SYMMETRIC FUNCTIONS

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Symmetric functions

We shall handle functions on different sets of indeterminates (called *alphabets*, though we shall mostly use commutative indeterminates).

A symmetric function of an alphabet $\mathbb{A}$ is a function of the letters which is invariant under permutation of the letters of $\mathbb{A}$.

The simpler symmetric functions are best defined through generating functions. We shall not use the classical notations for symmetric functions (as they can be found in Macdonald's book), except in the programs (paragraphs beginning with $ACE >$ and using typewriter characters) because it will become clear in the course of these lectures that we need to consider symmetric functions as *functors*, and connect them with operations on vector spaces and representations. It is a small burden imposed on the reader, but the compact notations that we propose greatly simplifies manipulations of symmetric functions. Notice that exponents are used for products, and that S^J is different from S_J, except if J is of length one (i.e. is an integer).

$$J = [j_1, j_2, \ldots] \Rightarrow \Lambda^J = \Lambda^{j_1}\Lambda^{j_2}\cdots \quad \& \quad S^J = S^{j_1}S^{j_2}\cdots \quad \& \quad \Psi^J = \Psi^{j_1}\Psi^{j_2}\cdots$$

are different from S_J, ψ_J etc.

Of course, when indices are of length 1, one has

$$S^j = S_j \ , \ \Lambda^j = \Lambda_j \ , \ \Psi^j = \Psi_j \ .$$

We need operations on alphabets, the first one being the *addition*, that is the disjoint union that we shall denote by a '+'-sign :

$$\Big(\mathbb{A} = \{a\} \ , \ \mathbb{B} = \{b\}\Big) \quad \mapsto \quad \mathbb{A} + \mathbb{B} := \{a\} \cup \{b\}$$

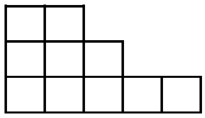
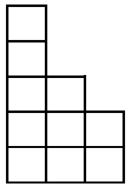
More operations will be introduced in the second section.

- **Partitions**

A weakly increasing sequence of strictly positive numbers $I = [i_1, i_2, \ldots, i_\ell]$ is called a *partition of the number* n of length $\ell(I) = \ell$, where $n = |I| := i_1 + \cdots + i_\ell$. One also uses weakly decreasing sequences instead of increasing ones, but to handle minors of matrices, it is preferable to use our convention.

A partition I has a graphical representation due to Ferrers, which is called its *diagram*: it is a diagram of square boxes left packed, i_1, $i_2, \ldots,$ i_ℓ being the number of boxes in the successive rows. Reading the number of boxes in the successive columns, one obtains another partition $I^{\sim}$ which is called the *conjugate partition*. Conjugating partitions is an involutive operation which can be interpreted as symmetry along the main diagonal, for what concerns diagrams.

For example, when $I = [2, 3, 5]$, then $I^{\sim} = [1, 1, 2, 3, 3]$ and their diagrams are


&

.

A partition I will be identified with any vector obtained by concatenating initial zeroes. This is coherent with identifying partitions and their diagrams, because one can start by reading empty rows!

Let $\mathfrak{P}art(n)$ be the set of partitions of n.

```
It is obtained by
ACE > ListPart(4);
  [[4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1]]
The diagrams are given by the following function, where one can choose
two symbols, one to represent the boxes the boxes of the diagram,
another for the outside :
ACE > [Part2Mat([5,3,2]), Part2Mat([5,3,2],'alphabet'=['#','.'])];
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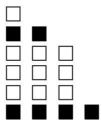
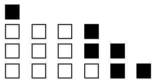
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} \# & \# & . & . & . \\ \# & \# & \# & . & . \\ \# & \# & \# & \# & \# \end{bmatrix}$$

There are two operations on partitions, '+' and '$\cup$' :

$$\mathfrak{P}art(m) \times \mathfrak{P}art(n) \to \mathfrak{P}art(m+n)$$

which are exchanged under conjugation: $(I, J) \to I + J$ is the addition of partitions, *normalizing them to belong to the same space* $\mathbb{N}^r$, and $I \cup J$ is the partition obtained by reordering the *concatenation* of I and J into a partition. With $I = [1, 3, 3, 3]$, $J = [2, 4]$, one has :

 , 

$I \cup J$, $I^{\sim} + J^{\sim}$

To order $\mathfrak{P}art(n)$, one uses, instead of partitions, their *cumulative sums* and one puts the *componentwise* order on these new vectors.

Definition 1 *Given a vector in $v \in N^n$, its* cumulative sum $\overline{\overline{v}}$ *is the vector*

$$\overline{\overline{v}} = [v_1,\, v_1{+}v_2,\, \ldots, v_1{+}v_2{+}\cdots{+}v_n] \ .$$

The n-th cumulative sum *(resp.* cumulative sum*) of a partition I of length $\ell(I) \le n$ is the cumulative sum of the vector obtained by concatenating $n - \ell(I)$ initial zeroes to I (resp. n being the weight of I).*

Given two partitions I, J of the same integer n, I is smaller than J for the dominance order *iff $\overline{\overline{I}}$ is smaller than $\overline{\overline{J}}$ componentwise, i.e.*

$$\overline{\overline{I}}_1 \le \overline{\overline{J}}_1,\, \overline{\overline{I}}_2 \le \overline{\overline{J}}_2,\, \ldots,\, \overline{\overline{I}}_n \le \overline{\overline{J}}_n \ .$$

Cumulative sums of partitions satisfy convexity inequalities that characterize them. It is immediate to check :

Lemma 2 *An element $u \in \mathbb{N}^n$ is the cumulative sum of a partition iff*

$$2u_i \le u_{i-1} + u_{i+1} \ , \ 1 < i < n \ . \tag{1}$$

But now the supremum (componentwise) of two vectors satisfying inequalities (1) also satisfies the same inequalities, and this allows to define the supremum of two partitions :

Definition 3 *The supremum $I \vee J$ of two partitions I, J of n is the only partition K such that $sup(\overline{\overline{I}}\,,\,\overline{\overline{J}})$ is the n-th cumulative sum of K.*

One could have taken the r-th cumulative sum of I, J, for any $r \ge \ell(I), \ell(J)$.

One defines the infimum $I \wedge J$ of two partitions of the same weight by using conjugation :

$$I \wedge J := \big(I^{\sim} \vee J^{\sim}\big)^{\sim} \tag{2}$$

Beware that the minimum componentwise of two cumulative sums is not necessarily the cumulative sum of a partition.

For example, take $I = [1,1,1,5]$, $J = [0,2,2,4]$. Then $\overline{\overline{I}} = [1,2,3,8]$, $\overline{\overline{J}} = [0,2,4,8]$, $sup(\overline{\overline{I}}, \overline{\overline{J}}) = [1,2,4,8] = \overline{\overline{K}}$, with $K = [1,1,2,4] = I \vee J$. On the other hand, $inf(\overline{\overline{I}}, \overline{\overline{J}}) = [0,2,3,8]$ is the cumulative sum of $[0,2,1,5]$, which is not a partition.

Conjugating, one has $I^{\sim} = [1,1,1,1,4]$, $J^{\sim} = [0,1,1,3,3]$, with cumulative sums $[1,2,3,4,8]$,$[0,1,2,5,8]$ and supremum $[1,2,3,5,8]$. Therefore $I^{\sim} \vee J^{\sim} = [1,1,1,2,3]$ and

$$I \wedge J = \big(I^{\sim} \vee J^{\sim}\big)^{\sim} = [1,1,1,2,3]^{\sim} = [1,2,5] \ .$$

One could have use the cumulative sums starting from the right, or equivalently, the cumulative sums on descending partitions.

The two operations $\vee$, $\wedge$, define a structure of lattice on $\mathfrak{P}art(n)$ (with minimum element $[n]$ and maximal element $[1^n]$).

The poset of partitions (*partially ordered set* – bad terminology, because every order is partial, unless otherwise specified !) is not a rank poset, i.e. maximal chains between two comparable elements do not have the same length.

For example, $\mathfrak{P}art(8)$ contains the following piece, which contains the four partitions that we have just considered, writing their cumulative sums on the right :

$$\begin{array}{ccccccccccc}
 & & 1124 & & & & & & 1248 & & \\
 & \swarrow & & \searrow & & & & \swarrow & & \searrow & \\
 & & & & 224 & & & & & & 0248 \\
1115 & & & & \downarrow & \leftrightarrow & 1238 & & & & \downarrow \quad . \\
 & & & & 134 & & & & & & 0148 \\
 & \searrow & & \swarrow & & & & \searrow & & \swarrow & \\
 & & 125 & & & & & & 0138 & &
\end{array}$$

However, it is easy to characterize consecutive elements :

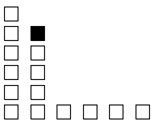
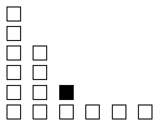
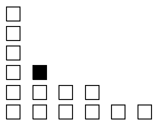
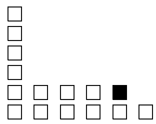
Lemma 4 *Let I, J be two partitions of the same number. If*

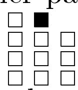
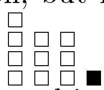
$$I < J \quad \& \quad \textit{there is no } K \textit{ such that } I < K < J \ ,$$

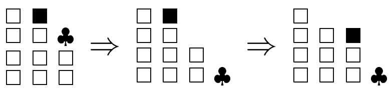
then either

$$\begin{gathered} \exists\, p,k \in \mathbb{N} : \ J = J'\, p^{k+2}\, J'' \quad \& \quad I = J',\, p\text{-}1,\, p^k,\, p\text{+}1,\ J'' \quad \textit{or} \\ \exists\, p,q \in \mathbb{N} : \ J = J'\, p\, q\, J'' \quad \& \quad I = J',\, p\text{-}1,\, q\text{+}1,\ J'' \ . \end{gathered}$$

Notice that the two cases are exchanged by conjugation of partition, so that the dominance order is reversed under conjugation. The possible pairs in Lemma 4 look like what follows :

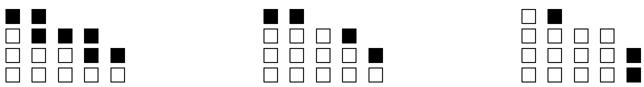
$J =$  $\Rightarrow I =$  , $J =$  $\Rightarrow I =$ 

Pushing a box down gives a smaller partition, but it is not true that it gives a pair of consecutive partitions :  and  are not consecutive, because the move of the black box can be performed in two steps:



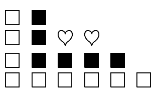
Let J, I be a pair of partitions such that the diagram of J contains the diagram of I. Then the set difference of the two diagrams is called a *skew diagram* and denoted J/I (adding common boxes to I and J does not change J/I. In some problems, one has to consider pairs (J, I) rather than J/I).

If J/I contains no 2×2 sub-diagram and is connected (resp. J/I contains no two boxes in the same column, res. no two boxes in the same row), then J/I is called a *ribbon* (resp. *horizontal strip*, resp. *vertical strip*). There are strips which are both vertical and horizontal, for example a single box.



ribbon horizontal strip vertical strip

A partition of the type $[1^\beta, \alpha{+}1]$ is called a *hook* and is denoted $(\alpha\&\beta)$. The decomposition of the diagram of a partition I into its diagonal hooks (i.e. hooks having their head on the diagonal) is called the *Frobenius code* of I and denoted $\mathfrak{F}rob(I) = (\alpha_1, \alpha_2, \ldots, \alpha_r \,\&\, \beta_1, \beta_2, \ldots, b_r)$ (where r, the number of boxes in the main diagonal, is called the *rank* of the partition).

$I = [2, 4, 5, 6] =$  gives $\mathfrak{F}rob([2, 4, 5, 6]) = (531 \,\&\, 320)$.

Given a box in a diagram, its *content* is its distance to the main diagonal. The multiset of contents allows to recover the diagram, hence the partition. Replacing each box by its content, one has, for example, that $I = [2, 4, 5, 6]$ has contents $\begin{smallmatrix} -3 & -2 & & & & \\ -2 & -1 & 0 & 1 & & \\ -1 & 0 & 1 & 2 & 3 & \\ 0 & 1 & 2 & 3 & 4 & 5 \end{smallmatrix}$ and multiset $\{(-3)^1, (-2)^2, (-1)^2, 0^3, 1^3, 2^2, 3^2, 4^1, 5^1\}$.

Finally (for the moment!), one can code a partition by the word obtained by reading the *border* of its diagram : 0 for an horizontal step, 1 for a vertical step.

$$I = [2,4,5,6] \Rightarrow \text{ border} = \begin{array}{ccccccc} 0 & 0 & 1 & & & & \\ \square & \square & 0 & 0 & 1 & & \\ \square & \square & \square & \square & 0 & 1 & \\ \square & \square & \square & \square & \square & 0 & 1 \\ \square & \square & \square & \square & \square & \square & \end{array} = [0,0,1,0,0,1,0,1,0,1]$$

Taking reverse words and exchanging 0 and 1 corresponds to taking conjugate partitions. In fact, all operations on words on two letters induce operations on partitions that we leave to the reader the pleasure to discover.

It is appropriate concatenate left 1's and right 0's to a border word. This amounts to consider that the partition is contained in a rectangular partition m^n (where m is the total number of 0's and n the total number of 1's). Since a 0, –1 sequence is specified by its length and the position of the 0's, one has the following lemma that one shall need in determinantal identities.

Lemma 5 *Let* $I = [i_1, \ldots, i_n] \in \mathbb{N}^n$ *be a partition contained in* m^n*, and* $J = [j_1, \ldots, j_m]$ *be its conjugate. Then*

$\{i_1{+}1,\ i_2{+}2,\ \ldots, i_n{+}n\}$ *and* $\{m{+}n{+}1{-}j_1{-}1,\ m{+}n{+}1{-}j_2{-}2,\ \ldots, m{+}n{+}1{-}j_m{-}m\}$

are complementary sets in $\{1,\ \ldots,\ m{+}n\}$.

For example, take $m = 5$, $n = 4$. Adding $[1,2,3,4]$ to $I = [0,0,2,4]$, one gets $[1,2,5,8]$; the partition $I^{\sim} = [0,1,1,2,2]$ gives under the same operation $[1,3,4,6,7]$, and $[10-1, 10-3, 10-4, 10-6, 10-7] = [9,7,6,4,3]$ is the complementary set of $1,2,5,8]$.

```
ACE > Part2Frob([6,5,4,2]);
[[5, 3, 1], [3, 2, 0]]
ACE > Frob2Part(");
[6, 5, 4, 2]
ACE > [Part2Border([6,5,4,2]), Part2Border(Part2Conjugate([6,5,4,2]))];
[[0, 0, 1, 0, 0, 1, 0, 1, 0, 1], [0, 1, 0, 1, 0, 1, 1, 0, 1, 1]]
```

Given a box $\square$ in the diagram of a partition J, let its *leg* be the set of boxes in the same column above it, its *arm* be the set of boxes in the same row, on its right. The *hook* relative to $\square$ is the union of the leg, the arm and the box itself, the total number of boxes being the *hook length* of $\square$ in the diagram. It is usual to directly write the hook length of a box in the box itself.

The *content* of a box in a diagram is its distance to the main diagonal, counted negative in the North-West sector.

For example, for $J = [2, 3, 3, 6, 6]$, one has the following hook lengths and contents :

$$\begin{array}{|c|c|c|c|c|c|}
\hline
2 & 1 \\
\cline{1-2}
4 & 3 & 1 \\
\cline{1-3}
5 & 4 & 2 \\
\hline
9 & 8 & 6 & 3 & 2 & 1 \\
\hline
10 & 9 & 7 & 4 & 3 & 2 \\
\hline
\end{array}$$

Hook lengths

$$\begin{array}{|c|c|c|c|c|c|}
\hline
-4 & -3 \\
\cline{1-2}
-3 & -2 & -1 \\
\cline{1-3}
-2 & -1 & 0 \\
\hline
-1 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 1 & 2 & 3 & 4 & 5 \\
\hline
\end{array}$$

Contents

These informations about a partition can also been read on the other codings of the partition, like its border, or its Frobenius decomposition in diagonal hooks.

One can relate the contents and hook lengths as follows. Let J be a partition in $\mathbb{N}^n$ (it can have initial zeros). Let $v := [j_1{+}0,\, j_1{+}1,\, \ldots,\, j_n{+}n{-}1]$. Taking row r of the diagram of J, one sees that the set of hook lengths in this row is such that

$$\{h_\square\} = \{1,\, 2,\, \ldots,\, v_r\} \setminus \{(v_r - v_i)\ :\ i < r\}\ ,$$

and therefore, in total, one has the following equality between multisets :

$$\{h_\square : \square \in Diagr(J)\} = \cup_{1\le r\le n}\{1,\, \ldots,\, v_r\} \setminus \cup\{(v_r{-}v_i) :\ 1 \le i < r \le n\}\ . \tag{3}$$

- **Generating Functions of symmetric functions**

Taking an extra indeterminate z, one has three fundamental series

$$\lambda_z(\mathbb{A}) := \prod_{a\in\mathbb{A}}(1+za)\ ,\ \sigma_z(\mathbb{A}) := \prod_{a\in\mathbb{A}}\frac{1}{1-za}\ ,\ \Psi_z(\mathbb{A}) := \sum_{i=1}^{\infty}\sum_{a\in\mathbb{A}} z^i a^i/i \tag{4}$$

the expansion of which gives the *elementary symmetric functions* $\Lambda^i(\mathbb{A})$ the *complete functions* $S^i(\mathbb{A})$, and the *power sums* $\Psi_i(\mathbb{A})$:

$$\lambda_z(\mathbb{A}) = \sum z^i\,\Lambda^i(\mathbb{A})\ ,\ \sigma_z(\mathbb{A}) = \sum z^i\,S^i(\mathbb{A})\ ,\ \boldsymbol{\Psi}_z(\mathbb{A}) = \sum_{i=1}^{\infty} z^i\Psi_i(\mathbb{A})/i\ . \tag{5}$$

Since $\log(1/(1-a) = \sum_{i>0} a^i/i$, one has

$$\sigma_z(\mathbb{A}) = \exp\left(\boldsymbol{\Psi}_z(\mathbb{A})\right) \quad , \quad \boldsymbol{\Psi}_z(\mathbb{A}) = \log\left(\sigma_z(\mathbb{A})\right) \tag{6}$$

Addition of alphabets implies product of generating series

$$\lambda_z(\mathbb{A}+\mathbb{B}) = \lambda_z(\mathbb{A})\,\lambda_z(\mathbb{B}) \quad , \quad \sigma_z(\mathbb{A}+\mathbb{B}) = \sigma_z(\mathbb{A})\,\sigma_z(\mathbb{B}) \ . \tag{7}$$

However, since one can invert formal series beginning by 1, or take any power of them, one can extend (4) by setting :

$$\sigma_z(\mathbb{A}-\mathbb{B}) := \frac{\prod_{b\in\mathbb{B}}(1-zb)}{\prod_{a\in\mathbb{A}}(1-za)} \quad , \quad \sigma_z(c\,\mathbb{A}) = (\sigma_z(\mathbb{A}))^c \ , \ c\in\mathbb{C} \ . \tag{8}$$

When $\mathbb{B}=0$, or $\mathbb{A}=0$, one recovers the two series $\sigma_z(\mathbb{A})$ and $\sigma_z(-\mathbb{B}) = \lambda_{-z}(\mathbb{B})$.

Notice that the addition of alphabets satisfy the usual properties of addition : $\sigma_z(-\mathbb{A})$ is the inverse of $\sigma_z(\mathbb{A})$ because $\mathbb{A}-\mathbb{A}=0$ and $\sigma_z(0)=1$. Similarly, the identity $(\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C}) = \mathbb{A}-\mathbb{B}$ translates, at the level of generating series, the fact that

$$\frac{\prod_b(1-zb)\prod_c(1-zc)}{\prod_a(1-za)\prod_c(1-zc)} = \frac{\prod_b(1-zb)}{\prod_a(1-za)} \ ,$$

and nobody will deny that one can simplify a factor common to the numerator and denominator of a rational function !

Formal series in z beginning by z^0 should be treated as generating series of complete or elementary symmetric functions of some formal alphabet, or of a difference of two alphabets (one alphabet is sufficient, but one has more flexibility with two alphabets), that one can manipulate without having access to the letters which compose them. This is indeed what one does with a polynomial, even when one is not able to factorize it. One writes a monic polynomial $P(x)$ as $P(x) = \prod_{a\in\mathbb{A}}(x-a)$, $\mathbb{A}$ being the alphabet of zeros of $P(x)$. Now, $\prod_{a\in\mathbb{A}}(x-a)^2$, $\prod_{a\in\mathbb{A}}(x-a^2)$, to show a few examples, are perfectly defined polynomials whose coefficients can be written in terms of the coefficients of P, though they have been defined in terms of the roots of P (but we respected the symmetry between the roots of P!).

Having written $\mathbb{A}+\mathbb{B}$ for the disjoint union of alphabets forces us to consider a finite alphabet as the sum of its sub-alphabets of cardinality 1, i.e to identify $\mathbb{A}$ and $\sum_{a\in\mathbb{A}} a$, and write

$$S^k(a_1+a_2+\cdots+a_n-b_1-\cdots-b_m)$$

instead of $S^k(\mathbb{A}-\mathbb{B})$ when we shall need the letters composing the finite alphabets $\mathbb{A}$ and $\mathbb{B}$.

Given a finite alphabet $\mathbb{A}$, let $\mathfrak{S}ym(\mathbb{A})$ be the ring of symmetric polynomials in $\mathbb{A}$ over the rational numbers. As a vector space, it has (multiplicative) bases

$$\left\{ \begin{array}{lcl} \Lambda^I(\mathbb{A}) & := & \Lambda^{i_1}(\mathbb{A})\Lambda^{i_2}(\mathbb{A})\cdots \\ S^I(\mathbb{A}) & := & S^{i_1}(\mathbb{A})S^{i_2}(\mathbb{A})\cdots \\ \Psi^I(\mathbb{A}) & := & \Psi^{i_1}(\mathbb{A})\Psi^{i_2}(\mathbb{A})\cdots \end{array} \right. , \tag{9}$$

sum over all k, all partitions $I = [i_1, \ldots, i_k]$, $i_k \le card(\mathbb{A})$.

The fact that Λ^I is a linear basis is called "Newton fundamental theorem". It is usually formulated as follows :

Theorem 6 *(Newton). Let $\mathbb{A}$ be an alphabet of cardinality n. Then $\mathfrak{S}ym(\mathbb{A})$ is a polynomial ring with generators $\Lambda^1(\mathbb{A}), \ldots, \Lambda^n(\mathbb{A})$.*

Because of relations (4), it is easy to deduce from Newton's theorem the two other statements in (9), in other words, that $S^1(\mathbb{A}), \ldots, S^n(\mathbb{A})$ and $\Psi^1(\mathbb{A}), \ldots, \Psi^n(\mathbb{A})$ are also algebraic bases of $\mathfrak{S}ym(\mathbb{A})$. In other words, the ring $\mathfrak{S}ym(\mathbb{A})$, with coefficients in $\mathbb{Q}$, is isomorphic to each of the three polynomial rings

$$\mathbb{Q}[\Lambda^1(\mathbb{A}), \ldots, \Lambda^n(\mathbb{A})] \ , \ \mathbb{Q}[S^1(\mathbb{A}), \ldots, S^n(\mathbb{A})] \ , \ \mathbb{Q}[\Psi^1(\mathbb{A}), \ldots, \Psi^n(\mathbb{A})] \ .$$

In the case of the elementary or symmetric functions, one does not need the condition that $Sym(\mathbb{A})$ contains the rationals. Newton's theorem is also valid for symmetric polynomials with coefficients in $\mathbb{Z}$, and consequently, for any ring of coefficients.

When working with *rational* symmetric functions, one can use other generators. For example, Cauchy shown that any symmetric polynomial is a rational function in the odd power sums $\Psi^1(\mathbb{A})$, $\Psi^3(\mathbb{A})$, $\ldots$, $\Psi^{2n-1}(\mathbb{A})$. This property has been rediscovered and extended many times, and we shall comment it in the exercises.

Many problems with symmetric functions involve changes of bases. We shall detail matrices of change of bases in another section, using different combinatorial objects such as Young tableaux, matrices with fixed row and column sums, etc.

The sum of all elements in the orbits of a monomial a^J under the action of the symmetric group $\mathfrak{S}(\mathbb{A})$ is of course a symmetric function, called *monomial function*[1] and we shall denote it $\Psi_J(\mathbb{A})$, (J partition), rather than m_J (except

[1] For the classics, monomial functions were the *symmetric functions.* Alphabets being defined as sets of roots of polynomials, the problem was to express the symmetric functions in terms of the $\Lambda^I(\mathbb{A})$, which were the data. Vandermonde solved this problem, without explaining his method, and published tables in the *Mémoires de l'Académie* for degree up to 10 – with no mistake, as controlled by D.Knuth.

in the programs, where we use the same conventions as Macdonald). They must not be mistaken with the product of power sums $\Psi^J(\mathbb{A})$.

It has been since long realized that one should use alphabets of infinite cardinality, and thus consider a universal ring $\mathfrak{S}ym$ from which one gets by specialization the rings $\mathfrak{S}ym(\mathbb{A})$, for specific alphabets $\mathbb{A}$ of finite cardinality (more generally, we shall use specializations such that *letters* are no more algebraically independent).

- **Matrix generating functions**

Let z stands now for the infinite matrix with diagonal $j-i=1$ filled with 1's, all other entries being 0's.

Since z^k, $k \in \mathbb{N}$, is the matrix with 1's in the k-th diagonal above the main diagonal, and 0 outside of it, we see that now $\sigma_z(\mathbb{A})$ is a Toeplitz matrix (i.e. a matrix with constant values in each diagonal) that we shall denote by $\mathbb{S}(\mathbb{A})$; similarly $\lambda_z(\mathbb{A})$ is a matrix denoted $\mathbb{L}(\mathbb{A})$:

$$\mathbb{S}(\mathbb{A}) = \Big(S^{j-i}(\mathbb{A})\Big)_{i,j\geq 0} \quad \& \quad \mathbb{L}(\mathbb{A}) = \Big(\Lambda^{j-i}(\mathbb{A})\Big)_{i,j\geq 0} . \tag{10}$$

$$\mathbb{S}(\mathbb{A}) = \begin{bmatrix} S^0(\mathbb{A}) & S^1(\mathbb{A}) & S^2(\mathbb{A}) & S^3(\mathbb{A}) & \cdots \\ S^{-1}(\mathbb{A}) & S^0(\mathbb{A}) & S^1(\mathbb{A}) & S^2(\mathbb{A}) & \cdots \\ S^{-2}(\mathbb{A}) & S^{-1}(\mathbb{A}) & S^0(\mathbb{A}) & S^1(\mathbb{A}) & \cdots \\ & \ddots & \ddots & \ddots & \end{bmatrix}$$

$$\mathbb{L}(\mathbb{A}) = \begin{bmatrix} \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) & \Lambda^3(\mathbb{A}) & \cdots \\ \Lambda^{-1}(\mathbb{A}) & \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) & \cdots \\ \Lambda^{-2}(\mathbb{A}) & \Lambda^{-1}(\mathbb{A}) & \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \cdots \\ & \ddots & \ddots & \ddots & \end{bmatrix} .$$

These matrices are upper triangular, but it is wiser to write entries S^{-k} rather than their value 0.

Addition or subtraction of alphabets still correspond to product of matrices, that z be an indeterminate or a matrix makes no difference :

$$\mathbb{S}(\mathbb{A} \pm \mathbb{B}) = \mathbb{S}(\mathbb{A})\,\mathbb{S}(\mathbb{B})^{\pm 1} \quad \& \quad \mathbb{L}(\mathbb{A} \pm \mathbb{B}) = \mathbb{L}(\mathbb{A})\,\mathbb{L}(\mathbb{B})^{\pm 1} . \tag{11}$$

The advantage of matrices, compared to formal series, is that they offer us their minors, that we shall index by (increasing) partitions, or more generally, by vectors with components in $\mathbb{Z}$. More precisely, given $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, $J = (j_1, \ldots, j_n) \in \mathbb{Z}^n$ one defines the *skew Schur function* $S_{J/I}(\mathbb{A})$ to be the minor of $\mathbb{S}(\mathbb{A})$ taken on rows $i_1+1, i_2+2, \ldots, i_n+n$ and columns $j_1+1, \ldots, j_n+n$ (we define the minor to be 0 if one of these numbers is < 0). When $I = 0^n$, the minor is called a *Schur function* and one writes $S_J(\mathbb{A})$ instead of $S_{J/0^n}(\mathbb{A})$.

In other words,

$$S_{J/I}(\mathbb{A}) = \left| S^{j_k - i_h + k - h}(\mathbb{A}) \right|_{1 \le h,k \le n} . \tag{12}$$

The expression of a Schur function as a determinant of complete functions is called the *Jacobi-Trudi* determinant (we shall see that there is also another expression in terms of elementary symmetric functions).

