Littlewood's formulas for characters of orthogonal and symplectic groups

Alain Lascoux

Institut Gaspard Monge, CNRS, Université de Marne-la-Vallée

77454 Marne-la-Vallée Cedex, France

e-mail: Alain.Lascoux @ univ-mlv.fr

Abstract

Litllewood gave expansions of products of the type $\prod 1/(1-a_ia_j)$. Several authors published generalizations involving a small number of extra factors $\prod 1/(1-xa_i)$. We show that the method of Littlewood (that we express in terms of λ-rings) covers these extensions. We also express Littlewood's coefficients in terms of orthogonal or symplectic Schur functions.

Given an infinite alphabet $\mathbb{A} = \{a_1, a_2, a_3, \ldots\}$, Littlewood [7] gave expansions of products of the type

$$\prod (1-a_i)^{\pm 1} \prod_{i<j} (1-a_ia_j)^{\pm 1} \quad \& \quad \prod (1-a_i)^{\pm 1} \prod_{i\leq j} (1-a_ia_j)^{\pm 1} \ , \qquad (1)$$

the case $\prod(1-a_i)^{-1} \prod_{i<j}(1-a_ia_j)^{-1}$ being due to Schur [10].

By functoriality of symmetric functions, Littlewood's formulas imply without further computations expansions of more complicated products like

$$\prod (1-a_i)^{\pm 1} \prod (1-b_i)^{\mp 1} \prod_{i\leq j} (1-a_ia_j)^{\pm 1} \prod_{i<j} (1-b_ib_j)^{\pm 1} \prod (1-a_ib_j)^{\mp 1} \quad (2)$$

involving a second alphabet $\mathbb{B} = \{b_1, b_2, \ldots\}$.

However, Littlewood's method does not restrict to expliciting the above products, and allows to expand products of the type

$$\prod_i \frac{1}{(1-x_1a_i)\cdots(1-x_na_i)} \prod_{i<j} \frac{1}{1-a_ia_j} \qquad (3)$$

involving a finite alphabet $\mathbb{X} = \{x_1, \ldots, x_n\}$, the alphabets $\mathbb{A}$ and $\mathbb{X}$ playing now a disymmetrical role.

To obtain his formulas, Littlewood describes in fact the adjoint of the multiplication by certain series of symmetric functions (see the theorem below). We shall state his results in terms of λ-rings.

The case $\mathbb{X} = \{x_1, x_2\}$ has been considered by Ishikawa [2], Ishikawa and Wakayama [3], Bressoud [1], and the case $\mathbb{X} = \{x_1, x_2, x_3\}$ by Jouhet and Zeng [4].

Before reinterpreting their formulas, let us recall some definitions, for which we refer to [8].

Given two alphabets $\mathbb{A}$, $\mathbb{B}$, the complete functions $S_j(\mathbb{A}\text{–}\mathbb{B})$ are defined by the generating series (with z an extra variable)

$$\sigma_z(\mathbb{A} - \mathbb{B}) := \prod_{b \in \mathbb{B}} (1 - zb) / \prod_{a \in \mathbb{A}} (1 - za) \tag{4}$$

Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{N}^r$, the Schur function $S_\lambda(\mathbb{A}\text{–}\mathbb{B})$ is

$$S_\lambda(\mathbb{A}\text{–}\mathbb{B}) = \left| S_{\lambda_{i+j-i}}(\mathbb{A}\text{–}\mathbb{B}) \right|_{1 \le i,j \le r} .$$

When λ is weakly decreasing, it is called a *partition.* In that case, following Frobenius, one uses another description of partitions, by decomposing their diagram into its diagonal hooks, and writing $\lambda = (\alpha \,|\, \beta)$ [8] p.3.

