_2 - \frac{ta-b}{a-b})$$

These relations (for any pair $i, i+1$), together with the trivial commutations $(\chi_i + a)(\chi_j + b) = (\chi_j + b)(\chi_i + a)$, $|i - j| > 1$, allow us to define $n!$ Yang-Baxter elements indexed by permutations in $\mathfrak{S}(n)$, taking arbitrary reduced decompositions of these permutations. One can use these elements to recover the classes of Schubert varieties in the cohomology ring, or the classes of their structure sheaves in the Grothendieck ring [**LLT**].

For the χ_y characteristic of Hirzebruch, one has to use the special parameters $1, t, t^2, \ldots t^{n-1}$. In other words, *Hirzebruch's characteristic for a relative flag manifold is the specialization of the Yang-Baxter element corresponding to the maximal permutation* $[n, \ldots, 1]$*, taking the "spectral parameters"* $1, t, \ldots, t^{n-1}$.

This is not yet the way that we shall choose to compute in flag manifolds.

5. Iwahori-Hecke algebra

Defining $T_i := \chi_i - 1$, one sees that these last elements satisfy the braid relations

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad \text{and} \quad T_iT_j = T_jT_i\ ,\ |i-j| > 1\ .$$

However, contrary to the two degenerate cases $\partial_i^2 = 0$, $\pi_i^2 = \pi_i$, they satisfy the *Hecke relation*

$$(T_i + 1)(T_i - t) = 0\ .$$

More general operators on the ring of polynomials representing the *Iwahori-Hecke algebra* can be found in [**LS1**]. The algebra generated by the T_i and x_j is usually called the *degenerate affine Hecke algebra*. With Schützenberger [**LS2**], we had called it the *Algèbre des différences divisées* since the relations between the T_i and x_j are no other that the (Leibniz) commutations between the ∂_i and x_j.

This still is not the way that we favor to compute with the χ_y characteristic , we shall rather have recourse to eigenfunctions.

6. Eigenfunctions

All the operators that we have used up to now commute with the multiplication with symmetric functions. A priori, their action on Hall-Littlewood polynomials presents no interest since these polynomials are symmetric. However, given a symmetric function, one can introduce a disymmetry, so that symmetrizing it afterwards provides identities.

Instead of using χ on the ring $\mathfrak{S}ym$, let us take the composite ∇:

$$\mathfrak{S}ym \ni f \mapsto f(0, x_2, x_3, \dots, x_n) \mapsto \chi(f(0, x_2, x_3, \dots, x_n))$$

It is an easy computation that for a dominant $I = [i_1, \dots, i_\ell, 0, \dots, 0]$, $i_\ell > 0$,

$$\nabla(Q_I) = (1 + t + \cdots + t^{n-\ell(I)})\, Q_I$$

We see that the eigenspaces of ∇ are not one-dimensional, and thus we cannot characterize Hall-Littlewood polynomials this way.

7. Macdonald operators

Macdonald eliminated multiplicities by introducing an extra parameter q and by considering the operator

$$\mathfrak{S}ym \ni f \mapsto f(qx_1, x_2, x_3, \dots, x_n) \mapsto \chi\left(f(qx_1, x_2, x_3, \dots, x_n)\right) .$$

However, taking into account that χ operates on a function which is symmetrical in $x_2, \dots, x_n$, one eliminates from an appropriate factorisation of χ the part which acts trivially, so that Macdonald's operator is ∇^q:

$$\mathfrak{S}ym \ni f \mapsto f(qx_1, x_2, x_3, \dots, x_n) \xrightarrow{\nabla^q} \partial_{n-1} \cdots \partial_1 \Big(\prod_{i>1} (tx_1 - x_i)\, f(qx_1, x_2, x_3, \dots, x_n)\Big) .$$

Let us notice that the operator $\partial_{n-1} \cdots \partial_1$ is the Gysin morphism for the cohomology of a projective bundle $\mathbb{P}(V)$, $rank(V) = n$.

Now one can check that eigenspaces of ∇^q are 1-dimensional. We shall give in the next section a more precise statement by computing the explicit action of ∇^q on Schur functions. By specialization, we shall thus be able to handle Hall-Littlewood polynomials and the χ_y characteristic . The following sections summarize the results of [**LLM**].