```
One can enter a (decreasing) partition or a skew partition :
```
$ACE >$ `SfJtMat([5,4,1]), SfJtMat([ [5,4,1],[2,1] ]);`

$$\begin{bmatrix} h5 & h6 & h7 \\ h3 & h4 & h5 \\ 0 & 1 & h1 \end{bmatrix} , \begin{bmatrix} h3 & h5 & h7 \\ h1 & h3 & h5 \\ 0 & 0 & h1 \end{bmatrix}$$

One can visualize the Schur function $S_{J/I}$ as being obtained from the initial minor of the same order, by shifting the columns by J, and the rows by I :

$$\begin{array}{ccc|c} 0 & 1 & 2 & -i_1 \\ -1 & 0 & 1 & -i_2 \\ -2 & -1 & 0 & -i_3 \\ \hline j_1 & j_2 & j_3 & \end{array} \Rightarrow \begin{vmatrix} 0+j_1-i_1 & 1+j_2-i_1 & 2+j_3-i_1 \\ -1+j_1-i_2 & 0+j_2-i_2 & 1+j_3-i_2 \\ -2+j_1-i_3 & -1+j_2-i_3 & 0+j_3-i_3 \end{vmatrix} .$$

It is convenient to also use determinants in elementary symmetric functions :

$$\Lambda_{J/I}(\mathbb{A}) = \left| \Lambda^{j_k - i_h + k - h}(\mathbb{A}) \right|_{1 \le h,k \le n} . \tag{13}$$

Of course, one must not forget that $\Lambda^i(\mathbb{A}) = (-1)^i S^i(-\mathbb{A})$, $i \in \mathbb{Z}$, and thus the $\Lambda_{J/I}(\mathbb{A})$ are also skew Schur functions in $-\mathbb{A}$ (we shall see that they also are Schur functions in $\mathbb{A}$, but indexed by "column lengths").

To write easily a Schur function $S_J(\mathbb{A})$, one first fill the diagonal, then complete the columns, increasing or decreasing indices by 1 when moving up or down :

$$J = [1,2,4] \Rightarrow \begin{vmatrix} S_1(\mathbb{A}) & & \\ & S_2(\mathbb{A}) & \\ & & S_4(\mathbb{A}) \end{vmatrix} \Rightarrow \begin{vmatrix} S_1(\mathbb{A}) & S_3(\mathbb{A}) & S_6(\mathbb{A}) \\ S_0(\mathbb{A}) & S_2(\mathbb{A}) & S_5(\mathbb{A}) \\ S_{-1}(\mathbb{A}) & S_1(\mathbb{A}) & S_4(\mathbb{A}) \end{vmatrix} = S_{124}(\mathbb{A})$$

We shall also need rectangular sub-matrices of $\mathbb{S}(\mathbb{A})$ that we shall continue to index the same way: $\mathbb{S}_{J/I}(\mathbb{A})$ is the sub-matrix of $\mathbb{S}(\mathbb{A})$ taken on rows $i_1+1, i_2+2, \ldots$, and columns $j_1+1, j_2+2, \ldots$.

Binet-Cauchy theorem for minors of the product of two matrices implies, in the case of $\mathbb{S}(\mathbb{A}+\mathbb{B})$, the following expansion of skew-Schur functions :

$$S_{J/I}(\mathbb{A}+\mathbb{B}) = \sum\nolimits_K S_{J/K}(\mathbb{A})\, S_{K/I}(\mathbb{B}) \ , \tag{14}$$

sum over all partitions (only those $K : I \subseteq K \subseteq J$ give a non-zero contribution).

Jacobi's theorem on minors of the inverse of a matrix gives, thanks to lemma 5

$$\Lambda_{J/I}(\mathbb{A}) = S_{J^\sim/I^\sim}(\mathbb{A}) = (-1)^{|J/I|} S_{J/I}(-\mathbb{A}) \tag{15}$$

One needs to enlarge the definition of a Schur function, to be able to play with different alphabets at the same time.

Given n, given two sets of alphabets $\{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_n\}$, $\{\mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_n\}$, and $I, J \in \mathbb{N}^n$, we define the *multi-Schur function*

$$S_{J/I}(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_n - \mathbb{B}_n) := \left| S_{j_k - i_h + k - h}(\mathbb{A}_k - \mathbb{B}_k) \right|_{1 \le h,k \le n} . \tag{16}$$

In the case where the alphabets are repeated, we indicate by a semicolon the corresponding block separation : given $H \in \mathbb{Z}^p$, $K \in \mathbb{Z}^q$, then $S_{H;K}(\mathbb{A} - \mathbb{B}; \mathbb{C} - \mathbb{D})$ stands for the multi-Schur function with index the concatenation of H and K, and alphabets $\mathbb{A}_1 = \cdots = \mathbb{A}_p = \mathbb{A}$, $\mathbb{B}_1 = \cdots = \mathbb{B}_p = \mathbb{B}$, $\mathbb{A}_{p+1} = \cdots = \mathbb{A}_{p+q} = \mathbb{C}$, $\mathbb{B}_{p+1} = \cdots = \mathbb{B}_{p+q} = \mathbb{D}$.

To write a multi-Schur function easily, one first fill the diagonal, then complete columns by keeping the same alphabet in each column :

$$S_{12;4}(\mathbb{A};\ \mathbb{B}) \Rightarrow \begin{vmatrix} S_1(\mathbb{A}) & & \\ & S_2(\mathbb{A}) & \\ & & S_4(\mathbb{B}) \end{vmatrix} \Rightarrow \begin{vmatrix} S_1(\mathbb{A}) & S_3(\mathbb{A}) & S_6(\mathbb{B}) \\ S_0(\mathbb{A}) & S_2(\mathbb{A}) & S_5(\mathbb{B}) \\ S_{-1}(\mathbb{A}) & S_1(\mathbb{A}) & S_4(\mathbb{B}) \end{vmatrix}$$

These functions are now sufficiently general to allow easy inductions, thanks to the following transformation lemma.

Lemma 7 *Let $S_J(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_n - \mathbb{B}_n)$ be a multi-Schur function, and $\mathbb{D}_0, \mathbb{D}_1, \ldots, \mathbb{D}_{n-1}$ be a family of finite alphabets such that $card(\mathbb{D}_i) \le i$, $0 \le i \le n-1$. Then $S_J(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_n - \mathbb{B}_n)$ is equal to the determinant*

$$\left| S_{j_k - i_h + k - h}(\mathbb{A}_k - \mathbb{B}_k - \mathbb{D}_{n-h}) \right|_{1 \le h,k \le n}$$

In other words, one does not change the value of a multi-Schur function S_J by replacing in row h the difference $\mathbb{A}-\mathbb{B}$ by $\mathbb{A}-\mathbb{B}-\mathbb{D}_{n-h}$. Indeed, thanks to the expansion (14) :

$$S_j(\mathbb{A}-\mathbb{B}-\mathbb{D}_h) = S_j(\mathbb{A}-\mathbb{B}) + S_1(-\mathbb{D}_h)\, S_{j-1}(\mathbb{A}-\mathbb{B}) + \cdots + S_h(-\mathbb{D}_h)\, S_{j-h}(\mathbb{A}-\mathbb{B}) \,,$$

the sum terminating because the $S_k(-\mathbb{D}_h)$ are null for $k > h$, we see that the determinant has been transformed by multiplication by a triangular matrix with 1's in the diagonal, and therefore has kept its value. QED

For example, taking $D_0 = \emptyset$, $\mathbb{D}_1 = \{x\}$, $\mathbb{D}_2 = \{y, z\}$, one has

$$\begin{bmatrix} S_i(\mathbb{A}_1 - y - z) & S_{j+1}(\mathbb{A}_2 - y - z) & S_{h+2}(\mathbb{A}_3 - y - z) \\ S_{i-1}(\mathbb{A}_1 - x) & S_j(\mathbb{A}_2 - x) & S_{h+1}(\mathbb{A}_3 - x) \\ S_{i-2}(\mathbb{A}_1) & S_{j-1}(\mathbb{A}_2) & S_h(\mathbb{A}_3) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & -y-z & yz \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} S_i(\mathbb{A}_1) & S_{j+1}(\mathbb{A}_2) & S_{h+2}(\mathbb{A}_3) \\ S_{i-1}(\mathbb{A}_1) & S_j(\mathbb{A}_2) & S_{h+1}(\mathbb{A}_3) \\ S_{i-2}(\mathbb{A}_1) & S_{j-1}(\mathbb{A}_2) & S_h(\mathbb{A}_3) \end{bmatrix}$$

and the determinant of the left matrix is equal to $S_{ijh}(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3)$.

To understand better the structure of such determinants, it is appropriate to use "umbral" notations and write alphabets on the border of the determinant: an entry k in position (i, j) will be interpreted as $S_k(\mathbb{A} \pm \mathbb{B})$ if $\mathbb{A}$ is written at the bottom of column j and $\pm\mathbb{B}$ on the right of row i.

$$\begin{array}{ccc|c} & \vdots & & \\ \cdots & k & \cdots & \pm\mathbb{B} \\ & \vdots & & \\ \hline & \mathbb{A} & & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} & \vdots & \\ \cdots & S_k(\mathbb{A} \pm \mathbb{B}) & \cdots \\ & \vdots & \end{array} .$$

For example, during a computation, one would rather write the preceding determinant :

$$\begin{array}{ccc|c} i & j+1 & h+2 & -y-z \\ i-1 & j & h+1 & -x \\ i-2 & j-1 & h & 0 \\ \hline \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{A}_3 & \end{array}$$

Taking $-\mathbb{A}_1, -\mathbb{A}_2, -\mathbb{A}_3$ instead of $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$, and getting rid of signs because of "isobarity", one gets by the same token

$$\Lambda_{i;j;h}(\mathbb{A}_1; \mathbb{A}_2; \mathbb{A}_3) = \begin{bmatrix} \Lambda_i(\mathbb{A}_1 + y + z) & \Lambda_{j+1}(\mathbb{A}_2 + y + z) & \Lambda_{h+2}(\mathbb{A}_3 + y + z) \\ \Lambda_{i-1}(\mathbb{A}_1 + x) & \Lambda_j(\mathbb{A}_2 + x) & \Lambda_{h+1}(\mathbb{A}_3 + x) \\ \Lambda_{i-2}(\mathbb{A}_1) & \Lambda_{j-1}(\mathbb{A}_2) & \Lambda_h(\mathbb{A}_3) \end{bmatrix}$$

In the preceding lemma, we needed only consecutive elements of a column to be complete functions of the same difference of alphabets, of consecutive degrees. A similar transformation can be performed in rows, when alphabets are repeated in some consecutive columns, for partitions having repeated parts.

Lemma 8 *Let j, n be two integers, $\mathbb{D}_0, \ldots, \mathbb{D}_{n-1}$ be a family of alphabets such that $card(\mathbb{D}_i) \leq i$, $0 \leq i \leq n-1$, and let $\mathbb{A}$, $\mathbb{B}$ be two arbitrary alphabets. Let $S_{\diamond; j^n; \circ}(\clubsuit;\ \mathbb{A}-\mathbb{B};\ \spadesuit)$ be a multi-Schur function of which we have specified only n columns.*

Then it is equal to the multi-Schur function

$$S_{\diamond; j,\ldots,j; \circ}(\clubsuit;\ \mathbb{A}-\mathbb{B}-\mathbb{D}_0,\ \mathbb{A}-\mathbb{B}-\mathbb{D}_1, \ldots,\ \mathbb{A}-\mathbb{B}-\mathbb{D}_{n-1};\ \spadesuit)\ .$$

For example,

$$S_{2;\,444}(\mathbb{A};\ \mathbb{B}) = S_{2;\,4;\,4;\,4}(\mathbb{A};\ \mathbb{B};\ \mathbb{B}-\mathbb{D}_1;\ \mathbb{B}-\mathbb{D}_2).$$

The above lemma implies many factorization properties, e.g. for $r \geq 0$,

$$S_J(\mathbb{A}-\mathbb{B}-x)\, x^r = S_{J,r}(\mathbb{A}-\mathbb{B}, x)\ , \tag{17}$$

since taking $\mathbb{D}_1 = \mathbb{D}_2 = \cdots = \{x\}$ factorizes the determinant $S_{J,r}(\mathbb{A}-\mathbb{B}, x)$.

More generally, for an alphabet $\mathbb{D}$ of cardinal $\leq r$ and $J \in \mathbb{N}^r$, one has

$$S_I(\mathbb{A}-\mathbb{B}-\mathbb{D})\, S_J(\mathbb{D}) = S_{I,J}(\mathbb{A}-\mathbb{B},\ \mathbb{D})\ . \tag{18}$$

Monomial themselves can be written as multi-Schur functions. Given a totally ordered alphabet $\mathbb{A} = \{a_1, a_2, \ldots\}$, denote, for any n, $\mathbb{A}_n := \{a_1, \ldots, a_n\}$. Then, for any $J = [j_1, \ldots, j_n]$, denoting $J^\omega := [j_n, \ldots, j_1]$, one has

$$a^J := a_1^{j_1} \cdots a_n^{j_n} = S_{J^\omega}(\mathbb{A}_n, \ldots, \mathbb{A}_2, \mathbb{A}_1)$$

Indeed, subtract the *flag* $0, \mathbb{A}_1, \mathbb{A}_2, \ldots$ in the successive rows, starting from the bottom one. One sees the monomial appearing in the diagonal, the upper part of the matrix vanishing because it is constituted of $S_k\big(-(\mathbb{A}_j - \mathbb{A}_i)\big)$ for $k > (j-i)$, $j \geq i$.

$$a^{532} = a_1^5 a_2^3 a_3^2 = S_{2;3;5}(a_1{+}a_2{+}a_3;\ a_1{+}a_2;\ a_1) =$$

$$= \begin{array}{ccc|c} 2 & 4 & 7 & -\mathbb{A}_2 \\ 1 & 3 & 6 & -\mathbb{A}_1 \\ 0 & 2 & 5 & 0 \\ \hline \mathbb{A}_3 & \mathbb{A}_2 & \mathbb{A}_1 & \end{array} = \begin{vmatrix} S_2(a_3) & S_4(0) & S_7(-a_2) \\ S_1(a_2{+}a_3) & S_3(a_2) & S_6(0) \\ S_0(a_1{+}a_2{+}a_3) & S_2(a_1{+}a_2) & S_5(a_1) \end{vmatrix}.$$

One could have put any order on the letters in $\mathbb{A}$, and in general, a monomial on an alphabet of n letters can be written in $n!$ different manners as a multi-Schur functions. However, because it is appropriate to restrict to the flag of alphabets $\mathbb{A}_1 \subset \mathbb{A}_2 \subset \mathbb{A}_3 \subset \cdots$, we have preferred the convention which gives as the index of the multi-Schur function the reverse of the exponent of the monomial.

Given two finite alphabets, the following factorization and vanishing properties implicitly appear in many classical 19-th century texts about elimination theory (modern reference is Berele-Regev, [4]).

Proposition 9 *Let $\mathbb{A}$, $\mathbb{B}$, be of cardinalities α, β, $p \in \mathbb{N}$, $I \in \mathbb{N}^p$, $J \in \mathbb{N}^\alpha$. Then*

$$S_{i_1,\ldots,i_p,\beta+j_1,\ldots,\beta+j_\alpha}(\mathbb{A}-\mathbb{B}) = S_I(-\mathbb{B})\, S_J(\mathbb{A}) \prod_{a\in\mathbb{A}, b\in\mathbb{B}} (a-b)\ .$$

Let J be a partition, $J \supseteq (\beta+1)^{\alpha+1}$. Then $S_J(\mathbb{A}-\mathbb{B}) = 0$.

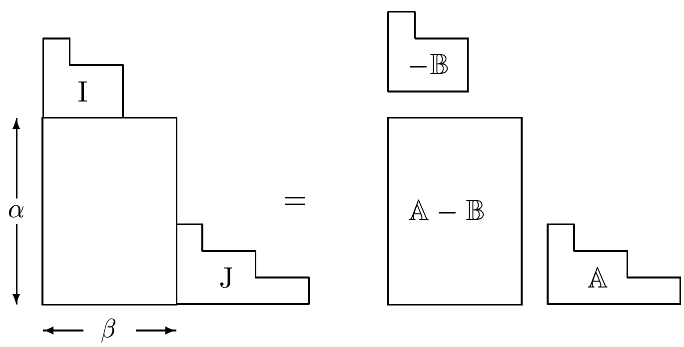
Proof. Subtract $\mathbb{A}$ in the first p rows. One gets the factorization

$$S_I(-\mathbb{B})\, S_{\beta+j_1,\ldots,\beta+j_\alpha}(\mathbb{A}-\mathbb{B})\ .$$

Now, using the partition K conjugate to $[\beta+j_1, \ldots, \beta+j_\alpha]$, one gets the factorization of $S_K(\mathbb{B}-\mathbb{A})$ into a Schur function of $\mathbb{A}$ and $S_{\alpha^\beta}(\mathbb{B}-\mathbb{A})$. This last function can be seen equal to the *resultant* $\prod_{a\in\mathbb{A}, b\in\mathbb{B}}(b-a)$ by subtracting the flag $0, \mathbb{B}_1, \mathbb{B}_2, \ldots$.

The case J too big can be treated by adding the same letters to $\mathbb{A}$ and $\mathbb{B}$, so that one is reduced to the preceding case. But now the factor $\prod(a-b)$ vanishes because $\mathbb{A}$ and $\mathbb{B}$ have a letter in common.

Pictorially, the relation is



Given a finite alphabet $\mathbb{A}$ (that one will totally order: $\mathbb{A} = \{a_1, \ldots, a_n\}$), Cauchy and Jacobi separately defined the Schur function $S_J(\mathbb{A})$ using the (infinite) *Vandermonde matrix*

$$V(\mathbb{A}) = \left[a_i^j\right]_{1\le i\le n\,;\,j\ge 0} = \begin{bmatrix} a_1^0 & a_1^1 & a_1^2 & \cdots \\ \vdots & \vdots & \vdots & \\ a_n^0 & a_n^1 & a_n^2 & \cdots \end{bmatrix}$$

and the *Vandermonde* (determinant)

$$\Delta(\mathbb{A}) := \prod_{i>j}(a_i - a_j) = \begin{vmatrix} a_1^0 & a_1^1 & \cdots & a_1^{n-1} \\ \vdots & \vdots & & \vdots \\ a_n^0 & a_n^1 & \cdots & a_n^{n-1} \end{vmatrix} .$$

Proposition 10 *Let $J \in \mathbb{N}^n$. Then $S_J(\mathbb{A})\,\Delta(\mathbb{A})$ is equal to the minor of index $(0^n, J)$ of the Vandermonde matrix $V(\mathbb{A})$.*

Proof. Let $\mathbb{S}_J(\mathbb{A})$ denotes the sub-matrix of $\mathbb{S}(A)$ taken on columns $j_1{+}1$, $j_2{+}2, \ldots, j_n{+}n$. Consider the product $V(\mathbb{A})\,\mathbb{S}(-A)\,\mathbb{S}_J(\mathbb{A})$. It can be factorized in two manners, using (11):

$$V(\mathbb{A})\,\mathbb{S}(-A)\,\mathbb{S}_J(\mathbb{A}) = \left[S_j(a_i - \mathbb{A})\right]_{1\le i\le n; j\ge 0}\ \mathbb{S}_J(\mathbb{A}) = V(\mathbb{A})\,\mathbb{S}_J(0) \ .$$

However, the $S_j(a_i - \mathbb{A})$ are null for $j \ge n$, because they are the elementary functions (up to sign) of alphabets of cardinality $n-1$. On the other hand, $S_j(0) = 0$, if $j \ne 0$. In both cases, we have obtained matrices such that only one minor of order n is different from 0. QED

For example, for $n = 3$, $J = [1, 3, 4]$, truncating the matrices, one has

$$\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^6 \\ 1 & a_2 & a_2^2 & \cdots & a_2^6 \\ 1 & a_3 & a_3^2 & \cdots & a_3^6 \end{bmatrix}
\begin{bmatrix} S_0(-\mathbb{A}) & S_1(-\mathbb{A}) & S_2(-\mathbb{A}) & S_3(-\mathbb{A}) & S_4(-\mathbb{A}) & S_5(-\mathbb{A}) & S_6(-\mathbb{A}) \\ 0 & S_0(-\mathbb{A}) & S_1(-\mathbb{A}) & S_2(-\mathbb{A}) & S_3(-\mathbb{A}) & S_4(-\mathbb{A}) & S_5(-\mathbb{A}) \\ 0 & 0 & S_0(-\mathbb{A}) & S_1(-\mathbb{A}) & S_2(-\mathbb{A}) & S_3(-\mathbb{A}) & S_4(-\mathbb{A}) \\ 0 & 0 & 0 & S_0(-\mathbb{A}) & S_1(-\mathbb{A}) & S_2(-\mathbb{A}) & S_3(-\mathbb{A}) \\ 0 & 0 & 0 & 0 & S_0(-\mathbb{A}) & S_1(-\mathbb{A}) & S_2(-\mathbb{A}) \\ 0 & 0 & 0 & 0 & 0 & S_0(-\mathbb{A}) & S_1(-\mathbb{A}) \\ 0 & 0 & 0 & 0 & 0 & 0 & S_0(-\mathbb{A}) \end{bmatrix}
\begin{bmatrix} S_1(\mathbb{A}) & S_4(\mathbb{A}) & S_6(\mathbb{A}) \\ S_0(\mathbb{A}) & S_3(\mathbb{A}) & S_5(\mathbb{A}) \\ 0 & S_2(\mathbb{A}) & S_4(\mathbb{A}) \\ 0 & S_1(\mathbb{A}) & S_3(\mathbb{A}) \\ 0 & S_0(\mathbb{A}) & S_2(\mathbb{A}) \\ 0 & 0 & S_1(\mathbb{A}) \\ 0 & 0 & S_0(\mathbb{A}) \end{bmatrix}$$

$$= \begin{bmatrix} S_0(a_1{-}\mathbb{A}) & S_1(a_1{-}\mathbb{A}) & S_2(a_1{-}\mathbb{A}) & 0 & 0 & 0 & 0 \\ S_0(a_2{-}\mathbb{A}) & S_1(a_2{-}\mathbb{A}) & S_2(a_2{-}\mathbb{A}) & 0 & 0 & 0 & 0 \\ S_0(a_3{-}\mathbb{A}) & S_1(a_1{-}\mathbb{A}) & S_2(a_1{-}\mathbb{A}) & 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} S_1(\mathbb{A}) & S_4(\mathbb{A}) & S_6(\mathbb{A}) \\ S_0(\mathbb{A}) & S_3(\mathbb{A}) & S_5(\mathbb{A}) \\ 0 & S_2(\mathbb{A}) & S_4(\mathbb{A}) \\ 0 & S_1(\mathbb{A}) & S_3(\mathbb{A}) \\ 0 & S_0(\mathbb{A}) & S_2(\mathbb{A}) \\ 0 & 0 & S_1(\mathbb{A}) \\ 0 & 0 & S_0(\mathbb{A}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a_1 & a_1^2 & a_1^3 & a_1^4 & a_1^5 & a_1^6 \\ 1 & a_2 & a_2^2 & a_2^3 & a_2^4 & a_2^5 & a_2^6 \\ 1 & a_3 & a_3^2 & a_3^3 & a_3^4 & a_3^5 & a_3^6 \end{bmatrix}
\begin{bmatrix} S_1(0) & S_4(0) & S_6(0) \\ S_0(0) & S_3(0) & S_5(0) \\ 0 & S_2(0) & S_4(0) \\ 0 & S_1(0) & S_3(0) \\ 0 & S_0(0) & S_2(0) \\ 0 & 0 & S_1(0) \\ 0 & 0 & S_0(0) \end{bmatrix} .$$

Still writing $\mathbb{A}_i = a_1 + \cdots + a_i$, one has an expression for Schur functions which "interpolates" between Jacobi-Trudi determinant and a minor of the Vandermonde matrix.

Lemma 11 *Let $\mathbb{A}$ be of cardinality n, and $K = [k_1, \ldots, k_n] \in \mathbb{N}^n$. Then*

$$S_K(\mathbb{A}) = \det\left|S_{k_j+j-i}(\mathbb{A}_i)\right|_{1\le i,j\le n} . \tag{19}$$

Proof. Subtract $0,\ a_n,\ a_n{+}a_{n-1},\ \ldots,\ a_n{+}\cdots{+}a_2$ in the successive rows of $S_K(\mathbb{A})$. QED

We shall later interpret such a determinant as a *discrete Wronskian*. Notice that the top row is made of powers of a_1, and the bottom row is the same as in Jacobi-Trudi determinant.

Given an alphabet $\mathbb{A}$ of finite cardinality n, one will need the alphabet of inverses $\mathbb{A}^\vee = \{a^{-1}\}$. Noticing that

$$\Lambda^i(\mathbb{A}^\vee) = \Lambda^{n-i}(\mathbb{A})/\Lambda^n(\mathbb{A}) \ ,\ 1 \le i \le n \ ,$$

and using the expression of a Schur function as a determinant of Λ^i, one has, for any r and any partition $I \subseteq \Box = r^n$:

$$S_I(\mathbb{A}^\vee) = S_{\Box/I}(\mathbb{A})/S_\Box(\mathbb{A}) \ . \tag{20}$$

• Cauchy formula

The most important formula in the theory of symmetric functions is the following expansion, due to Cauchy.

Let $\mathbb{A}, \mathbb{B}$ be two alphabets. Then :

$$K(\mathbb{A},\mathbb{B}) := \sigma_1(\mathbb{A}\mathbb{B}) = \prod_{a\in\mathbb{A}}\prod_{b\in\mathbb{B}}(1-ab)^{-1} = \sum\nolimits_J S_J(\mathbb{A})S_J(\mathbb{B}) \ , \tag{21}$$

sum over all partitions J (the terms for which $\ell(J) > min\big(card(\mathbb{A})\,,\ card(\mathbb{B})\big)$ vanish).

One will find later a proof of Cauchy formula, using symmetrizing operators, starting from the straightforward identity

$$\frac{1}{1-a_1b_1}\frac{1}{1-a_1a_2b_1b_2}\frac{1}{1-a_1a_2a_3b_1b_2b_3}\cdots = \sum\nolimits_\lambda a^\lambda b^\lambda \ ,$$

sum over all weakly decreasing exponents λ.

For the moment, let us sketch a proof that one can find in the literature, supposing that the two alphabets have cardinality n.

Consider the Cauchy matrix $\big[1/(1{-}ab)\big]_{a\in\mathbb{A},b\in\mathbb{B}}$. Each entry $1/(1{-}ab)$ can be considered as the scalar product of the two infinite vectors $[1, a, a^2, \ldots]$

and $[1, b, b^2, \ldots]$, and therefore the Cauchy matrix is equal to the product $V(\mathbb{A})\, V(\mathbb{B})^{tr}$ of two Vandermonde matrices.

Since minors of each matrix are Schur functions multiplied by a Vandermonde, Binet-Cauchy expansion gives :

$$\left|1/(1-ab)\right|_{a\in\mathbb{A}, b\in\mathbb{B}} = \Delta(\mathbb{A})\Delta(\mathbb{B}) \sum\nolimits_J S_J(\mathbb{A}) S_J(\mathbb{B}) \ .$$

Now, the determinant itself is equal to the sum ($\ell(\sigma)$ denoting the *length* of a permutation σ)

$$\sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)} \frac{1}{\Big((1-a_1b_1)\cdots(1-a_nb_n)\Big)^{\sigma}} \ .$$

Extracting the full denominator $\prod_{a\in\mathbb{A},\, b\in\mathbb{B}} 1/(1-ab)$, one has to compute the sum

$$\sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)} \Big(S^{n-1}(1+b_1a_1-b_1\mathbb{A})\cdots S^{n-1}(1+b_na_n-b_n\mathbb{A})\Big)^{\sigma}$$

This sum is divisible by $\Delta(\mathbb{A})\,\Delta(\mathbb{B})$, because it vanishes when two of the a's, or two of the b's coincide. The degree in each a or b is at most $n-1$, and therefore the quotient is of degree zero in each variable. One has to check that it is equal to 1. QED

The last step in the above sketch of a demonstration misses the crucial fact that what is really involved is *Jacobi symmetrizer*

$$\mathbb{C}[a_1, \ldots, a_n] \ni f \mapsto \left(\sum_{\sigma\in\mathfrak{S}(\mathbb{A})} (-1)^{\ell(\sigma)} f^{\sigma}\right) \frac{1}{\Delta(\mathbb{A})} \ ,$$

sum over all permutations σ of the letters of $\mathbb{A}$. Jacobi's symmetrizer provides a connection with the theory of characters (and extends to Weyl's character formula for the classical groups). We postpone this point of view to another section, where we shall express Jacobi's symmetrizer as a product of divided differences.

There are other forms of Cauchy's formula, for two alphabets of finite cardinalities :

$$\prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (1-ab) = \sigma_1(-\mathbb{A}\mathbb{B}) \ = \ \sum_I (-1)^{|I|} S_I(\mathbb{A}) S_{I^\sim}(\mathbb{B}) \tag{22}$$

$$R(\mathbb{A},\mathbb{B}) := \prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (a-b) \ = \ \sum_I S_I(\mathbb{A}) S_{\square/I}(-\mathbb{B}) \tag{23}$$

$$= \ \sum_I (-1)^{|\square/I|} S_I(\mathbb{A}) S_{\square^\sim/I^\sim}(\mathbb{B}) \ ,$$

where $\square = \beta^\alpha$, $\alpha = card(\mathbb{A})$, $\beta = card(\mathbb{B})$.

Formula (22) is equivalent to (21), changing $\mathbb{B}$ into $-\mathbb{B}$, and using (15). Changing $\mathbb{A}$ into $\mathbb{A}^\vee = \{a^{-1}\}$, one gets (23), thanks to the relations (20) between the Schur functions of $\mathbb{A}$ and those of $\mathbb{A}^\vee$.

Thus, the equivalence of the three forms of Cauchy formulas is purely formal, once one proves them for any cardinality. However, we already have stated a property implying (23) in theorem 9 and using only the transformation lemma 7. Let us repeat the proof in detail.

The right hand side of (23) is the expansion of $S_\square(\mathbb{A} - \mathbb{B})$, according to (12). One subtracts the alphabet $\mathbb{A}-a_1$ in the first row of $S_\square(\mathbb{A} - \mathbb{B})$, and a_1 in all the columns, except the first one. Now, the first row of the determinant has become

$$S_\beta(a_1 - \mathbb{B}),\ S_{\beta+1}(-\mathbb{B}),\ \ldots,\ S_{\beta+\alpha-1}(-\mathbb{B})\ .$$

Since $S_{\beta+1}(-\mathbb{B}), \ldots$ are null, the new determinant factorizes into

$$S_\beta(a_1 - \mathbb{B})\, S_{\beta^{\alpha-1}}((\mathbb{A}-a_1) - \mathbb{B})\ ,$$

and this gives (23) by induction on α.

- **Scalar Product**

There are other decompositions of $K(\mathbb{A}, \mathbb{B})$ as a sum of products of symmetric functions in $\mathbb{A}$ and in $\mathbb{B}$. However, there is only one of the type $\sum P(\mathbb{A})P(\mathbb{B})$ over $\mathbb{Z}$: up to signs, the P's are all the Schur functions indexed by partitions in $\mathbb{N}$. Thus $K(\mathbb{A}, \mathbb{B})$, that we shall call *Cauchy kernel*, determines the Schur functions, because this is the only $\mathbb{Z}$-basis in which $K(\mathbb{A}, \mathbb{B})$ is diagonal.

One can interpret differently the kernel, as defining a scalar product on the space of symmetric functions, the Schur functions constituting the only orthogonal basis. Now, any expansion of the type

$$\mathbf{K}(\mathbb{A}, \mathbb{B}) = \sum P_J(\mathbb{A})\, Q_J(\mathbb{B}) \tag{24}$$

define a pair of adjoint bases $\{P_J\}$, $\{Q_J\}$, with respect to the *canonical scalar product* $(\ ,\ )$ induced by $\mathbf{K}(\mathbb{A}, \mathbb{B})$, i.e. the scalar product such that $(S_J, S_J) = 1$, for all partition J. In other words (24) is equivalent to

$$(P_J(\mathbb{A})\,,\, Q_J(\mathbb{A})) = 1 \qquad \& \qquad (P_I(\mathbb{A})\,,\, Q_J(\mathbb{A})) = 0 \text{ for } I \neq J\ . \tag{25}$$

There are some difficulties for what concerns scalar products when taking finite alphabets, and in the rest of the section, we shall take only infinite alphabets.

The expansion

$$\mathbf{K}(\mathbb{A},\mathbb{B}) = \prod_b \left(\prod_a \frac{1}{1-ab}\right) = \prod_b \left(\sum b^i S^i(\mathbb{A})\right) = \sum_I \Psi_I(\mathbb{B})\, S^I(\mathbb{A}) \quad (26)$$

shows that the basis adjoint to S^I, I partition, is the monomial basis Ψ_I.

From the identity (5), one gets

$$\begin{aligned} \sigma_1(\mathbb{A}) &= \exp\big(\sum_{i=1}^{\infty} \Psi^i(\mathbb{A})/i\big) = \prod_{i=1}^{\infty} \exp(\Psi^i(\mathbb{A})/i) \\ &= \sum\nolimits_I \Psi^I(\mathbb{A})/z_I \ , \end{aligned} \quad (27)$$

sum over all partitions $I = [1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots]$, defining

$$z_I = m_1! 1^{m_1}\, m_2! 2^{m_2}\, m_3! 3^{m_3} \cdots \quad (28)$$

Since $\Psi^i(\mathbb{A}\mathbb{B}) = \Psi^i(\mathbb{A})\, \Psi^i(\mathbb{B})$, it implies the expansion

$$\mathbf{K}(\mathbb{A},\mathbb{B}) = \sigma_1(\mathbb{A}\mathbb{B}) = \sum_I \Psi^I(\mathbb{A})\Psi^I(\mathbb{B})/z_I \ , \quad (29)$$

which shows that the basis of products of power sums is orthogonal, with scalar product $(\Psi_I, \Psi_I) = z_I$.

The name kernel is justified by the following property, which is just another way of stating that $\mathbf{K}(\mathbb{A},\mathbb{B})$ defines a scalar product:

Lemma 12 *Let f be a symmetric function and $(\ ,\ )$ be the canonical scalar product on symmetric functions in $\mathbb{A}$. Then*

$$(\mathbf{K}(\mathbb{A},\mathbb{B})\,,\ f(\mathbb{A})) = f(\mathbb{B})$$

Proof. The identity is linear in $f \in \mathfrak{S}ym(\mathbb{A})$. We check it on the basis of Schur functions :

$$\Big(\sum\nolimits_I S_I(\mathbb{A}) S_I(\mathbb{B})\,,\ S_J(\mathbb{A})\Big) = S_J(\mathbb{B}) \ .$$

QED

One can use simultaneoulsy several alphabets $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}, \ldots$, as well as the scalar products corresponding to them. We shall specify in which symmetric ring we are evaluating the scalar product by indexing it by the alphabet : $(\ ,\ )_{\mathbb{A}}$.

Let us give a first example. Let $f, g, h \in \mathfrak{S}ym$. Then one has

$$\Big(\sigma_1\big((\mathbb{A}+\mathbb{B})\mathbb{C}\big)\ ,\ f(\mathbb{A})\, g(\mathbb{B})\, h(\mathbb{C})\Big)_{\mathbb{A}} = \sigma_1(\mathbb{B}\mathbb{C})\, f(\mathbb{C})\, g(\mathbb{B})\, h(\mathbb{C})\ . \tag{30}$$

Proof. One factors out $\sigma_1(\mathbb{B}\mathbb{C})$ and $g(\mathbb{B})h(\mathbb{C})$, which are scalars in $\mathfrak{S}ym(\mathbb{A})$. One is left with $(\sigma_1(\mathbb{A}\mathbb{C})\ ,\ f(\mathbb{A}))_{\mathbb{A}} = f(\mathbb{C})$. QED

- **Differential calculus**

Having a scalar product, one can now obtain operators *adjoint* to some simple ones. We did not use the multiplicative structure of $\mathfrak{S}ym$ up to now, but only the vector space structure of $\mathfrak{S}ym$. Now we shall use that any symmetric function f can be thought of as the operator

"multiplication by f" .

Definition 13 *For $f \in \mathfrak{S}ym$, D_f is the operator adjoint to the multiplication by f, i.e. for every $S', S'' \in \mathfrak{S}ym$ one has*

$$(D_f(S')\ ,\ S'') = (S'\ ,\ f\, S'')\ .$$

The following lemma shows that Schur functions play a special rôle (cf. several exercises; the best proof of the following lemma is interpreting $\mathfrak{S}ym$ as a ring of shifting operators on isobaric determinants, as explained in the appendix).

Lemma 14 *For every $I, J \in \mathbb{N}^n$, one has*

$$D_{S_I}(S_J) = S_{J/I}\ , \tag{31}$$

$$D_{S^I}(\Psi_J) = \begin{cases} \Psi_K & \text{if } J = K \cup I \\ 0 & \text{otherwise} \end{cases}\ . \tag{32}$$

Proof. Instead of a single $S_{J/I}$, let us introduce two other alphabets $\mathbb{B}$, $\mathbb{C}$ and take the generating function

$$\sigma_1((\mathbb{A}+\mathbb{B})\,\mathbb{C}) = \sum\nolimits_{J,H} S_{J/H}(\mathbb{A})\, S_H(\mathbb{B})\, S_J(\mathbb{C})\ .$$

The scalar product (in $\mathfrak{S}ym(\mathbb{B})$) of this function with $S_I(\mathbb{B})$ is equal to $\sum_J S_{J/I}(\mathbb{A})\, S_J(\mathbb{C})$, which is, according to (30), equal to $\sigma_1(\mathbb{A}\mathbb{C})\, S_I(\mathbb{C})$. Therefore

$$\begin{aligned} S_{J/I}(\mathbb{A}) &= \big(\sigma_1(\mathbb{A}\mathbb{C}) S_I(\mathbb{C})\ ,\ S_J(\mathbb{C})\big)_{\mathbb{C}} \\ &= \big(\sigma_1(\mathbb{A}\mathbb{C})\ ,\ D_{S_I} S_J(\mathbb{C})\big)_{\mathbb{C}} \end{aligned}$$

and, because $\sigma_1(\mathbb{A}\mathbb{C})$ is a reproducing kernel, one gets the wanted identity $S_{J/I}(\mathbb{A}) = D_{S_I} S_J(\mathbb{A})$.

As for the second identity, it is just another way of stating that monomial functions Ψ_J and product of complete functions S^J constitute two adjoint bases. Indeed,

$$(D_{S^I}(\Psi_J)\,,\, S^K) = (\Psi_J\,,\, S^{I\cup K})$$

allows to conclude. QED

```
ACE > SfDiff(s[2,1], s[5,4,2]);
    s[5,3] +s[4,4] +s[5,2,1] +2 s[4,3,1] +s[4,2,2] +s[3,3,2]
ACE > Tos(det( SfJtMat([ [5,4,2],[2,1]])));
    s[5,3] +s[4,4] +s[5,2,1] +2 s[4,3,1] +s[4,2,2] +s[3,3,2]
ACE > Tom(SfDiff( h2*h4, a*m[4,3,2,2] +b*m[3,3,2,2] ), collect);
    a m[3, 2]
```

The operators D_{Ψ_i} are differential operators. Indeed, for every integer $i > 0$, one has

$$D_{\Psi_i} = i\partial_{\Psi_i} \tag{33}$$

as can be checked by operating on the basis Ψ^J: the scalar product is compatible with the tensor decomposition

$$\mathfrak{S}ym \simeq \mathbb{C}[\Psi_1] \otimes \mathbb{C}[\Psi_2] \otimes \mathbb{C}[\Psi_3] \otimes \cdots \ ,$$

and the equality

$$\big(\Psi_i(\Psi_i)^k\,,\,(\Psi_i)^m\big) = \delta_{k,m-1}\, i^m\, m! = \delta_{k,m-1}\, z_{i^m}$$

proves the assertion.