On the space $\mathfrak{S}ym(\mathbb{A})$ of symmetric functions in $\mathbb{A}$, there exists a scalar product $(\,,\,)_{\mathbb{A}}$, and Schur functions indexed by partitions constitute an orthormal basis with respect to it :

$$\big(S_\lambda(\mathbb{A}),\, S_\lambda(\mathbb{A}) \big)_{\mathbb{A}} = 1 \qquad \& \qquad \big(S_\lambda(\mathbb{A}),\, S_\nu(\mathbb{A}) \big)_{\mathbb{A}} = 0 \quad \lambda \neq \nu \; . \tag{5}$$

Another way to state (5) is to assert the existence of a *reproducing kernel*

$$K(\mathbb{A},\, \mathbb{B}) := \sigma_1(\mathbb{A}\mathbb{B}) = \prod_{a \in \mathbb{A},\, b \in \mathbb{B}} (1 - ab)^{-1}$$

which satisfies

$$\forall f \in \mathfrak{S}ym \;,\; \big(K(\mathbb{A},\, \mathbb{B}),\, f(\mathbb{A}) \big)_{\mathbb{A}} = f(\mathbb{B}) \;, \tag{6}$$

because the Cauchy formula states that

$$K(\mathbb{A},\, \mathbb{B}) = \sum_\lambda S_\lambda(\mathbb{A})\, S_\lambda(\mathbb{B}) \;, \tag{7}$$

sum over all partitions.

Given any symmetric function f, the adjoint $D(f;\bullet)$ of the multiplication by f is defined by

$$\forall f,\, g,\, h \in \mathfrak{S}ym\ ,\ \big(D(f;g)\,,\, h\big) = (g\,,\, fh) \tag{8}$$

The formulas

$$\sigma_1(\Lambda^2(\mathbb{A})) = \sum S_\mu(\mathbb{A})\ ,\ \text{all } \mu \text{ with columns of even lengths} \tag{9}$$

and

$$\sigma_1(\mathbb{A} + \Lambda^2(\mathbb{A})) = \sum S_\lambda(\mathbb{A})\ ,\ \text{all partitions } \lambda \tag{10}$$

are equivalent, because $\Lambda^2(\mathbb{A}+1) = \mathbb{A}+\Lambda^2(\mathbb{A})$. Combinatorially, considering the letter 1 to be bigger than all the letters in $\mathbb{A}$, it is equivalent to enumerate all tableaux in $\mathbb{A}$, or all tableaux in the alphabet $\mathbb{A}+1 := \mathbb{A} \cup \{1\}$ with columns of even length (we write unions of alphabets with a '+'). Indeed, the 1's occupy the horizontal strip such that the inner shape μ has columns of even lengths.

The computations of Littlewood in chapter XI of his book [7] can be rephrased in the λ-rings paradigm as follows :

Theorem 1 (Littlewood). *Given two alphabets $\mathbb{A}$, $\mathbb{X}$, then one has the identities*

$$\begin{aligned}
\sigma_1(\mathbb{A}\mathbb{X} + \Lambda^2(\mathbb{A})) &= \sum_\mu S_\mu(\mathbb{A})\, D\big(\sigma_1(\Lambda^2); S_\mu)\big)(\mathbb{X}) && (11)\\
\sigma_1(\mathbb{A}\mathbb{X} + S^2(\mathbb{A})) &= \sum_\mu S_\mu(\mathbb{A})\, D\big(\sigma_1(S^2); S_\mu)\big)(\mathbb{X}) && (12)\\
\sigma_1(\mathbb{A}\mathbb{X} - \Lambda^2(\mathbb{A})) &= \sum_\mu S_\mu(\mathbb{A})\, D\big(\sigma_1(-\Lambda^2); S_\mu)\big)(\mathbb{X}) && (13)\\
\sigma_1(\mathbb{A}\mathbb{X} - S^2(\mathbb{A})) &= \sum_\mu S_\mu(\mathbb{A})\, D\big(\sigma_1(-S^2); S_\mu)\big)(\mathbb{X})\ , && (14)
\end{aligned}$$

sum over all partitions μ.

Proof. All four generating series factorize : $\sigma_1(\mathbb{A}\mathbb{X}+\Lambda^2(\mathbb{A})) = \sigma_1(\mathbb{A}\mathbb{X})\sigma_1(\Lambda^2(\mathbb{A}))$. Since $\sigma_1(\mathbb{A}\mathbb{X})$ is a reproducing kernel, one has for any μ :

$$\begin{aligned}
\big(\sigma_1(\mathbb{A}\mathbb{X} + \Lambda^2(\mathbb{A}))\,,\, S_\mu(\mathbb{A})\big)_{\mathbb{A}} &= \big(\sigma_1(\mathbb{A}\mathbb{X})\,,\, D(\sigma_1(\Lambda^2); S_\mu)(\mathbb{A})\big)_{\mathbb{A}}\\
&= D(\sigma_1(\Lambda^2); S_\mu)(\mathbb{X})\ .
\end{aligned}$$

and this proves the first identity.