8. Action of Macdonald operators on Schur functions

We must first adapt notation to λ-rings. A symmetric function f of $x_1, x_2, \dots, x_n$ will be noted $f(X)$, with $X = x_1 + x_2 + \cdots + x_n$. The two parameters t and q will be considered as classes of line bundles. Now an element like $X^{tq} := X \frac{1-t}{1-q}$ makes sense as a formal power series in q. The generating function $\lambda_{-1}(X^{tq})$ for all the exterior powers is by definition equal to

$$\lambda_{-1}(X^{tq}) = \prod_i \frac{1 - tx_i}{\prod_{j \geq 0} 1 - q^j x_i}$$

Now the first step in Macdonald's operator is replacing x_1 by qx_1, which in a λ-ring must be written

$$X \mapsto X - x_1 + qx_1$$

However, if we start from a function of X^{tq}, we get

$$X^{tq} \mapsto (X - x_1 + qx_1)\frac{t-1}{q-1} = X^{tq} + x_1(t-1)$$

that one can as well write

$$(X^{tq} - X) + x_1 + \cdots + x_n \mapsto (X^{tq} - X) + tx_1 + x_2 + \cdots + x_n \ .$$

Let us write $X^{tq} - X = Y$ to indicate that this element will be invariant under all the operators we shall use. Our first step is

$$Y + x_1 + \cdots + x_n \mapsto Y + tx_1 + x_2 + \cdots + x_n \ .$$

Now the second step consists in a multiplication by the factor $\prod_{i>1}(tx_1 - x_i)$. One can interchange the operations, the first step being now

$$f(X^{tq}) = f(Y + x_1 + \cdots + x_n) \mapsto f(Y + x_1 + \cdots + x_n)\prod_{i>1}(x_1 - x_i)$$

followed by the operation $x_1 \mapsto tx_1$.

However, if one starts from a Schur function f, then its product by the Vandermonde $\prod_{i<j}(x_i - x_j)$ is, according to Jacobi, a determinant of powers in the variables $x_1, x_2, \ldots, x_n$, each of them being in different columns. We still have x_1 separated from the other variables in the case of a Schur function of $Y + X$ multiplied by the factor $\prod_{i>1}(x_1 - x_i)$. We are in fact reduced to operate on a determinant where all the columns are invariant, except the first one whose entries are polynomials in x_1 with constant coefficients. The only ingredient which is finally needed was known to Newton, that is, the image of $1/(1 - tx_1)$ under the sequence of divided differences $\partial_1, \partial_2, \ldots, \partial_{n-1}$ is equal to

$$t^{n-1}/(1 - tx_1)\cdots(1 - tx_n)$$

(one can also say that the Gysin push-down of a power of the first Chern class of the tautological line bundle on $\mathbb{P}(V)$ is equal to a *Segre* class of V, i.e. a *complete function* in the x_i).

To be more precise, let symmetric powers of any element Y of the Grothendieck ring be $S^i(Y) := (-1)^i\lambda^i(-Y)$, and denote by Ξ_j the "translation operator" which transforms a determinant of classes of symmetric powers $S^i(Y)$ by changing in the j-th column each entry $S^i(Y)$ into $S^i(Y + X(t-1))$, leaving the other columns invariant. Then

THEOREM. *The action of Macdonald's operator ∇^q on a Schur function of X^{tq} coincides with the operator*

$$t^{n-1}\Xi_1 + t^{n-2}\Xi_2 + \cdots + t^0\Xi_n$$

For example, for $n=3$

$$S_{632}(X^{tq}) = \begin{vmatrix} S^6(X^{tq}) & S^7(X^{tq}) & S^8(X^{tq}) \\ S^2(X^{tq}) & S^3(X^{tq}) & S^4(X^{tq}) \\ S^0(X^{tq}) & S^1(X^{tq}) & S^2(X^{tq}) \end{vmatrix}$$

Its image under ∇^q is equal to the following sum of three determinants taking into account that $X^{tq} + X(t-1) = qX^{tq}$, and writing S^i for $S^i(X^{tq})$:

$$t^2 \begin{vmatrix} q^6S^6 & S^7 & S^8 \\ q^2S^2 & S^3 & S^4 \\ q^0S^0 & S^1 & S^2 \end{vmatrix} + t \begin{vmatrix} S^6 & q^7S^7 & S^8 \\ S^2 & q^3S^3 & S^4 \\ S^0 & q^1S^1 & S^2 \end{vmatrix} + \begin{vmatrix} S^6 & S^7 & q^8S^8 \\ S^2 & S^3 & q^4S^4 \\ S^0 & S^1 & q^2S^2 \end{vmatrix}$$