```
ACE > SfDiff( p1∧2*(p3/3)∧2, s[4,4,3,1]);
               1/9 s[2, 2] + 1/9 s[4] + 2/9 s[3, 1]
ACE > Tos(diff(Top(s[4,4,3,1]), p1,p1,p3,p3));
                1/9 s[2,2] + 1/9 s[4] + 2/9 s[3, 1]
```

In the usual differential calculus on polynomials, one can define derivatives without having recourse to vanishing ϵ's, but just as the successive coefficients in Taylors's formula, in other words, just using $f(y) \to f(y+x)$.

The same property is true in the ring $\mathfrak{S}ym$.

Proposition 15 *For any pair of adjoint bases $\{P_I\}$, $\{Q_I\}$, and any element $f \in \mathfrak{S}ym$, one has the decomposition*

$$f(\mathbb{A}+\mathbb{B}) = \sum_I D_{P_I}\big(f(\mathbb{A})\big)\, Q_I(\mathbb{B}) \ . \tag{34}$$

Proof. This is a linear statement that it is sufficient to prove for one pair of adjoint bases, for example for the Schur basis, and for a f a generic Schur function.

But in that case, the identity to be proven is the expansion of $S_J(\mathbb{A}+\mathbb{B})$ given in (14). QED

We shall use very often the identity

$$\forall f \in \mathfrak{S}ym\,,\ f(\mathbb{B}+\mathbb{A}) = \sum\nolimits_I D_{S_I}\big(f(\mathbb{B})\big)\, S_I(\mathbb{A}) \tag{35}$$

when A is a single letter a or when it is $-a$. In that case it becomes

$$\begin{aligned} f(\mathbb{B}+a) &= \sum\nolimits_i D_{S^i}\big(f(\mathbb{B})\big)\, a^i & (36)\\ f(\mathbb{B}-a) &= \sum\nolimits_i D_{\Lambda^i}\big(f(\mathbb{B})\big)\, (-a)^i \ . & (37)\end{aligned}$$

Another corollary of (34) is :

$$\begin{aligned} \forall f \in \mathfrak{S}ym\,,\ f(\mathbb{A}+\mathbb{B}) &= \sum_I \Psi_I(\mathbb{A})\, D_{S^I}\big(f(\mathbb{B})\big)\ , & (38)\\ &= \sum_J \frac{1}{(\Psi^I\,,\,\Psi^I)} \Psi^I(\mathbb{A})\, D_{\Psi^I}\big(f(\mathbb{B})\big)\ . & (39)\end{aligned}$$

The operators D_{S_I} are usually called *Hammond operators*. They satisfy a kind of Leibnitz[2] formula :

Lemma 16 *For every $f, g \in \mathfrak{S}ym$, and every partition J, one has*

$$D_{S_J}(fg) = \sum\nolimits_I D_{S_{J/I}}(f)\, D_{S_I}(g) \ . \tag{40}$$

Proof. $D_{S_J}(fg)(\mathbb{A})$ is the coefficient of $S_J(\mathbb{B})$ in

$$f(\mathbb{A}+\mathbb{B})\, g(\mathbb{A}+\mathbb{B}) = \sum\nolimits_{H,K} (S_H(\mathbb{B}) S_K(\mathbb{B})\,,\ S_J(\mathbb{B}))_{\mathbb{B}}\, D_{S_H(\mathbb{A})} f(\mathbb{A})\, D_{S_K}(g)(\mathbb{A}) \ .$$

The lemma follows from the fact that $\sum_H (S_H\, S_K\,,\, S_J)\, S_H = S_{J/K}$. QED

[2] Leibnitz was spelling his name with a 't' which is ignored by most anglo-saxons authors.

Operators on isobaric determinants

We have not yet used an evident property of the different determinants that we have written up to now, *that all terms in their expansion have the same total degree*. We shall say that such a determinant is *isobaric* (in fact, more generally, we are only dealing with *homogeneous polynomials*).

But now, given an isobaric determinant, how to operate on it obtaining only isobaric determinants ? A simpler question even :

how to increase degrees by 1 ?

It is appropriate to consider more general "weights" than degree, and we shall begin by vectors before considering determinants.

Let V be a vector space of functions of a variable $x \in \mathbb{C}$, with values in a commutative ring. Given two integers $1 \le k \le n$, define T_k^+ (resp. T_k^-) to be the operator on $V^{\otimes n}$ sending $f_1(x_1) \otimes \cdots \otimes f_n(x_n)$ onto

$$\sum_{\epsilon \in \{0,1\}^n,\, \epsilon_1+\cdots+\epsilon_n=k} f_1(x_1 \pm \epsilon_1) \otimes \cdots \otimes f_n(x_n \pm \epsilon_n) \ .$$

For example,

$$T_2^+\big(f_1(x_1) \otimes f_2(x_2) \otimes f_3(x_3)\big) = f_1(x_1+1) \otimes f_2(x_2+1) \otimes f_3(x_3) \\ + f_1(x_1+1) \otimes f_2(x_2) \otimes f_3(x_3+1) + f_1(x_1) \otimes f_2(x_2+1) \otimes f_3(x_3+1) \ .$$

The following theorem is not difficult to check. It amounts to the fact that the T_i's, $i = 1, \ldots, n$ commute between themselves, and are algebraically independent (they have been my first encounter with the theory of symmetric functions, through manipulations of isobaric determinants ([27]).

Theorem 17 *Let n be a positive integer. Then $T_0^+ = 1, T_1^+, \ldots, T_n^+$ $\big($resp. $T_0^- = 1, T_1^-, \ldots, T_n^-\big)$ generate a commutative algebra isomorphic to $\mathfrak{S}ym(n)$, the algebra of symmetric polynomials in n variables, the image of $T_k^{\pm}$ being Λ_k.*

Any symmetric polynomial S gives rise to two operators T_S^+ and T_S^-. Decomposing S into the basis Λ^J: $S = \sum_{J \in \mathbb{N}^n} c_J\, \Lambda^J$, one has

$$T_S^+ = \sum\nolimits_J c_J\, T_{j_1}^+ \cdots T_{j_n}^+ \quad \& \quad T_S^- = \sum\nolimits_J c_J\, T_{j_1}^- \cdots T_{j_n}^- \tag{41}$$

The operators T_k^+, T_k^- can be made to act on $V^{\otimes n^2}$, considered as a space of matrices $\big[f_{ij}(x_{ij})\big]_{1 \le i,j \le n}$, but now there are two natural ways to define their action, either by columns :

$${}^cT_k^{\pm}\big([f_{ij}(x_{ij})]\big) = \sum_{\epsilon \in \{0,1\}^n,\, |\epsilon|=k} \big[f_{ij}(x_{ij} \pm \epsilon_j)\big] \ ,$$

or by rows:

$$ {}^{r}T_k^{\pm}\big([f_{ij}(x_{ij})]\big) = \sum_{\epsilon\in\{0,1\}^n,\,|\epsilon|=k} \big[f_{ij}(x_{ij}\pm\epsilon_i)\big] \ . $$

Symmetry between rows and columns of a matrix entails the following lemma, when evaluating the determinants appearing in ${}^{c}T$ or ${}^{r}T$:

Lemma 18 *Given any matrix* $\big[f_{ij}(x_{ij})\big]_{1\le i,j\le n}$, *fixing a sign* $\pm$, *then one has*

$$ \sum_{\epsilon\in\{0,1\}^n,\,|\epsilon|=k} \big|f_{ij}(x_{ij}\pm\epsilon_j)\big| = \sum_{\epsilon\in\{0,1\}^n,\,|\epsilon|=k} \big|f_{ij}(x_{ij}\pm\epsilon_i)\big| \ . \tag{42} $$

Indeed, any term in the expansion of the determinant of the original matrix will be transformed according to $T_k^{\pm}$ and will be found in the expansion of one of the determinants of each side of equation (42).

By abuse of language, we shall write ${}^{c}T_k^{\pm}\big(|f_{ij}(x_{ij})|\big)$ and ${}^{r}T_k^{\pm}\big(|f_{ij}(x_{ij})|\big)$ for the above two sums of determinants.

For example, writing only the shifts in the x_{ij}'s, the equality ${}^{c}T_1^{+} = {}^{r}T_1^{+}$ reads, for a determinant of order 3 :

$$ \begin{vmatrix} +1 & \cdot & \cdot \\ +1 & \cdot & \cdot \\ +1 & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & +1 & \cdot \\ \cdot & +1 & \cdot \\ \cdot & +1 & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & +1 \\ \cdot & \cdot & +1 \\ \cdot & \cdot & +1 \end{vmatrix} = \begin{vmatrix} +1 & +1 & +1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & \cdot \\ +1 & +1 & +1 \\ \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ +1 & +1 & +1 \end{vmatrix} \ . $$

Notice that the determinants appearing on the right and on the left are different, though their sums are equal.

We could have written the sum of all T_k, $k = 0, \ldots, n$, in one stroke :

$$ \big(T_0 + zT_1 + \cdots + z^n T_n\big)\left(\Big|f_{ij}(x_{ij})\Big|\right) = \Big|f_{ij}(x_{ij}) + z\, f_{ij}(x_{ij}{+}1)\Big| , \tag{43} $$

and this renders symmetry between rows and columns even more evident.

Let $\mathbb{A}$ be of cardinality n. Then the two algebras generated by the $T_k^{\pm}$ can be made act on $Sym(\mathbb{A})$, by making them operate on the Jacobi-Trudi determinants of complete functions expressing Schur functions, and extending the action by linearity.

Theorem 19 *Let* $\mathbb{A}$ *be of cardinality* n *and* S *belong to* $\mathfrak{S}ym(n)$. *Then* T_S^{+} *is the operator "multiplication by* $S(\mathbb{A})$*" and* T_S^{-} *acts by* $f(\mathbb{A}) \mapsto D_S(f)(\mathbb{A})$.

Proof. We have to test the action of the $T_k^{\pm} = T_{\Lambda^k}^{\pm}$ on the linear basis of Schur functions $S_J(\mathbb{A})$, but we shall rather take the Vandermonde matrices $\mathbb{V}_J$. Shifting the exponents uniformly by r in row i of $\mathbb{V}_J$ has the effect of multiplying its determinant by a_i^r, and more generally, multiplying $\mathbb{V}_J$

by a monomial a^K can be realized by increasing exponents by $k_1, \ldots, k_n$ in successive rows. Therefore

$$\Lambda^k(\mathbb{A}) V_J(\mathbb{A}) = {}^r T_k^+(V_J(\mathbb{A})) \ . \tag{44}$$

Notice that

$${}^c T_k^+(V_J(\mathbb{A})) = \sum_{\epsilon \in \{0,1\}^n,\, |\epsilon| = k} V_{J+\epsilon}(\mathbb{A}) = \Delta(\mathbb{A}) \sum_{\epsilon} S_{J+\epsilon}(\mathbb{A}) = \Delta(\mathbb{A}) {}^c T_k^+(S_J(\mathbb{A})) \tag{45}$$

gives a description of the product of a Schur function by an elementary symmetric function that is called *Pieri formula*, and that we shall comment with more details in the next section.

Some care is needed to identify T_k^- by using its action on $\mathbb{V}_J(\mathbb{A})$ rather than $S_J(\mathbb{A})$. The operator is not "multiplication by $\Lambda^k(\mathbb{A}^\vee)$", because the operation of decreasing degrees by 1 sends a_i^0 onto 0, and not onto $1/a_i$, if we want this action to be coherent with $S_j = 0$ for $j < 0$. However, if $J \supseteq 1^n$, then

$$\Lambda^k(\mathbb{A}^\vee)\, S_J(\mathbb{A}) = \Lambda^{n-k}(\mathbb{A})\, S_{J/1^n}(\mathbb{A}) = T_k^-(S_J(\mathbb{A})) \ .$$

To identify T_k^-, we use ${}^r T_k^-$ operating on Schur functions. The image of $S_j(\mathbb{A})$ is a sum of $\binom{n}{k}$ determinants, each of which is null except for the last one

$$S_{J/0^{n-k}1^k}(\mathbb{A}) = D_{\Lambda^k} S_J(\mathbb{A}) \ .$$

The operator ${}^c T_k^-$ allows us to recover the dual Pieri formula :

$$S_{J/0^{n-k}1^k}(\mathbb{A}) = {}^c T_k^-(S_J(\mathbb{A})) = {}^r T_k^-(S_J(\mathbb{A})) = \sum_{\epsilon \in \{0,1\}^n,\, |\epsilon| = k} S_{J-\epsilon}(A) \ ,$$

because one can restrict the preceding sum to terms such that $J - \epsilon$ is a partition, the other terms are 0 (having two identical columns). QED

For example, the action of T_2^- on Schur functions of order 3 reads :

$$\begin{vmatrix} \cdot & \cdot & \cdot \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -1 & \cdot \\ -1 & -1 & \cdot \\ -1 & -1 & \cdot \end{vmatrix} + \begin{vmatrix} -1 & \cdot & -1 \\ -1 & \cdot & -1 \\ -1 & \cdot & -1 \end{vmatrix} + \begin{vmatrix} \cdot & -1 & -1 \\ \cdot & -1 & -1 \\ \cdot & -1 & -1 \end{vmatrix} \ .$$

The operators $T_{\Psi_K}^+$ have been in particular considered by Muir, who obtained the following corollary, that one can check directly on minors of the Vandermonde matrix (and that we also give for $T_{\Psi_K}^-$ at the same time) :

Corollary 20 *Let K, J be two partitions in $\mathbb{N}^n$ and $\mathbb{A}$ be of cardinality n. Then*

$$\Psi_K(\mathbb{A})\, S_J(\mathbb{A}) \;=\; \sum_{H=perm(K)} S_{J+H}(\mathbb{A}) \ , \tag{46}$$

$$D_{\Psi_K}\big(S_J(\mathbb{A})\big) \;=\; \sum_{H=perm(K)} S_{J-H}(\mathbb{A}) \ , \tag{47}$$

sum over all different permutations H of K.

For example, for $\mathbb{A}=\{a,b,c\}$, the product of $\Delta(\mathbb{A})S_{125}(\mathbb{A})$ by $\Psi_{22}(\mathbb{A})$ is :

$$\begin{matrix} a^2 \\ b^2 \\ \ \end{matrix} \begin{vmatrix} a^1 & a^3 & a^7 \\ b^1 & b^3 & b^7 \\ c^1 & c^3 & c^7 \end{vmatrix} + \begin{matrix} a^2 \\ \ \\ c^2 \end{matrix} \begin{vmatrix} a^1 & a^3 & a^7 \\ b^1 & b^3 & b^7 \\ c^1 & c^3 & c^7 \end{vmatrix} + \begin{matrix} \ \\ b^2 \\ c^2 \end{matrix} \begin{vmatrix} a^1 & a^3 & a^7 \\ b^1 & b^3 & b^7 \\ c^1 & c^3 & c^7 \end{vmatrix}$$

$$= \begin{vmatrix} a^{1+2} & a^{3+2} & a^7 \\ b^{1+2} & b^{3+2} & b^7 \\ c^{1+2} & c^{3+2} & c^7 \end{vmatrix} + \begin{vmatrix} a^{1+2} & a^3 & a^{7+2} \\ b^{1+2} & b^3 & b^{7+2} \\ c^{1+2} & c^3 & c^{7+2} \end{vmatrix} + \begin{vmatrix} a^1 & a^{3+2} & a^{7+2} \\ b^1 & b^{3+2} & b^{7+2} \\ c^1 & c^{3+2} & c^{7+2} \end{vmatrix} .$$

As in the case of elementary functions seen above, if there exists ℓ such that $K \subseteq \square := \ell^n \subseteq J$, then

$$T^-_{\Psi_K}(\mathbb{V}_J(\mathbb{A})) = T^-_{\Psi_{\square-K}}(\mathbb{V}_{J-\square}(\mathbb{A})) = \frac{\Psi_{\square-K}(\mathbb{A})}{\Psi_\square(\mathbb{A})}\, V_J(\mathbb{A}) = \Psi_K(\mathbb{A}^\vee)\, V_J(\mathbb{A}) \ ,$$

and in that case $T^-_{\Psi_K}$ is also realized by a multiplication.

The special case of Muir's rule for $K=[k,0^{n-1}]$ is called *Murnaghan-Nakayama rule.* The action of D_{Ψ_k} on a Schur function S_J consists in subtracting in all possible manners k to a part of J. To get Schur functions indexed by partitions, one must reorder the columns of the determinant $S_{J+[0\cdots 0\,k\,0\cdots 0]}$. This correspond to subtracting in all possible manners a connected ribbon of length k to the diagram of J, taking as a sign $(-1)^{h-1}$, h being the height of the ribbon (counting 1 for an horizontal ribbon). This rule is iterated to compute values of irreducible characters of symmetric groups.

Conversely, multiplication by $\Psi_k(\mathbb{A})$ is realized by adding connected ribbons of length k to the diagram of J.

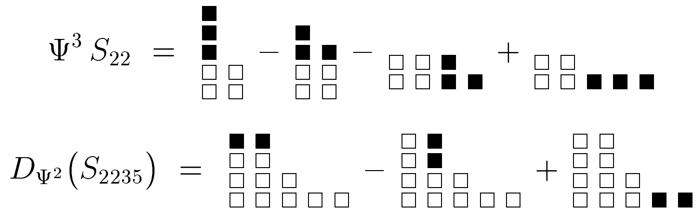
Corollary 21 *(Murnaghan-Nakayama). Let k be an integer, J be a partition. Then*

$$\Psi^k(\mathbb{A})\, S_J(\mathbb{A}) \;=\; \sum\nolimits_H (-1)^{h-1} S_H(\mathbb{A}) \tag{48}$$

$$D_{\Psi^k}\big(S_J(\mathbb{A})\big) \;=\; \sum\nolimits_H (-1)^{h-1} S_H(\mathbb{A}) \tag{49}$$

sum over all partitions H such that H/J (resp. J/H) is a connected ribbon of length k, h being its height.

For example,



$ACE >$ `Tos(p3*s[2,2]);`

```
    s[2,2,1,1,1] - s[2,2,2,1] - s[4,3] + s[5,2]
```

$ACE >$ `SfDiff(p2, s[5,3,2,2]);`

```
    s[5,3,2] - s[5,3,1,1] + s[3,3,2,2]
```

Formula (46) can be used to express a monomial function into the basis of Schur functions :

$$\Psi_K = \Psi_K \, S_{0^{|K|}} = \sum_{H=perm(K)} S_H \ . \tag{50}$$

For example, writing only the non-zero terms,

$$\Psi_{22} = S_{000022} + S_{000202} + S_{002020} = S_{22} - S_{112} + S_{1111} \ .$$

We have seen that ${}^rT_k^{\pm}(S_J(\mathbb{A}))$ is restricted to a single non-zero determinant. This property is in fact true for any Schur function :

Lemma 22 *Let $\mathbb{A}$ be arbitrary, and K, L be two partitions in $\mathbb{N}^n$. Then*

$$ {}^rT^+_{S_L}\Big(S_K(\mathbb{A})\Big) \;=\; det\Big(S_{k_j+j-i+\ell_{n+1-i}}(\mathbb{A})\Big) \ , \tag{51}$$

$$ {}^rT^-_{S_L}\Big(S_K(\mathbb{A})\Big) \;=\; det\Big(S_{k_j+j-i-\ell_i}(\mathbb{A})\Big) = S_{K/L}(\mathbb{A}) \ . \tag{52}$$

Proof. The terms in the action of ${}^rT^+_{S_L}$ on S_K are in bijection with the terms in the action of ${}^cT^+_{S_L}$ on the Vandermonde matrix $\mathbb{V}_{0^n}(\mathbb{B})$, where $\mathbb{B}$ is of cardinality n. Two terms equal, up to a sign, in the second sum, are also equal, up to a sign, in the first sum. Since the second sum reduces to a single term, the first one also does. One can pass from the first statement to the second one, by taking complementary partitions. QED

For example, because $S_{12} = \Psi_{12} + 2\Psi_{111}$, the action of ${}^rT^+_{S_{12}}$ on a determinant of order 4 will be, writing the shifts in successive rows as a vector, and eliminating the null determinants :

$$[2100] + [1020] + [0210] + 2[1110] \ .$$

The terms [1020] and [0210] are each obtained from [1110] by a transposition of rows, and therefore the sum reduces to the single term [2100].

Notice that if A is of cardinality equal to n, then the action of $T^+_{S_L}$ is "multiplication by $S_L(\mathbb{A})$". If, on the contrary, A is of cardinality bigger than n, one still has, for $K, L \in \mathbb{N}^n$

$${}^rT^+_{S_L}\Big(S_K(\mathbb{A})\Big) = {}^cT^+_{S_L}\Big(S_K(\mathbb{A})\Big) = \sum_{H\in\mathbb{N}^n} (S_K S_L \, , \, S_H)\, S_H(\mathbb{A}) \ ,$$

but $T^+_{S_L}(S_K(\mathbb{A}))$ is not equal to $S_L(\mathbb{A})\, S_K(\mathbb{A})$.

For example,

$${}^rT^+_1(S_{25}(\mathbb{A})) = \left[\begin{smallmatrix} S_3 & S_7 \\ S_1 & S_5 \end{smallmatrix}\right] = S_{36/01}(\mathbb{A}) = S_{35}(\mathbb{A}) + S_{26}(\mathbb{A}) = S_1(\mathbb{A})S_{25}(\mathbb{A}) - S_{125}(\mathbb{A}) \ .$$

The T_k operators can be applied to other determinants than Jacobi-Trudi determinants, or Vandermondes. For example, let $f(x)$ be the function

$$f(x) = 1/x! \text{ if } x \in \mathbb{N}^n \text{ and } 0 \text{ otherwise}$$

To a vector $x = [x_1, \ldots, x_n]$, associate the matrix

$$D(x) := [f(x_j + j - i)] \ .$$

Then

$${}^rT^+_2\big(D([2,4,4,6])\big) = \begin{vmatrix} \frac{1}{3!} & \frac{1}{6!} & \frac{1}{7!} & \frac{1}{10!} \\ \frac{1}{2!} & \frac{1}{5!} & \frac{1}{6!} & \frac{1}{9!} \\ \frac{1}{0!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{8!} \\ 0 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{7!} \end{vmatrix} = {}^cT^+_2\big(D([2,4,4,6])\big) =$$
$$= D([3,4,5,6]) + D([3,4,4,7]) + D([2,5,5,6]) + D([2,4,5,7]) \ ,$$

the zero determinants like $D([3,5,4,6])$ not having been written.

In that case, the operator T^+_2 does not correspond to a multiplication. In fact $D(x)$ is the dimension of the irreducible representation of index x of the symmetric group, and the equality of dimension can be obtained by specialization of the Schur functions

$$S_{3557/0011} = S_{3456} + S_{3447} + S_{2556} + S_{2457} \ .$$

We leave it to the reader to use the T operators to show that, for $\mathbb{A}$ of cardinality n, and two partitions $K=[k_1,\ldots,k_n]$, $L=[\ell_1,\ldots,\ell_n]$, one has

$$\det\Big(\Psi_{k_i+\ell_j+i+j-2}(\mathbb{A})\Big)_{1\le i,j\le n} = S_K(\mathbb{A})\, S_L(\mathbb{A})\ \det\Big(\Psi_{i+j}(\mathbb{A})\Big)\ . \tag{53}$$

For example, for $n=3$, one has

$$\begin{vmatrix} \Psi_1 & \Psi_2 & \Psi_5 \\ \Psi_4 & \Psi_5 & \Psi_8 \\ \Psi_7 & \Psi_8 & \Psi_{11} \end{vmatrix} = S_{023}(\mathbb{A})\, S_{113}(\mathbb{A}) \begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix}\ .$$

- **Pieri formulas**

We are mostly using the linear basis of Schur functions. Still, $\mathfrak{S}ym$ is a ring, and to recover its multiplicative structure, one needs to describe the product of two Schur functions. This is given by the so-called *Littlewood-Richardson Rule*, which is better stated, and proved, in terms of non-commutative symmetric functions (cf. the relevant section). Since products Λ^I or S^I also constitute linear bases, we shall content ourselves, for the moment, to describe the product of general Schur functions by a complete or elementary symmetric function to determine the multiplicative structure. The products $S^r S_J$ and $\Lambda^r S_J$, $r\in\mathbb{N}$, J partition, were in fact determined by the Italian geometer Pieri in 1873. They have the remarkable property of being multiplicity free.

In fact, since an elementary symmetric function is a monomial function, we already know, from (45) how it acts by multiplication on Schur functions. We shall however reinterpret this multiplication.

First, let us introduce two notations for family of partitions deduced from a given one $I=[i_1,\ldots,i_n]$. The symbols $\{I\otimes k\}$ and $\{I\otimes 1^k\}$ respectively denote all the partitions J of weight $|J|=|I|+k$ such that

$$\begin{aligned} \{I\otimes k\} &:= \{J=[j_0,\ldots,j_n], 0\le j_0\le i_1\le j_1\le\cdots\le i_n\le j_n\}, \quad (54)\\ \{I\otimes 1^k\} &:= \{J : J^\sim \in \{I^\sim\otimes k\}\}\ , \quad (55)\end{aligned}$$

that is, the families of partitions obtained from I by adding horizontal (resp. vertical) strips of k boxes.

The two above sets of partitions satisfy the property of being distributive sublattices of the lattice of partitions of a given weight. To show it, we first must code differently the partitions $J\in\{I\otimes k\}$. Let us write $J^\diamond := J-I$ (as vectors). Instead of J, we use the cumulative sum $\overline{\overline{J^\diamond}}$.

For example, for $I=[0,1,3,4]$, $k=2$, we shall have the following lattice (with three labellings) :

$$\begin{tikzcd}
& & 1234 \arrow[dl] \arrow[dr] & \\
& 1144 \arrow[dl] \arrow[dr] & & 0334 \arrow[dl] \\
1135 \arrow[dr] & & 0244 \arrow[dl] & \\
& 0235 \arrow[d] & & \\
& 0145 \arrow[d] & J & \\
& 0136 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & 1100 \arrow[dl] \arrow[dr] & \\
& 1010 \arrow[dl] \arrow[dr] & & 0200 \arrow[dl] \\
1001 \arrow[dr] & & 0110 \arrow[dl] & \\
& 0101 \arrow[d] & & \\
& 0011 \arrow[d] & J^\diamond & \\
& 0002 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & 1222 \arrow[dl] \arrow[dr] & \\
& 1122 \arrow[dl] \arrow[dr] & & 0222 \arrow[dl] \\
1112 \arrow[dr] & & 0122 \arrow[dl] & \\
& 0112 \arrow[d] & & \\
& 0012 \arrow[d] & \overline{\overline{J^\diamond}} & \\
& 0002 & &
\end{tikzcd}$$

Lemma 23 *Given a partition I and $k\in\mathbb{N}$, then $\{I\otimes k\}$ is a distributive sublattice of the lattice of partitions, with minimal element $I+[0\cdots 0k]$ and maximal element $I\cup k$.*

Proof. The lattice structure is seen on the vectors $\overline{\overline{J^\diamond}}$ (notice that $\overline{\overline{J^\diamond}}=\overline{\overline{J}}-\overline{\overline{I}}$). The condition that J belongs to $\{I\otimes k\}$ is equivalent to the fact that $J^\diamond$ is integral vector, with $|J^\diamond|=k$, componentwise majorized by $[i_1-0,\, i_2-i_1,\, i_3-i_2,\, \ldots,\, i_\ell-i_{\ell-1},\, \infty]$. QED

Lemma 24 *Let k be a positive integer, I be a partition. Then one has the following decompositions :*

$$S^k\, S_I \;=\; \sum\nolimits_{J\in\{I\otimes k\}} S_J\ , \tag{56}$$

$$\Lambda^k\, S_I \;=\; \sum\nolimits_{J\in\{I\otimes 1^k\}} S_J\ . \tag{57}$$

Replacing I by $[0^k, I]$, the last sum can be written $\sum_H S_H$, sum over all $H\in\mathbb{N}^{n+k}$ such that $h_1-i_1,\ldots,h_{n+k}-i_{n+k}$ are 0 or 1, and $|H|=|I|+k$, because the extra terms such that H is not a partition index null determinants. Now (45) states that this last sum is equal to the product $\Lambda^k\, S_I$.

The involution $\mathbb{A}\mapsto -\mathbb{A}$ exchanges the two Pieri formulas (and graphically, conjugation exchanges horizontal strips with vertical ones). QED

Pieri rules also occur in the non commutative world, as shows the following proposition.

Proposition 25 *Let $\mathcal{R}$ be a ring containing a family of elements s_J, $J=$ all partitions, satisfying (56) for all r, J. Then the $\mathbb{Z}$-module generated by the s_J is a subring, which is a quotient of the ring of symmetric polynomials with coefficient in $\mathbb{Z}$.*

The proof follows from the fact that the leading term (for the dominance order) in a product S_rS_J is $S_{(r,J)}$, with (r,J) := increasing reordering of

$[r, j_1, j_2, \ldots]$. By recursion, it allows to obtain all Schur functions, starting from the S_r's, it proves that they commute and can be expressed as determinants in the S_r's .

In the commutative case, the proposition is due to Giambelli, who used it to characterize the cohomology ring of a Grassmann manifold as a quotient of $\mathfrak{S}ym$[3]. Schensted algorithm satisfy the non commutative product identity $S_2 S_J = \sum S_K$, and this imply the non commutative Pieri rules. Starting from the geometrical problem of intersecting Schubert cycles, one is led to the non commutative world as the simplest way to describe these intersection in a cohomology ring.

Exercises

Ex. 1 Let $\mathbb{A}$ be arbitrary, k be a positive integer. Expand the product $S^k(1 - z\mathbb{A})\, \sigma_z(\mathbb{A})$ in the basis of Schur functions.

Corr.Ex.1. Take $\mathbb{B}$ of cardinality k. Then

$$S^k(1 - z\mathbb{B})\, \sigma_z(\mathbb{A}) = \sigma_z(\mathbb{A}-\mathbb{B}) = (-1)^k \sum_j z^j S_{1^k;\, j-k}(\mathbb{B}; \mathbb{A}) \ .$$

The specialization $S^i(-\mathbb{B}) = S^i(-\mathbb{A})$, $i = 0, \ldots, k$ consists in replacing $\mathbb{B}$ by $\mathbb{A}$ in the determinants $S_{1^k;j-k}(\mathbb{B}; \mathbb{A})$, because $\mathbb{B}$ occurs in degree k at most in them. In summary,

$$S^k(1 - z\mathbb{B})\, \sigma_z(\mathbb{A}) = (-1)^k \sum_j z^j S_{1^k,\, j-k}(\mathbb{A}) \ .$$

```
Ex1:=proc(k,n) local i,pol;
pol:=1+convert([seq((-z)^i*e.i,i=1..k)],‘+‘);
map(Tos,taylor( pol*(1+convert([seq(z^i*h.i,i=1..n)],‘+‘)),z, n+1), collect);
end:
ACE > Ex1(3,6);
s[] -s[1,1,1,1]z^4 -s[2,1,1,1]z^5 -s[3,1,1,1]z^6 +s[]O(1)z^7
```

[3] In the non commutative case, because multiplication is an associative operation, one needs only to describe products $xy\,t$, x, y letters, t tableau, to get all products of tableaux by increasing or decreasing words.

Ex. 2 Let $I \in \mathbb{N}^m$, $J \in \mathbb{N}^n$. Given $H \in \mathbb{N}^n$, denote $H^\omega + J := (j_1 + h_n, \ldots, j_n + h_1)$. Check that

$$S_{I;J}(\mathbb{A};\mathbb{B}) = \sum_H (-1)^{|H|} S_{I/H^\sim}(\mathbb{A})\, S_{J+H^\omega}(\mathbb{B}) \ ,$$

sum over all partitions $H \in \mathbb{N}^n$, $H \subseteq m^n$.

In particular, for $n = 1$, one has

$$S_{I,j}(\mathbb{A},\mathbb{B}) = \sum_{0 \le h \le m} (-1)^h S_{I/1^h}(\mathbb{A})\, S_{j+h}(\mathbb{B}) \ .$$

Ex. 3 Let $I = [i_1, \ldots, i_r]$ be a partition, n an integer : $n \ge i_r$. Check that

$$S_{I;0^n}(\mathbb{A},\mathbb{B}) = S_I(\mathbb{A} - \mathbb{B}) \ .$$

Corr.Ex.3. Since $\mathbb{B}$ is arbitrary, one cannot directly apply the transformation lemma (7) which requires cardinality conditions on alphabets to be subtracted . However, the Laplace expansion along the last n columns imply the result, because the list of partitions contained in I is a subset of the list of partitions obtained by taking minors in the last n columns.

```
# incr=increasing partition, lA=list of alphabets
MultiSchur:=proc(incr,lA) local n,i,j,ma;
n:=nops(incr);
transpose(array([seq(
map(proc(x,i,ll) if (x>0) then s[x](op(i,ll)) elif (x=0)
then 1 else 0 fi end,[seq(incr[i]-j+i,j=1..n)],i,lA),i=1..n)] ));
end;
ExSchur2:=proc(pa,n) local i,v; # pa=decreasing partition
v:=[seq(pa[nops(pa)-i],i=0..nops(pa)-1) ,0$n];
MultiSchur(v, [A1$nops(pa), A2$n])
end:
```

$ACE >$ `aa:=ExSchur2([2,1], 3);`

$$aa := \begin{bmatrix} s[1](A1) & s[3](A1) & s[2](A2) & s[3](A2) & s[4](A2) \\ 1 & s[2](A1) & s[1](A2) & s[2](A2) & s[3](A2) \\ 0 & s[1](A1) & 1 & s[1](A2) & s[2](A2) \\ 0 & 1 & 0 & 1 & s[1](A2) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$ACE >$ `SfAExpand(det(aa));`

```
s[2,1](A1)-s[1](A2) s[1,1](A1) -s[1](A2) s[2](A1) +s[1](A1) s[1,1](A2)
+ s[1](A1) s[2](A2) - s[2,1](A2)
```

$ACE >$ `SfAExpand(s[2,1](A1-A2));`

```
s[2,1](A1)-s[1](A2) s[1,1](A1) -s[1](A2) s[2](A1) +s[1](A1) s[1,1](A2)
+ s[1](A1) s[2](A2) - s[2,1](A2)
```

There is, however, a better reasoning. Because $i_r \le n$, the Laplace expansion will involve only Schur functions $S_J(\mathbb{B})$ with $\ell(J) \le n$. Therefore, one can suppose that $\mathbb{B}$ be of cardinality n, in which case subtracting $\mathbb{B}$ in the first r rows produces the block $S_I(\mathbb{A}-\mathbb{B})$.

Ex. 4 Let n be an integer. Build a $n \times n$ matrix with a first column of indeterminates y_i, and entries $[i,i] = x$, entries $[i-1,i] = -1$, for $i = 2, \ldots, n$, all other entries being 0, and compute its determinant without expanding it. For example, for $n = 4$, the matrix is $\begin{vmatrix} y_1 & -1 & 0 & 0 \\ y_2 & x & -1 & 0 \\ y_3 & 0 & x & -1 \\ y_4 & 0 & 0 & x \end{vmatrix}$.

Corr.Ex.4. Permuting rows and columns, putting signs, one recognizes in the determinant $\Lambda_{n;0^n}(\mathbb{A}; b)$, with $b = -x$, a rank 1 element, and $\Lambda^i(\mathbb{A}) = y_i$. Therefore it is equal to the polynomial $\Lambda_n(\mathbb{A} - b)$. One could as well have seen that the determinant is the resultant of $\Lambda_n(\mathbb{A} - z)$ and $z - x$.

Ex. 5 Let m, n be two integers. Compute, after Composto (1916; MuirV, p.349), the value of

$$\det \left(\binom{m+i+j-2}{j} \right)_{1 \le i,j \le n} .$$

Corr.Ex.5. To see that one can apply the transformation lemma (7), let $\mathbb{A}$ be of cardinality m, and let $\mathbb{X}_i$, $\mathbb{Y}_i$, $1 \le i \le n$ be of cardinality i–1.

Consider the determinant

$$M(\mathbb{A}, \mathbb{X}, \mathbb{Y}) := \det \left(\Lambda^j(\mathbb{A} + \mathbb{X}_j + \mathbb{Y}_i) \right)_{1 \le i,j \le n} .$$

From (7) one can replace the $\mathbb{X}_j$ by 0. Expanding by linearity in $\mathbb{Y}_i$, we similarly get that the resulting matrix is the product of $\left[\Lambda^j(\mathbb{A})\right]$ by $\begin{bmatrix} \Lambda^0(\mathbb{Y}_1) & 0 & 0 & \cdots \\ \Lambda^0(\mathbb{Y}_2) & \Lambda^1(\mathbb{Y}_2) & 0 & \cdots \\ \Lambda^0(\mathbb{Y}_3) & \Lambda^1(\mathbb{Y}_3) & \Lambda^2(\mathbb{Y}_3) & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$.

Therefore, $M(\mathbb{A}, \mathbb{X}, \mathbb{Y}) = \Lambda_{1^n}(\mathbb{A})\, \Lambda^1(\mathbb{Y}_2) \cdots \Lambda^{n-1}(\mathbb{Y}_n)$, and one gets the value of the original determinant by specializing all the letters in the different alphabets to 1, that is one gets

$$\Lambda_{1^n}(m) = S_n(m) = \binom{n+m-1}{n} ,$$

writing m for the specialization of $\mathbb{A}$. Notice that the essential fact which allowed us to simplify the determinant was that it had regularly increasing alphabets in rows and columns. The same computation would have led to the evaluation, given $k \in \mathbb{N}$, of $\det \left(\Lambda^{j+k}(\mathbb{A} + \mathbb{X}_j + \mathbb{Y}_i) \right)$.

Ex. 6 Express $S_{J;r}(\mathbb{A};x)$ as a determinant of smaller order when $r<0$ (when $r=0$, we have expressed it as $S_J(\mathbb{A}\text{-}x)$ in(17)).

Ex. 7 Let $\mathbb{A}$, $\mathbb{B}$ be arbitrary, and let k be a positive integer. Show that

$$(-1)^{k+2}S_{1^k;2}(\mathbb{A};\mathbb{A}\text{-}\mathbb{B}) = S_{1;k+1}(\mathbb{B}\text{-}\mathbb{A};-\mathbb{A}) + S_{k+2}(-\mathbb{B}) \ .$$

Corr.Ex.7. The involution $\mathbb{A} \mapsto \text{-}\mathbb{A}$ in the argument of a Schur function corresponds to conjugation of partitions. It is more complex to follow it in a multi-Schur function. The above identity is a case where there is only an extra term apart from the one indexed by $(1^k,2)^\sim = (1,k+1)$.

Expanding both sides of the required identity with respect to $\mathbb{B}$, we transform them into

$$(-1)^{k+2}\big(S_{1^k2}(\mathbb{A}) + S_{1^k1}(\mathbb{A})S_1(\text{-}\mathbb{B}) + \cdots + S_{1^k,-k}(\mathbb{A})S_{k+2}(\text{-}\mathbb{B})\big) =$$
$$S_{1,k+1}(\text{-}\mathbb{A}) + S_{k+1}(\text{-}\mathbb{A})S_1(-\mathbb{B}) + S_{k+2}(-\mathbb{B}) \ ,$$

the terms not written being null (=Schur functions with two identical columns). QED

Ex. 8 Using the factorization property (9), compute the Schur functions $S_J(\mathbb{A}-\mathbb{B})$ when $\mathbb{A}$ and $\mathbb{B}$ are of cardinality 1 or 2.

Corr.Ex.8. Let us take four letters, and $A = x_1+x_2$, $B = y_1+y_2$. Specializing some of them allow to recover the smaller cardinalities.

We have nullities $S_I(\mathbb{A}-\mathbb{B}) = 0$ when $I \supseteq [3,3,3]$. In the case where $I \supseteq [2,2]$, $I \not\supseteq [3,3,3]$, on has a factorization of $S_I(\mathbb{A}-\mathbb{B})$ in three factors given by (9). There remains to explicit the case where I is a hook.