The same proof would work for $\sigma_1(\mathbb{A}\mathbb{X})\, f(\mathbb{A})$, f being any symmetric function. Indeed property (6) entails

$$\sigma_1(\mathbb{A}\mathbb{X})\, f(\mathbb{A}) = \sum_\mu S_\mu(\mathbb{A})\, D(f; S_\mu)(\mathbb{X}) \ , \tag{15}$$

and formulas (12, 13, 14) are a special case of it. QED

Let us mention that other expansions are possible. Indeed,

$$\begin{aligned} S^2(\mathbb{A} \pm \mathbb{X}) &= S^2(\mathbb{A}) \pm S^1(\mathbb{A})S^1(\mathbb{X}) + S^2(\pm\mathbb{X}) \\ \Lambda^2(\mathbb{A} \pm \mathbb{X}) &= \Lambda^2(\mathbb{A}) \pm \Lambda^1(\mathbb{A})\Lambda^1(\mathbb{X}) + \Lambda^2(\pm\mathbb{X}) \ . \end{aligned}$$

It implies, for example, that

$$\begin{aligned} \sigma_1(\mathbb{A}\mathbb{X} + \Lambda^2(\mathbb{A})) &= \sigma_1(-\Lambda^2(\mathbb{X}))\, \sigma_1(\Lambda^2(\mathbb{A} + \mathbb{X})) \\ &= \sigma_1(-\Lambda^2(\mathbb{X})) \sum\nolimits_\nu S_\nu(\mathbb{A} + \mathbb{X}) \ , \end{aligned} \tag{16}$$

sum over all partitions ν with columns of even lengths. To go further, one would decompose $S_\nu(\mathbb{A} + \mathbb{X})$ in the basis of Schur functions in $\mathbb{A}$, and use in general Littlewood-Richardson's rule to collect functions in $\mathbb{X}$. But in the case where $\mathbb{X}$ is of small cardinality, the expansion can be written directly. For example ([1], th.IV), if $\mathbb{X} = x_1 + x_2$, equation (16) can be rewritten as

$$\sigma_1\big((x_1 + x_2)\mathbb{A} + \Lambda^2(\mathbb{A})\big) = (1 - x_1 x_2) \sum\nolimits_\nu S_\nu(\mathbb{A} + x_1 + x_2) \tag{17}$$

and is easy to expand in terms of the $S_\mu(\mathbb{A})$, interpreting it as the sum of all Young tableaux with even columns, on an alphabet with last two letters x_1, x_2.

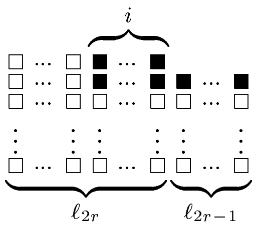
Let us show in more details how to make explicit the coefficients in Littlewood's identities, for $card(\mathbb{X}) = n = 1, 2, 3$, recovering the related results of [1, 2, 3, 4]. We shall treat only the case of $\sigma_1(\mathbb{X}\mathbb{A} + \Lambda^2\mathbb{A})$, the other being similar.

When $n = 1$, $\mathbb{X} = \{x_1\}$. Then $D(S_\nu; S_\mu)(\mathbb{X})$ is non zero iff the diagram of μ/ν is an horizontal strip. Therefore, given any partition μ, there is one and only one partition ν with even columns such that $D(S_\nu; S_\mu)(\mathbb{X})$ is non zero. Equation (11) specializes to

$$\sigma_1(x_1\mathbb{A} + \Lambda^2(\mathbb{A})) = \sum_\mu x_1^{odd(\mu)}\, S_\mu(\mathbb{A}) \ , \tag{18}$$

where $odd(\mu)$ is the number of columns of μ of odd length.