9. Determinantal expression of a Macdonald polynomial

If one takes the diagonals of all the determinants in the sum expressing the image of a Schur function, one obtains that the image of $S_I(X^{tq})$ under Macdonald's operator is

$$t^{n-1}(q^{i_1}S^{i_1})S^{i_2}S^{i_3}\cdots + t^{n-2}S^{i_1}(q^{i_2}S^{i_2})S^{i_3}\cdots + t^{n-3}S^{i_1}S^{i_2}(q^{i_3}S^{i_3})\cdots + \cdots$$

i.e. that the leading term (with respect to the so-called *natural order* on partitions) of $\nabla^q(S_I(X^{tq})$ is

$$(t^{n-1}q^{i_1} + t^{n-2}q^{i_2} + \cdots + t^0q^{i_n})S_I(X^{tq}) \ .$$

Therefore, puting

$$[|I|] := t^{n-1}q^{i_1} + t^{n-2}q^{i_2} + \cdots + t^0q^{i_n} \ , \ I \in N^n \ ,$$

one has that there exists a family of symmetric functions indexed by partitions, which are eigenvalues of ∇^q with eigenvalues the $[|I|]$, and are obtained by a triangular elimination from the S_I^{tq}.

The *Macdonald polynomial* M_I, $I \in N^n$ dominant, is defined to be the eigenfunction of ∇^q with eigenvalue $[|I|]$, satisfying a certain normalisation that we shall ignore here. Many properties of Macdonald's polynomials have been obtained during these last ten years, but no satisfactory explicit expression of them. Using the structure of λ-ring of the Grothendieck ring, we propose a determinantal expression.

Indeed, the elimination process induced from the order on partitions produces a determinant of the type

$$\begin{vmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ S & S & S & S \end{vmatrix}$$

that is, a triangular matrix of linear combinations of eigenvalues $\star$, apart from the last row which is constituted of Schur functions.

For example, for $\lambda = (2,2,1)$, one has

$$M_{(2,2,1)} = \begin{array}{c} \begin{vmatrix} [5,0,0] & -[0,5,0] & 0 & [0,0,5] & 0 \\ 0 & [4,1,0] & -[1,4,0] & -[0,4,1] & [1,0,4] \\ 0 & 0 & [3,2,0] & -[3,0,2] & -[2,0,3] \\ 0 & 0 & 0 & [3,1,1] & -[1,3,1] \\ S_5 & S_{4,1} & S_{3,2} & S_{3,1,1} & S_{2,2,1} \end{vmatrix} \end{array} \begin{array}{l} {\scriptstyle (5,0,0)} \\ {\scriptstyle (4,1,0)} \\ {\scriptstyle (3,2,0)} \\ {\scriptstyle (3,1,1)} \\ \end{array}$$

Let us explain how to write the formal determinant before interpreting it. The bottom row is the list of all Schur functions whose index is smaller or equal to the partition I. In the column of S_J, one writes the (ordered) expansion of the Jacobi-Trudi determinant in the complete functions S^k expressing S_J, puting on the same row all the vectors equal up to permutation. For example, the column of $S_{3,1,1}$ corresponds to the expansion (ordered from left to right) of $S_{3,1,1} = \begin{vmatrix} S^3 & S^4 & S^5 \\ S^0 & S^1 & S^2 \\ 0 & S^0 & S^1 \end{vmatrix}$ which is graphically represented by the following diagram

$$\begin{tikzcd} & {-[041]} \arrow[r] & {[005]} \arrow[dr] & \\ {[311]} \arrow[ur] \arrow[dr] & & & 0 \\ & {-[302]} \arrow[r] & 0 \arrow[ur] & \end{tikzcd}$$

In the rightmost column of the determinant expressing M_I, one does not take the top element S_I. Let us remark that these ordered expansions are also used to compute Kostant's partition function.