```
ACE > factor(SfEval(s[4,2,1], x1+x2-y1-y2));
    -(-x2+y2) (-x2+y1) (y1+y2) (x1x1+x1x2+x2x2) (-y2+x1) (x1-y1)
ACE > factor(SfEval(s[6,1$3],x1-y1));
      - x1^5 y1^3 (x1 - y1)
```

Ex. 9 Let I, J be partitions such that the diagram of J/I decomposes into disjoint blocks K_1/H_1, K_2/H_2, $\ldots$, K_ℓ/H_ℓ. Show that, for any $\mathbb{A}$, $S_{J/I}(\mathbb{A})$ factorizes into

$$S_{J/I}(\mathbb{A}) = S_{K_1/H_1}(\mathbb{A})\cdots S_{K_\ell/H_\ell}(\mathbb{A}) \ .$$

Ex. 10 Let $\mathbb{A}$ be of cardinality n, and $I, J \in \mathbb{N}^n$. Show that the determinant

$$\left| S_{j_k+k-h+i_{n-h+1}}(\mathbb{A}) \right|$$

factorizes into $S_J(\mathbb{A})\, S_I(\mathbb{A})$. What can be said if n is not the cardinality of $\mathbb{A}$?

For example, for $n = 3$, $I = [1, 3, 3]$, $J = [2, 4, 6]$, one has

$$\begin{array}{ccc|c} 0 & 1 & 2 & 3\\ -1 & 0 & 1 & 3\\ -2 & -1 & 0 & 1\\ \hline 2 & 4 & 6 & \end{array} = \begin{vmatrix} S_5 & S_8 & S_{11}\\ S_4 & S_7 & S_{10}\\ S_1 & S_4 & S_7 \end{vmatrix} = S_{133}(\mathbb{A})\, S_{246}(\mathbb{A}) \ .$$

hint: Use the expression in terms of Λ^i.

Corr.Ex.10.

ACE > `aa:= ProdSchur([2,6], [0,2]);`

$$aa := \begin{bmatrix} h4 & h9\\ h1 & h6 \end{bmatrix}$$

ACE > `factor(`Toe_n`(det(aa));`

$e2^2$ `(- e2 + ` $e1^2$ `) (`$e1^4$ `- 3 e2` $e1^2$ `+` $e2^2$ `)`

`# Preceding determinant is s[ 84/20 ].  Take conjugate partitions`

ACE > `map(`Toe_n`, SfJtMat([[2$4,1$4],[1,1]], 'e'));`

$$\begin{bmatrix} e1 & e2 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & e1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & e2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & e1 & e2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & e1 & e2 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & e1 & e2 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & e1 & e2\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & e1 \end{bmatrix}$$

Ex. 11 Compute the adjoint matrix of a Jacobi-Trudi matrix. For example

ACE > `SfJtMat([5,4,1]), map(Tos, adj(SfJtMat([5,4,1])));`

$$\begin{bmatrix} h5 & h6 & h7\\ h3 & h4 & h5\\ 0 & 1 & h1 \end{bmatrix} , \begin{bmatrix} s[4,1] & -s[6,1] & s[6,5]\\ -s[3,1]-s[4] & s[5,1]+s[6] & -s[5,5]-s[6,4]\\ s[3] & -s[5] & s[5,4] \end{bmatrix}$$

Ex. 12 Let $\mathbb{A}$ be an alphabet of cardinality n. Describe an algorithm to express the $S^k(\mathbb{A})$ and $\Psi_k(\mathbb{A})$, $k > n$, in terms of respectively $S^1(\mathbb{A}), \ldots, S^n(\mathbb{A})$, $\Psi_1(\mathbb{A}), \ldots, \Psi_n(\mathbb{A})$.

Ex. 13 Ribbon functions.

Check that the Jacobi-Trudi matrix of a ribbon of height r is built from its diagonal $S_{c_1}, \ldots, S_{c_r}$ as follows: the diagonal below the main diagonal is filled with S_0's. The entries below are 0. Each entry $[i, j]$, $j \geq i$, is equal to $S_{c_i + \cdots + c_{j-1}}$.

Show that the product of two ribbon Schur functions is equal to the sum of two ribbon Schur functions.

Writing $\mathcal{R}_k$ for the ribbon $S_{12\ldots k/12\ldots k-2}$, show that

$$S_{123456} = \begin{vmatrix} \mathcal{R}_2 & \mathcal{R}_3 & \mathcal{R}_4 \\ \mathcal{R}_3 & \mathcal{R}_4 & \mathcal{R}_5 \\ \mathcal{R}_4 & \mathcal{R}_5 & \mathcal{R}_6 \end{vmatrix} ,$$

using the preceding property ($\mathcal{R}_0 = S_0$, $\mathcal{R}_1 = S_1$, $\mathcal{R}_2 = S_{12}$, $\mathcal{R}_3 = S_{123/1}, \ldots$).

When $\mathbb{A} = \{a_1, \ldots, a_n\}$ is finite, show that if a ribbon $\mathcal{R}$ has a column of height $> n$, then $S_{\mathcal{R}}(\mathbb{A})$ vanishes; if it has a column of height n, then $S_{\mathcal{R}}(\mathbb{A})$ factorizes into the product of two ribbons multiplied by $a_1 \cdots a_n$.

Ex. 14 Let $\mathcal{Z} = z\Psi^1 - z^3\Psi^3 + z^5\Psi^5 - \cdots$. Show that

$$\tan(\mathcal{Z}) = \frac{z\Lambda^1 - z^3\Lambda^3 + z^5\Lambda^5 - \cdots}{1 - z^2\Lambda^2 + z^4\Lambda^4 + \cdots} = zS_1 + z^3 S_{12} + z^5 S_{123/1} + \cdots + z^{2n-1} S_{1\ldots n/1\ldots n-2} + \cdots$$

Deduce from the above, after Cauchy [5], Laguerre [26] and many other mathematicians, that the elements of $\mathfrak{S}ym(\mathbb{A})$, when $\mathbb{A}$ is of cardinality n, are rational functions of

$$\Psi^1(\mathbb{A}), \Psi^3(\mathbb{A}), \ldots, \Psi^{2n-1}(\mathbb{A}) .$$

Beware that it is not true that elements of $\mathfrak{S}ym(\mathbb{A})$ can be expressed as rational functions of $S^1(\mathbb{A}), S^3(\mathbb{A}), \ldots, S^{2n-1}(\mathbb{A})$, for $n \geq 3$. The rationality property is specific to the basis of power sums.

Corr.Ex.14. From the definition of tan(), and from $\exp\left(\sum_{i\geq 1}(-z)^i \Psi^i/i\right) = \sum z^i \Lambda^i$, one gets the first equality. Using the determinantal expression of the coefficients of the quotient $\lambda_{-y}(\mathbb{B})/\lambda_{-y}(\mathbb{A})$ of two series, for $y = z^2$, and replacing

$\Lambda^i(\mathbb{A})$ by Λ^{2i}, $\Lambda^i(\mathbb{B})$ by Λ^{2i+1}/Λ^1 , one recognizes in them the specified Schur functions. For example,

$$S_{1234/12} = \Lambda_{1234/12} = \begin{vmatrix} \Lambda^1 & \Lambda^3 & \Lambda^5 & \Lambda^7 \\ \Lambda^0 & \Lambda^2 & \Lambda^4 & \Lambda^6 \\ 0 & \Lambda^0 & \Lambda^2 & \Lambda^4 \\ 0 & 0 & \Lambda^0 & \Lambda^2 \end{vmatrix} = \Lambda^1 \begin{vmatrix} \Lambda^0(\mathbb{B}) & \Lambda^1(\mathbb{B}) & \Lambda^2(\mathbb{B}) & \Lambda^3(\mathbb{B}) \\ \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) & \Lambda^3(\mathbb{A}) \\ 0 & \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) \\ 0 & 0 & \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) \end{vmatrix} .$$

Given a formal series f and two other series g, h such that $f = g/h$, then g, h are of course not uniquely determined by f. However, one has here the constraint that the identity is valid for $\mathbb{A}$ of any cardinality. In other words, each truncation $(z\Lambda^1 - \cdots \pm z^{2n+1}\Lambda^{2n+1})/(1 - \cdots \pm z^{2n}\Lambda^{2n})$ or $(z\Lambda^1 + \cdots \pm z^{2n+1}\Lambda^{2n+1})/(1 - \cdots \pm z^{2n+2}\Lambda^{2n+2})$ is a rational approximation of $\tan(\mathcal{Z})$, and now the coefficients are uniquely determined if the approximation exists (normalizing the approximation as $(z + \cdots)/(1 + \cdots)$).

Since $\mathcal{Z}$ involves only the odd power sums, the elementary symmetric functions are rational functions of the odd power sums. If $\mathbb{A}$ is of finite cardinality n, then the right member is a rational function, which is determined by the Taylor expansion of $\tan(\mathcal{Z})$ to the order involving the same number of parameters. Thus $\Lambda^1, \ldots, \Lambda^n$ are rational functions of $\Psi^1, \Psi^3, \ldots, \Psi^{2n+1}$, and so is any symmetric function of $\mathbb{A}$.

Ex. 15 Let $\mathbb{A}$ be of cardinality n. Show, after Foulkes [11], that the explicit expression of $\Lambda^{n-k}(\mathbb{A})$, $0 \le k \le n$ as a rational function in the odd power sums is

$$\Lambda^{n-k}(\mathbb{A}) = S_{1\ldots n/1^k}(\mathbb{A})/S_{1\ldots n-1}(\mathbb{A}) ,$$

the two Schur functions being expressed in terms of power sums.

Corr.Ex.15. Take the determinantal expression of $S_{1\ldots n/1^k} = \Lambda_{1\ldots n/k}$ in terms of the Λ^i. It factorizes, a factor being equal to Λ^{n-k}. One moreover knows that $\mathfrak{S}_{1\ldots n}$ depends only on the odd power sums $\Psi^1, \Psi^3, \ldots, \Psi^{2,n-1}$, because one cannot remove of ribbon of even length from the diagram of $[1, \ldots, n]$ (cf. (49)). On can obtain the skew functions $S_{1\ldots n/I}$, any I, by derivation with respect to power sums, and therefore they also involve only odd power sums.

```
Foulkes:=proc(card)   local i,k;
 option remember;
 [p1,seq(Top(SfDiff(cat(e,(card-k)), s[seq(card-i,i=0..card-1)]))/
                Top(s[seq(card-i,i=1..card-1)]),k=2..card)]
end:
SfByOddPsi:=proc(sf0,card)  local v,i,sf;
  CLG_n(card); sf:=Toe_n(sf0); v:=Foulkes(card);
  subs(seq(cat(e,i)=v[i],i=Sf2TableVar(sf,'e')), sf)
```

```
end:
ACE> factor(SfByOddPsi(p2,3));
                                          2      5
                                 5 p3 p1   + p1   - 6 p5
                             1/5 ---------------------
                                            3
                                          p1  - p3
ACE> simplify(SfEval(numer(%),a+b+c)/SfEval(denom(%),a+b+c)) ;
                                 2    2    2
                                b  + a  + c
```

Ex. 16 Let $\mathbb{A}$ be any alphabet, m, n be two integers. Check that the adjoint matrix of $\mathbb{S}_{m^n}(\mathbb{A})$ is the matrix of the quadratic form $Q(x,y) := S_{(m+1)^{n-1}}(\mathbb{A} - x - y)$.

Corr.Ex.16. Minors of $S_{m^n}(\mathbb{A})$ are skew Schur functions. The expansion of $Q(x,y)$ also produces skew Schur functions that one has to check to be the same.

For example, the adjoint to $\mathbb{S}_{6666}(\mathbb{A})$ is

$$\begin{bmatrix} S_{666} & -S_{667} & S_{677} & -S_{777} \\ S_{666/001} & -S_{667/001} & S_{677/001} & -S_{777/001} \\ S_{666/011} & -S_{667/011} & S_{677/011} & -S_{777/011} \\ S_{666/111} & -S_{667/111} & S_{677/111} & -S_{777/111} \end{bmatrix} = \begin{bmatrix} S_{666} & -S_{667} & S_{677} & -S_{777} \\ -S_{566} & S_{567}+S_{666} & -S_{667}-S_{577} & S_{677} \\ S_{556} & -S_{566}-S_{557} & S_{567}+S_{666} & -S_{667} \\ -S_{555} & S_{556} & -S_{566} & S_{666} \end{bmatrix}$$

Ex. 17 Define a vertex operator on $\mathfrak{Sym}$ as a formal series in z by

$$\nabla := \exp\Big(\sum_{n\geq 1} z^n\, \Psi_n/n\Big)\exp\Big(\sum_{n\geq 1} z^{-n}\, D_{\Psi_n}/n\Big)$$

and expand it according to powers of z:

$$\nabla = \sum_{-\infty}^{+\infty} \nabla_n \ .$$

For any $\ell \in \mathbb{N}$, any $v = [v_1, \ldots, v_\ell] \in \mathbb{Z}^\ell$, show that $\nabla_{v_\ell} \cdots \nabla_{v_1}(1)$ is equal to the Schur function of index v.

See **??** for more general operators, due to Jing, "creating" Hall-Littlewood polynomials.

Corr.Ex.17. Identify $\mathfrak{Sym}$ with $\mathfrak{Sym}(\mathbb{A})$. The right factor of ∇ is the operator $f \mapsto f(\mathbb{A} - 1/z)$, as is checked on the basis $\Psi^K(\mathbb{A})$. The second factor is "Multiplication by $\sigma_z(\mathbb{A})$". Assuming the property to be true for length $< \ell$, we have, for any $I \in \mathbb{Z}^{\ell-1}$, any $k \in \mathbb{Z}$

$$\nabla_k\big(S_I(\mathbb{A})\big) = S_i(\mathbb{A} - 1/z)\sigma_z(\mathbb{A}) \cap z^k = \sum_{i,j} (-z)^{-j} S_{I/1^j}(\mathbb{A})\, z^i S^i(\mathbb{A}) \cap z^k$$

and this is the expansion of the determinant $S_{Ik}(\mathbb{A})$ along its last column. QED

Ex. 18 Let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be three alphabets, k a positive integer. Express the following determinant of order 10 : $\begin{array}{|c|c|}\hline M & N \\ \hline P & Q \\ \hline \end{array}$ as a Schur fonction, where $Q = 0$,

$$M = \begin{bmatrix} S_k(\mathbb{C}) & \cdots & S_{k+7}(\mathbb{C}) \\ \vdots & & \vdots \\ S_{k-6}(\mathbb{C}) & \cdots & S_{k+1}(\mathbb{C}) \end{bmatrix}, N = \begin{bmatrix} S_5(\mathbb{A}) & S_6(\mathbb{A}) \\ \vdots & \vdots \\ S_{-1}(\mathbb{A}) & S_0(\mathbb{A}) \end{bmatrix}, P = \begin{bmatrix} S_0(\mathbb{B}) & \cdots & S_7(\mathbb{B}) \\ S_{-1}(\mathbb{B}) & \cdots & S_6(\mathbb{B}) \\ S_{-2}(\mathbb{B}) & \cdots & S_5(\mathbb{B}) \end{bmatrix}$$

Generalize to any order.

Corr.Ex.18. Expanding the determinant along the last two rows, one would get Schur functions of $\mathbb{A}$ indexed by partitions of length ≤ 2. One thus loses no information by supposing that $\mathbb{A}$ is of cardinality 2, and similarly, that $\mathbb{B}$ is of cardinality 3.

Subtract now $\mathbb{A}$ in rows $1, \ldots, 5$ and $\mathbb{B}$ in columns $4, \ldots, 8$. The new determinant factorizes into

$$S_{0^3}(\mathbb{B})\, S_{(k+3)^5}(\mathbb{C} - \mathbb{A} - \mathbb{B})\, S_{0^2}(\mathbb{A}) \ .$$

Ex. 19 Let $\mathbb{A}$ be finite and $\mathbb{B}$ arbitrary. Following Mattia (MuirV, p.204), show that one can replace, in the Vandermonde $\Delta(\mathbb{A})$ of $\mathbb{A}$, each entry a^j by $S^j(\mathbb{A} + \mathbb{B})$ without changing the value of $\Delta(\mathbb{A})$. Show that it implies that one can replace each a^j by the complete function $\Lambda^j(\mathbb{A} - a)$.

Corr.Ex.19. The Vandermonde $\Delta(\mathbb{A})$ is a Schur function $S_{n-1,n-2,\ldots,0}(a_1, \ldots, a_n)$, but the transformation is not given by lemma (7) because one modifies each row. However, it also is given by multiplication by a triangular matrix with unit diagonal. One could have even added or subtracted different alphabets, $\mathbb{A}_1, \mathbb{A}_2, \ldots$ in successive columns.

$ACE >$ `SfAVars({{a}});`

```
{{a}}
```

$ACE >$ `ma:=matrix([seq( [1,seq(s[j-1](a.i-A.j), j=2..3)],i=1..3)]);`

$$ma := \begin{bmatrix} 1 & s[1](a1 - A2) & s[2](a1 - A3) \\ 1 & s[1](a2 - A2) & s[2](a2 - A3) \\ 1 & s[1](a3 - A2) & s[2](a3 - A3) \end{bmatrix}$$

$ACE >$ `factor(SfAExpand(det(ma)));`

```
- (- a3 + a2) (- a3 + a1) (a1 - a2)
```

In the special case where $B = -\mathbb{A}$, getting rid of signs which are uniform, one obtains a determinant of elementary symmetric functions in the subalphabets $\mathbb{A} - a_i$.

Ex. 20 Let $\mathbb{A}$ be of cardinality n, and let $x_1, \ldots, x_n$ be n indeterminates. Compute the determinant

$$\det\Bigl(\sum_k \Lambda^{n+j-2-2k}(\mathbb{A}) x_i^{2k}\Bigr)_{1\le i,j\le n} .$$

Corr.Ex.20. The matrix is the product of the Vandermonde matrix in the x_i^2, and of the matrix $\bigl[\Lambda^{n-i+j-1}(\mathbb{A})\bigr])_{1\le i,j\le n}$. Therefore, its determinant is equal to

$$\pm \prod (x_i^2 - x_j^2) \Lambda_{12\ldots n-1}(\mathbb{A}) = \pm \prod (x_i - x_j) \Lambda_{12\ldots n-1}(x_1 + \cdots + x_n)\, \Lambda_{12\ldots n-1}(\mathbb{A}) .$$

Ex. 21 Let n be a positive integer and $\mathbb{A} = \{1, \ldots, n\}$. Show, after Theisinger (1915; MuirV p. 205) that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \frac{1}{1! \cdots n!} \det(a^0, a^2, \ldots, a^n)_{a\in\mathbb{A}} .$$

Corr.Ex.21. The determinant is equal to $\Lambda^{n-1}(\mathbb{A})\,\Delta(\mathbb{A})$, and thus proportional to the first elementary symmetric function of the alphabet $\{1/1, \ldots, 1/n\}$.

Ex. 22 Let n be an integer. Evaluate

$$\frac{1}{S_1} + \frac{S_2}{S_1 S_{11}} + \frac{S_{22}}{S_{11} S_{111}} + \cdots + \frac{S_{2^n}}{S_{1^n} S_{1^{n+1}}} .$$

Corr.Ex.22. Better use Λ^i. By induction on n, with Pieri formula, one checks that the sum is equal to Λ^n/Λ^{n-1}.

```
Whittaker:=proc(n) local i;
    1/e1 + convert([seq( s[2$i]/e.i/e.(i+1),i=1..n)],`+`)
end:
```

$ACE >$ `aa:= Whittaker(4);`

$$aa := \frac{1}{e1} + \frac{s[2]}{e1\, e2} + \frac{s[2,2]}{e2\, e3} + \frac{s[2,2,2]}{e3\, e4} + \frac{s[2,2,2,2]}{e4\, e5}$$

$ACE >$ `simplify(Toe(numer(aa))/denom(aa));`

```
e4/ e5
```

This expression is used by Whittaker (MuirV, p.272) as an approximation of the smallest root of $\sigma_x(\mathbb{A}) = 0$.

Ex. 23 Use Muir's formula to expand the monomial functions $\Psi_{1^\alpha\, 2^\beta}$, $\alpha, \beta \in \mathbb{N}$, in the basis of Schur functions.

For example,

$$\Psi_{1222} = S_{1222} - 2S_{11122} + 3S_{1^5\,2} - 4S_{1^7}\ .$$

Ex. 24 Given two positive integers k, n, show that Ψ_{k^n} expands as a sum of Schur functions with coefficients ± 1.

Ex. 25 Using shift operators on Jacobi-Trudi determinants, reprove Pieri formulas for multiplication by Λ^k, by S^2 and by S^3.

Corr.Ex.25. For the mutiplication of S_I by Λ^k, one first concat 0^k to I. Now ${}^rT_k^+(S_{0^k\, I})$ reduces to a single non-zero determinant, the one where indices have been increased in the first k rows. This determinant factorizes into $\Lambda^k\, S_I$, and therefore ${}^cT_k^+$ gives the multiplication by Λ^k.

For the multiplication by S^2 or S^3, one writes $S_2 = \Psi_2 + \Psi_{11}$, $S_3 = \Psi_3 + \Psi_{12} + \Psi_{111}$. One has to check that the terms S_{I+H} , with H a permutation of $[0\cdots 02]$, $[0\cdots 011]$; $[0\cdots 03]$, $[0\cdots 012]$, $[0\cdots 0111]$ are zero functions, or cancel two by two, and that the terms which survive are exactly those such that J/I is an horizontal strip.

Ex. 26 Show that the set of partitions $\{[0038] \otimes 5\}$ is equal to the disjoint union

$$\{[0035] \otimes 8\} \cup \{[0068] \otimes 2\} \ .$$

Relate this property to the vanishing of the determinant $\begin{vmatrix} S_3 & S_6 & S_9 \\ S_2 & S_5 & S_8 \\ S_2 & S_5 & S_8 \end{vmatrix}$.

Recurrent Sequences

Given a positive integer k, a sequence $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2, \ldots$ is *recurrent of order* k if there exists constants $c_1, \ldots, c_k$ such that, for $n \geq k$,

$$\mathcal{S}_n + c_1\, \mathcal{S}_{n-1} + \cdots + c_k\, \mathcal{S}_{n-k} = 0 \ . \tag{58}$$

The polynomial

$$x^k + c_1 x^{k-1} + c_2 x^{k-2} + \cdots + c_k$$

is called the *characteristic polynomial* of the recurrence. Writing it $R(x, \mathbb{A})$, with $\mathbb{A}$ the alphabet of zeroes of the characteristic polynomial, one has that

$$c_i = S^i(-\mathbb{A}) = (-1)^i \Lambda^i(\mathbb{A}) \ . \tag{59}$$

Expanding $S^n(\mathbb{A} - \mathbb{A}) = 0$, $n > 0$, one sees that for any $r \geq -k{+}1$,

$$\big(S^r(\mathbb{A}),\ S^{r+1}(\mathbb{A}),\ S^{r+2}(\mathbb{A}),\ S^{r+3}(\mathbb{A}), \ldots\big)$$

is a recurrent sequence with characteristic polynomial $R(x, \mathbb{A})$. A linear combination of such sequences is still a recurrent sequence with same characteristic polynomial.

Let us normalize the sequence by imposing $\mathcal{S}_0 = 1$. One has $k{-}1$ free parameters $\mathcal{S}_1, \ldots, \mathcal{S}_{k-1}$ to determine the sequence, and this is best realized by introducing an alphabet $\mathbb{B}$ of cardinality $\leq k{-}1$ such that

$$\mathcal{S}_1 = S^1(\mathbb{A}{-}\mathbb{B}), \ldots, \mathcal{S}_{k-1} = S^{k-1}(\mathbb{A}{-}\mathbb{B}) \ . \tag{60}$$

Indeed, the equations

$$\sum_{i=0}^{k} S^{n-i}(\mathbb{A}{-}\mathbb{B}) S^i(-\mathbb{A}) = S^n\big((\mathbb{A}{-}\mathbb{B}) - \mathbb{A}\big) = S^n(-\mathbb{B}) = 0 \text{ for } n \geq k \tag{61}$$

are just a rewriting of (58). The alphabet $\mathbb{B}$ is determined by the equations

$$S^j(-\mathbb{B}) = S^j\big((\mathbb{A}{-}\mathbb{B}) - \mathbb{A}\big) \ ,\ 1 \leq j \leq k{-}1 \ . \tag{62}$$

Therefore, any recurrent sequence with characteristic polynomial $R(x, \mathbb{A})$ (such that $\mathcal{S}_0 = 1$) is of the type $\mathcal{S}_n = S^n(\mathbb{A}{-}\mathbb{B})$, and the space of sequences of characteristic polynomial $R(x, \mathbb{A})$, with arbitrary $\mathcal{S}_0$, is of dimension k.

Because $S^n(\mathbb{A}\text{-}\mathbb{B}) = S^n(\mathbb{A}) + S^1(\text{-}\mathbb{B})S^{n-1}(\mathbb{A}) + \cdots + S^{k-1}(\text{-}\mathbb{B})S^{n-k+1}(\mathbb{A})$, our writing of the general solution of equations (58) with $\mathcal{S}_0 = 1$, as $\mathcal{S}_n = S^n(\mathbb{A}\text{-}\mathbb{B})$, is equivalent to saying that we have taken as fundamental solutions

$$\mathcal{S}_n = S^n(\mathbb{A}), \ldots, \mathcal{S}_n = S^{n-k+1}(\mathbb{A}) \ .$$

Let us indeed check that every solution is a linear combination of them. Consider the determinant of order $k+1$:

$$\begin{vmatrix} \mathcal{S}_n & S^{n-k+1}(\mathbb{A}) & \cdots & S^n(\mathbb{A}) \\ \vdots & \vdots & & \vdots \\ \mathcal{S}_{n-k} & S^{n-2k+1}(\mathbb{A}) & \cdots & S^{n-k}(\mathbb{A}) \end{vmatrix} .$$

Subtracting $\mathbb{A}$ in the first row factorizes the determinant, $\mathcal{S}_n$ being replaced by $\mathcal{S}_n + S^1(\text{-}\mathbb{A})\mathcal{S}_{n-1} + \cdots + S^k(\text{-}\mathbb{A})\mathcal{S}_{n-k}$ which is null. Therefore the determinant is null, and its columns are linearly dependent. QED

Suppose that all elements of $\mathbb{A}$ are distinct. In that case, the sequences $(1,\, a,\, a^2,\, a^3, \ldots)$, $a \in \mathbb{A}$, are k linearly independent solutions of (58). This implies that for any $\mathbb{B}$ of cardinality $k\text{-}1$ (smaller cardinalities are obtained by specializing some letters to 0), there exists constants m_a, for $a \in \mathbb{A}$ such that

$$S^n(\mathbb{A}\text{-}\mathbb{B}) = \sum\nolimits_{a\in\mathbb{A}} m_a\, a^n \ , \ n \geq 0 \ . \tag{63}$$

The explicit relations between these two parametrizations of the general solution of (58) satisfying $\mathcal{S}_0 = 1$ is given by Lagrange interpolation. Indeed, the $S^n(\mathbb{A}\text{-}\mathbb{B})$ are the successive coefficients in the Taylor expansion of the rational function $\prod_b(1-zb)/\prod_a(1-za)$, that one can decompose into rational functions having a simple pole (defining $x = 1/z$) :

$$\begin{aligned} \sum_n z^n S^n(\mathbb{A}\text{-}\mathbb{B}) = \frac{\prod_{b\in\mathbb{B}} 1 - zb}{\prod_{a\in\mathbb{A}} 1 - za} = \frac{xR(x,\mathbb{B})}{R(x,\mathbb{A})} &= \sum_{a\in\mathbb{A}} \frac{R(a,\mathbb{B})}{(1-za)\, R(a,\ \mathbb{A}\text{-}a)} \\ &= \sum_n z^n \sum_a \frac{a^n\, R(a,\mathbb{B})}{R(a,\ \mathbb{A}\text{-}a)} \end{aligned} \tag{64}$$

Therefore, the coefficient m_a is a residue in $x = a$:

$$m_a = \frac{R(a,\mathbb{B})}{R(a,\ \mathbb{A}\text{-}a)} = \frac{S^{k-1}(a-\mathbb{B})}{S^{k-1}(2a-\mathbb{A})} \ . \tag{65}$$

Once more, one can control the fact that all solutions are a linear combination of $\mathcal{S} = (1,\, a,\, a^2,\, a^3, \ldots)$, $a \in \mathbb{A}$, by checking the vanishing of the following determinant of order k (we have numbered the elements of $\mathbb{A}$ to write the determinant) :

$$\begin{vmatrix} \mathcal{S}_n & a_1^n & \cdots & a_k^n \\ \vdots & \vdots & & \vdots \\ \mathcal{S}_{n-k} & a_1^{n-k} & \cdots & a_k^{n-k} \end{vmatrix} .$$

Indeed, its Laplace expansion along the first column gives

$$\sum_0^k \mathcal{S}_{n-i}\,(-1)^i \Lambda^i(\mathbb{A})$$

multiplied by the cofactor of $\mathcal{S}_n$ (which is equal to det $|a_j^{n-i}|_{1\le i,j\le k}$), and thus is just a multiple of the defining equation of the recurrence. QED

Notice that writing $\mathcal{S}_n = S^n(\mathbb{A}\text{–}\mathbb{B})$ is also valid when several elements of $\mathbb{A}$ coincide, contrary to the present situation where the expression must be transformed in that case.

The case $\mathcal{S}_n = \Psi_n/k$ is specially interesting. We know that we can write it $\mathcal{S}_n = S^n(\mathbb{A}\text{–}\mathbb{B})$ for some $\mathbb{B}$. In fact, this alphabet $\mathbb{B}$ is found by taking the logarithmic derivative of $S^k(x - \mathbb{A})$, which is

$$\frac{S^{k-1}(x{+}x - \mathbb{A})}{S^k(x - \mathbb{A})} = \sum_a \frac{1}{x - a} = \sum_n x^{-1-n}\Psi_n(\mathbb{A}) \ .$$

Therefore the numerator of the above rational function gives

$$S^{k-1}(x{+}x - \mathbb{A}) = k\, S^{k-1}(x - \mathbb{B})$$

and, more explicitely,

$$\Lambda^i(\mathbb{B}) = \frac{k - i}{k}\,\Lambda^i(\mathbb{A}) \ , \ i = 0, \ldots, k \ . \tag{66}$$

Given the Taylor expansion of a rational function $R(x,\mathbb{B})/R(x,\mathbb{A})$, we know from proposition 9 how to recover symmetric functions of $\mathbb{A}$, or symmetric functions of $\mathbb{B}$. Taking partitions $I = n^k$, one thus get the following property which was already known to Euler in the case $k = 2$, and in general, to Brioschi (1854) and Sylvester (1862; MuirIII p.316).

Lemma 26 *Given a recurrent sequence of order k, writing it $\mathcal{S}_n = S^n(\mathbb{A}\text{–}\mathbb{B})$, one has*

$$\det\left(\mathcal{S}_{n+j-i}\right)_{1\le i,j\le k} = S_{n^k}(\mathbb{A}\text{–}\mathbb{B}) = S_{(k-1)^k}(\mathbb{A}\text{–}\mathbb{B})\left(\Lambda^k(\mathbb{A})\right)^{n-k+1} \ , \ \textit{for}\ n \ge k\text{–}1 \ .$$

In other words, $\big(S_{n^k}(\mathbb{A}-\mathbb{B}),\ n \geq k-1\big)$ is a recurrent sequence of order 1 and $S_{n^k}(\mathbb{A}-\mathbb{B})/S_{(n-1)^k}(\mathbb{A}-\mathbb{B})$ is an invariant of the sequence.

Once can easily generalize the preceding property.

Proposition 27 *Let $\big(S^n(\mathbb{A}-\mathbb{B})\big)_{n\geq 0}$ be a recurrent sequence of order k. Let $J, I \in \mathbb{N}^k$. Then, for n big enough, one has that*

$$S_{(J+(n+1)^k)/I}(\mathbb{A}-\mathbb{B})/S_{(J+n^k)/I}(\mathbb{A}-\mathbb{B})$$

is an invariant of the sequence (which is equal to $\Lambda^k(\mathbb{A})$).

Proof. We have given a factorization property (9) for diagrams of partitions containing $(k-1)^k$. The reasoning must be adapted to skew diagrams. Instead of trying to transform the determinant expressing $S_{(J+n^k)/I}(\mathbb{A}-\mathbb{B})$, to see that one can extract $\Lambda^k(\mathbb{A})$, it is easier to introduce a third alphabet $\mathbb{C}$, of cardinality r such that $I \subseteq r^k$. Then

$$S_{J+(n+1)^k}(\mathbb{A}-\mathbb{B}-\mathbb{C}) = \Lambda^k(\mathbb{A})\, S_{J+n^k}(\mathbb{A}-\mathbb{B}-\mathbb{C})$$

as soon as $n \geq card(\mathbb{B}+\mathbb{C})$ — one has even that $S_{J+n^k}(\mathbb{A}-\mathbb{B}-\mathbb{C}) = S_{(k-1+r)^k}(\mathbb{A}-\mathbb{B}-\mathbb{C})\, S_{J+(n+1-k-r)^k}(\mathbb{A})$. Taking the coefficient of $S_I(\mathbb{C})$, one gets the required property. QED

Recurrent sequences are sensitive to the initial conditions, but also to the choice of the "origin"; shifting indices by 1, i.e. putting

$$S^n(\mathbb{A}-\mathbb{B}') = \mathcal{S}'_n = \mathcal{S}_{n+1}/\mathcal{S}_1 = S^{n+1}(\mathbb{A}-\mathbb{B})/S^1(\mathbb{A}-\mathbb{B})$$

amounts to a transformation $\mathbb{B} \to \mathbb{B}'$ which is not straightforward to explicit.

It is easier to do it at the level of the generating function $\lambda_z(\mathbb{B}-\mathbb{A})$, because, by definition :

$$\lambda_z(\mathbb{B}'-\mathbb{A}) = \frac{1}{z\Lambda^1(\mathbb{B}-\mathbb{A})}\left(\lambda_z(\mathbb{B}-\mathbb{A}) - 1\right) \ .$$

This resolves into

$$\Lambda^j(\mathbb{B}') = \big(\Lambda^{j+1}\mathbb{A} - \Lambda^{j+}\mathbb{B}\big) \,/\, \big(\Lambda^1\mathbb{A} - \Lambda^1\mathbb{B}\big) \ ,\ j = 0, \ldots,\ k-1 \ .$$

The transformation $\mathcal{S}'_n = \mathcal{S}_{n+1}/\mathcal{S}_1$ can also be seen when writing $\mathcal{S}_n = \sum_{a\in\mathbb{A}} m_a\, a^n$; indeed

$$\mathcal{S}'_n = \mathcal{S}_{n+1}/\mathcal{S}_1 = \sum\nolimits_{a\in\mathbb{A}} (m_a a/\mathcal{S}_1)\ a^n \ .$$

Wronski had been the first to notice that the initial conditions for the sequence of complete functions (that he called *aleph functions* $\aleph$) could be simplified, and that the sequence $\mathcal{S}_n$ with characteristic polynomial $S^k(x-\mathbb{A})$ and initial conditions $\mathcal{S}_0 = 0 = \cdots = \mathcal{S}_{k-1},\ \mathcal{S}_k = 1$, had solution

$$\mathcal{S}_n = S^{n-k+1}(\mathbb{A}) \ .$$

Wronski further noticed that equation (58) could be used to extend the sequence of aleph functions to all $n \in \mathbb{Z}$:

Lemma 28 *(Wronski). Let $\mathbb{A}$ be of cardinality k, $\mathbb{A}^\vee = \{1/a \ : \ a \in \mathbb{A}\}$. Then the sequence*

$$\mathcal{S}_n = (-1)^{k-1} S^{-n-k}(\mathbb{A}^\vee)\, \Lambda^k(\mathbb{A}^\vee)\,,\ n<0 \quad \& \quad \mathcal{S}_n = S^n(\mathbb{A})\,,\ n \geq 0 \qquad (67)$$

satisfies

$$\mathcal{S}_n - \Lambda^1(\mathbb{A})\, \mathcal{S}_{n-1} + \Lambda^2(\mathbb{A})\, \mathcal{S}_{n-2} - \cdots + (-1)^k \Lambda^k(\mathbb{A})\, \mathcal{S}_{n-k} = 0 \ ,\ n \in \mathbb{Z} \ .$$

In particular, $\mathcal{S}_{-1} = 0 = \cdots = \mathcal{S}_{-k-1}$, $\mathcal{S}_{-k} = (-1)^{k-1}\big(\Lambda^k(\mathbb{A})\big)^{-1}$.

Proof. The equations

$$0 = S^n(\mathbb{A} - \mathbb{A}) = \sum_{i=0}^{k} S^{n-i}(\mathbb{A})\, S^i(-\mathbb{A}) \ ,\ n \geq 1$$

show that $\big(\mathcal{S}_n = S_n(\mathbb{A})\,,\ n \geq -k{+}1\big)$ is the sequence with characteristic polynomial $S^k(x-\mathbb{A})$ and initial conditions $\mathcal{S}_{-k+1} = 0 = \cdots = \mathcal{S}_{-1}$, $\mathcal{S}_0 = 1$. The sequence $\big(S^{n-k+1}(\mathbb{A}^\vee)\,,\ n \geq 0\big)$ has characteristic polynomial $S^k(x - \mathbb{A}^\vee) = x^k S^k(-\mathbb{A}^\vee)\, S^k(1/x - \mathbb{A})$, and initial conditions $0, \ldots, 0, 1$. But if we reverse it, by indexing it with negative indices, the new sequence has also characteristic polynomial $S^k(x-\mathbb{A})$.

$$\begin{array}{cccccccccc}
\cdots & S^2(\mathbb{A}^\vee) & S^1(\mathbb{A}^\vee) & S^0(\mathbb{A}^\vee) & 0 & \cdots\ 0 & & & & \\
 & & & & 0 & \cdots\ 0 & S^0(\mathbb{A}) & S^1(\mathbb{A}) & S^2(\mathbb{A}) \cdots & \\
\scriptstyle position & \scriptstyle -k-2 & \scriptstyle -k-1 & \scriptstyle -k & \scriptstyle -k+1 & \scriptstyle -1 & \scriptstyle 0 & \scriptstyle 1 & \scriptstyle 2 &
\end{array}$$

To glue the two sequences into a single one, we have just to take into account the overlapping of initial conditions in positions $-k$ and 0. We have to multiply the sequence $S^j(\mathbb{A}^\vee)$ by the factor $(-1)^{k-1}\Lambda^k(\mathbb{A}^\vee)$, to get a bi-infinite sequence

$$\ldots,\ (-1)^{k-1}\Lambda^k(\mathbb{A}^\vee) S^1(\mathbb{A}^\vee),\ (-1)^{k-1}\Lambda^k(\mathbb{A}^\vee), \underbrace{0, \ldots, 0}_{k-1},\ S^0(\mathbb{A}),\ S^1(\mathbb{A}), \ldots$$

having characteristic polynomial $S^k(x-\mathbb{A})$. This proves Wronski's statement. QED

More generally, *recurrent sequences with characteristic polynomial* $S^k(x{-}\mathbb{A})$ *are sequences* $(\mathcal{S}_n : n \in \mathbb{Z})$ *satisfying*

$$\mathcal{S}_n + S^1(-\mathbb{A})\, \mathcal{S}_{n-1} + \cdots + S^k(-\mathbb{A})\, \mathcal{S}_{n-k} = 0 \quad \forall n \in \mathbb{Z} \ , \tag{68}$$

with "initial" conditions any set of k *consecutive elements.*

We have already met the identity, for any $j \geq 0$,

$$S^j(1 - z\mathbb{A})\, \sigma_z(\mathbb{A}) = 1 + (-z)^j \sum_{i=1}^{\infty} z^i S_{1^j\, i}(\mathbb{A}) \ . \tag{69}$$

It can be used to get a recurrent sequence with characteristic polynomial $S^k(x{-}\mathbb{A})$ and initial conditions : $\mathcal{S}_0$ arbitrary, $\mathcal{S}_1 = 0 = \cdots = \mathcal{S}_j$, $\mathcal{S}_{j+1}, \ldots, \mathcal{S}_{k-1}$ arbitrary. Indeed :

Lemma 29 *Let* k, j *be integers:* $k > j > 0$*, and* $\mathbb{A}$ *be of cardinality* k*. Let* $c_j, \ldots, c_{k-1}$ *be constants. Then*

$$\mathcal{S}_n := c_j\, S_{1^j, n-j}(\mathbb{A}) + \cdots + c_{k-1}\, S_{1^{k-1}, n-k+1}(\mathbb{A})$$

is a recurrent sequence with characteristic polynomial $S^k(x - \mathbb{A})$*, satisfying the conditions*

$$\mathcal{S}_1 = 0, \ldots, \mathcal{S}_j = 0 \ .$$

Proof. Equation (69) can be interpreted as the fact that $(-1)^j S_{1^j, n-j}(\mathbb{A})$ is a recurrent sequence with characteristic polynomial $S^k(x - \mathbb{A})$, satisfying the conditions $\mathcal{S}_0 = 1$, $\mathcal{S}_1 = 0 = \cdots = \mathcal{S}_j$. Taking a linear combination of such sequences gives the lemma. QED

The recurrence (68) can be put into matrix form:

$$\begin{bmatrix} \mathcal{S}_n \\ \mathcal{S}_{n-1} \\ \vdots \\ \vdots \\ \mathcal{S}_{n-k+1} \end{bmatrix} = \begin{bmatrix} \Lambda^1 & -\Lambda^2 & \Lambda^3 & \cdots & \mp\Lambda^{k-1} & \pm\Lambda^k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ & \ddots & & & & \vdots \\ & & & \ddots & & \vdots \\ 0 & 0 & & & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{S}_{n-1} \\ \mathcal{S}_{n-2} \\ \vdots \\ \vdots \\ \mathcal{S}_{n-k} \end{bmatrix} \tag{70}$$

One has just added the relations $\mathcal{S}_{n-i} = \mathcal{S}_{n-i}$, $i = 1, \ldots, k$ to the original one!

The above matrix is called the *companion matrix* of the polynomial $S^k(x - \mathbb{A})$. It is convenient to write it like

$$\mathcal{C} := \begin{bmatrix} S_1 & S_{2,0} & S_{3,0,0} & \cdots & S_{k,0^{k-1}} \\ S_0 & S_{1,0} & S_{2,0,0} & \cdots & S_{k-1,0^{k-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ S_{-k+2} & S_{-k+3,0} & S_{-k+4,0,0} & & S_{1,0^{k-1}} \end{bmatrix} . \tag{71}$$

The matrix $\mathcal{C}$ is the matrix expressing the morphism "multiplication by x" in the space of polynomials in x modulo $S^k(x-\mathbb{A})$ (in the basis of powers of x). Its powers are easy to compute :

$$\mathcal{C} := \begin{bmatrix} S_n & S_{n+1,0} & \cdots & S_{n+k-1,0^{k-1}} \\ S_{n-1} & S_{n,0} & \cdots & S_{n+k-2,0^{k-1}} \\ \vdots & \vdots & & \vdots \\ S_{n-k+1} & S_{n-k+2,0} & & S_{n,0^{k-1}} \end{bmatrix} . \tag{72}$$

For example, for $k = 4$, the first powers are (not writing $\mathcal{C}^0 = 1$) :

$$\begin{bmatrix} S_1 & S_{20} & S_{300} & S_{4000} \\ 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \end{bmatrix}, \begin{bmatrix} S_2 & S_{30} & S_{400} & S_{5000} \\ S_1 & S_{20} & S_{300} & S_{4000} \\ 1 & . & . & . \\ . & 1 & . & . \end{bmatrix}, \begin{bmatrix} S_3 & S_{40} & S_{500} & S_{6000} \\ S_2 & S_{30} & S_{400} & S_{5000} \\ S_1 & S_{20} & S_{300} & S_{4000} \\ 1 & . & . & . \end{bmatrix}, \begin{bmatrix} S_4 & S_{50} & S_{600} & S_{7000} \\ S_3 & S_{40} & S_{500} & S_{6000} \\ S_2 & S_{30} & S_{400} & S_{5000} \\ S_1 & S_{20} & S_{300} & S_{4000} \end{bmatrix}$$

Each entry in position $[i, j]$ of the successive powers $\mathcal{C}^n$ is by construction a recurrent sequence with characteristic polynomial $S^k(x-\mathbb{A})$. The successive entries $[4, 1]$ for example, are

$$0,\, 0,\, 0,\, 1,\, S_1, \ldots$$

The entries $[4, 2]$ are

$$0,\, 0,\, 1,\, 0,\; S_{2,0} = -S_{11},\, S_{3,0} = -S_{12},\, \ldots$$

The entries belonging to the same column correspond to the same sequence, but with a shift of indices.

In other words, the companion matrix allows to recover the fact that the hook Schur functions $(-1)^j S_{1^j,n}$ (for j fix), are a recursive sequence with all initial elements equal to 0, except one which is equal to 1. Explicitely,

$$S_{-k+1+j,\,0^j} = 0 = \cdots = S_{-1,\,0^j},\, 1 = S_{0,\,0^j},\, S_{1,\,0^j} = 0 = \cdots = S_{j,\,0^j} \ .$$

Conversely, from relations (69) (completed towards negative indices) one recovers the powers of the companion matrix, negative powers included.

We have still to identify the entries of the negative powers of $\mathcal{C}$. We already know that going towards $-\infty$ corresponds to taking the characteristic polynomial $S^k(x-\mathbb{A}^\vee)$. However, because for each column of the powers of $\mathcal{C}$, we have initial conditions consisting of $k-1$ zeros and a 1, then then we know that we shall find the hook Schur functions of $\mathbb{A}^\vee$, up to sign and powers of $\Lambda^k(\mathbb{A})$. Putting all this together, one gets :

Lemma 30 *The inverse of the companion matrix of $S^k(x-\mathbb{A})$ is the companion matrix of $S^k(x-\mathbb{A}^\vee)$, up to symmetry with respect to the center of the matrix. The entries of the negative powers of $\mathcal{C}$ are the hook Schur functions of $\mathbb{A}^\vee$.*

Still continuing to illustrate the case $n = 4$, one has the following infinite matrix, such that the powers of $\mathcal{C}$ are the submatrices on consecutive rows. To stress regularity, it is appropriate to use the indexing of representations of the linear group: given any $v = [v_1, v_2, v_3, v_4] \in \mathbb{Z}^4$, let $\aleph$ be any positive integer such all $\aleph + v_i - i + 1$ are positive. Then v codes for $S_{\aleph+v_1,\,\aleph+v_2,\,\aleph+v_3,\,\aleph+v_4}(\mathbb{A})\,(\Lambda^4(\mathbb{A}))^{-\aleph}$. For example, the last written line, taking $\aleph = 5$, codes the Schur functions

$$S_{555,\,-3} = -S_{0444},\ S_{55,\,-2,5} = S_{0445},\ S_{5,\,-1,55} = -S_{0455},\ S_{0555}\ ,$$

and this agrees with the experiment below.

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ [0004] & [0050] & [0600] & [7000] \\ [0003] & [0040] & [0500] & [6000] \\ [0002] & [0030] & [0400] & [5000] \\ [0001] & [0020] & [0300] & [4000] \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ [000\bar{4}] & [00\bar{3}0] & [0\bar{2}00] & [\bar{1}000] \\ [000\bar{5}] & [00\bar{4}0] & [0\bar{3}00] & [\bar{2}000] \\ [000\bar{6}] & [00\bar{5}0] & [0\bar{4}00] & [\bar{3}000] \\ [000\bar{7}] & [00\bar{6}0] & [0\bar{5}00] & [\bar{4}000] \\ [000\bar{8}] & [00\bar{7}0] & [0\bar{6}00] & [\bar{5}000] \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