Take now $n = 2$. A skew diagram μ/ν gives a non-zero function $S_{\mu/\nu}(x_1 + x_2)$ iff it is made of horizontal strips of height 1 and 2. Decompose the partition into blocks $\mathcal{B}_k$, $k = 1, 2, \ldots$ of columns of the same length k (denoting ℓ_k its width). If k is odd, then the restriction of μ/ν to this block must be an horizontal strip of length ℓ_k, giving a factor $S_{\ell_k}(x_1 + x_2)$. If k is even, the restriction of μ/ν is a $2 \times i$ rectangle, with $0 \le i \le \ell_k$. The figure shows the restriction of a strip to two consecutive blocks :

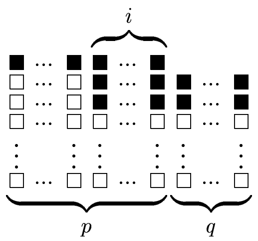


The sum over all possible i gives a factor $S_{\ell_k}(1 + x_1x_2)$. Moreover, there is no interference between the different blocks, and therefore, one gets that the coefficient of $S_\mu(\mathbb{A})$ in $\sigma_1\big((x_1 + x_2)\mathbb{A} + \Lambda^2(\mathbb{A})\big)$ is

$$D(\sigma_1(\Lambda^2); S_\mu)(x_1 + x_2) = \prod_{k=0}^{\infty} S_{\ell_{2k}}(1 + x_1x_2)\, S_{\ell_{2k+1}}(x_1 + x_2)\ . \tag{19}$$

The case $n = 3$ is treated by Jouhet and Zeng [4]. The independent blocks are now the blocks $\mathcal{B}_k$ of columns of length $2k+1$, $2k$.

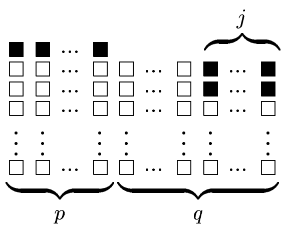
There are two possible configurations for those μ/ν giving a non-zero contribution :



$$, \quad p = \ell_{2k+1},\ q = \ell_{2k}, \quad i \ge 1$$

The sum over all $i = 1, \ldots, p$ gives

$$x_1x_2x_3S_{p-1}(1 + x_1x_2x_3)S_q(\Lambda^2(x_1 + x_2 + x_3))\ .$$



$$, \ i = 0,\ 0 \le j \le q\ .$$

The sum over all $j = 0, \dots, q$ is

$$S_p(x_1 + x_2 + x_3)\, S_q(1 + \Lambda^2(\mathbb{X}))\ .$$

For example, for $\mu = [7,7,4]$, the possible configurations, together with their factors, are :

$$\underset{S_{411}(\mathbb{X})\ S_{33}(\mathbb{X})}{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \square\square\square\blacksquare\blacksquare\blacksquare\blacksquare\\ \square\square\square\blacksquare\blacksquare\blacksquare\blacksquare\end{array}} + \underset{S_{422}(\mathbb{X})\ S_{33}(\mathbb{X})}{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \square\square\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\\ \square\square\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\end{array}} + \underset{S_{433}(\mathbb{X})\ S_{33}(\mathbb{X})}{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \square\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\\ \square\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\end{array}} + \underset{S_{444}(\mathbb{X})\ S_{33}(\mathbb{X})}{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\\ \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\end{array}}$$

$$+ \underset{S_{4}(\mathbb{X})\ S_{00}(\mathbb{X})}{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \square\square\square\square\square\square\square\\ \square\square\square\square\square\square\square\end{array}} + \underset{S_{4}(\mathbb{X})\ S_{11}(\mathbb{X})}{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \square\square\square\square\square\square\blacksquare\\ \square\square\square\square\square\square\blacksquare\end{array}} + \underset{S_{4}(\mathbb{X})\ S_{22}(\mathbb{X})}{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \square\square\square\square\square\blacksquare\blacksquare\\ \square\square\square\square\square\blacksquare\blacksquare\end{array}} + \underset{S_{4}(\mathbb{X})\ S_{33}(\mathbb{X})}{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \square\square\square\square\blacksquare\blacksquare\blacksquare\\ \square\square\square\square\blacksquare\blacksquare\blacksquare\end{array}}\ .$$