A bigger example, with multiplicities, for $M_{2,2,2}$:

$$\begin{vmatrix} [6,0,0] & -[0,6,0] & 0 & [0,0,6] & 0 & 0 & 0 \\ 0 & [5,1,0] & -[1,5,0] & -[0,5,1] & 0 & [1,0,5] & 0 \\ 0 & 0 & [4,2,0] & -[4,0,2] & -[2,4,0] & 0 & -[0,2,4] \\ 0 & 0 & 0 & [4,1,1] & 0 & -[1,4,1] & [1,1,4] \\ 0 & 0 & 0 & 0 & [3,3,0] & -[3,0,3] & [0,3,3] \\ 0 & 0 & 0 & 0 & 0 & [3,2,1] & -[2,1,3]-[1,3,2] \\ S_6 & S_{5,1} & S_{4,2} & S_{4,1,1} & S_{3,3} & S_{3,2,1} & S_{2,2,2} \end{vmatrix}$$

How to interpret the determinant meant to express the Macdonald polynomial M_I? Each entry $[J]$ must be simply replaced by the difference $[|J|] - [|I|]$, and S_J must be interpreted as the Schur function $S_J(X^{tq})$.

For example,

$$M_{2,2} = \begin{vmatrix} [4,0] & -[0,4] & 0 \\ 0 & [3,1] & -[1,3] \\ S_4 & S_{3,1} & S_{2,2} \end{vmatrix} = \begin{vmatrix} [|2,2|]-[|4,0|] & [|0,4|]-[|2,2|] & 0 \\ 0 & [|2,2|]-[|3,1|] & [|1,3|]-[|2,2|] \\ S_4(X^{tq}) & S_{3,1}(X^{tq}) & S_{2,2}(X^{tq}) \end{vmatrix}$$

and more explicitely

$$M_{2,2} = \begin{vmatrix} (1-q^2)(q^2t-1) & (1-q^2)(t-q^2) & 0 \\ 0 & q(1-q)(qt-1) & q(q-1)(t-1) \\ S_4(X^{tq}) & S_{3,1}(X^{tq}) & S_{2,2}(X^{tq}) \end{vmatrix}$$

10. Associativity again

As for Hall-Littlewood polynomials, one wants to retain only the associative algebra underlying the indices of a choosen basis of symmetric polynomials, at this stage, Macdonald polynomials (that we shall index by the conjugate increasing partitions and note N_J, in this section).

Let I be an increasing vector: $I = [i_1, \ldots, i_n]$, $0 \le i_1 \le \cdots \le i_n$. We have to interpret I as a sequence of operators such that the image of $1 = N_0$ under B_{i_1} is N_{i_1}, the image of N_{i_1} under B_{i_2} is N_{i_1,i_2}, and finally the image of $N_{i_1,i_2,\ldots,i_{n-1}}$ under B_{i_n} is N_I.

It happens that there are algebraic operators on the ring of polynomials satisfying such properties, they have been defined and baptized *creation operators* by [**LV**] and [**KN**].

These operators were originally defined as symmetric functions in some distinguished elements of the affine Hecke algebra due to Cherednik [**Ch**]. However, as Macdonald's operators, they can be factorised, one of the factor being the Euler-Poincaré characteristic. The degree raising step just consists into multiplication by a factor $x_1 \cdots x_k$. One needs moreover to combine the operation $x_1 \mapsto qx_1$ with circular permutations of variables. Thus let

$$\zeta_k : [x_1, \ldots, x_k, x_{k+1}, \ldots, x_n] \mapsto [qx_k, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n]$$

and define, for every positive integer k the operator B_k (due to [**LV**])

$$\mathfrak{S}ym \ni f \xrightarrow{B_k} \chi\Big((x_1 \cdots x_k)(1 - t\zeta_k) \cdots (1 - t^k\zeta_k) f\Big) .$$

The same type of computations as for Macdonald's operators give

THEOREM. *Let k be a positive integer, I a partition in N^k, J its (increasing) conjugate. Then*

1) $B_k(S_I(X^{tq}))$ is the determinant obtained from the one expressing $S_I(X^{tq})$ by replacing in each entry $S^i(X^{tq})$ in column h, $h = 1, \ldots, k$, by $(1-t^{k-h}q^{i+1})S^(X^{tq})$.

2) $B_k(N_J) = N_{J,k}$, i.e. the image is the Macdonald polynomial indexed by the concatenation of J and k.