```
CompanionMat:=proc(k) local i;
matrix([ [seq((-1)^(i-1)*e.i,i=1..k)], seq([0$(i-1),1,0$(k-i)],i=1..k-1)])
end:
```

$ACE >$ $CLG_n(4)$: `aa:=CompanionMat(4):`

$ACE >$ `map(`Tos_n`,linalg[multiply](aa$5));`

$$\begin{bmatrix} s[5] & -s[5,1] & s[5,1,1] & -s[5,1,1,1] \\ s[4] & -s[4,1] & s[4,1,1] & -s[4,1,1,1] \\ s[3] & -s[3,1] & s[3,1,1] & -s[3,1,1,1] \\ s[2] & -s[2,1] & s[2,1,1] & -s[2,1,1,1] \end{bmatrix}$$

$ACE >$ `bb:=linalg[adj](aa):[map(`Tos_n`,eval(bb)),det(bb)];`

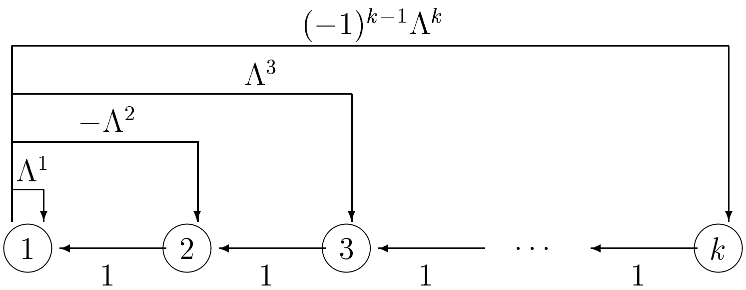
$$\left[\begin{bmatrix} 0 & s[1,1,1,1] & 0 & 0 \\ 0 & 0 & s[1,1,1,1] & 0 \\ 0 & 0 & 0 & s[1,1,1,1] \\ -1 & s[1] & -s[1,1] & s[1,1,1] \end{bmatrix}, e4\verb|^|3 \right]$$

$ACE >$ `map(`Tos_n`,linalg[multiply](bb$5));`

$$\begin{bmatrix} -s[4,4,4,3] & s[5,4,4,3] & -s[5,5,4,3] & s[5,5,5,3]] \\ -s[4,4,4,2] & s[5,4,4,2] & -s[5,5,4,2] & s[5,5,5,2] \\ -s[4,4,4,1] & s[5,4,4,1] & -s[5,5,4,1] & s[5,5,5,1] \\ -s[4,4,4] & s[5,4,4] & -s[5,5,4] & s[5,5,5] \end{bmatrix}$$

Following Chen and Louck [7], one can use the automaton associated to $\mathcal{C}$ to describe the entries of the powers of $\mathcal{C}$, in particular, the first entry (which is S_n).

Proposition 31 *(Chen-Louck). The complete function $S_n(\mathbb{A})$ is the sum of all paths of length n, from the origin to the origin, in the following automaton :*



For example, there are 4 paths of length 3,

$$\Lambda^1\Lambda^1\Lambda^1,\ \Lambda^1\,(-\Lambda^2)\,1,\ (-\Lambda^2)\,\Lambda^1\,1,\ \Lambda^3\,1\,1\,1\ ,$$

and this gives $S_3 = \Lambda^{111} - 2\Lambda^{12} + \Lambda^3$.

Summing on all paths differing by a permutation, one gets the expression of a complete function in the basis of elementary symmetric ones :

Corollary 32 *Let n be a positive integer. Then*

$$S^n = \sum_I (-1)^{m_2+m_4+\cdots} \binom{\ell(I)}{m_1,\, m_2,\, m_3,\, \ldots} S_I$$

sum over all partitions $I = [1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots]$, $|I| = n$.

The most famous recurrent sequence is the *Fibonacci sequence*:

$$F(n+2) = F(n+1) + F(n) \quad \& \quad F(0) = 0,\, F(1) = 1\ . \tag{73}$$

In our conventions, it is the sequence $F(n+1) = S^n(\mathbb{A}-\mathbb{B})$, with $\mathbb{B} = 0$, $\mathbb{A}$ such that $\Lambda^1(\mathbb{A}) = 1$, $\Lambda^2(\mathbb{A}) = -1$. Explicitely,

$$\mathbb{A} = \left\{ \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \right\}\ ,$$

is the alphabet composed of the *golden number* and its conjugate. The general term of the Fibonacci sequence is therefore

$$F(n+1) = S^n(\mathbb{A}) = \left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right) \Big/ \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \right)\ .$$

Exercises

Ex. 27 (Fibonacci numbers of order k).

Let $2 \leq k \in \mathbb{N}$. Define numbers $F(n,k)$ by the recurrence of order k

$$F(n,k) = F(n-1,k) + \cdots + F(n-k+1,k)$$

together with the initial conditions

$$F(n,k) = 0, -k+2 \leq n < 0 \quad \& \quad F(1,k) = 1\ .$$

Write $F(n,k)$ as a sum of multinomial coefficients.

Show that

$$F(n,k) = \big(F(n+1,k) + F(n-k,k)\big)/2\ .$$

Corr.Ex.27. Initial conditions, as for usual Fibonacci numbers, imply that $\mathbb{B}=0$, and that the alphabet $\mathbb{A}_k$ such that $S_n(\mathbb{A}_k)=F(n{+}1,k)$ is such that $\Lambda^j(\mathbb{A}_k)=(-1)^{j-1}$, $j=1,\ldots,k$. Therefore, formula (32) translates into

$$F(n{+}1,k)=\sum_I \binom{\ell(I)}{m_1,\,m_2,\,m_3,\,\ldots} S_I \ ,$$

sum over all partitions $I=[1^{m_1},2^{m_2},3^{m_3},\ldots]$, $|I|=n$, formula given by A. Philippou, Fibo. Quat. **21**(1983) 82-86.

Ex. 28 (Fibonacci numbers of infinite order).

They are defined by the recurrence

$$F(n,\infty)=F(n-1,\infty)+\cdots+F(1,\infty),\ n\geq 1\ ,$$

with initial condition $F(1,\infty)=1$.

Compute the generating function $\sum_{n=0}^{\infty} z^n F(n{+}1,\infty)$. Deduce from it that, with $\mathbb{A}_k$ defined by $S_n(\mathbb{A}_k)=F(n,k)$ one has

$$\Psi_n(\mathbb{A}_k)=2^n-1=\Psi_n(\mathbb{A})\ ,\ \forall n\leq k\ .$$

Prove the identity

$$2^{n-1}=\sum_J\prod_i\left(\frac{2^i-1}{i}\right)^{m_i}\frac{1}{m_i!}\ ,$$

sum over all partitions $J=1^{m_1}\,2^{m_2}\,3^{m_3}\cdots$ of n.

Corr.Ex.28. One sees that, with $\mathbb{A}=limit(\mathbb{A}_k)$, i.e. $\mathbb{A}$ such that $\Lambda^j(\mathbb{A})=(-1)^{j-1}$, $\forall j>0$, then $F(n,\infty)=S_n(\mathbb{A})$ and

$$\sigma_z(\mathbb{A})=\sum_{n=0} z^n F(n,\infty)=z(1-z)/(1-2z)\ .$$

This implies that $F(n,\infty)=2^{n-1}$, fact that we could expect from the recurrence! At the level of power sums, it is more interesting. From $\sigma_z(\mathbb{A}_k)=(1-z)/(1-2z+z^{k+1})$, one finds that $\Psi_n(\mathbb{A}_k)$ and $\Psi_n(\mathbb{A})$ agree up to degree n, and are equal, with $a=2$, $b=1$, to $a^n-b^n=2^n-1$, $n=1,\ldots,k$. The expression of a complete function in terms of elementary ones gives the last required identity.

As a matter of fact, keeping general a and b instead of 2 and 1, one has the more general identity

$$S_n(a-b)=(a-b)a^{n-1}=\sum_J\prod_i\left(\frac{a^i-b^i}{i}\right)^{m_i}\frac{1}{m_i!}\ .$$

Ex. 29 Define *Fibonacci polynomials* $F_n(x)$ by

$$F_n(x) - xF_{n-1}(x) - F_{n-2}(x) = 0 \quad , \; F_0(x) = 1, \; F_1(x) = x \; .$$

Show that

$$F_n(x) = \left(\left(\frac{x + \sqrt{x^2+4}}{2} \right)^{n+1} - \left(\frac{x - \sqrt{x^2+4}}{2} \right)^{n+1} \right) \frac{1}{\sqrt{x^2+4}} \; ,$$

and

$$F_n(x) = \sum_0^\infty \binom{j}{2j-n} x^{2j-n} \; .$$

Corr.Ex.29. One finds $F_n(x) = S^n(\mathbb{A})$, with $\Lambda^1(\mathbb{A}) = x$, $\Lambda^2(-\mathbb{A}) = -1$. Therefore, the two roots of $S^2(x-\mathbb{A})$ are $(x + \sqrt{x^2+4})/2$ and $(x - \sqrt{x^2+4})/2$, which gives the first expression of $F_n(x)$. The second one comes from the expression of S^n in the basis Λ^I. These expressions have been obtained by Scott ((1878), MuirIII, p. 420).

Ex. 30 (Lucas sequence).

The Lucas sequence satisfy the same recursion as Fibonacci sequence

$$L_n = L_{n-1} + L_{n-2} \; ,$$

but the initial conditions are $L_0 = 2$, $L_1 = 1$. Express the Lucas sequence in terms of the alphabet $\mathbb{A} = \{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$.

Corr.Ex.30. Lucas sequence corresponds to the same alphabet $\mathbb{A}$ as Fibonacci. The initial conditions are such that $L_0 = \Psi_0(\mathbb{A})$, $L_1 = \Psi_1(\mathbb{A})$. Therefore $L_n = \Psi_n(\mathbb{A})$.

One could also write $L_n = 2S^n(\mathbb{A}-\mathbb{B})$, with $\mathbb{B}$ of cardinality 1 such that $\Lambda^1(\mathbb{B}) = 1/2$.

Ex. 31 Let $k \in \mathbb{N}$. Take a recurrence of order k

$$S_n = S_{n-1} + \cdots + S_{n-k} \; ,$$

with arbitrary initial conditions $S_0, \ldots, S_{k-1}$. Show that the limit of S_n, for $n = \infty$, exists and is equal to

$$\frac{2}{k(k+1)} \left(S_0 + 2S_1 + 3S_2 + \cdots + kS_{k-1} \right) \; .$$

Corr.Ex.31. (Gelfond, Différences finies, Dunod, Paris (1963)). As usual, $S_n = S^n(\mathbb{A}-\mathbb{B})$. The characteristic polynomial (in $z = 1/x$) is

$$\lambda_{-z}(\mathbb{A}) = 1 - (z^1 + \cdots + z^k)/k \ .$$

Since $z = 1$ is a root, let us define $\mathbb{C} := \mathbb{A} - 1$. Therefore,

$\Lambda^1(\mathbb{C}) = -(k-1)/k$, $\Lambda^2(\mathbb{C}) = (k-2)/k$, $\Lambda^3(\mathbb{C}) = -(k-3)/k$, $\ldots$, $\Lambda^{k-1}(\mathbb{C}) = (-1)^{k-1}/k$.

One recognizes that

$$(S_0 + 2S_1 + 3S_2 + \cdots + kS_{k-1})/k = S^{k-1}(\mathbb{A}-\mathbb{B} - \mathbb{C}) = S^{k-1}(1 - \mathbb{B}) \ .$$

But $S^n(\mathbb{A}-\mathbb{B}) = S^n(1 + (\mathbb{C}-\mathbb{B}))$ tends towards $\sum_0^\infty S^j(\mathbb{C}-\mathbb{B})$ and the conclusion follows, once checked that

$$\sigma_1(\mathbb{C}) = \big(\lambda_{-1}(\mathbb{C})\big)^{-1} = \big(\frac{k}{k} + \frac{k-1}{k} + \cdots + \frac{1}{k}\big)^{-1} = \frac{2}{k+1} \ .$$

Ex. 32 Define the *Tchebychef polynomials of the first kind* T_n by

$$T_n(x) - 2xT_{n-1}(x) + T_{n-2}(x) = 0 \quad , \ T_0(x) = 1, \, T_1(x) = x \ .$$

and the *Tchebychef polynomials of the second kind* U_n by

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0 \quad , \ U_0(x) = 1, \, U_1(x) = 2x \ .$$

Given a parameter β, define the β-Tchebychef polynomial P_n^β by the same recursion, with $P_0^\beta = 1$, $P_1^\beta = 2x - \beta$.

Write $x = \cos(\theta)$ and express T_n, U_n, P_n^β as trigonometric functions of θ. Express also T_n, U_n P_n^β as symmetric functions of the alphabet $\mathbb{A}$ of cardinality 2 such that $\Lambda^1(\mathbb{A}) = 2x$, $\Lambda^2(\mathbb{A}) = 1$.

Prove that

$$nU_n(x)/2 = U_{n-1}(x)T_1(x) + U_{n-2}(x)T_2(x) + \cdots + T_n(x) \quad , \ n \geq 0 \ .$$

Ex. 33 Define Legendre polynomials $P_n(x)$ by the generating function

$$1/\sqrt{(1 - 2xz + z^2)} = \sum P_n z^n \ .$$

Show that they can be written

$$P_n(x) = S_n((n+1)x - nA)/2^n \quad \& \quad P_n(x) = \Lambda^n(n - (n+1)u) \ ,$$

u, x being rank 1 element, $\mathbb{A}$ being of cardinality 2, u being specialized to $(1-x)/2$ and $\mathbb{A}$ to $\{1, -1\}$.

Deduce the expansion

$$P_n(x) = \sum_{i=0}^{n} \frac{(n-i+1)\cdots(n+i-1)}{i!\, i!} \left(\frac{x-1}{2}\right)^i \ .$$

Corr.Ex.33.

```
ACE > with(orthopoly):
ACE > LEGENDRE:= proc(n) SfAVars(x,z);
simplify(subs(z=-1,SfAExpand( s[n]((n+1)*x-n-n*z)))/2∧n)
end:
ACE > LEGENDRE2:= proc(n) SfAVars(u);
simplify(subs(u=(1-x)/2,SfAExpand( e[n](n-(n+1)*u))))
end:
ACE > [P(4,x),LEGENDRE(4), LEGENDRE2(4)];
[35/8 x∧4-15/4 x∧2+3/8,35/8 x∧4-15/4 x∧2+3/8,35/8 x∧4-15/4 x∧2+ 3/8]
```

Ex. 34 Let f be a function of one variable of exponential type, i.e. such that there exists an integer k and constants $c_1, \ldots, c_k$; $a_1, \ldots, a_k$ in $\mathbb{C}$ (all different) :

$$f(x) = c_1 a_1^x + \cdots + c_k a_k^x \ .$$

Express the parameters c_i, a_i in terms of the values $f(0)$, $f(1)$, ..., $f(2k-1)$.

Corr.Ex.34. (Levy & Lessman, *Finite Difference Equations*, Pitman (1959)). The expression of $f(x)$ shows that $f(0)$, $f(1)\, f(2)$, ... is a recurrent sequence of order n. We have just changed notations! Therefore $a_1, \ldots, a_n$ are solutions of the equation (in z) :

$$\begin{vmatrix} 1 & z & \cdots & z^n \\ f(0) & f(1) & \cdots & f(n) \\ \vdots & \vdots & & \vdots \\ f(n-1) & f(n) & \cdots & f(2n-1) \end{vmatrix} = 0 \ ,$$

and $c_1, \ldots, c_n$, 1 are the multiplicators of the columns which give a null combination. Note that we have to suppose that $\begin{bmatrix} f(0) & \cdots & f(n-1) \\ \vdots & & \vdots \\ f(n-1) & \cdots & f(2n-2) \end{bmatrix}$ be different from 0.

Change of Basis

Many problems involving symmetric functions involve change of bases, the most fundamental basis being the basis of Schur functions. Of course, since we already know the adjoint basis of S_J, S^I, Ψ_J, Ψ^I, we can express any symmetric function in one of these bases by computing scalar products. However, we shall see that the mutual change of bases between these bases can be described combinatorially.

Complete to Schur : $(\mathbf{S}^I\,,\,\mathbf{S}_J)$.

Pieri formula describes the product of a Schur function by a complete function as an addition of an horizontal strip to a diagram. Iterating, one has thus a description of the product by S^I as *chains* (sequence of diagrams cotained into each other) of diagrams differing by an horizontal strip.

For example, the product of S_{23} by $S^{4365} = S_4S_3S_6S_5$ can be represented by :

$$\begin{array}{l}
\diamondsuit \\
\heartsuit\ \clubsuit \\
\spadesuit\ \heartsuit\ \clubsuit\ \diamondsuit\ \diamondsuit \\
\square\ \square\ \spadesuit\ \clubsuit\ \clubsuit \\
\square\ \square\ \square\ \spadesuit\ \spadesuit\ \heartsuit\ \clubsuit\ \clubsuit\ \clubsuit\ \diamondsuit\ \diamondsuit
\end{array}$$

Of course, for a longer chain of multiplication, it would be inconvenient to use graphical symbols, and one rather takes $1, 2, 3, \ldots$.

Definition 33 *A* skew Young tableau *is a chain of partitions differing by horizontal strips. Equivalently, it is the filling of the boxes of a skew diagram by positive integers (considered as letters) in such a way that rows are weakly increasing, and columns strictly decreasing (from top to bottom). The* weight *of a tableau filled with i_1 times 1, i_2 times 2, &c. is $[i_1, i_2, \ldots]$ and its* evaluation *is the monomial $1^{i_1}2^{i_2}\cdots$. A* Young tableau *or, simply,* tableau *is a skew Young tableau with void inner partition.*

We shall later consider tableaux as words in non commutative letters, the evaluation being the monomial obtained by allowing variables to commute, the weight being the exponent of the monomial.

The preceding example reads :

$$\begin{array}{l}
6 \\
4\ 5 \\
3\ 4\ 5\ 6\ 6 \\
\square\ \square\ 3\ 5\ 5 \\
\square\ \square\ \square\ 3\ 3\ 4\ 5\ 5\ 5\ 6\ 6
\end{array} \quad , \quad \text{weight} = [0,0,4,3,7,5]\ , \quad \text{evaluation} = 1^0 2^0 3^4 4^3 5^7 6^5\ .$$

In this language, one has the following description of the scalar product $(S^I S_H \,,\, S_J)$.

Proposition 34 *Given partitions J, H, and an integral vector I, then the scalar product $(S^I S_H \,,\, S_J)$ is equal to the number of skew Young tableaux of shape J/H and weight I.*

The numbers (S^I, S_J) are called *Kostka numbers.* They can be interpreted in two ways, either giving the expansion of a product of complete functions in the basis of Schur functions, or as giving the expansion of a Schur function in the basis of monomial functions :

$$S^I = \sum\nolimits_J (S^I, S_J)\, S_J \quad \& \quad S_J = \sum\nolimits_I (S^I, S_J)\, \Psi_I \ . \tag{74}$$

Kostka numbers for functions of degree n can be put into a triangular matrix which is called the *Kostka matrix.*

ACE > `SfMat(4,'h','s') , SfMat(4,'s','m');`

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The partitions indexing rows and columns are given by ListPart(4); expansions of complete functions (in terms of Schur) are obtained by reading rows; expansions of Schur functions (in terms of monomial) are obtained by reading columns of the left matrix, or rows of the right one.

The scalar products $(S^{1\cdots 1}, S_J)$ have an interpretation as *dimensions of irreducible representations of the symmetric group* and can be computed directly from the diagram of J.

Monomial to Schur : $(\boldsymbol{\Psi}_I \,,\, \mathbf{S}_J)$.

The coefficients of the inverse of the Kostka matrix are the scalar products $(\Psi_I \,,\, S_J)$ which appear in the two expansions :

$$\Psi_I = \sum\nolimits_J (\Psi_I, S_J)\, S_J \quad \& \quad S_J = \sum\nolimits_I (\Psi_I, S_J)\, S^I \ . \tag{75}$$

From Muir's formula for the multiplication of the Schur function S_{000} by Ψ_I, we know that we can write

$$\Psi_I = \sum_{H=perm(I)} S_H \ , \tag{76}$$

sum over all different permutations H of I.

Each element of the sum is equal to $\pm$ a Schur function, or to 0. One has to collect all $H \in \mathbb{N}^n$, which give the same Schur function S_J, J partition in $\mathbb{N}^n$. They are such that, writing $\rho := [0, 1, 2, \ldots, n{-}1]$, there exists a permutation σ such that $H + \rho = (J + \rho)^\sigma$.

$$(\Psi_I\,,\, S_J) = \sum_{H=perm(I)} (S_H,\,,\, S_J) = \sum_{\sigma:\, H=(J+\rho)^\sigma, H=perm(I)} (-1)^{\ell(\sigma)} \,. \qquad (77)$$

Instead of adding ρ (which is just a way to get the indices on the top row of the determinant expressing S_H from H), one can order the development of the determinant S_J :

$$S_J = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)}\, S^{J+\rho-\rho^\sigma} \,. \qquad (78)$$

For example, for $n = 3$, $J = [a, b, c]$, writing only exponents, together with the sign of the permutation, one has the following expansion of the Schur function S_{abc} (writing on the right the case $J = [000]$, that is the vectors $\rho - \rho^\sigma$) :

$$\begin{tikzcd}
 & {[a,b,c]} \arrow[dl] \arrow[dr] & \\
-[a{-}1, b{+}1, c] \arrow[d] & & -[a, b{-}1, c{+}1] \arrow[d] \\
{[a{-}1, b{-}1, c{+}2]} \arrow[dr] & & {[a{-}2, b{+}1, c{+}1]} \arrow[dl] \\
 & -[a{-}2, b, c{+}2] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & {[012]} \arrow[dl] \arrow[dr] & \\
-\,[\bar{1}10] \arrow[d] & & -[0\bar{1}1] \arrow[d] \\
{[\bar{1}\bar{1}2]} \arrow[dr] & & {[\bar{2}11]} \arrow[dl] \\
 & -[\bar{2}02] &
\end{tikzcd}$$

The left hexagon corresponds to expanding the Jacobi-Trudi determinant from left to right. Expanding it from top to bottom instead, amounts using the action of the symmetric group on vectors such that the transposition s_i acts on the components v_i, v_{i+1} of a vector v as follows :

$$v = [\ldots, \underbrace{a,\, b,}_{i,\, i+1} \ldots] \to v^{<i>} := [\ldots,\, b{+}1,\, a{-}1,\, \ldots] \,.$$

The expansion of the determinant now becomes

$$S_J = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)}\, S^{J^{<\sigma>}} = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)}\, S^{(J+\rho)^\sigma - \rho} \,. \qquad (79)$$

The preceding hexagon is replaced by

$$\begin{tikzcd}[column sep=small]
 & & {[a,b,c]} \arrow[dll] \arrow[drr] & & \\
-[b{+}1, a{-}1, c] \arrow[d] & & & & -[a, c{+}1, b{-}1] \arrow[d] \\
{[b{+}1, c{+}1, a{-}2]} \arrow[drr] & & & & {[c{+}2, a{-}1, c{-}1]} \arrow[dll] \\
 & & -[c{+}2, b, a{-}2] & &
\end{tikzcd}$$

Note that there is no multiplicity if $a \neq c$.

Double Kostka matrices.

Since the Kostka matrix and its inverse are unitriangular, one can glue them into a single matrix that we shall call *double Kostka matrix.* This was done by Kostka, who commented in many articles, from 1875 to 1918, the beauties of it.

$ACE >$ `mm:=SfMat(5,'h','s'):add(add(mm,SfMat(5,'m','s')),diag((-1)$7));`

$$\begin{array}{cccc}
 & & \Psi_I \text{ to } S_J & & \\
\begin{array}{c} S_J \\ \text{to} \\ \psi_I \end{array} & \Bigg\downarrow & \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -2 \\ 1 & 1 & 1 & -1 & -1 & 2 & -2 \\ 1 & 2 & 1 & 1 & -1 & -1 & 3 \\ 1 & 2 & 2 & 1 & 1 & -2 & 3 \\ 1 & 3 & 3 & 3 & 2 & 1 & -4 \\ 1 & 4 & 5 & 6 & 5 & 4 & 1 \end{bmatrix} & \Bigg\uparrow & \begin{array}{c} S_J \\ \text{to} \\ S^I \end{array} \\
 & & S^I \text{ to } S_J & &
\end{array} .$$

The involution $\mathbb{A} \to -\mathbb{A}$ preserves the scalar product. Therefore, $(\Lambda^I\,,\, S_J) = (S^I\,,\, S_{J^\sim})$, and the bottom rows of Kostka double matrix give the expansion of the products of elementary symmetric functions in the basis of Schur functions. However, we have not yet met the adjoint basis of $\{\Lambda^I\}$, which is the image of the monomial basis under the involution which exchanges the elementary and complete functions. It is called the *forgotten basis*, the best reason for it being that people forgot that Kostka had defined it.