In summary, the contribution of the block $\mathcal{B}_k$ is

$$c(\mathcal{B}_k) = \sum_{i=0}^{p} S_{p-i}(\mathbb{X}) S_{qq}(\mathbb{X}) + S_p\mathbb{X})\Big(1 + S_{11}(\mathbb{X}) + \cdots + S_{nn}(\mathbb{X})\Big) \tag{20}$$

and the coefficient of $S_\mu(\mathbb{A})$ in $\sigma_1\big((x_1 + x_2 + x_3)\mathbb{A} + \Lambda^2(\mathbb{A})\big)$ is the product of all the factors $c(\mathcal{B}_k)$ given by the decomposition of μ into blocks. This expression is equivalent to formula 8 of Jouhet and Zeng.

Littlewood's expansions were motivated by the description of characters of orthogonal and symplectic groups. In turn, these characters allow to expand Littlewood's formulas (13), (14).

Following [7], p.240, using a second alphabet $\mathbb{X}$, define the *orthogonal Schur function* $\mathcal{O}_\lambda(\mathbb{A})$ by the generating series :

$$\prod_{i \le j} (1 - a_i a_j) \prod_i \prod_j (1 - a_i x_j)^{-1} = \sum_\lambda \mathcal{O}_\lambda(\mathbb{X})\, S_\lambda(\mathbb{A})\ , \tag{21}$$

that is,

$$\sigma_1\big(\mathbb{A}\mathbb{X} - S^2(\mathbb{A})\big) = \sum_\lambda \mathcal{O}_\lambda(\mathbb{A})\, S_\lambda(\mathbb{X})\ . \tag{22}$$

Formula (14) can now be rewritten :

$$\mathcal{O}_\lambda = D\big(\sigma_1(-S^2)\,;S_\lambda\big) \tag{23}$$

and the inverse formula is

$$S_\lambda = subs\Big(S = \mathcal{O}, D(\sigma_1(S^2)\,;S_\lambda)\Big) \tag{24}$$

which means that in that last case, one uses the operator $D\big(\sigma_1(S^2)\,;\bullet\big)$ as a formal operator on a vector space, the basis of which is indexed by partitions (and the action being obtained by identifying this basis with Schur functions).

For example, Littlewood (p. 241) gives (writing in the expansion of $\sigma_1(-S^2)$ and $\sigma_1(S^2)$ only the terms which give a non-zero contribution) :

$$\mathcal{O}_{332} = D(S_0 - S_2 + S_{31} - S_{33}\,;\, S_{332}) = S_{332} - S_{332/2} + S_{332/31} - S_2$$

and the identity

$$D(S_0 + S_2 + S_{22} + S_{222}\,;\, S_{332}) = S_{332} + S_{33} + S_{321} + S_{211} + S_{31} + S_{11}$$

implies, by formal change of S into $\mathcal{O}$:

$$S_{332} = \mathcal{O}_{332} + \mathcal{O}_{33} + S_{321} + \mathcal{O}_{211} + \mathcal{O}_{31} + \mathcal{O}_{11} \ .$$

Ishikawa and Wakayama [3] give the expansion of $\sigma_1((x_1+x_2)\mathbb{A} - S^2(\mathbb{A}))$, with $x_1x_2 = 1$. It means computing the specialization of orthogonal Schur functions for such $\mathbb{X}$. In fact the complete functions (resp. power sums) of $\mathbb{X}$ are Tchebychef polynomials of the first and second kind in the variable $(x_1+x_2)/2$. Ishikawa and Wakayama characterize those partitions for which $\mathcal{O}_\lambda(\mathbb{X}) \neq 0$ in terms of their Frobenius decomposition, and obtains that in that case $\pm\mathcal{O}_\lambda(\mathbb{X})$ is equal to a Tchebychef polynomial of the second kind.