For example, still noting S^i instead of $S^i(X^{tq})$, one has

$$B_3(S_{421}) = B_3 \begin{pmatrix} S^4 & S^5 & S^6 \\ S^1 & S^2 & S^3 \\ S^{-1} & S^0 & S^1 \end{pmatrix} = \begin{vmatrix} (1-t^2q^5)S^5 & (1-tq^6)S^6 & (1-q^7)S^7 \\ (1-t^2q^2)S^2 & (1-tq^3)S^3 & (1-q^4)S^4 \\ (1-t^2q^0)S^0 & (1-tq^1)S^1 & (1-q^2)S^2 \end{vmatrix}$$

11. Back to the χ_y characteristic : q=0

In the preceeding determinants, we have to put $q = 0$ or $t = 0$ (after dividing in each row by the minimum power of q) to recover the cases of interest for the χ_y characteristic . In the first case, it gives the expansion of Hal-Littlewood polynomials in the basis of "modified Schur functions", in the second, the expansion in the basis of Schur functions.

For example,

$$M_{221} = \begin{vmatrix} [5,0,0] & -[0,5,0] & 0 & [0,0,5] & 0 \\ 0 & [4,1,0] & -[1,4,0] & -[0,4,1] & [1,0,4] \\ 0 & 0 & [3,2,0] & -[3,0,2] & -[2,0,3] \\ 0 & 0 & 0 & [3,1,1] & -[1,3,1] \\ S_5 & S_{4,1} & S_{3,2} & S_{3,1,1} & S_{2,2,1} \end{vmatrix}$$

specializes in $q=0$ to

$$\begin{vmatrix} -t-1 & t^2+1 & 0 & -t-t^2 & 0 \\ 0 & -1 & 1 & t^2 & -t \\ 0 & 0 & -1 & t & t \\ 0 & 0 & 0 & -t & t^2 \\ S_5 & S_{4,1} & S_{3,2} & S_{3,1,1} & S_{2,2,1} \end{vmatrix} = \begin{array}{c} (t^2+t)S_{2,2,1} + t^2(t+1)^2 S_{3,2} + \\ +(t^2+t^3)S_{3,1,1} + t^3(t+1)^2 S_{4,1}(t^5+t^6)S_5 \end{array}$$

The specialized determinant can be written *à vue*: the powers of t just record the positions of the minimum components in the vectors $[J]$ and $[I]$. For example, $[8,1,8,8,1,8]$ would give $t+t^4$ (counting the positions of the 1's from the right, starting from 0).

However, the preceeding determinant is not equal to the Hall-Littlewood function Q_{221}, because there is a parasitic factor ($=t^2+t$), which however is easily eliminated, knowing that $Q_I = S_I + \cdots$.

Of course, instead of specializing q to 0 ("crystalizing" the flag variety), it would be better to explain why geometry needs the extra parameter q.

This, I cannot explain.

12. Reading tables

Let $G_{q,r}(V)$ be a Grassmann variety, Q its tautological quotient vector bundle, R its tautological kernel. The relative cotangent bundle is isomorphic to $R\otimes Q^\vee$, and $\mathcal{O}(1) := \wedge^q(Q)$ is the line bundle associated to the Plücker embedding. Cauchy formula allows to decompose the exterior powers of the cotangent bundle:

$$\wedge^j(R\otimes Q^\vee) \simeq \oplus_{J:|J|=j}\, S_J(R)\otimes S_{J'}(Q^\vee)\ ,$$

where J' denotes the partition conjugate to J.

Bott's theorem gives the cohomology of each individual module $S_J(R)\otimes S_{J'}(Q^\vee)\otimes \mathcal{O}(k)$: it is zero or an irreducible representation of $Gl(q+r)$. Computing in the Grothendieck ring of the base variety $\mathcal{M}$, simultaneously for all partitions J, one loses no information. Now, with the same notation as for the flag manifold, one has the following equality in the Grothendieck ring of $G_{q,r}(V)$:

$$\sum_j y^j[\wedge^j(R\otimes Q^\vee)] = \prod_{1\le i\le q,\, q+1\le h\le q+r} (1+y\frac{x_h}{x_i})$$

and $[\mathcal{O}(k)] = (x_1\cdots x_q)^k$. Therefore,

$$\sum_{i,j} y^j H^i(\Omega^j\otimes\mathcal{O}(k)) = \sum_{i,J} y^j H^i(S_J(R)\otimes S_{J'}(Q^\vee)\otimes\mathcal{O}(k)) = \chi((x_1\cdots x_q)^k) = Q_{k^q}$$