```
Forgotten:=proc(pa) SfOmega(m[op(pa)]) end:
ACE > Forgotten([5$4]); ## rectangular partitions are the only
                                         simple case (see exercise).
    m[20] + m[15, 5] + m[5, 5, 5, 5] + m[10, 10] + m[10, 5, 5]
```

Complete to Monomials: $(S^I\,,\,S^J)$

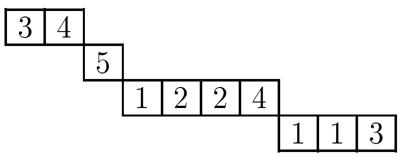
Pieri's formula has allowed us to compute the scalar products $(S^I S_H\,,\,S_K) = (S^I\,,\,S_{K/H})$, as the number of skew tableaux of shape K/H, of weight I. Let us apply it to the case where K/H is a product of independent rows of lengths $j_1, j_2, \ldots$. It gives that $(S^I\,,\,S^J)$ is equal to the number of tableaux with weight I and shape $\{j_1 \otimes j_2 \otimes \cdots\}$.

However, one can interpret these tableaux in a different manner which takes into account the symmetry between I and J. Let us call *matrix with row sums $J \in \mathbb{N}^p$ and columns sums $I \in \mathbb{N}^q$* a $p \times q$ matrix of non negative integers such that j_1 is the sum of the entries in the first row, j_2 in the second row, &c. , and similarly for columns (the term matrix is inappropriate, because multiplication of such objects has no properties; we shall put them into a box).

Interpreting now the entry $M[r,c]$ of a matrix M as giving the number of occurrences of the letter c in row r, one sees that row sums give the shape $\{j_1 \otimes j_2 \otimes \cdots\}$ and column sums give the number of occurrences of $1, 2, \ldots$.

Instead of a matrix, one can equivalently write the sequence of increasing words $1^{m[r,1]}\,2^{m[r,2]}\cdots$. But this is exactly the sequence of a skew tableau with independent rows, and we are back to our starting point.

For example, the two following objects are equivalent ($I = [3,2,2,2,1]$, $J = [2,1,4,3]$) :



$\leftrightarrow$

values		1	2	3	4	5	
row	1	0	0	1	1	0	2
	2	0	0	0	0	0	1
	3	1	2	0	1	0	4
	4	2	0	1	0	0	3
		3	2	2	2	1	*sums*

There is still another interpretation, as *double coset representatives of the symmetric group.*

Indeed, given a positive integer, and J such that $|J| = n$, let $\mathfrak{S}_n/\mathfrak{S}_J$ be the quotient of the symmetric group $\mathfrak{S}_n$ by the direct product of symmetric groups $\mathfrak{S}_{j_1} \otimes \mathfrak{S}_{j_2} \otimes \cdots$. It means that two permutations of $\mathfrak{S}_n$ are equivalent if, cutting them into blocks of successive lengths i_1, $i_2, \ldots$, one obtains one from the other by permuting letters inside blocks only. Thus one can decide that the *canonical* representatives of classes of $\mathfrak{S}_n/\mathfrak{S}_J$ are the permutations which are increasing in each block (they are the elements of minimum length in each coset). Similarly, two permutations have the same image in $\mathfrak{S}_I \backslash \mathfrak{S}_n$

if, when one replaces $1,\ldots,\ i_1$ by x_1, $i_1{+}1,\ldots,\ i_1{+}i_2$ by x_2, &c., one gets the same word. Here again, a canonical representative will be a permutation which is a shuffle of $[1,\ldots,\ i_1]$, $[i_1{+}1,\ldots,\ i_1{+}i_2], [i_1{+}i_2{+}1,\ldots,\ i_1{+}i_2{+}i_3],\ldots$.

Combining the two descriptions, one has that double cosets $\mathfrak{S}_I\backslash\mathfrak{S}_n/S_J$ are in bijection with Young tableaux of shape $j_1\otimes j_2\otimes\cdots$, and weight I, and that given such a tableau, one gets the canonical representative of the associated double coset by replacing successive occurrences of 1 by $1,2,\ldots,\ i_1$, occurrences of 2 by $i_1{+}1,\ldots,\ i_1{+}i_2$, &.c , (this operation on skew tableaux, or on words, is called *standardization*) and reading the new skew tableau as a word.

For example, the above skew tableau gives the following canonical permutation :

$$\begin{array}{cccccccccc} 6&8&&&&&&&&\\ &&10&&&&&&&\\ &&&1&4&5&9&&&\\ &&&&&&&2&3&7 \end{array} \longrightarrow [6,8,10,1,4,5,9,2,3,7]\ ,$$

which is a double coset representative of a class in $\mathfrak{S}_{[3,2,2,2,1]}\backslash\mathfrak{S}_{10}/\mathfrak{S}_{[2,1,4,3]}$, the starting tableau being obtained by replacing the values $1,2,3$ by $1,1,1$, the values $4,5$ by $2,2,\ \ldots$, the value 10 by 5.

To summarize, one has the following proposition ;

Proposition 35 *Given two compositions I,J of the same number n, then the scalar product $(S^I,\ ,\ S^J)$ is equal to the number of skew tableaux with weight I, shape $j_1\otimes j_2\otimes\cdots$. It is also equal to the number of matrices with row sums J and column sums I, and to the number of double cosets of $\mathfrak{S}_I\backslash\mathfrak{S}_n/\mathfrak{S}_J$.*

For example, one has $(S^{023},S^{113})=4$, because there are four matrices :

$$\begin{array}{|ccc|}\hline 0&1&0\\ 0&1&0\\ 0&0&3\\ \hline\end{array}\ ,\qquad \begin{array}{|ccc|}\hline 0&1&0\\ 0&0&1\\ 0&1&2\\ \hline\end{array}\ ,\qquad \begin{array}{|ccc|}\hline 0&0&1\\ 0&1&0\\ 0&1&2\\ \hline\end{array}\ ,\qquad \begin{array}{|ccc|}\hline 0&0&1\\ 0&0&1\\ 0&2&1\\ \hline\end{array}$$

corresponding to the four skew tableaux

$$\begin{array}{ccccc} 2&&&&\\ &2&&&\\ &&3&3&3 \end{array}\quad \begin{array}{ccccc} 2&&&&\\ &3&&&\\ &&2&3&3 \end{array}\quad \begin{array}{ccccc} 3&&&&\\ &2&&&\\ &&2&3&3 \end{array}\quad \begin{array}{ccccc} 3&&&&\\ &3&&&\\ &&2&2&3 \end{array}\ .$$

The list of matrices with given row and column sums, as well as their number without enumerating them, can be accessed by a function (to be added to ACE) :

$ACE >$ `GenMat([1,3,2], [4,2]);`

$$\begin{bmatrix}1&0\\3&0\\0&2\end{bmatrix}\quad\begin{bmatrix}1&0\\2&1\\1&1\end{bmatrix}\quad\begin{bmatrix}1&0\\1&2\\2&0\end{bmatrix}\quad\begin{bmatrix}0&1\\3&0\\1&1\end{bmatrix}\quad\begin{bmatrix}0&1\\2&1\\2&0\end{bmatrix}$$

$ACE >$ `GenMat([1$9], [3,1,3,2],nb);`

```
5040
```

Power sums to Schur: $(\Psi^I\,,\,S_J)$

The scalar products $(\Psi^I\,,\,S_J)$ have a fundamental importance in the theory of representation of the symmetric group : $(\Psi^I\,,\,S_J)$ is the value, denoted $\chi_J(I)$, of the irreducible character of the representation of index J at a permutation whose cycles are of successive lengths $i_1, i_2, \ldots$.

We know how to compute such scalar products by using the operators adjoint to multiplication by Ψ^k:

$$(\Psi^k\,\Psi^I\,,\,S_J) = (\Psi^I\,,\,D_{\Psi^i}(S_J)) = \Big(\Psi^I\,,\,\sum\nolimits_K S_{J-K}\Big) \tag{80}$$

sum over all vectors K which are permutations of $[0^{\ell(J)-1},\,k]$. Equivalently, the sum can be written as $\sum \pm S_H$, summation over all partitions such that J/H is a connected ribbon of length k, the sign being the height of the ribbon minus 1.

Iterating, one gets decompositions of the diagram of J into ribbons of successive lengths $i_1, i_2, \ldots$. Let us call such a decomposition an *even decomposition into ribbons of lengths I* if the product of signs is 1, and *odd decomposition* otherwise.

Decompositions into ribbons thus give a way to evaluate characters, and we state again the Murnaghan-Nakayama rule seen in Corollary **??** :

Proposition 36 *(Murnaghan-Nakayama rule). Given two partitions I, J of the same integer, the character $\chi_J(I) = (\Psi^I\,,\,S_J)$ is equal to the number of even decompositions of J into ribbons of lengths I, minus the number of odd decompositions.*

There are several ways to compute the *table of characters of* $\mathfrak{S}_n$, which, in ACE, can be accessed by two commands (giving transposed matrices)
$ACE >$ `SfMat(5,p,s), SgCharTable(5);`

$$
\begin{bmatrix}
1 & -1 & 0 & 1 & 0 & -1 & 1 \\
1 & 0 & -1 & 0 & 1 & 0 & -1 \\
1 & -1 & 1 & 0 & -1 & 1 & -1 \\
1 & 1 & -1 & 0 & -1 & 1 & 1 \\
1 & 0 & 1 & -2 & 1 & 0 & 1 \\
1 & 2 & 1 & 0 & -1 & -2 & -1 \\
1 & 4 & 5 & 6 & 5 & 4 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 0 & -1 & 1 & 0 & 2 & 4 \\
0 & -1 & 1 & -1 & 1 & 1 & 5 \\
1 & 0 & 0 & 0 & -2 & 0 & 6 \\
0 & 1 & -1 & -1 & 1 & -1 & 5 \\
-1 & 0 & 1 & 1 & 0 & -2 & 4 \\
1 & -1 & -1 & 1 & 1 & -1 & 1
\end{bmatrix}
$$

Because the basis Ψ^i is orthogonal, the table of characters is equal to the transpose of its inverse, multiplied by the diagonal matrix of scalar products (Ψ^I, Ψ^I).

$ACE >$ `aa:=diag(seq(SfZee(pa),pa=ListPart(4))):`

$ACE >$ `transpose(SfMat(4,p,s)),multiply(SfMat(4,s,p),aa);`

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
-1 & 0 & -1 & 1 & 3 \\
0 & -1 & 2 & 0 & 2 \\
1 & 0 & -1 & -1 & 3 \\
-1 & 1 & 1 & -1 & 1
\end{bmatrix}
\qquad
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
-1 & 0 & -1 & 1 & 3 \\
0 & -1 & 2 & 0 & 2 \\
1 & 0 & -1 & -1 & 3 \\
-1 & 1 & 1 & -1 & 1
\end{bmatrix}
$$

Moreover, the product of the Kostka matrix by the table of characters is triangular (expressing the characters of some "Yang elements" in the group algebra of $\mathfrak{S}_n$).

$ACE >$ `multiply(SfMat(7,h,s),SgCharTable(7));`

$$
\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&2&0&1&3&1&0&2&4&1&3&5&7\\
0&0&1&1&0&1&3&0&2&2&6&3&5&11&21\\
0&0&0&2&0&0&6&0&0&2&12&0&6&20&42\\
0&0&0&0&1&1&1&2&1&3&5&3&7&15&35\\
0&0&0&0&0&1&3&0&0&2&12&3&9&35&105\\
0&0&0&0&0&0&6&0&0&0&24&0&6&60&210\\
0&0&0&0&0&0&0&2&0&4&8&0&12&40&140\\
0&0&0&0&0&0&0&0&2&2&6&6&14&50&210\\
0&0&0&0&0&0&0&0&0&2&12&0&12&80&420\\
0&0&0&0&0&0&0&0&0&0&24&0&0&120&840\\
0&0&0&0&0&0&0&0&0&0&0&6&18&90&630\\
0&0&0&0&0&0&0&0&0&0&0&0&12&120&1260\\
0&0&0&0&0&0&0&0&0&0&0&0&0&120&2520\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&5040
\end{bmatrix}
$$

$ACE >$`SfMat(4,p,m),multiply(SfMat(4,h,p),diag(seq(SfZee(pa),pa=ListPart(4))));`

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 \\
1 & 2 & 2 & 2 & 0 \\
1 & 4 & 6 & 12 & 24
\end{bmatrix}
\qquad
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 2 & 4 \\
0 & 0 & 2 & 2 & 6 \\
0 & 0 & 0 & 2 & 12 \\
0 & 0 & 0 & 0 & 24
\end{bmatrix}
$$

Newton relations and Waring formula

Recall that

$$\log\left(\sigma_z(\mathbb{A})\right) = \sum\nolimits_{i=1}^{\infty} z^i \Psi^i(\mathbb{A})/i \ .$$

Derivating with respect to z, one gets

$$\sum\nolimits_{i=1}^{\infty} i z^i S^i(-\mathbb{A}) = \lambda_{-z}(\mathbb{A}) \sum\nolimits_{i=1}^{\infty} z^i \Psi^i(\mathbb{A}) \ , \tag{81}$$

which is equivalent to the following system of equations due to Newton (but see also Girard, "Invention Nouvelle en Algèbre", Amsterdam 1629).

$$\begin{array}{rcl} \Lambda^1 & = & \Psi_1 \\ 2\Lambda^2 & = & \Lambda^1\Psi_1 - \Psi_2 \\ 3\Lambda^3 & = & \Lambda^2\Psi_1 - \Lambda^1\Psi_2 + \Psi_3 \\ \cdots & & \cdots \\ n\Lambda^n & = & \Lambda^{n-1}\Psi_1 - \Lambda^{n-2}\Psi_2 + \cdots + (-1)^{n-1}\Psi_n \\ \cdots & & \cdots \end{array} \tag{82}$$

The image of Newton's relations under the involution $\mathbb{A} \to -\mathbb{A}$ is due to Brioschi :

$$\begin{array}{rcl} S^1 & = & \Psi_1 \\ 2S^2 & = & S^1\Psi_1 + \Psi_2 \\ \cdots & & \cdots \\ nS^n & = & S^{n-1}\Psi_1 + S^{n-2}\Psi_2 + \cdots + \Psi_n \,. \end{array} \tag{83}$$

We could have reasoned differently, using the fact that $(\Psi^0, \Psi^1, \Psi^2, \ldots)$ is a recursive sequence of characteristic polynomial $S^n(x-\mathbb{A})$ when $\mathbb{A}$ is of cardinality n. This gives in particular

$$\Psi^n(\mathbb{A}) - \Lambda^1(\mathbb{A})\,\Psi^{n-1}(\mathbb{A}) + \cdots \pm \Lambda^{n-1}(\mathbb{A})\,\Psi^1(\mathbb{A}) \mp \Lambda^n(\mathbb{A})\,n = 0 \ , \tag{84}$$

but this relation is valid for any $\mathbb{A}$ because its degree is not higher than the cardinality of the alphabet (there are no relations for $\Psi^1(\mathbb{A}), \ldots, \Psi^{n-1}(\mathbb{A})$, these power sums are initial conditions). Therefore one recovers Newton's relations from the fact that $(\Psi^i,\ i = 0, 1, 2\ldots)$ is a recursive sequence.

From Newton's and Brioschi's relations, one gets

$$\Psi_n = (-1)^{n-1} \begin{vmatrix} n\,\Lambda^n & \Lambda^1 & \ldots & \Lambda^n \\ (n{-}1)\Lambda^{n-1} & \Lambda^0 & \ldots & \Lambda^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0\,\Lambda^0 & . & \ldots & \Lambda^0 \end{vmatrix} = \begin{vmatrix} n\,S^n & S^1 & \ldots & S^n \\ (n{-}1)S^{n-1} & S^0 & \ldots & S^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0\,S_0 & . & \ldots & S^0 \end{vmatrix} .$$

Conversely,

$$n!\,\Lambda^n = \begin{vmatrix} \Psi_1 & 1 & 0 & \dots & 0 \\ \Psi_2 & \Psi_1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Psi_{n-1} & \Psi_{n-2} & \Psi_{n-3} & \dots & n{-}1 \\ \Psi_n & \Psi_{n-1} & \Psi_{n-2} & \dots & \Psi_1 \end{vmatrix}, \; n!\,S^n = \begin{vmatrix} \Psi_1 & -1 & 0 & \dots & 0 \\ \Psi_2 & \Psi_1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Psi_{n-1} & \Psi_{n-2} & \Psi_{n-3} & \dots & 1{-}n \\ \Psi_n & \Psi_{n-1} & \Psi_{n-2} & \dots & \Psi_1 \end{vmatrix}.$$

Taking the Laplace expansion along the first column of the preceding determinants, one gets

$$\Psi_n = S^{n-1}\,\Lambda^1 - 2S^{n-2}\,\Lambda^2 + \cdots + (-1)^{n-1} n S^0\,\Lambda^n\ , \tag{85}$$

which, using Pieri's formula (56), gives

$$\Psi_n = S_n - S_{1,n-1} + S_{1,1,n-2} - S_{1,1,1,n-3} + \cdots\ . \tag{86}$$

Because $S^n(\mathbb{A}-\mathbb{A})=0$, $n>0$, one can shift by any integer $k\in\mathbb{Z}$ relation (87) :

$$\Psi_n = (n{+}k)\,S^n - (n{+}k{-}1)S^{n-1}\,\Lambda^1 + (n{+}k{-}2)S^{n-2}\,\Lambda^2 + \cdots + (-1)^n k S^0\,\Lambda^n\ . \tag{87}$$

Expanding the determinant expressing Ψ_i/i in terms of the complete functions, one gets

$$\begin{aligned} \Psi_2/2 &= S^2 - (1!/2!)S^{11}, \\ \Psi_3/3 &= S^3 - S^{12} + (2!/3!)S^{111}, \\ \Psi_4/4 &= S^4 - S^{13} - (1/2!)S^{22} + (2!/2!)S^{112} - (3!/4!)S^{1111}\,. \end{aligned}$$

and, in general, for $|I|=n$, writing partitions exponentially, the coefficients are :

$$(\Psi_I\,,\,\Psi_n/n) = (-1)^{l(I)-1}\frac{(l(I)-1)!}{m_1!\,m_2!\,\dots}\ . \tag{88}$$

These scalar products can in fact be obtained, through the involution $\mathbb{A}\to -\mathbb{A}$, from the celebrated *Waring formula* [52]

$$\Psi_k/k = \sum_{I:\,|I|=k} (-1)^{l(I)+k}\frac{(l(I)-1)!}{m_1!\,m_2!\,\cdots}\Lambda^I\ . \tag{89}$$

Monomial to Power sums : $(\psi_J\,,\,\Psi^I)$

To determine the expansion of monomial functions in the basis of power sums, it is simpler to normalize them differently. Let $n \in \mathbb{N}$,
$J = 0^{m_0} 1^{m_1} 2^{m_2} \ldots \in \mathbb{N}^n$. Then

$$\Phi_J := m_0!\, m_1! \cdots \Psi_J \tag{90}$$

is the *augmented monomial function* of index J.

When $\mathbb{A}$ is of cardinality n, then $\Phi_J(\mathbb{A}) = \sum_{\sigma \in \mathfrak{S}_n} u^\sigma$, where u is any monomial in the expansion of $\Psi_J(\mathbb{A})$.

Let us identify Φ_J and Φ_I if I is a permutation of J. Then, for any $k \geq 1$, any $\mathbb{A} :\ card(\mathbb{A}) = n$, one has

$$\Psi_k(\mathbb{A})\, \Phi_J(\mathbb{A}) = \sum_{i=1}^{n} \Phi_{J+[0^{i-1} k\, 0^{n-i}]}(\mathbb{A}) \ . \tag{91}$$

These equations can be inverted and give the expansion of Φ_J in the basis Ψ^I. One first need to introduce the *lattice of set-partitions* $\mathcal{P}art(n)$ of the set $\{1, \ldots, n\}$ We shall avoid using the word "partition" and say that $\nu = \{\nu_1, \ldots, \nu_r\}$ is a *decomposition* of $\{1, \ldots, n\}$ iff $\nu_1 \cup \ldots \cup \nu_r$ is a decomposition into disjoints subsets. Remember that we used the notation $\mathfrak{P}art(n)$ for the set of partitions of the integer n.

The Möbius function of $\mathcal{P}art(n)$ has been computed by many authors: Faa de Bruno[4] [9], p.9, M.P. Schützenberger [48], G.C. Rota [45]; it can also be extracted from E. West [53].

$$\mu(\nu) = \prod_{i=1}^{r} (-1)^{card(\nu_i)-1} \big(card(\nu_i) - 1\big)! \tag{92}$$

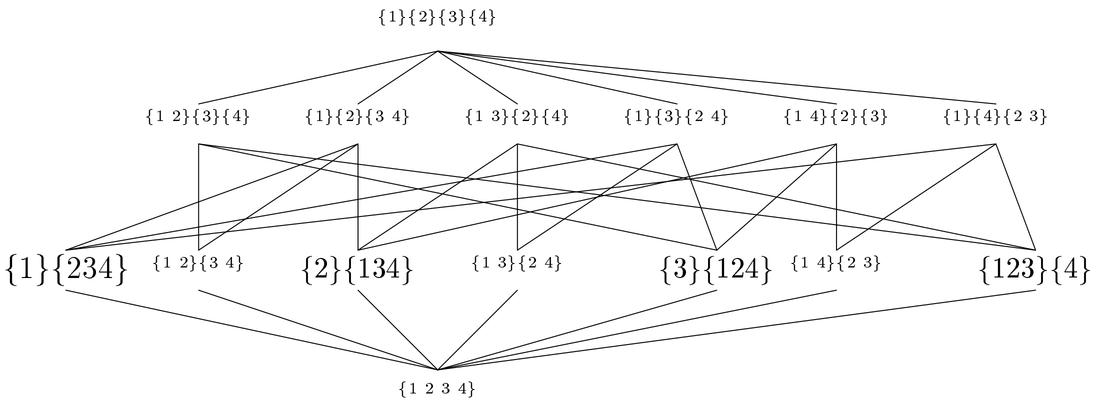
To any decomposition ν, any $J \in \mathbb{N}^n$, let us associate the vector

$$\nu(J) = \left[\sum_{i \in \nu_1} j_i,\ \ldots,\ \sum_{i \in \nu_r} j_i \right] \ .$$

Equations (91) imply :

$$\Phi_J = \sum_{\nu \in \mathcal{P}art(n)} \mu(\nu)\, \Psi^{\nu(J)} \ . \tag{93}$$

[4]only mathematician which was made "beatus" by the Pope. Elimination theory can lead to Paradise.



For example, for $n = 4$, writing $1, 2, 3, 4$ instead of j_1, j_2, j_3, j_4, one reads from the preceding figure the expansion

$$\begin{aligned}\Phi_{1234} = {}& \Psi^1\Psi^2\Psi^3\Psi^4 - \Psi^{1+2}\Psi^3\Psi^4 - \Psi^{1+3}\Psi^2\Psi^4 - \Psi^{1+4}\Psi^2\Psi^3 - \Psi^1\Psi^{2+3}\Psi^3\Psi^4 \\ & - \Psi^1\Psi^{2+4}\Psi^3 - \Psi^1\Psi^2\Psi^{3+4} + \Psi^{1+2}\Psi^{3+4} + \Psi^{1+3}\Psi^{2+4} + \Psi^{1+4}\Psi^{2+3} \\ & + 2\Psi^{1+2+3}\Psi^4 + 2\Psi^{1+2+4}\Psi^3 + 2\Psi^{1+3+4}\Psi^2 + 2\Psi^1\Psi^{2+3+4} - 6\Psi^{1+2+3+4}\end{aligned}$$

(vertices where Möbius function takes value 2 use a bigger font).

One can in fact recover the value of the Möbius function by evaluating the number (that we shall note $\Phi_J(N)$) of monomials in $\Phi_J(\mathbb{A})$, with $N = card(\mathbb{A})$. Indeed, (93) becomes

$$\Phi_J(N) = N(N-1)\cdots(N-n+1) = \sum_{\nu} \mu(\nu)\, N^{\ell(\nu)} \ .$$

It proves that

$$\mu(\{1, \ldots, n\}) = (-1)\cdots(-n+1) \ ,$$

from which one deduces all values of the Möbius function, because the interval between ν and $\{1, \ldots, n\}$ is a direct product of intervals for smaller n if $\ell(\nu) > 1$.

The summands in (93) for a given J are not necessarily distinct, and the final expression of the coefficients (Ψ_J, Ψ^I) is given by the following lemma.

Lemma 37 *Let $J = 1^{m_1}\cdots p^{m_p}$, $\ell(J) = n$, and I be a partition. Then*

$$(\Psi_J\,,\ \Psi^I) = \frac{(\Psi^I, \Psi^I)}{m_1!\cdots m_p!} \sum_{\nu} \mu(\nu) \ , \tag{94}$$

sum over all $\nu \in \mathcal{P}art(n)$ such that $\nu(J)$ is a permutation of I.

For example, $(\Psi_{1112223}, \Psi^{444}) = \frac{4^3 3!}{3!3!} 9 times 2 = 192$, because $J = 1^3 2^3 3^1$, $n = 7$, $(\Psi^{444}, \Psi^{444}) = 4^3 3!$, and there are 9 decompositions of the type (1+1+2, 2+2, 1+3), the Möbius function taking value 2 in each of these decompositions.

Exercises

Ex. 35 Given n, compute the determinant of the matrix ma such that $ma[i,j] = \Psi_{i+1-j}$, $1 \le j \le i \le n-1$, $ma[i,i+1] = i$, $1 \le i < n$, $ma[n,j] = x^{n-j}$, $1 \le j \le n$, the other entries being zero.

Corr.Ex.35. This is the determinant expressing $n!\,\Lambda^n$, except for the last row. Expanding along this last row, putting an alphabet $\mathbb{A}$ for clarity (that is, $\Psi^i = \Psi^i(\mathbb{A})$, for all i, one finds $(n-1)!\Lambda^{n-1}(\mathbb{A} - x)$.

```
ModifMat_e2p:=proc(n) local i,j;
matrix([ seq([ seq(p.(i+1-j),j=1..i), i, 0$(n-i-1)],i=1..n-1),
[seq(x∧(n-j),j=1..n)] ])
end:
```

$ACE >$ `ma:= ModifMat_e2p(4);`

$$\begin{bmatrix} p1 & 1 & 0 & 0 \\ p2 & p1 & 2 & 0 \\ p3 & p2 & p1 & 3 \\ x^3 & x^2 & x & 1 \end{bmatrix}$$

$ACE >$ `Toe(det(ma), collect);`

$$6 \text{ e3} - 6 \text{ x e2} + 6 \text{ x}^2 \text{ e1} - 6 \text{ x}^3$$

Ex. 36 Compute the following determinant of inverse of factorials, with extra parameters in the first column:

$$\begin{vmatrix} x_4/4! & 1/3! & 1/2! & 1/1! \\ x_3/3! & 1/2! & 1/1! & 1/0! \\ x_2/2! & 1/1! & 1/0! & 0 \\ x_1/1! & 1/0! & 0 & 0 \end{vmatrix}$$

Generalize it to any order.

Corr.Ex.36. The determinant can be interpreted as

$$\Lambda_{3;\,000}(\mathbb{A};\,\mathbb{B}) = \Lambda_3(\mathbb{A}-\mathbb{B}),$$

with $\Lambda^i(\mathbb{A}) = x_i/i!$, $\Lambda^i(\mathbb{B}) = 1/i!$ (forcing us to take $x_1 = 1$, but it is easy to reintroduce a general x_1 by homogeneity). Its generalization to any order is now evident.

In the expansion of $\Lambda_n(\mathbb{A}-\mathbb{B})$, the coefficients of the $\Lambda^k(\mathbb{A})$ are, up to signs, the $S^j(\mathbb{B}) = 1/j!$

Ex. 37 Given two positive integers n, k and two partitions $I, J \in \mathbb{N}^n$, with $|J| = |I|+k$, show that the coefficient $c^J_{k,I}$ in the product $S^k\,\Psi_I = \sum_J c^J_{k,I}\,S_J$ is equal to the number of permutations H of I which are (componentwise) majorized by J.

Use this property to express monomial functions in the basis of products of complete functions.

Corr.Ex.37. (A. Kohnert).