Using symmetrizing operators, one easily expands $\sigma_1(-S^2)$ as :

$$\begin{aligned} \sigma_1(-S^2) &= \sum \pm S_{\epsilon_1,\epsilon_2,\epsilon_3,\dots} = S_{000\dots} - S_{200\dots} - S_{040\dots} + S_{240\dots} + \cdots && (25) \\ &= S_0 - S_2 + S_{31} - S_{33} + \cdots \ . && (26) \end{aligned}$$

In the first summation, Schur functions are indexed by vectors $[\epsilon_1, \epsilon_2, \epsilon_3, \dots]$ such that $\epsilon_i \in \{0, 2i\}$, and not by partitions, and the sign alternates with the parity of the number of non-zero components.

It is remarkable that the operator $D\big(\sigma_1(-S^2); \bullet\big)$ can also be written as an enumeration of vectors :

$$S_\lambda \to \mathcal{O}_\lambda = D\big(\sigma_1(-S^2), S_\lambda\big) = \sum_{\epsilon_1,\epsilon_2,\dots} \pm S_{\lambda/[\epsilon_1,\epsilon_2,\dots]} \qquad (27)$$

(recall that, for any two integral vectors λ, ν, $S_{\lambda/\nu}$ stands for the determinant $\big|S_{\lambda_i+j-i-\nu_j}\big|$).

Indeed, the fact that $D(S_\nu; S_\lambda) = S_{\lambda/\nu}$ for two partitions, extends, by reordering the rows of the determinantal expression of S_ν and the columns of $S_{\lambda/\nu}$, to the case where ν is not a partition, i.e. is not decreasing.

Thus $\mathcal{O}_{332}$ can also be written

$$\begin{aligned} \mathcal{O}_{332} &= S_{332/000} - S_{332/200} - S_{332/040} + S_{332/240} \\ &= \begin{vmatrix} S_3 & S_4 & S_5 \\ S_2 & S_3 & S_4 \\ S_0 & S_1 & S_2 \end{vmatrix} - \begin{vmatrix} S_1 & S_4 & S_5 \\ S_0 & S_3 & S_4 \\ 0 & S_1 & S_2 \end{vmatrix} - \begin{vmatrix} S_3 & S_0 & S_5 \\ S_2 & 0 & S_4 \\ S_0 & 0 & S_2 \end{vmatrix} + \begin{vmatrix} S_1 & S_0 & S_5 \\ S_0 & 0 & S_4 \\ 0 & 0 & S_2 \end{vmatrix} \ . \end{aligned}$$

By mutilinearity of determinants, one can write the above sum as a single determinant, due to Weyl ([11], th.7.9.A), that we display for the order 3 :

$$\mathcal{O}_{\lambda_1\lambda_2\lambda_3} = \begin{vmatrix} S_{\lambda_1} - S_{\lambda_1-2} & S_{\lambda_1+1} - S_{\lambda_1+1-4} & S_{\lambda_1+2} - S_{\lambda_1+2-6} \\ S_{\lambda_2-1} - S_{\lambda_2-1-2} & S_{\lambda_2} - S_{\lambda_2-4} & S_{\lambda_2+1} - S_{\lambda_2+1-6} \\ S_{\lambda_3-2} - S_{\lambda_3-2} & S_{\lambda_3-1} - S_{\lambda_3-1-4} & S_{\lambda_3} - S_{\lambda_3-6} \end{vmatrix} \tag{28}$$

For example, $\mathcal{O}_{332}$, again, is equal to

$$\begin{vmatrix} S_3 - S_1 & S_4 - S_0 & S_5 - 0 \\ S_2 - S_0 & S_3 - 0 & S_4 - 0 \\ S_0 - 0 & S_1 - 0 & S_2 - 0 \end{vmatrix}$$

We shall refer to [5] for determinantal expressions of orthogonal or symplectic characters. The case of "nearly rectangular shape" leads to an interesting combinatorics obtained by Krattenthaler [6].

Similarly, define the *symplectic Schur function* $\mathcal{S}p_\lambda$ by

$$\mathcal{S}p_\lambda = D(\sigma_1(-\Lambda^2)\,; S_\lambda) \ , \tag{29}$$

but this time, it will be evaluated into the alphabet
$\mathbb{Y} := \{x_1, 1/x_1, x_2, 1/x_2, \ldots, x_n, 1/x_n\}$ instead of $\mathbb{X}$.