Because the Hall-Littlewood functions occuring in this problem are special, one has in fact a direct algorithm to obtain them (in general, as we already said, one has to specialize t to 0 in the expression of Macdonald polynomials to get the expansion in the basis of Schur functions). It amounts to enumerate all possible words w of length $q+r$ in the two letters alphabet $\{0,k\}$ satisfying the condition that there exists no i such that $w_i=0, w_{i+k}=k$ (otherwise the corresponding Schur determinant would have two identical rows and thus would vanish).

For example, if $q = r = k = 3$, then the tables give

$$Q_{333} = S_{333} - yS_{3321} + y^2 S_{3222} + y^2 S_{33111} - y^4 S_{22221} - y^4 S_{321111} + y^5 S_{222111} \ .$$

The coefficient of $(-y)^4$ is $S_{22221} + S_{321111}$, which means that the space of sections of differential forms of order 4, twisted by $\mathcal{O}(3)$, is a representation of $Gl(6)$ which breaks into two irreducible components indexed by the partitions 22221 and 321111.

However the above expansion exactly corresponds to the easier enumeration of all words with three "0" and three "3" which do not possess any factor of length 4 of the type $[0, \star, \star, 3]$, that is to the set

$$[3,3,3,0,0,0], [[3,3,0,3,0,0], [3,0,3,3,0,0], [3,0,3,0,0,3], [0,3,3,0,3,0], [0,3,3,0,0,3] \, .$$

This list of words has to be interpreted as enumerating Schur functions, or irreducible representations of the linear group.

To obtain easily such sets of words, one can have recourse to the non-commutative technics of automata theory.

References

[At] M. F. Atiyah, *Collected Works*, Oxford Science Publications, Clarendon, Oxford, 1988.

[Ch] I. Cherednik, *Non symmetric Macdonald polynomials*, Intern. Math. Res. Notices **10** (1995), 483–515.

[DKLLST] G. Duchamp, D. Krob, A. Lascoux, B. Leclerc, T. Scharf, J.Y. Thibon, *Euler-Poincaré characteristics and polynomial representations of Iwahori-Hecke algebras*, Pub RIMS (Kyoto) **31** (1995), 179–201.

[Hi] F. Hirzebruch, *Gesammelte Abhandlungen*, Springer, Berlin, 1987.

[HBJ] F. Hirzebruch, T. Berger, R. Jung, *Manifolds and modular forms, Aspects of Mathematics*, Vieweg, Wiesbaden, 1992.

[KN] A. Kirillov, M. Noumi, *Affine Hecke algebras and raising operators for Macdonald polynomials*, Duke Math *(to appear)* (1998).

[LLM] L. Lapointe, A. Lascoux, J. Morse, *Determinantal expressions for Macdonald polynomials*, Intern. Math. Res. Notices *(to appear)* (1998).

[LV] L. Lapointe, L. Vinet, *A short proof of the integrality of the Macdonald (q,t)-Kostka coefficients*, Duke Math. **91** (1998), 205–214.

[LLT] A. Lascoux, B. Leclerc et J.Y. Thibon, *Flag Varieties and the Yang-Baxter Equation*, Letters in Math. Phys. **40** (1997), 75–90.

[LS1] A. Lascoux, M.P. Schützenberger, *Symmetry and Flag manifolds*, Invariant Theory, Springer L.N., vol. 996, Springer, Berlin, 1983, pp. 118–144.

[LS2] A. Lascoux, M.P. Schützenberger, *Algèbre des différences divisées*, Discrete Maths **99** (1992), 165–179.

[Li] D.E. Littlewood, *On certain symmetric functions*, Proc. London M.S. **43** (1961), 485–498.

[Ma] I.G. Macdonald, *Symmetric functions and Hall Polynomials (2^{nd})*, Oxford Math. Monograph, Clarendon, Oxford, 1995.

[Sn] D. M. Snow, *Cohomology of twisted holomorphic forms on Grassmann manifolds and quadric hypersurfaces*, Math. Ann. **276** (1986), 159–176.

[Ya] C. N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett **19** (1967), 1312–1315.

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