```
IsMajorized:=proc(small,big) local i;
for i from 1 to nops(small) do if small[i]>big[i] then RETURN(false)
fi; od; RETURN(true)
end:
CoeffProdCompleteMonomial:=proc(input,output) local lp:
if nops(input)>nops(output) then RETURN(0) fi;
lp:=ListPerm([op(input),0$(nops(output)-nops(input))]);
select(IsMajorized, lp,output)
end:
ACE > aa:= CoeffProdCompleteMonomial([4,2],[4,4,2,1]);
   aa :=[[4, 2, 0, 0], [4, 0, 2, 0], [2, 4, 0, 0], [0, 4, 2, 0]]
ACE > coeff(Tom( h5*m[4,2]), m[4,4,2,1]);
                                       4
Of course, an efficient program will not enumerate all permutations!
```

Ex. 38 Let $\mathbb{A}$ be of cardinality n. Compute the symmetric function

$$\prod_{i=1}^{n}(a_1 + \cdots + a_{i-1} - a_i + a_{i+1} + \cdots + a_n)$$

in your favourite basis.

Corr.Ex.38. The product is $\prod_a (\Lambda^1 - 2a)$, and therefore, expands in the $\Lambda^I(\mathbb{A})$ basis as

$$(\Lambda^1)^n - 2\Lambda^1\,(\Lambda^1)^{n-1} + 4\Lambda^2\,(\Lambda^1)^{n-1} - \cdots \pm 2^n \Lambda^n\,(\Lambda^1)^0$$

(only hook partitions appear, with coefficients $\pm$ a power of 2).

Ex. 39 Let $\mathbb{A} = \{a_1, \ldots, a_n\}$, and let $f \in \mathfrak{S}ym(\mathbb{A})$ be of degree $\leq n$. Show that the determinant

$$\left| 1\,,\ a_i\,,\ \ldots\ a_i^{n-2}\,,\ \frac{d}{da_i}(f) \right|_{1\leq i\leq n}$$

is equal to $(f\,,\,\Psi^n)\,\Delta(\mathbb{A})$.

Corr.Ex.39. The formula being linear in f, it is sufficient to test it on the basis $\Psi^I(\mathbb{A})$, $|I| \leq n$. The determinant is clearly null, except when $I = [n]$, and so does the scalar product.

Ex. 40 Let $\mathbb{A} = \{a, b, c, d\}$. Show the equality

$$\begin{vmatrix} 1 & S^1(\mathbb{A}) & S^2(\mathbb{A}) & S^3(\mathbb{A}) & S^4(\mathbb{A}) \\ 0 & S^0(\mathbb{A}) & S^1(\mathbb{A}) & S^2(\mathbb{A}) & S^3(\mathbb{A}) \\ -1 & 0 & S^0(\mathbb{A}) & S^1(\mathbb{A}) & S^2(\mathbb{A}) \\ -2 & 0 & 0 & S^0(\mathbb{A}) & S^1(\mathbb{A}) \\ -3 & 0 & 0 & 0 & S^0(\mathbb{A}) \end{vmatrix} = \begin{vmatrix} 1 & b & c & d \\ a & 1 & c & d \\ a & b & 1 & d \\ a & b & c & 1 \end{vmatrix}$$

Corr.Ex.40. Let us evaluate separately both determinants. Replacing the first column of the left one by $z^1, z^0, \ldots, z^{-3}$, and taling its derivative in $z = 1$, one gets

$$1 - \Lambda^2(\mathbb{A}) + 2\Lambda^3(\mathbb{A}) - 3\Lambda^3(\mathbb{A})\ .$$

On the other hand, dividing the successive columns of the other determinant by a, b, c, d successively, one gets a determinant filled with 1's, except for the diagonal equal to $[1/a,\ 1/b,\ 1/c,\ 1/d]$. Therefore, it is equal to

$$abcd(a^{-1}-1)(b^{-1}-1)(c^{-1}-1)(d^{-1}-1)\left(1 + \frac{1}{a^{-1}-1} + \frac{1}{b^{-1}-1} + \frac{1}{c^{-1}-1} + \frac{1}{d^{-1}-1}\right)$$
$$= \Big(1 - \Lambda^1(\mathbb{A}) + \Lambda^2(\mathbb{A}) - \Lambda^3(\mathbb{A}) + \Lambda^4(\mathbb{A})\Big)\Big(1 + \sum_{i\geq 1} \Psi^i(\mathbb{A})\Big)$$

and one can conclude using Newton's formula.

Of course, a similar result is true for any cardinality.

Ex. 41 Let $f \in \mathfrak{S}ym(n)$ be homogenous of degree $p \leq n$. Let $\mathbb{A}$ be the set of roots of the polynomial $x^n - x^{n-1} + x^{n-2} - \cdots + (-1)^n$. Show that the scalar product (f, Λ^p) is equal to $f(\mathbb{A})$.

Corr.Ex.41. (Muir (1899), MuirIV, p.198). Since $f = \sum(f, \Lambda_I)\Lambda_I$, one has to show that

$$\forall I : |I| = p,\ \Lambda_I(\mathbb{A}) = 0\ ,\ \text{except } \Lambda_p(\mathbb{A}) = 1\ .$$

Indeed, if $\ell(I) > 1$ and $|I| \leq n$, then the determinant $\Lambda_I(\mathbb{A})$ has its last two columns identical and vanishes. Moreover, all $\Lambda_p(\mathbb{A})$, $0 \leq p \leq n$ are equal to 1.

Ex. 42 Show that

$$n!\, S^n = \begin{vmatrix} 2\Lambda^1 & \Lambda^0 & 0 & \cdots & 0 \\ 4\Lambda^2 & 3\Lambda^1 & 2\Lambda^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (2n-2)\Lambda^{n-1} & (2n-3)\Lambda^{n-2} & (2n-4)\Lambda^{n-3} & \cdots & (n-1)\Lambda^0 \\ n\Lambda^n & (2n-2)\Lambda^{n-1} & (n-2)\Lambda^{n-2} & \cdots & \Lambda^1 \end{vmatrix}$$

(only the last row is irregular).

Corr.Ex.42. (Mangeot (1897), MuirIV, p.239). Thanks to Newton's relations, the above determinant is the product of $\begin{vmatrix} \Psi^1 & 1 & 0 & 0 & \cdots \\ -\Psi^2 & \Psi^1 & 2 & 0 & \cdots \\ \Psi^3 & -\Psi^2 & \Psi^1 & 3 & \cdots \\ \vdots & \vdots & \vdots & & \end{vmatrix}$ by the matrix Λ^{j-i} of determinant equal to 1.

Ex. 43 Express the term of degree n in the series $f := (1 - \Lambda^2 - 2\Lambda^3 - 3\Lambda^4 - \cdots)^{-1}$ as a determinant in the elementary symmetric functions, then as a determinant in the power sums.

Corr.Ex.43. The first development is obtained, from the determinantal expression of S_n in terms of the Λ^i, through the transformation $\Lambda^1 \to 0$, $\Lambda^2 \to -1\Lambda^2$, $\Lambda^3 \to -2\Lambda^3$, &c. It is therefore equal to

$$\begin{vmatrix} 0 & -\Lambda^2 & 2\Lambda^3 & -3\Lambda^4 & \cdots \\ 1 & 0 & -\Lambda^2 & 2\Lambda^3 & \cdots \\ 0 & 1 & 0 & -\Lambda^2 & \cdots \\ \vdots & & \ddots & \ddots & \end{vmatrix} .$$

Since the denominator of f is the difference $(1+\Lambda^1+\Lambda^2+\Lambda^3+\cdots)-(\Lambda^1+2\Lambda^2+3\Lambda^3+\cdots)$, one can transform it using Newton's relations, and write it as

$$(1+\Lambda^1+\Lambda^2+\Lambda^3+\cdots)\,(1-\Psi^1+\Psi^2-\Psi^3+\cdots)\ .$$

Therefore

$$f = (1-S^1+S^2-S^3+\cdots)\,(1-\Psi^1+\Psi^2-\Psi^3+\cdots)^{-1}\ ,$$

and the term of degree n of f is equal to

$$\begin{vmatrix} \Psi^1 & \Psi^2 & \cdots & \Psi^n & S^n \\ 1 & \Psi^1 & \cdots & \Psi^{n-1} & S^{n-1} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & & & 1 & S^0 \end{vmatrix} .$$

One transforms the last column with the help of Brioschi relations (83). Defining

$$\Phi_1 = 0,\ \Phi_2 = \Psi^1\Psi^1, \ldots, \Phi_j = \Psi^{j-1}\Psi^1 + \Psi^{j-2}\Psi^2 + \cdots + \Psi^1\Psi^{j-1}, \ldots$$

on can finally write $n!$ times the determinant as follows :

$$\begin{vmatrix} (n-1)\Psi^1 & (n-1)\Psi^2-\Phi_2 & (n-1)\Psi^3-\Phi_3 & \cdots & (n-1)\Psi^n-\Phi_n \\ (n-1) & (n-2)\Psi^1 & (n-2)\Psi^2-\Phi_2 & \cdots & (n-2)\Psi^{n-1}-\Phi_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \end{vmatrix} ,$$

and this gives the expression of the term of degree n of f in the basis of power sums.

For example, for $n=4$, one has

$$\Lambda^{22}+3\Lambda^4 = \begin{vmatrix} 0 & -\Lambda^2 & 2\Lambda^3 & -3\Lambda^4 \\ 1 & 0 & -\Lambda^2 & 2\Lambda^3 \\ 0 & 1 & 0 & -\Lambda^2 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \frac{1}{4!}\begin{vmatrix} 3\Psi^1 & 3\Psi^2-\Phi_2 & 3\Psi^3-\Phi_3 & 3\Psi^4-\Phi_4 \\ 3 & 2\Psi^1 & 2\Psi^2-\Phi_2 & 2\Psi^3-\Phi_3 \\ 0 & 2 & \Psi^1 & \Psi^2-\Phi_2 \\ 0 & 0 & 1 & 0 \end{vmatrix} .$$

Symmetric Functions as Operators and λ-Rings

We have used algebraic operations on alphabets, like addition, subtraction and multiplication of two alphabets, without expliciting the underlying algebraic structure. We shall show in this section that we have in fact used λ-rings, a structure due to Alexandre Grothendieck, without knowing it!

Already we met an intepretation of symmetric functions, as *operators* on isobaric determinants, but λ-rings are a more powerful way of considering symmetric functions as "functors".

Let us first consider the product of an alphabet by a positive integer. This, we defined by taking powers of generating functions :

$$\sigma_z(k\,\mathbb{A}) := (\sigma_z(\mathbb{A}))^k \ .$$

However, it can be given a more concrete interpretation. In the simplest case, passing from $\sigma_z(\mathbb{A})$ to $\sigma_z(2\mathbb{A}) := (\sigma_z(\mathbb{A}))^2$ can be interpreted as doubling the alphabet $\mathbb{A} = \{a\} \mapsto 2\mathbb{A} := \{a'\} \cup \{a''\}$, computing the symmetric functions in $2\mathbb{A}$, and eventually erasing the diacritics on the letters a's. Similarly, one introduces, for each letter $a \in \mathbb{A}$ other letters a', $a''\ a'''$, ..., $a^{(k)}$, compute the symmetric functions in this new alphabet, and erase the diacritics, obtaining functions of $\mathbb{A}$ again.

However, the inverse operations, say $\mathbb{A} = \{a\} \mapsto \frac{1}{2}\mathbb{A}$, cannot be described in the same way, which means that one must not stick to considering alphabets as composed of letters.

More explicitely, at the level of power sums, multiplication by 2 is just

$$\Psi_i(2\mathbb{A}) = \sum (a')^i + \sum (a'')^i = 2\,\Psi_i(\mathbb{A}) \ .$$

Therefore the "alphabet" $\mathbb{B} = k\,\mathbb{A}$, $k \in \mathbb{C}$ will be defined by

$$\Psi_i(\mathbb{B}) := k\,\Psi_i(\mathbb{A}) \ , \ i \geq 1 \ , \tag{95}$$

and now, k needs no more be an integer, we can take $k = 1/2$ as we desired.

Instead of only multiplying alphabets by constants, one can go one step further and realize that every polynomial can play the rôle of an alphabet, i.e. that *the* Ψ_i *are operators on polynomials* ($\alpha \in \mathbb{C}$, u monom):

$$P = \sum_{\alpha,u} \alpha\, u \Rightarrow \Psi_i(P) = \sum_{\alpha,u} \alpha\, u^i \ . \tag{96}$$

The ring $\mathfrak{S}ym$, being generated by the Ψ_i, $i = 1, 2, \ldots$, formulas (96) extend to an action of $\mathfrak{S}ym$ on the ring of polynomials.

Thus, the generating functions of elementary and symmetric functions become operators, their explicit action (determined by $\sigma_z = 1/\lambda_{-z}$ $= \exp\left(\sum_{i\geq 1} z^i \Psi^i / i\right)$) being

$$P = \sum_{\alpha,u} \alpha\, u \mapsto \lambda_z(P) = \prod (1 + zu)^\alpha \ , \tag{97}$$

$$P = \sum_{\alpha,u} \alpha\, u \mapsto \sigma_z(P) = \prod (1 - zu)^{-\alpha} \ . \tag{98}$$

Each of formulas (96), (97) or (98) can be chosen at will to define the action of symmetric polynomials and implies the two others, and the ring of polynomials, with $\mathfrak{S}ym$ operating on it, is called a *λ-ring*.

Grothendieck choosed *lambda operations*, that is, the *exterior powers* of vector spaces, or more generally, fiber bundles, to introduce λ-rings. In the same interpretation, the S^i are the symmetric powers. Algebraic topologists prefers the *Adams operations* Ψ_i.

Having taken polynomials as our building blocks, rather than more sophisticated mathematical entities, we needed only the single axiom (96). The general theory of λ-rings require three axioms: the compatibility of the λ-operations with addition, product and composition.

Let us remark the different rôle played by constants α and monomials u :

$$\begin{cases} \Psi_i(\alpha) = \alpha \ , & S^i(\alpha) = \binom{\alpha+i-1}{i} \ , \qquad \Lambda^i(\alpha) = \binom{\alpha}{i} \\ \Psi_i(u) = u^i \ , & S^i(u) = u^i \ , \qquad \Lambda^i(u) = 0, \ i > 1, \quad \Lambda^1(u) = u \end{cases} \tag{99}$$

When implementing λ-rings, one must distinguish between indeterminate coefficients α and variables u. The preceding terminology is not satisfactory; it is preferable, instead of using the term "monomial" to say *element of rank 1* (i.e. non zero element x such that $\Lambda^i(x) = 0 \ \forall i > 1$), and avoid the term "constant" for the elements invariant under the Ψ_i, but rather say *binomial element* as a tribute to Rota, because these elements are such that their images $\Lambda^i(\alpha) = \binom{\alpha}{i}$ are binomial coefficients.

Because $\Psi_i(1) = 1 = S^i(1)$, one can specialize rank 1-elements to 1 (the only other specialization compatible with λ-rings is specialization to 0). It has the consequence that, for a positive integer n, $\Psi_I(n)$, $S^i(n)$, $\Lambda^i(n)$ are the number of terms in the expansion, in terms of monomials, of $\Psi_I(\mathbb{A})$, $S^i(\mathbb{A})$, $\Lambda^i(\mathbb{A})$ when $card(\mathbb{A}) = n$. The identity

$$\Psi_I(n) = \frac{n(n-1)\cdots(n-\ell(I)+1)}{m_1!\, m_2!\, m_3! \cdots} \quad \text{with } I = 1^{m_1} 2^{m_2} 3^{m_3} \cdots \ , \tag{100}$$

being true for any positive integer n, is true for any complex number $n \in \mathbb{C}$.

Polynomials in one indeterminate are conveniently coded in a λ-ring. Let $\mathbb{A}$ be an alphabet of cardinality n. Then

$$\prod_{a\in\mathbb{A}}(x-a) = S^n(x-\mathbb{A}) = \sum x^{n-i}S^i(-\mathbb{A})$$

Now, expanding $S^{n+k}(x-\mathbb{A})$, $k \in \mathbb{N}$, one sees that

$$S^{n+k}(x-\mathbb{A}) = x^k\, S^n(x-\mathbb{A}) \ . \tag{101}$$

On the other hand, $S^{n-k}(x-\mathbb{A})$ is the component of positive degree of the Laurent polynomial

$$x^{-k}\, S^n(x\text{-}\mathbb{A}) = x^{-k}S^n(\text{-}\mathbb{A})+\cdots+x^{-1}S^{n-k+1}(\text{-}\mathbb{A})+x^0 S^{n-k}(\text{-}\mathbb{A})+\cdots+x^{n-k}S^0(\text{-}\mathbb{A}) \ .$$

Derivating with respect to x the generating function $\sigma_z(x\mathbb{A})$ of the $S^k(x\text{-}\mathbb{A})$, one sees that for any $k \in \mathbb{N}$, the successive derivatives of $S^k(x\text{-}\mathbb{A})$ are

$$S^{k-1}(2x\text{-}\mathbb{A})\,,\ 2!\, S^{k-2}(3x\text{-}\mathbb{A})\,,\ 3!\, S^{k-3}(4x\text{-}\mathbb{A})\,, \ldots, j!\, S^{k-j}((j+1)x\text{-}\mathbb{A}), \ldots$$

There are other codings. For example, given any $\mathbb{A}$, any $n \in \mathbb{N}$, and an element x of rank 1,

$$\Lambda^n(nx+\mathbb{A}) = \sum \binom{n}{i} x^{n-i}\, \Lambda^i(\mathbb{A}) \tag{102}$$

is a polynomial of degree n.

Derivating with respect to x the generating function $\lambda_z(nx+\mathbb{A}) = (1+zx)^n\lambda_z(\mathbb{A})$, one sees that the successive derivatives are now

$$n\,\Lambda^{n-1}((n\text{-}1)x+\mathbb{A})\,,\ n(n-1)\,\Lambda^{n-2}((n\text{-}2)x+\mathbb{A})\,,\ \ldots$$

Therefore, the integral of $\Lambda^n(nx+\mathbb{A})$ is $\frac{1}{n+1}\Lambda^{n+1}((n\text{+}1)x+\mathbb{A})$, up to the addition of a constant, and more generally, the k-th integral will be

$$\frac{1}{(n+1)\cdots(n+k)}\Lambda^{n+k}((n\text{+}k)x+\mathbb{A}) \ .$$

We shall see in another section that orthogonal polynomials are also conveniently expressed in λ-rings.

One can extends the action of symmetric polynomials to rational functions or formal power series :

$$\Psi_i\left(\frac{\sum \alpha u}{\sum \beta v}\right) = \frac{\sum \alpha u^i}{\sum \beta v^i}\ , \tag{103}$$

If q is of rank 1, then one has now $\frac{1}{1-q} = 1 + q + q^2 + \cdots$, i.e. $\frac{1}{1-q}$ is the infinite alphabet $\{1, q, q^2, \ldots\}$ and $\Psi_i(\frac{1}{1-q}) = \frac{1}{1-q^i}$, $i \geq 1$. From

$$\sigma_x\left(\frac{1}{1-q}\right) = \exp\left(\sum_{i=1}^{\infty} \frac{x^i}{i\,(1-q^i)}\right) ,$$

Cauchy obtained that :

$$S_i\left(\frac{1}{1-q}\right) = \frac{1}{(1-q)\cdots(1-q^i)} . \tag{104}$$

(we shall more generally determine in equation (126) the values of all Schur functions on $1/(1-q)$).

Therefore

$$\sigma_x\left(\frac{1}{1-q}\right) = \sum \frac{x^i}{(1-q)\cdots(1-q^i)} \tag{105}$$

is the q-*exponential*, equality $\frac{1}{1-q} = 1+q+q^2+\cdots$ implying the factorization

$$\sigma_x\left(\frac{1}{1-q}\right) = \prod_{i=1}^{\infty} \frac{1}{1-xq^i} \tag{106}$$

Such a factorization renders the q-exponential sometimes easier to use than the classical exponential.

Some classical identities on q-series come from addition in λ-rings. For example, take the q-*binomial identity*

$$\sum_{i \geq 0} z^i \frac{(1-a)(1-aq)\cdots(1-aq^{i-1})}{(1-q)(1-q^2)\cdots(1-q^i)} = \prod_{i \geq 0} \frac{1-zaq^i}{1-zq^i} . \tag{107}$$

Considering z, a, q to be of rank 1, one recognizes the left hand side to be $\sigma_1(z\frac{1-a}{1-q})$ (we shall in (127) determine more generally all Schur functions of $(1-a)/(1-q)$), and the right one to be $\sigma_1(\frac{z}{1-q})\,\sigma_1(\frac{-za}{1-q})$. Therefore, the identity does not need a proof, because it is just

$$z\,\frac{1-a}{1-q} = \frac{z}{1-q} + \frac{-za}{1-q} . \tag{108}$$

Lagrange inversion

Let us illustrate the advantages of λ-rings on another very classical exemple, the Lagrange inversion of formal series in one variable. Given $f(z) = z + f_1 z^2 + f_2 z^3 + \cdots$, one has to find $g(z) = z + g_1 z^2 + g_2 z^3 + \cdots$ such that

$$f(g(z)) = z \quad \& \quad g(f(z)) = z \ .$$

Lagrange inversion has many applications in classical analysis and combinatorics. Many extensions (q-generalizations, or multivariate extensions, &c.) have been proposed. We shall refer in particular to Gessel [13].

Lagrange and Bürman solved the original problem by expressing the coefficients of g in terms of coefficients of derivatives of powers of f.

Another powerful approach is due to Jabotinsky [18] who associates to f a matrix $\mathfrak{Jab}(f)$, whose entries are all the coefficients of all the integral powers of f. Now, composition or inversion of series becomes multiplication or inversion of the associated Jabotinsky's matrices.

Lagrange's solution uses only the fact that the logarithmic derivative of a series has no residue. More generally, one has the following lemma :

Lemma 38 *Given an alphabet $\mathbb{A}$, let $f(z) := z\sigma_{-z}(\mathbb{A})$. Then for any $m, n \in \mathbb{Z}$, the residue of $f^{-n}\frac{d}{dz}(f^m)$ is null if $m \neq n$, and equal to n if $m = n$.*

Equivalently, one has the relations

$$\sum\nolimits_{k\in\mathbb{Z}} k\, S^{n-k}(-n\,\mathbb{A})\, S^{k-m}(m\,\mathbb{A}) = n\delta_{m,n} \ . \tag{109}$$

Proof. Because $f^m = z^m\sigma_{-z}(m\mathbb{A})$ and $f^{-n} = z^{-n}\sigma_{-z}(-n\mathbb{A})$, the two statements are equivalent.

The residue to determine is equal to the coefficient of z^{n-1} in $\sigma_{-z}(-n\mathbb{A})\,\frac{d}{dz}\left(z^m\sigma_{-z}(m\mathbb{A})\right)$. If $m \neq n$, it is the coefficient of z^{n-1} in

$$\frac{m}{m-n}\frac{d}{dz}\left(z^m\sigma_{-z}((m-n)\mathbb{A})\right) + \left(m - \frac{m^2}{m-n}\right) z^{m-1}\sigma_{-z}((m-n)\mathbb{A}) \ ,$$

which is

$$(-1)^{n-m}\frac{m}{m-n} nS^{n-m}((m-n)\mathbb{A}) + \left(m - \frac{m^2}{m-n}\right)(-1)^{n-m}S^{n-1-m+1}((m-n)\mathbb{A}) \ .$$

It is indeed null; the case $m = n$ comes from a similar computation. QED

Equations (109) determine the coefficients of the powers of the series inverse to f :

Theorem 39 (Lagrange-Bürman) *Let $f(z) = z\sigma_{-z}(\mathbb{A})$, and $g(z)$ its inverse series. Then for any $k \neq 0$, one has*

$$g^k(z) = k\, z^k \sum \frac{z^i}{i+k} \Lambda^i\big((i+k)\mathbb{A}\big) \ , \tag{110}$$

$$\log\big(g(z)\big) = \sum \frac{z^i}{i} \Lambda^i\big(i\,\mathbb{A}\big) \ . \tag{111}$$

Proof. Writing $\big(g(f(z)\big)^k = z^k$, one sees that the expressions given by the theorem satisfy the right recursions thanks to relations (109). One gets the logarithm as the limit $k \to 0$ of g^k/k. QED

Since Lagrange's expression involves taking elementary symmetric functions in the product of two alphabets ($(i{+}k)$ and $\mathbb{A}$), then the Cauchy formula gives an expansion of it in any pair of adjoint bases of symmetric functions, without further computations. In particular, one has

$$\begin{array}{rclr}
\Lambda^i((i+k)\mathbb{A}) & = & (-1)^i \sum_{|J|=i} \Psi_J(-(i+k))S^J(\mathbb{A}) & (a)\\
& = & \sum_{|J|=i} \Psi_J(i+k)\Lambda^J(\mathbb{A}) & (b)\\
& = & \sum_{|J|=i} S_J(i+k)S_{J^\sim}(\mathbb{A}) & (c)\\
& = & \sum_{|J|=i} \Lambda^J(i+k)\Psi_J(\mathbb{A}) & (d)\\
& = & \sum_{|J|=i} S^J(i+k)F_J(\mathbb{A}) & (e)\\
& = & (-1)^i \sum_{|J|=i} \frac{(-i-k)^{\ell(J)}}{m_1!m_2!\cdots}(\frac{\Psi_1}{1})^{m_1}(\frac{\Psi_2}{2})^{m_2}(\frac{\Psi_3}{3})^{m_3}\cdots & (f)
\end{array} \tag{112}$$

using forgotten symmetric functions F_J in equation (e) and using exponential notations for partitions in the last one. It is interesting to note that all these equations (except the one involving forgotten functions) can be found in the mathematical litterature, each time obtained from a new computation.

For example, for $i = 3$, the different expressions of Lagrange coefficient $\Lambda^3((3{+}k)\mathbb{A})$ are :

```
ACE> sfa:=SfAExpand(e[3]((3+k)*A1)):
ACE> sf:=SfA2Sf(sfa);   # to get rid of the name A1
                        3
  1/6 (k+3)(k+2)(k+1) e1  +(k+3) e3 +(k+3)(k+2) e2 e1


ACE> map(factor,Toh(sf,collect));
                                                  3
 (k+3) h3 -(4+k)(k+3) h1 h2 +1/6 (k+5)(4+k)(k+3) h1


ACE> map(factor,Tos(sf,collect));
(k+3)(k+2)(k+1)/6s[3]+(4+k)(k+3)(k+2)/3s[2,1]+(k+5)(4+k)(k+3)/6s[1,1,1]
```

```
ACE> map(factor,Tom(sf,collect));
                                              2            3
 1/6(k+3)(k+2)(k+1) m[3] +1/2(k+2)(k+3)  m[2,1] +(k+3) m[1,1,1]

ACE> map(factor, subs(m=F, Tom(SfOmega(sf),collect)));
                                               2          3
1/6 (k+5)(4+k)(k+3) F[3] +1/2 (4+k)(k+3) F[2,1] +(k+3) F[1,1,1]

ACE> map(factor, Top(sf,collect));
                       2                                 3   3
          - 1/2 (k+3) p1 p2 + 1/3 (k+3) p3 + 1/6 (k+3)  p1
```

One can also write Lagrange coefficients as determinants, as shows the next lemma.

Lemma 40 *Let $\mathbb{A}$ be an alphabet, $k \in \mathbb{C}$, $k \neq 0$, $n \in \mathbb{N}$. Then*

$$n!\, S_n(k\,\mathbb{A}) = \det\Big|\big((j-i+1)k + 1 - i\big)\, S_{j-i+1}(\mathbb{A})\Big|_{1\le i,j\le n}$$
$$= \det\Big|\big((j-i+1)k - j + n\delta_{j,n}\big)\, S_{j-i+1}(\mathbb{A})\Big|_{1\le i,j\le n} \qquad (113)$$

Proof. Each of the two matrices is the product, on the right or on the left respectively, of the matrix $\Big[S_{j-i}(\mathbb{A})\Big]$ (of determinant 1) by the Newton matrix[5] $\Big[\Psi^{j-i+1}(k\mathbb{A})\Big]_{1\le i,j\le n}$, which is of determinant $n!\,S_n(k\mathbb{A})$. We already used this factorization in Ex. 42. QED

For example, $4!\,S_4(k\mathbb{A})$ is equal to each of the following determinants :

$$\begin{vmatrix} kS_1 & 2kS_2 & 3S_3k & 4kS_4 \\ -1 & (k-1)S_1 & (2k-1)S_2 & (3k-1)S_3 \\ 0 & -2 & (k-2)S_1 & (2k-2)S_2 \\ 0 & 0 & -3 & (k-3)S_1 \end{vmatrix} = \begin{vmatrix} (k-1)S_1 & (2k-2)S_2 & (3k-3)S_3 & 4kS_4 \\ -1 & (k-2)S_1 & (2k-3)S_2 & 3S_3k \\ 0 & -2 & (k-3)S_1 & 2kS_2 \\ 0 & 0 & -3 & kS_1 \end{vmatrix}$$

Replacing k by $-i-k$, and n by i, one gets two determinantal expressions of Lagrange coefficient $\Lambda^i((i+k)\mathbb{A})$.

Exercises

Ex. 44 Let $x_1, \ldots, x_n$, y be rank 1-elements. Compute $\sum_i 1/(y - x_i)$.

[5] with subdiagonal $-1, -2, \ldots, -n+1$, and not Ψ^0. Recall also that $\Psi^j(k\mathbb{A}) = k\Psi^j(\mathbb{A})$.

Corr.Ex.44. The sum is the logarithmic derivative of $S^n(y-\mathbb{X})$ with respect to y (with $\mathbb{X}=x_1+\cdots+x_n$). Therefore $\sum_i 1/(y-x_i) = S^{n-1}(2y-\mathbb{X})/S^n(y-\mathbb{X})$.

Ex. 45 Let $\mathbb{A}$ be arbitrary, x of rank 1, $n,p\in\mathbb{N}$. Show that

$$S^n(\mathbb{A})-\binom{k}{1}S^n(\mathbb{A}-x)+\binom{k}{2}S^n(\mathbb{A}-2x)+\cdots+(-1)^k\binom{k}{k}S^n(\mathbb{A}-kx) = x^k S^{n-k}(\mathbb{A})\ .$$

Corr.Ex.45. Both sides are linear in $S^j(\mathbb{A})$; one can take $\mathbb{A}=a$ of rank 1, in which case the identity is $a^n-\binom{k}{1}a^{n-1}(a-x)+\binom{k}{2}a^{n-2}(a-x)^2+\cdots = a^{n-k}(a-(a-x))^k$

Ex. 46 Let q be rank-1 elements, and $\mathbb{A}$ be such that $S_n(\mathbb{A})=(1-\alpha q^n)^{-1}$, $n\geq 1$. Let I be a partition in $\mathbb{N}^n$ such that $I\supseteq n^n$. Evaluate $S_I(\mathbb{A})$.

Corr.Ex.46. We do not need α, q to be rank-1 elements, but just formal parameters. We recognize that the Jacobi-Trudi determinant expressing S_I is a specialization of Cauchy determinant $\left|(1-xy)^{-1}\right|$, for $\mathbb{X}=\{\alpha,\alpha q,\ldots,\alpha q^{n-1}\}$, $\mathbb{Y}=\{q^{i_1-n+1},\ldots,q^{i_n}\}$, and therefore $S_I(\mathbb{A})$ is equal to the product of all the entries of the determinant times $\Delta(\mathbb{X})\,\Delta(\mathbb{Y})$.

When $\alpha=1$, one has the limit case $S_n(\mathbb{A})=1/n$ for $q\to 1$.

Ex. 47 Let α,β be elements of binomial type. Define $\mathbb{A}$ by $S_n(\mathbb{A})=\alpha(\alpha+n\beta)^{-1}\,S^n(\alpha+n\beta)$, $n>0$. Compute $\Lambda^n(\mathbb{A})$ and $\Psi^n(\mathbb{A})$.

Corr.Ex.47. cf. Riordan, [42], p.169. By recursion, one shows that

$$\Lambda^n(\mathbb{A}) = \alpha(\alpha-n\beta)^{-1}\,S^n(\alpha-n\beta)\quad,\quad \Psi^n(\mathbb{A})=\alpha(n\beta-1)\cdots(n\beta-n+1)/(n-1)!\ .$$

Ex. 48 Given two positive integers m,n, border the matrix $\mathbb{S}_{m^n}(\mathbb{A})$ by a first column $[x^n,\ldots,x^0,0]$ and a bottom row $[0,y^0,\ldots,y^n]$. Show that the determinant of this new matrix is equal to $S_{(m+1)^{n-1}}(\mathbb{A}-x-y)$, taking x,y to be rank 1-elements.

Corr.Ex.48. Subtract x in all the rows, except the last one, and subtract y in all columns, but the first one. Then the determinant factorizes, one block being a Schur function of $\mathbb{A}-x-y$, the other being $\begin{vmatrix} S^0(0) & \star \\ 0 & S^0(0)\end{vmatrix}=1$.

Ex. 49 Let $\mathbb{A}$ be an alphabet of cardinality n, $\{m_a \in \mathbb{N}\}_{a\in\mathbb{A}}$ be a family of "multiplicities". Define $\mathbb{X} = \sum_a m_a\, a$, the letters a being rank-1 elements.

Show that minors of order $> n$ of the matrix $\left[\Psi^{i+j}(\mathbb{X})\right]_{i,j\geq 0}$ are null, and compute the minors of order n.

Corr.Ex.49. Decomposing each column as a sum of columns involving only one letter, acccording to $\Psi^k = \sum m_a a^k$, one gets a sum of determinants, but the only ones which are non zero are those for which no two columns contain the same letter. This proves nullity for order $> n$. For order n, the remaining determinants are in bijection with permutations of the letters, and their sum is just $\prod_a m_a$ times the sum obtained when all m_a are equal to 1. Therefore, a minor of index I, J, $I, J \in \mathbb{N}^n$ is equal to

$$\prod_a m_a\, S_I(\mathbb{A})\, S_J(\mathbb{A})\, \Delta(\mathbb{A})^2 \ .$$

Bellavitis (1857), MuirII, p.353, gives the case $I = 0^n$, $J = k^n$.

Ex. 50 To each sequence of alphabets $\mathcal{A} = (\mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_5, \ldots, \mathbb{A}_p, \ldots)$, all primes p, associate a *Moëbius function* on the positive integers :

$$\mu_{\mathcal{A}}(n) := \prod_p \bigl(S_k(\mathbb{A}_p)\bigr) \ ,$$

where k is the order of p in n (p^k is the greatest power dividing n).

Given two sequences $\mathcal{A}$, $\mathcal{B}$, show that $\mu_{\mathcal{A}+\mathcal{B}}$ is the Dirichlet convolution product of $\mu_{\mathcal{A}}$ and $\mu_{\mathcal{B}}$, i.e.

$$\mu_{\mathcal{A}+\mathcal{B}}(n) = \sum_{d|n} \mu_{\mathcal{A}}(d)\mu_{\mathcal{B}}(n/d) \ ,$$

summation over all divisors of n, where the sum $\mathcal{A}+\mathcal{B}$ is componentwise.

Show that the Moebius function μ, and the Euler functions φ, τ, σ (Recall that $\mu(n)$ is 0 if n is not square-free, and ± 1 otherwise, the *totient function* $\varphi(n)$ is the number of positive integers less than n and prime to n, $\tau(n)$ is the number of divisors of n and $\sigma(n)$ is their sum) correspond respectively to

$$(-1, \ldots, -1, \ldots)\, ,\ (\mathbb{A}_2 - 1, \ldots, \mathbb{A}_p - 1, \ldots)\, ,\ (2, \ldots, 2, \ldots)\, ,\ (\mathbb{A}_2 + 1, \ldots, \mathbb{A}_p + 1, \ldots)\, ,$$

where $\mathbb{A}_p$ is such that $S_k(\mathbb{A}_p) = p^k$.

Corr.Ex.50.

First notice that, for the properties that we examine, the set of integers decomposes according to each prime.

The statement involving convolution comes from the identity

$$S^k(\mathbb{A}_p + \mathbb{B}_p) = \sum S^i(\mathbb{A}_p)\, S^{k-i}(\mathbb{B}_p)\ .$$

Products of $S_k(-1)$ are 0 if one of the k is bigger than 1, and the sign is the parity of the number of prime divisors.

For all n, $\mu_{1,\ldots,1,\ldots}(n) = 1$, and the convolution square of this function is τ.

The sequence $\mathcal{A} = (\mathbb{A}_2, \ldots, \mathbb{A}_p, \ldots)$ gives the function $\mu_{\mathcal{A}}$ such that $\mu_{\mathcal{A}}(n) = n$, i.e. gives the identity function.

One checks that φ is the convolution of μ and $\mu_{\mathcal{A}}$, and that σ is the convolution of $\mu_{\mathcal{A}}$ and μ_1 (with $\mu_1(n) = 1$ for all n).

For more properties of the Moebius function on the integers, and of multiplicative functions, we refer to Dixon [8] and Apostol [2].

Ex. 51 Let $\mathbb{A}$ be an alphabet. Grothendieck defined functions $\gamma^i(\mathbb{A})$ by the generating function

$$\gamma_z(\mathbb{A}) = \sum z^i\, \gamma^i(\mathbb{A}) := \lambda_{z/(1-z)}(\mathbb{A})\ .$$

Define accordingly the alphabet $\mathbb{A}^\diamond$ by $\Lambda^i(\mathbb{A}^\diamond) = \gamma^i(\mathbb{A})$, $i \geq 0$.

Show that

$$\begin{aligned}
\Lambda^i(\mathbb{A}^\diamond) &= \Lambda^i(\mathbb{A} + i - 1)\\
S^i(\mathbb{A}^\diamond) &= S^i(\mathbb{A} + i - 1)\\
\Psi^i(\mathbb{A}^\diamond) &= \sum_{j=0}^{i-1} (-1)^j \binom{i}{j}\, \Psi^{i-j}(\mathbb{A})\ .
\end{aligned}$$

Show that the transformation $\mathbb{A} \mapsto \mathbb{A}^\diamond$ satisfies $(-\mathbb{A})^\diamond = -\mathbb{A}^\diamond$, $(\mathbb{A}+\mathbb{B})^\diamond = \mathbb{A}^\diamond + \mathbb{B}^\diamond$, but that in general, $(\mathbb{A}\mathbb{B})^\diamond \neq \mathbb{A}^\diamond\mathbb{B}^\diamond$.

Ex. 52 Let the Gauss polynomial $\begin{bmatrix} n \\ i \end{bmatrix}$ be $S^i((1-q^{n-i+1})/(1-q))$, with q of rank 1. Show that

$$\begin{bmatrix} m+n+1 \\ n \end{bmatrix} = \sum_0^n q^j \begin{bmatrix} m+j \\ j \end{bmatrix}$$

$$\begin{bmatrix} m+n \\ k \end{bmatrix} = \sum_j q^{(n-j)(k-j)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m \\ k-j \end{bmatrix}\ .$$

Corr.Ex.52. The first equation results from the expansion of $S^n(1+q\,\frac{1-q^{m+1}}{1-q})$, the second one, from the expansion of $q^{-k(k-1)/2}\,\Lambda^k\Big((1+\cdots+q^{n-1})+(q^n+\cdots+q^{n+m-1})\Big)$.

Ex. 53 Let $\mathbb{A}$ be arbitrary, p binomial, $n\in\mathbb{N}$. Show that

$$nS^n(p\mathbb{A}) - (p+n-1)S^{n-1}(p\mathbb{A})\,\Lambda^1(\mathbb{A}) + (2p+n-2)S^{n-2}(p\mathbb{A})\,\Lambda^2(\mathbb{A}) - \cdots$$
$$\cdots \pm ((n-1)p+1)S^0(p\mathbb{A})\,\Lambda^n(\mathbb{A}) = 0\ .$$

Corr.Ex.53. Introduce two elements x,y of rank 1. Recognize that the above expression is the coefficient of x^n in $\frac{d}{dy}f(x,y)\Big|_{y=1}$, where
$f(x,y) := \sum_i (xy)^i S^i(p\mathbb{A})\sum_j(-xy^p)\Lambda^j(\mathbb{A})$. Moreover, one can take $\mathbb{A}$ to be a sum $\sum a$ of rank 1 elements; in that case $f(x,y) = \prod_a(1-axy^p)/(1-axy)^p$. The logarithmic derivative $\frac{d}{dy}\log f(x,y)$ is equal to $\sum_a \frac{-p(-ax)}{1-axy}+\frac{p(-axy^{p-1})}{1-axy^p}$, from which one sees the required nullity in $y=1$.

Ex. 54 Let $\mathbb{A}$ be arbitrary. Show that the product of $\Big[(-1)^{i+j}\Lambda^j(i)\Big]_{0\le i,j\le n}$ by $\Big[S_j(i\mathbb{A})\Big]_{0\le i,j\le n}$ is a triangular matrix. Express its entries in the $S^I(\mathbb{A})$ basis.

Ex. 55 Let $\mathbb{A}$ be of cardinality n, and $\mathbb{A}^{der}$ be the alphabet of roots of $S^{n-1}(2x-\mathbb{A})$ (we shall call it the *derived alphabet*). Show that

$$S^k(\mathbb{A}-\mathbb{A}^{der}) = \Psi_k(\mathbb{A})/n\ ,\ \ k=1,2,\ldots\ .$$

Show that, for any $a\in\mathbb{A}$,

$$S_{n^{n-1}}(\mathbb{A}-\mathbb{A}^{der}-2a) = (-1)^{n-1}\Delta(\mathbb{A})^2\ ,$$

i.e. is equal to the discriminant $\prod_{i<j}(a_i-a_j)^2$, up to a sign.

Corr.Ex.55. $S^{n-1}(2x-\mathbb{A})/S^n(x-A)$ is the logarithmic derivative of $S^n(x-A)$, and this settles the first point.

One recognizes in $S_{n^{n-1}}(A-a-(\mathbb{A}^{der}+a))$ a resultant $R(A-a,\mathbb{A}^{der}+a)$. However, for any $a'\in\mathbb{A}$, one has $R(\mathbb{A}^{der},a') = R(\mathbb{A}-a',a')$, and therefore

$$R(A-a,\mathbb{A}^{der}+a) = \pm R(A-a,a)\prod_{a'\in\mathbb{A}\backslash a} R(A-a',a') = \pm\Delta(\mathbb{A})^2\ .$$

Ex. 56 Let $\mathbb{A}$ be an alphabet, $\mathbb{A}^{der}$ the alphabet of roots of the derivative of $R(x,\mathbb{A})$. For any integer k, express $S^k(x-\mathbb{A}^{der})$ in the basis $\{S^i(x-\mathbb{A}),\ i\geq 0\}$.

Corr.Ex.56. Write $-\mathbb{A}^{der} = (\mathbb{A}-\mathbb{A}^{der})-\mathbb{A}$. Therefore $S^k(x-\mathbb{A}^{der}) = \sum S^{k-i}(x-\mathbb{A})S^i(\mathbb{A}-\mathbb{A}^{der}) = \sum S^{k-i}(x-\mathbb{A})\,\Psi_i(\mathbb{A})/\Psi_0(\mathbb{A})$.

Ex. 57 (Sylvester, [50] **1**, p.502). Let $\mathbb{A}$ be of cardinality n, and let $\mathbb{A}^{der}$ its derived alphabet. For any positive integer k, and x of rank 1, let $P_k(x) := S_{k^k}(\mathbb{A}-\mathbb{A}^{der}-x)$. Show that for any $k>0$, any $0\leq j<k$, one has

$$\sum\nolimits_{a\in\mathbb{A}} a^j\,P_k(a) = 0\ ,$$

and that

$$\Big|\,P_0(a)^2\,,\,P_1(a)^2\,,\,\ldots,\,P_{n-1}(a)^2\Big|_{a\in\mathbb{A}} = 0\ .$$

Corr.Ex.57. Since $P_k(x) := S_{k^k,0}(\mathbb{A}-\mathbb{A}^{der},x)$, using the preceding exercise, it can be written as

$$\begin{vmatrix} \frac{1}{n}\Psi_k(\mathbb{A}) & \cdots & \frac{1}{n}\Psi_{2k-1}(\mathbb{A}) & x^{k+j} \\ \vdots & & \vdots & \vdots \\ \frac{1}{n}\Psi_0(\mathbb{A}) & \cdots & \frac{1}{n}\Psi_{k-1}(\mathbb{A}) & x^j \end{vmatrix}\ .$$

Summation on $x=a\in\mathbb{A}$ transforms the last column into a multiple of a previous one, hence the sum vanishes.

Using the quadratic relations (?), one can write

$$P_k(x)^2 = S_{k^{k-1}}(\mathbb{B}-2x)S_{k^{k+1}}(\mathbb{B}) + S_{k+1^{k-1}}(\mathbb{B}-2x)S_{k-1^{k+1}}(\mathbb{B})\ ,$$

with $\mathbb{B}=\mathbb{A}-\mathbb{A}^{der}$.

Take now $n=4$, to simplify indices. The candidate to nullity is

$$\Big|\,1,\ S_{11}(\mathbb{B})+S_2(\mathbb{B}-2a),\ S_2(\mathbb{B}-2a)S_{222}(\mathbb{B})+S_{33}(\mathbb{B}-2a)S_{11}(\mathbb{B}),$$
$$S_{33}(\mathbb{B}-2a)S_{3333}(\mathbb{B})+S_{444}(\mathbb{B}-2a)S_{222}(\mathbb{B})\,\Big|_{a\in\mathbb{A}}.$$

By successive subtractions, it simplifies into

$$\Big|\,1,\ S_2(\mathbb{B}-2a),\ S_{33}(\mathbb{B}-2a)S_{11}(\mathbb{B}),\ S_{444}(\mathbb{B}-2a)S_{222}(\mathbb{B})\,\Big|\ .$$

However, the last exercise shows that the elements $S_{444}(\mathbb{B}-2a)$ are constant, and thus the last column is a multiple of the first one and the determinant vanishes. QED

Ex. 58 Let x be such that $\forall n \in \mathbb{N}$, $S^n(1+x) = 1 + nx$. Compute the Schur function $S_J(1+x)$, J partition, without expansion of determinants.

Corr.Ex.58. Put $z := 1-x$. Then z is a rank 1 element, and thus $S_J(1+x) = S_J(2-z) = \sum(-z)^i S_{J/1^i}(2)$. Apart from one part-partitions, the only J for which $S_J(1+x)$ can be non zero are of the type $J = [1^m, h, k]$, $h \geq 1, m \geq 0$. The corresponding summation reduces to $\Big((-z)^m\Lambda^2(2) + (-z)^{m+1}\Lambda^1(2) + (-z)^{m+2}\Big) S_{h-1,k-1}(2)$. Finally, for such J, one has

$$S_{1^m,h,k}(1+x) = (k-h+1)\,(x-1)^m x^2 \ .$$

Ex. 59 Let $\mathbb{A}$ be arbitrary, β be of binomial type. Show, after Han Guo Niu that

$$\frac{n}{\beta+1} S_n\big((\beta{+}1)\mathbb{A}\big) = \sum_{j=0}^{n} j\, S_j(\mathbb{A}) S_{n-j}(\beta\mathbb{A}) \ .$$

Corr.Ex.59. Taking generating series, one has to show that

$$\frac{u}{\beta+1}\sum n u^{n-1} S_n\big((\beta{+}1)\mathbb{A}\big) = \frac{d}{dz}\left(u^n S_n\big((\beta{+}z)\mathbb{A}\big)\right)\Big|_{z=1} \ ,$$

but both sides are equal to $\prod_{a\in\mathbb{A}}(1-ua)^{-\beta-1}\sum_a ua\,(1-ua)^{-1}$.

Ex. 60 Show that, for any $r \in \mathbb{N}$,

$$x(x+1)\cdots(x+r-1) = \sum_{J:|J|=r} x^{\ell(J)} \frac{r!}{(\Psi^J, \Psi^J)} \ .$$

What becomes the left hand side when the summation is restricted to partitions with all parts odd ? For example, one gets, for $n = 2, \ldots, 6$ the following values x^2, x^3+2x, x^4+8x^2, x^5+20x^3+24x, $x^6+40x^4+184x^2$.

Corr.Ex.60. The left hand side is $r!\, S_r(x)$, with x of binomial type, the right side being its expansion in the basis Ψ^J. This gives the first equality. To take into account the restriction on parity, one introduces a rank 1 element ξ which will be specialized to -1. Then $\Psi_i\big((1-\xi)\,x/2\big) = x$ if i is odd, and $= 0$ otherwise. The left hand side becomes $r! S_r\big((1-\xi)\,x/2\big)$, and one finds its values by taking the generating series

$$\sigma_z\big((1-\xi)\,x/2\big) = \left(\frac{1-z\xi}{1-z}\right)^{x/2} = \left(\frac{1+z}{1-z}\right)^{x/2} \ .$$

```
ACE> simplify(6!*coeff(taylor(((1+z)/(1-z))^(x/2),z,7),z,6));
                                          2         4     6
                                     184 x   + 40 x   + x
```

The same method would allow to explicit the summation $\sum_J x^{\ell(J)} 1/(\Psi^J, \Psi^J)$ restricted to partitions with all parts multiple of a fixed integer p. One has to take ξ to be a primitive p-th root of unity instead of being -1.

Ex. 61 Let x be of binomial type, and sf be a symmetric function. Express $sf(x)$ in the basis of Newton's polynomials $N_i := x(x-1)\cdots(x-i+1)$.

Corr.Ex.61. Since a monomial function $\Psi_J(x)$ is proportional to $N_{\ell(J)}$, according to Formula (100), one has just to expand sf in the basis of monomial functions.

$$sf(x) = \sum_J \frac{(sf\,,\,S^J)}{m_1!\,m_2!\cdots} N_{\ell(J)} \ ,$$

sum over all partitions $J = 1^{m_1} 2^{m_2} \cdots$.