The identity

$$\sigma_1(-\Lambda^2) = \sum S_{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3,\ldots} \ , \tag{30}$$

sum over all $\epsilon_i \in \{0, 2i\}$, with $\epsilon_0 = 0$, extends, as before, to

$$\mathcal{S}p_\lambda = D(\sigma_1(-\Lambda^2)\,; S_\lambda) = \sum_{\epsilon_1,\epsilon_2,\ldots} S_{\lambda/[0,\epsilon_1,\epsilon_2,\ldots]} \ , \tag{31}$$

and this sum can be written as a single determinant due to Weyl ([11], th.7.8.E) :

$$\begin{vmatrix} S_{\lambda_1} & S_{\lambda_1+1} - S_{\lambda_1+1-2} & S_{\lambda_1+2} - S_{\lambda_1+2-4} \\ S_{\lambda_2-1} & S_{\lambda_2} - S_{\lambda_2-2} & S_{\lambda_2+1} - S_{\lambda_2+1-4} \\ S_{\lambda_3-2} & S_{\lambda_3-1} - S_{\lambda_3-1-2} & S_{\lambda_3} - S_{\lambda_3-4} \end{vmatrix} \tag{32}$$

Lovers of λ-rings will prefer the expression

$$\mathcal{S}p_\lambda = S_\lambda(\mathbb{Y} - 0, \mathbb{Y} - 2, \mathbb{Y} - 4, \ldots) := \Big| S_{\lambda_i+j-i}(\mathbb{Y} - 2j + 2) \Big| \ , \tag{33}$$

with, for any alphabet $\mathbb{Y}$, any number k,

$$\sigma_z(\mathbb{Y} - k) := \sigma_z(\mathbb{Y} - 1 - \cdots - 1) = (1 - z)^k \, \sigma_z(\mathbb{Y}) \ .$$

Indeed, because

$$S_i(\mathbb{Y}-k) = S_i(\mathbb{Y}) - \binom{k}{1} S_{i-1}(\mathbb{Y}) + \binom{k}{2} S_{i-2}(\mathbb{Y}) - \cdots \pm S_{i-k}(\mathbb{Y}) \ ,$$

expanding each term of the determinant $\left| S_{\lambda_i+j-i}(\mathbb{Y}-2j+2) \right|$, one obtains Weyl's determinant by eliminating in each column multiples of the columns on its left.

In particular, when $x_1, \ldots, x_n$ are specialized to 1, the determinant specializes to a determinant of binomial coefficients

$$\left| \binom{2n+\lambda_i-i-j+1}{\lambda_i+j-i} \right|_{1\le i,j\le n} \tag{34}$$

which is one of the many expressions of the dimension of a representation of $\mathfrak{S}p_{2n}$.

Let us mention that the interpretation of the orthogonal and symplectic Schur functions as characters depends on the parity of the number of variables. See Proctor [9] for the "'odd symplectic groups".

References

[1] D. Bressoud. *Identities for Schur functions and plane partitions*, Ramanujan J. **4** (2000) 69–80.

[2] M. Ishikawa *Minor summation formula of Pfaffians and Schur functions identities*, FPSAC 95 (Formal Power Series and Algebraic Combinatorics), Université de Marne La Vallée (1995) 379–386.

[3] M. Ishikawa, M. Wakayama. *Applications of Minor-Summation Formula II. Pfaffians and Schur Polynomials*, J. Combin. Th., Ser. A 88 (1999) 136-157.

[4] F. Jouhet, J. Zeng. *Some new identities for Schur functions* preprint (2001).

[5] K. Koike, I. Terada. *Young diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n*, J. Algebra **107** (1987) 466–511.

[6] C. Krattenthaler. *Identities for classical group characters of nearly rectangular shape*, J. Algebra **209** (1998) 1–64.

[7] D.E. Littlewood. *The theory of group characters*, Oxford University Press (1950).

[8] I.G. Macdonald. *Symmetric functions and Hall polynomials*, Clarendon Press, second edition, Oxford, (1995).

[9] R. Proctor. *Interconnections between symplectic and orthogonal characters*, Contemp. Math. **88** (1989) 145–162.

[10] I. Schur. *Gesammelte Abhandlungen*, vol III, p.456, Springer (1973).

[11] H. Weyl. *The classical groups*, Princeton Univ. Press (1939).