```
Sf2NewtonBym:=proc(sf)  local tt,pa;
  tt:=Sf2Table(Tom(sf),'m');
  convert([seq(tt[pa]/convert(map(i->i!,Part2Exp(pa)),`*`)* cat(N,nops(pa)),
          pa=map(op,[indices(tt)]) )],`+`)
end:
```

Ex. 62 Let x be of binomial type, z be a rank-1 element. Compute the specializations $z = -1$ of the monomial functions and Schur functions in $\mathbb{A} = x(1+z)$.

Give as a corollary Gillis's formula :

$$S^{2n}(x(1+z))\Big|_{z=-1} = S^n(x) \ .$$

Corr.Ex.62. The power sums $\Psi^i(\mathbb{A}) = \Psi^i(x)\Psi^i(1+z) = x(1+z^i)$ specialize into $2x$ if i is even, 0 if i is odd. The expression of monomial functions in terms of power sums is compatible with doubling the parts of a partition I. Denote this operation by $I \to 2\star I$. Then

$$\Psi_J(\mathbb{A})\Big|_{z=-1} = \Psi_I(2x) \quad \text{if } J = 2\star I \quad \text{or 0 if } J\ texthasanoddpart \ .$$

The determination of Schur functions is a little more subtle. One needs to use the factorization, due to Littlewood [35], of $S_I(\Omega\,\mathbb{B})$, for Ω is the "alphabet of

p-th roots of unity" (here p=2), and $\mathbb{B}$ is arbitrary. The outcome in our case is that the specialization of a Schur function S_I is equal to $\pm S_{I'}(x)\, S_{I''}(x)$ if I is a partition without 2-core, and with 2-quotient $[I', I'']$, and is null if I has a core. In particular, if $I = [2n]$, then $I' = [n]$, $I'' = 0$, and this gives Gillis'identity

$$\sum_{i=0}^{2n} (-1)^i \binom{x}{i} \binom{x}{2n-i} = \binom{x}{n} .$$

Ex. 63 *Séries de Facultés* (Nördlund, Leçons sur les séries d'interpolation, Gauthiers-Villars, Paris(1926)). A "série de facultés" is a series of the type

$$f(z, \mathbb{A}) := 1 + \frac{S^1(\mathbb{A})}{z} + \frac{S^2(\mathbb{A})}{z(z+1)} + \frac{2!\, S^3(\mathbb{A})}{z(z+1)(z+2)} + \cdots = \sum + \frac{S^n(\mathbb{A})}{(n+1)S^{n+1}(z)} ,$$

where z is of binomial type and $\mathbb{A}$ arbitrary.

Show that for every binomial element y, one has $f(z, \mathbb{A}) = f(z+y, \mathbb{A}+y)$. Deduce that

$$\frac{d}{dz} f(z, \mathbb{A}) = -\sum \Big(S^{n-1}(\mathbb{A})/1 + \cdots + S^0(\mathbb{A})/n \Big) / (n+1) S^{n+1}(z) .$$

Express the product $f(z, \mathbb{A})\, f(z, \mathbb{B})$ as a série de facultés.

Corr.Ex.63. Because $f(z, \mathbb{A})$ is linear in the $S^n(\mathbb{A})$, one can restrict to $\mathbb{A} = \alpha$, with α binomial. In that case

$$f(z, \alpha) = \frac{1}{z - \alpha} = 1 + \frac{1}{z} + \frac{\alpha}{z(z+1)} + \frac{\alpha(\alpha+1)}{z(z+1)(z+2)} + \frac{\alpha(\alpha+1)(\alpha+2)}{z(z+1)(z+2)(z+3)} + \cdots$$

identity that is a special case of Newton's interpolation formula. Now everybody agrees that $z - \alpha = (z+y) - (\alpha+y)$ and this settles the first point.

As for the product $\left(\frac{1}{z-\alpha}, \frac{1}{z-\beta}\right) \mapsto \frac{1}{(z-\alpha)(z-\beta)} = \left(\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right)\frac{1}{\alpha-\beta}$, one is reduced to evaluate all the $\big(S^n(\alpha) - S^n(\beta)\big)/(\alpha-\beta)$, $n \in \mathbb{N}$. The expansion of $S^n(\alpha)$ involves Stirling numbers of the first kind, that is, the elementary symmetric functions of an alphabet specialized in $\{0, 1, \ldots, n-1\}$. It remains to express the $(\alpha^k - \beta^k)/(\alpha-\beta)$ in terms of products of $S^i(\alpha)$, $S^j(\beta)$

Ex. 64 Let x be of binomial type, r be an integer, $J \in \mathbb{N}^r$ be a partition. Compute the generating series $\sum_n z^n S_{Jn}(x)$, where Jn means the concatenation of J and n.

Corr.Ex.64. Taking advantage that the interval of summation has not been specified, one takes $-r \le n < \infty$. Taking the Laplace expansion of the determinantal expression of $S_{Jn}(x)$ along its last column, one finds

$$\begin{aligned}
\sum_n z^n S_{Jn}(x) &= \sum_n \sum_{i=0}^{r} (-1)^i z^n S_{J/1^i}(x) S_{n+i}(x) \\
&= \sum_i (-z)^{-i} S_{J/1^i}(x) \sum_n z^{n+i} S_{n+i}(x) \\
&= \sum_i (-z)^{-i} S_{J/1^i}(x)\, (1-z)^{-x} \ .
\end{aligned}$$

Notice that the summation in i is restricted to $0 \le i \le \ell(J)$. For example,

$$(1-z)^x \sum_{n \ge -1} z^n S_{1n}(x) = S_1(x) - z^{-1} S_{1/1}(x) = x - z^{-1} \ ,$$

$$(1-z)^x \sum_{n \ge -2} z^n S_{23n}(x) = S_{23}(x) - z^{-1} S_{23/1}(x) + z^{-2} S_{23/11}(x)$$

$$= \frac{1}{24} x^2 (x+1)(x+2)(x-1) - \frac{1}{24} x(x-1)(5x+6)(x+1) z^{-1} + \frac{1}{3} x(x+1)(x-1) z^{-2}$$

Ex. 65 Let $k \in \mathbb{N}$. Writing partitions exponentially, $J = 1^{m_1} 2^{m_2} \cdots$, show that

$$\sum_J m_k \frac{\Psi^J}{(\Psi^J, \Psi^J)}$$

is a sum of Schur functions with coefficients ± 1.

Corr.Ex.65. For all positive integers n, k, one has $D_{\Psi_k} S_{n+k} = S_n$. On another hand, expressing S_{n+k} in the basis Ψ^I, one gets

$$D_{\Psi_k} S_{n+k} = k \frac{d}{d_{\Psi^k}} \sum \frac{\Psi^I}{(\Psi^I, \Psi^I)} = k \sum m_k \frac{\Psi^I}{(\Psi^I, \Psi^I) \Psi^k} \ .$$

The unknown sum is therefore equal to $\sum \Psi^k S_n / k$. Using multiplication of Schur function by monomial functions, one gets a sum on Schur functions indexed by vectors of the type $[k,\, 0^i] + [0^i,\, n]$, $i = 0, \ldots, k-1$, which by reordering give finally

$$S_{n+k} + S_{k,n} - S_{1,\,k-1,\,n} + S_{1,\,1,\,k-2,\,n} - \cdots$$

Ex. 66 Let $f(x)$ be a formal series $f(x) = f_0 + xf_1 + x^2 f_2 + \cdots$. Let y be another indeterminate. For any positive integer m, define, after Catalan,

$$\Omega = (f_0 + xf_1 + x^2 f_2 + \cdots) + \binom{m+1}{1} y(xf_1 + x^2 f_2 + \cdots) \\ + \binom{m+2}{2} y(x^2 f_2 + x^3 f_3 + \cdots) + \cdots$$

Let $\phi(x,y) := y^m \Big(f(x) - yf(xy) \Big)/(1-y)$. Show that

$$\Omega = \frac{1}{m!} \frac{d}{dy^m} \phi(x,y) \ .$$

Corr.Ex.66. Derivations of polynomials $S^n(x-\mathbb{A})$ rather than of series $\sigma_x(\mathbb{A})$ are easier to write. Truncating the series to an arbitrary big order, one can write it $f(x) = S^n(x-\mathbb{A})$. Now, Ω becomes

$$\begin{aligned} \Omega &= S^n(x-\mathbb{A}) + \binom{m+1}{1} yxS^{n-1}(x-\mathbb{A}) + \binom{m+2}{2} y^2x^2S^{n-2}(x-\mathbb{A}) + \cdots \\ &= S^n(x - \mathbb{A} + (m{+}1)xy) \ . \end{aligned}$$

On the other hand, a direct computation shows that

$$\frac{xf(x) - xyf(xy)}{x - xy} = S^n(x + xy - \mathbb{A}) \text{ and } \frac{1}{m!} \frac{d}{dz^m} z^m S^n(z{+}\mathbb{B}) = S^n((m{+}1)z + \mathbb{B}) \ ,$$

for all $\mathbb{B}$ and all rank 1 element z. One concludes by putting $z = xy$, $\mathbb{B} = x - \mathbb{A}$.

Ex. 67 Define the Hermite polynomials $H_n(x)$ and the alphabet $\mathbb{A}$ by

$$S_n(\mathbb{A}) = H_n/n! := \sum_{r=0}^{n} (-1)^r (2x)^{n-2r}/r!(n-2r)! \ .$$

Compute the forgotten symmetric functions in $\mathbb{A}$.

Corr.Ex.67. First, the explicit expression of Hermite polynomials is equivalent to the generating function $\sum z^n H_n/n! = \exp(-2xz - z^2)$. Therefore, $\mathbb{A}$ is such that $\Psi^1(\mathbb{A}) = -2x$, $\Psi^2(\mathbb{A}) = -2$, $\Psi^i(\mathbb{A}) = 0$ for $i > 2$. The forgotten functions for $\mathbb{A}$ are equal, by definition, to the monomial functions for $\mathbb{A}'$, such that $\Psi^1(\mathbb{A}') = -2x$, $\Psi^2(\mathbb{A}') = 2$, $\Psi^i(\mathbb{A}') = 0$ for $i > 2$. In particular, the only surviving functions are those indexed by partitions of the type $1^n 2^k$. Because

$$2\Psi_{1^n 2^k}(\mathbb{A}') = \Psi^2(\mathbb{A}')\Psi_{1^n 2^k}(\mathbb{A}') = \Psi_{1^n 2^{k+1}}(\mathbb{A}') \ ,$$

one can ignore the parts equal to 2, and therefore, one has only to compute for the indices 1^n, in which case one obtains $H_n/n!$.

```
SpecHermite:=proc(sf0)  local i,sf;
  sf:=Top(sf0);
  subs(seq(cat(p,i)=0, i=Sf2TableVar(sf,p)minus {1,2}), p1=2*x,p2=-2, sf)
end:
ACE> simplify(orthopoly[H](4,x)/ SpecHermite(SfOmega(m[2,2,1,1,1,1])));
                                 12
```

Ex. 68 Define the *Gegenbauer polynomials* $G(n, \alpha, x)$ by the generating function

$$\sum_{n \geq 0} z^n G(n, \alpha, x) = (1 - 2xz + z^2)^{-\alpha} .$$

Let $k \in \mathbb{C}$. Evaluate the determinant of order n with first row $[\, jkG(j, \alpha, x),\ j = 1 \ldots n]$, entries $[i, j] = (j - i + 1)k + i - 1)\, G(j - i + 1, \alpha, x)$, $2 \leq i \leq j \leq n$, subdiagonal equal to $[1, 2, \ldots, n{-}1]$, and other entries 0.

For example, for $n = 4$, evaluate the determinant

$$\begin{vmatrix} kG(1,\alpha,x) & 2\,kG(2,\alpha,x) & 3\,kG(3,\alpha,x) & 4\,kG(4,\alpha,x) \\ 1 & (k+1)\,G(1,\alpha,x) & (2\,k+1)\,G(2,\alpha,x) & (3\,k+1)\,G(3,\alpha,x) \\ 0 & 2 & (k+2)\,G(1,\alpha,x) & (2+2\,k)\,G(2,\alpha,x) \\ 0 & 0 & 3 & (k+3)\,G(1,\alpha,x) \end{vmatrix} .$$

Corr.Ex.68. This determinant is a special case of the first determinant in Eq. 113. Therefore, the determinant is to $(-1)^n n!$ times a Gegenbauer of order n, but for the parameter $-k\alpha$.

```
with(orthopoly):
MatGegenBauer:=proc(n,k,a) local i,j;
  matrix([ [seq(j*k*G(j,a,x),j=1..n)], seq([0$(i-2),i-1,
     seq( ((j-i+1)*k+i-1)*G(j-i+1,a,x), j=i..n)],i=2..n)])
end:
ACE> aa:= det(MatGegenBauer(4,k,a)):
ACE> simplify(aa/ G(4,-k*a,x);
                                        24
```

In particular, Gegenbauer polynomials can be expressed as determinants in the Tchebychef polynomials $G(n, 1, x)$, but more simply, one can choose to express it in terms of the coefficients of $(1 - 2xz + z^2)$ (taking $\alpha = -1$).

In that case, the determinant is tridiagonal, with main diagonal $[2\alpha x,\ 2(\alpha{+}1)x, \ldots, 2(\alpha{+}n{-}1)x]$, above diagonal $[2\alpha, 2\alpha{+}1, \ldots, 2\alpha{+}n{-}2]$, subdiagonal $[1, 2, \ldots, n{-}1]$.

```
GegenTridiag:=proc(n,a,x)  local i,ma;
  ma:=diag(seq( 2*(a+i)*x,i=0..n-1));
  for i from 1 to n-1 do ma[i,i+1]:= 2*a+i-1; ma[i+1,i]:=i; od;
 eval(ma)
end:
ACE> simplify( det(GegenTridiag(4,a,x))/G(4,a,x));
                                        24
```

Ex. 69 Let $\mathbb{A}$ be an alphabet, and n a positive integer. Show that there exists a unique polynomial $F(x)$ of degree n such that

$$F\big(z^{-1}\lambda_z(\mathbb{A})\big) = z^{-n} + z(\cdots)$$

i.e. such that its evaluation in $x = z^{-1}\lambda_z(\mathbb{A})$ has no term of degree $-n+1, \ldots, 0$.

Corr.Ex.69. cf. Schur [47] III, p.361, Prosper [41]. The equations defining $F(x)$ are

$$(-z)^n \Psi^n(\mathbb{A}) + \sum \frac{n}{i} \Lambda^{n-i}(-iz\mathbb{A})\lambda_z(i\mathbb{A}) \;\cap\; \{z, \ldots, z^n\} = \{0, \ldots, 0\} \ ,$$

writing $f \cap z^i$ for the coefficient of z^i in f.

These equations are very similar to those involved in Lagrange inversion, and admits the solution

$$F(x) = (-1)^n \Psi^n(\mathbb{A}) + \sum_{i=1}^{n} \frac{n}{i} z^i\, \Lambda^{n-i}(-i\mathbb{A}) \ .$$

Transformation of alphabets

Having, with Cauchy formula, a scalar product together with pairs of adjoint bases on the space of symmetric functions, having, with lemma (7) a tool to transform determinants and, with Pieri formula, a rule for multiplication, we needed only to formalize in a structure of λ-ring the algebraic operations on alphabets : "$+, -, \cdot$" and the multiplication of alphabets by constants, to be able to handle efficiently symmetric functions.

All our definitions involved *generic* alphabets composed of algebraically independent letters, but of course, as in the case of the alphabet of roots of a polynomial, we shall be interested in *specialized* alphabets. We shall use the operations on generic series $\sigma_z(\mathbb{A})$ that we defined in the preceding sections to get informations on every formal series $1 + zc_1 + z^2c_2 + \cdots$. By abuse of language, we shall still call a set like

$$\mathbb{A} = \{0,\, (1/2),\, (\sqrt{-1}),\, (-\sqrt{-1})\}$$

as an "alphabet" of cardinality 4, meaning that we we specialize, in an identity like $\Psi^2(\mathbb{A}) = S_2(\mathbb{A}) - S_{11}(\mathbb{A})$, the symmetric functions of $\{a_1, a_2, a_3, a_4\}$ in $a_1 = 0$, $a_2 = 1/2$, $a_3 = \sqrt{-1}, a_4 = -\sqrt{-1}$. Thus we must be careful in distinguishing the element of binomial type $1/2$ and the specialization $x = (1/2)$ of a rank 1 element.

Beware that specialization forces us to leave λ-rings, remember that, inside a λ-ring, an element of rank 1 can only be specialized to 1 or 0 ! In other words, we use λ-rings to get algebraic identities, and then freely specialize in these identities rank 1-elements and elements of binomial type, forgetting about their original status.

Bernoulli numbers

Let us illustrate on the example of Bernoulli numbers why it is interesting to use λ-rings to treat some topics of classical combinatorics (but we do not give, for the moment, any new property of Bernoulli numbers!).

Define an alphabet $\mathbb{B}^+$ by the equations :

$$S_n((n{+}1)\,\mathbb{B}^+) = 1\ ,\ n = 1, 2, \ldots \tag{114}$$

Thus $1 = S_1(2\mathbb{B}^+)$ implies $S_1(\mathbb{B}^+) = 1/2$,

$$1 = S_2(3\mathbb{B}^+) = \Psi_2(3)S^2(\mathbb{B}^+) + \Psi_{11}(3)S^{11}(\mathbb{B}^+) = 3S^2(\mathbb{B}^+) + 3(1/2)^2\ ,$$

gives $S^2(\mathbb{B}) = 1/12$. More generally, the equation for order n has leading term $\Psi_n(3)S^n(\mathbb{B}^+)$ and allows to determine $S^n(\mathbb{B}^+)$, knowing the complete functions of smaller degree.

Then, from the formula giving the Lagrange inverse of a series (see exercises for more details), we get that

$$\sigma_z(\mathbb{B}^+) = z/\big(1-\exp(-z)\big) = 1 + z/2 + \sum_{n=2}^{\infty} z^n B_n/n!\ , \tag{115}$$

where B_n is the n-th Bernoulli number (B_1 is equal to $-1/2$, but we prefered here taking $1/2$).

It is remarkable that the simple Equations (114) define so fundamental numbers as Bernoulli numbers[6].

To recover the classical case, let us define the *Bernoulli alphabet* $\mathbb{B}$ by

$$\sigma_z(\mathbb{B}) = z/\big(\exp(z)-1\big) = \sum_{n=0}^{\infty} z^n B_n/n!\ . \tag{116}$$

What benefice to draw from identifying the Bernoulli number B_n to $n!\,S^n(\mathbb{B})$? Well, we can now specialize any symmetric function S to $S(\mathbb{B})$, and therefore get from the expression of this function in terms of complete functions identities involving Bernoulli numbers.

For example, the inverse series $\sigma_z(-\mathbb{B}) = (\exp(z)-1)/z$ is simpler, and gives

$$\Lambda^i(\mathbb{B}) = (-1)^i/(i{+}1)!\ ,\ i = 1, 2, \ldots \tag{117}$$

Now, any skew Schur function $S_{J/I}(\mathbb{B})$ will be a determinant of Bernoulli numbers, as well as a determinant of inverses of factorial, by taking conjugate partitions. Because Bernoulli numbers of odd index are 0 (except B_1), one can find a skew partition such that the Jacobi-Trudi determinant factorizes into $\pm S_{2n}(\mathbb{B})$ times a determinant equal to 1, the determinant in $\Lambda^i(\mathbb{B})$ having at first glance no special property. Here such an example, for $n = 5$. We first write a function which evaluates any symmetric function on the alphabet $\mathbb{B}$, passing through its expansion in the bases S^I, Λ^I or Ψ^I according to the choice of the user.

ACE > `[bernoulli(1), seq(bernoulli(2*i),i=1..10)];`

$$[-1/2, 1/6, -1/30, 1/42, -1/30, \frac{5}{66}, \frac{-691}{2730}, \frac{7}{6}, \frac{-3617}{510}, \frac{43867}{798}, \frac{-174611}{330}]$$

[6] These equations say that the cohomology of the projectif space $P(\mathbb{C}^{n+1})$ is of dimension 1 in degree n, and play a fundamental role in Hirzebruch's version of Riemann-Roch theorem [17].

```
ACE > SfSpecialBernoulli:=proc(sf,b) local sf2,i;
if 'b'='e' then sf2:=Toe(sf);
subs(seq(cat(e,i)= (-1)∧i/(i+1)!, i=Sf2TableVar(sf2,'e')), sf2);
elif 'b'='h' then sf2:=Toh(sf);
subs(seq(cat(h,i)= bernoulli(i)/i!, i=Sf2TableVar(sf2,'h')), sf2)
elif 'b'='p' then sf2:=subs(p1=-1/2,Top(sf));
subs(seq(cat(p,i)= -bernoulli(i)/i!, i=Sf2TableVar(sf2,'p')), sf2)
fi;
end:
ACE > pa:=[[6,6,5,4,3], [5,4,3,2]]:
ACE > pa2:=map(Part2Conjugate, pa);
              pa2:=[[5,5,5,4,3,2],[4,4,3,2,1]];
ACE > deuxMat:=[ SfJtMat(pa), SfJtMat(pa2,'e')];
```

$$\begin{bmatrix} h1 & h3 & h5 & h7 & h10 \\ 1 & h2 & h4 & h6 & h9 \\ 0 & 1 & h2 & h4 & h7 \\ 0 & 0 & 1 & h2 & h5 \\ 0 & 0 & 0 & 1 & h3 \end{bmatrix}, \begin{bmatrix} e1 & e2 & e4 & e6 & e8 & e10 \\ 1 & e1 & e3 & e5 & e7 & e9 \\ 0 & 1 & e2 & e4 & e6 & e8 \\ 0 & 0 & 1 & e2 & e4 & e6 \\ 0 & 0 & 0 & 1 & e2 & e4 \\ 0 & 0 & 0 & 0 & 1 & e2 \end{bmatrix}$$

```
ACE > map(SfSpecialBernoulli,op(1,deuxMat), h),
         map(SfSpecialBernoulli,op(2,deuxMat), e);
```

$$\begin{vmatrix} -1/2 & 0 & 0 & 0 & \frac{1}{47900160} \\ 1 & \frac{1}{12} & \frac{-1}{720} & \frac{1}{30240} & 0 \\ 0 & 1 & \frac{1}{12} & \frac{-1}{720} & 0 \\ 0 & 0 & 1 & \frac{1}{12} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} \frac{-1}{2} & \frac{1}{6} & \frac{1}{120} & \frac{1}{5040} & \frac{1}{362880} & \frac{1}{39916800} \\ 1 & \frac{-1}{2} & \frac{-1}{24} & \frac{-1}{720} & \frac{-1}{40320} & \frac{-1}{3628800} \\ 0 & 1 & \frac{1}{6} & \frac{1}{120} & \frac{1}{5040} & \frac{1}{362880} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{120} & \frac{1}{5040} \\ 0 & 0 & 0 & 1 & \frac{1}{6} & \frac{1}{120} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{6} \end{vmatrix}$$

```
ACE > map(det, %), bernoulli(10)/10!;;
              [1/47900160, 1/47900160] 1/47900160
```

The general rule to obtain a skew partition such that $S_{J/I}(\mathbb{B})$ reduces to a single entry is given by the following lemma.

Lemma 41 *For any positive integer n, the Bernoulli number B_{2n} is given by the following determinant of inverses of factorials :*

$$(-1)^{n-1} B_{2n}/(2n)! = \Lambda_{[2,3,\ldots,n-1,n,n,n]/[1,\ldots,n-1,n-1]}(\mathbb{B}) \ .$$

Proof. The pair of conjugate partitions is

$$[3,\,4,\,\ldots,\,n{+}1,\,n{+}1]/[2,\,3,\,\ldots,\,n]$$

and the associated determinant in the $S_k\mathbb{B}$ is such that indices in its top row are all odd integers ≥ 3, except the last one equal to $2n$ (beware that *ACE* uses decreasing partitions). QED

Our defining relations for the Bernoulli alphabet can be expanded using the Cauchy formula (with $B_1 = 1/2$) :

$$S_n((n{+}1)\mathbb{B}^+) = \sum_{J:|J|=n} \Psi_J(n{+}1) \prod_{i\in J} B_i/i! = 1\ . \tag{118}$$

```
# for functions of even degree, no difference between 𝔹+ and 𝔹
ACE >aa:=[seq(SfEval(m[op(pa)],7)*convert(map(i->bernoulli(i)/i!,pa),'*'),
          pa=ListPart(6))];
```

$$[\frac{1}{4320},\,0,\,\frac{-7}{1440},\,\frac{-7}{192},\,0,\,0,\,0,\,\frac{35}{1728},\,\frac{35}{96},\,\frac{35}{64},\,\frac{7}{64}]$$

```
ACE > convert(aa,'+');
                                1
```

The identity $\frac{d}{dx}\exp(x) = \exp(x)$ implies $\psi_i(\mathbb{A}) = -S^i(A)\ \forall i > 1$, and therefore the Newton relations between complete functions S^i and power sums Ψ_k

$$nS^n = \Psi_1 S^{n-1} + \Psi_2 S^{n-2} + \cdots \Psi_n S^0$$

give the following recursion between Bernoulli numbers :

$$-(n{+}1)S^n(\mathbb{B}) = \frac{-n{-}1}{n!} B_n = S_2(\mathbb{B})S_{n-2}(\mathbb{B}) {+} S_4(\mathbb{B})S_{n-4}(\mathbb{B}) {+}\cdots{+} S_{n-2}(\mathbb{B})S_2(\mathbb{B})\ . \tag{119}$$

None of the above identities for Bernoulli numbers is difficult to prove directly, but,in fact, *they require no proof, since they are a consequence of identities on symmetric functions.*

Uniform shift on alphabets, and binomial determinants

Before plunging into specializations, let us look at transformations $\mathbb{A} \to \mathbb{B}$, where $\mathbb{B}$ is an alphabet obtained from $\mathbb{B}$ such that symmetric functions of $\mathbb{B}$ are also symmetric functions of $\mathbb{A}$.

For example, one can take a function of one variable $\varphi(x)$ and define $\mathbb{A}^\varphi$ to be the alphabet $\{\varphi(a)\,:\, a \in \mathbb{A}\}$.

The simplest of these functions $x \to x+1$ already gives an interesting operation on $\mathfrak{S}ym(n)$. Let us denote $\mathbb{A}^+$ the shifted alphabet $\mathbb{A}^+ := \{(a{+}1) : a \in \mathbb{A}\}$.

Schur functions of $\mathbb{A}^+$ are easy to evaluate, taking their expression in terms of a minor of the Vandermonde matrix. Indeed, let $n = card(\mathbb{A})$. Then the Vandermonde matrix $\mathbb{V}(\mathbb{A}^+)$ factorizes into the product of the Vandermonde matrix $\mathbb{V}(\mathbb{A})$ by the infinite matrix of binomial coefficients $\mathcal{B}inom := \left[\binom{j}{i}\right]_{i,j\geq 0}$.

Therefore, for any partition $J \in \mathbb{N}^n$, one has a factorization of the matrix $\mathbb{V}_J(\mathbb{A}^+)$, and, remarking that $V_{0^n}(\mathbb{A}^+) = V_0(\mathbb{A})$, one obtains from Binet-Cauchy formula the following expression of $S_J(\mathbb{A}^+)$:

$$S_J(\mathbb{A}^+) = \sum_{I\subseteq J} S_I(\mathbb{A})\, \mathcal{B}inom(I,J) \ , \tag{120}$$

denoting by $\mathcal{B}inom(I,J)$ the submatrix taken on rows $i_1, i_2+1, \ldots, i_n{+}n{-}1$, columns $j_1, j_2+1, \ldots, j_n{+}n{-}1$ (numbering start from 0!).

```
ACE > ShiftSchur:=proc(pa,card) CLG_n(card);
Tos_n(x2m_n( expand(subs(seq(cat(x,i)=cat(x,i)+1,i=1..card),
Tox_n(s[op(pa)])))))
end:
# input two decreasing partitions, and cardinal
# one can truncate the matrix to the length of the bigger partition
ACE > DetBinom:=proc(out,inside,card) local i,j,k,big,small;
k:=nops(out); big:=[seq(out[i]+card-i,i=1..k)];
small:=[seq(inside[i]+card-i,i=1..nops(inside)),
           seq(card-nops(inside)-i,i=1..k-nops(inside))];
matrix([seq([seq(binomial(big[i],small[j]),j=1..k)],i=1..k)])
end:
ACE > aa:=DetBinom([4,3,1],[2,1],4);
```

$$aa := \begin{bmatrix} 21 & 35 & 7 \\ 1 & 10 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

```
ACE > det(aa),coeff(ShiftSchur([4,3,1],4), s[2,1]);
                              350 , 350
```

It is specially appropriate, in Tianjin, to give properties of the alphabet $\mathbb{A}^+$, because $\mathbb{A}^+$ is needed in the expansion of the *Chern classes of a tensor product of two vector bundles.* In non geometrical words, given two finite alphabets $\mathbb{A}$, $\mathbb{B}$, one wants to expand the product

$$\prod\nolimits_{a\in\mathbb{A},\, b\in\mathbb{B}} (1+a+b)\ .$$

Writing it $\prod_{a\in\mathbb{A},\, b\in\mathbb{B}} \Big((1+a)+b\Big)$, and introducing by commodity a rank 1 element z, one has solved the problem, because

$$\prod\nolimits_{a\in\mathbb{A},\, b\in\mathbb{B}} \Big((1+a)-zb\Big) = S_{\square}(\mathbb{A}^{+}-z\mathbb{B}) = \sum\nolimits_{I} (-z)^{|\square/I|} S_I(\mathbb{A}^{+})\, S_{\square^{\sim}/I^{\sim}}(\mathbb{B})\ , \tag{121}$$

where $\square = (card(\mathbb{B}))^{card(\mathbb{A})}$. Now, one can, leaving λ-rings, put $z = -1$ in the preceding identity which becomes

$$\prod\nolimits_{a\in\mathbb{A},\, b\in\mathbb{B}} (1+a+b) = \sum\nolimits_{I} S_I(\mathbb{A}^{+})\, S_{\square^{\sim}/I^{\sim}}(\mathbb{B})\ . \tag{122}$$

Geometry also requires the expansion of

$$\prod\nolimits_{a\in\mathbb{A},\, b\in\mathbb{B}} (1-(a+b))^{-1}\ .$$

The same reasoning as in the preceding case gives the coefficients of the expansion in the Schur basis, as minors of a matrix of binomial coefficients.

Geometry, still, needs the alphabet. $\mathbb{A}^{\diamond} := \{\left(\frac{1}{1-a}\right) : a\in\mathbb{A}\}$. The Vandermonde matrix $\mathbb{V}(\mathbb{A}^{\diamond})$ is the product of $\mathbb{V}(\mathbb{A})$ by a matrix of binomial coefficients. However, one has an extra factor coming from the Vandermonde :

$$\Delta(\mathbb{A}^{\diamond}) = \mathbb{V}_{0^n}(\mathbb{A}^{\diamond}) = \Delta(\mathbb{A}) \prod_{a\in\mathbb{A}} (1-a)^{1-n}\ ,$$

because $1/(1-a) - 1/(1-a') = (a-a')/(1-a)(1-a')$, with $n = card(\mathbb{A})$.

It implies that, for any $J\in\mathbb{N}^n$, $\Delta(\mathbb{A})\, S_J(\mathbb{A}^{\diamond})$ is equal to the minor of index J of the matrix

$$\Big[(1-a)^{n-1},\ ,\ (1-a)^{n-2}, \ldots, (1-a)^{-\infty}\Big]_{a\in\mathbb{A}}\ .$$

This last matrix factorizes into the product of $\mathbb{V}(\mathbb{A})$ by the matrix of binomial coefficients

$$\mathcal{B}inom^{\diamond n} := \Big[S_r(c-n+1)\Big]_{r,c\geq 0} = \left[\binom{c+r-n}{r}\right]_{r,c\geq 0}\ ,$$

and finally, with once more the help of Binet & Cauchy, one gets :

$$S_J(\mathbb{A}^{\diamond}) = \sum_{I\subseteq J} S_I(\mathbb{A})\, \mathcal{B}inom^{\diamond n}(I,J)\ . \tag{123}$$

Notice that, at the level of power sums,

$$\Psi_k(\mathbb{A}^\diamond) = \sum_a (1-a)^{-k} = \sum_{i\geq 0} S_i(k)\, \Psi_{(i)}(\mathbb{A}) \ .$$

One also needs

$$\mathbb{A}^\heartsuit := \left\{ \left(\frac{a}{1+a}\right) \ : \ a \in \mathbb{A} \right\} \ ,$$

but this requires only a minor adpatation of the preceding case, and the coefficients are minors of the matrix of binomial coefficients

$$\mathcal{B}inom^{\heartsuit n} := \Big[(-1)^{r-c} S_{r-c}(c-n+1)\Big]_{r,c\geq 0} = \left[(-1)^{r-c}\binom{r-n}{r}\right]_{r,c\geq 0} \ .$$

Alphabet of successive powers of q

In this section, we want to explicit the symmetric functions of the alphabet $\frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$, n being any positive integer, and q being a rank-1 element, and of the alphabet $\frac{1}{1-q} = 1 + q + q^2 + \cdots$. Littlewood [34] has consacred a chapter of his book to this subject.

Let us start with the most important functions, the Schur functions. Let J be a partition in $\mathbb{N}^n$, and $v := [j_1{+}0,\, j_1{+}1,\, \ldots,\, j_n{+}n{-}1]$. Using the Vandermonde matrix $\mathbb{V} := \mathbb{V}((1-q^n)/(1-q))$, one has that

$$S_J\left(\frac{1-q^n}{1-q}\right) = \mathbb{V}_J/\mathbb{V}_{0^n} = \det\Big((q^i)^{v_j}\Big)_{0\leq i<n,\, 1\leq j\leq n} / \mathbb{V}_{0^n} \ .$$

Interchanging the powers, one sees that $\mathbb{V}_J = \det\big((q^{v_j})^i\big)$, that is, is equal to $\Delta(q^{v_1}, \ldots, q^{v_n})$.

Therefore, up to a power of q, one has that

$$S_J\left(\frac{1-q^n}{1-q}\right) = q^\star \frac{\prod_{1\leq i<j\leq n} 1 - q^{v_j - v_i}}{\prod_{1\leq i<j\leq n} 1 - q^{j-i}} \ .$$

However, there can be many factors in common in the numerator and denominator.

Equation (3) allows to get rid of the factors $(1-q^{v_j-v_i})$ and gives

$$S_J\left(\frac{1-q^n}{1-q}\right) = q^\star \ \frac{\prod_i (v_i)!_q}{1!_q \cdots (n{-}1)!_q} \ , \tag{124}$$

writing $k!_q$ for the q-factorial $(1-q)\cdots(1-q^k)$. Simplifying now the factorials, and computing the value of $\star$, one finally gets a formula, published by Robinson[43], but that he attributes to Hall :

$$S_J\left(\frac{1}{1-q}\right) = q^{\sum_{i=1}^{\ell(J)}(\ell-i)j_i} \prod_{\square\in Diagr(J)} \frac{1-q^{n+c_\square}}{1-q^{h_\square}} \ . \tag{125}$$

Passing to the limit on n, one gets

$$S_J\left(\frac{1-q^n}{1-q}\right) = q^{\sum(n-i)j_i} \prod_{\square\in Diagr(J)} \frac{1}{1-q^{h_\square}} \ . \tag{126}$$

Realizing that, with a a rank-element, $S_J((1-a)/(1-q))$ is a rational function in a, with a appearing only in numerator[7], one also gets

$$S_J\left(\frac{1-a}{1-q}\right) = q^{\sum(n-i)j_i} \prod_{\square\in Diagr(J)} \frac{1-a\,q^{c_\square}}{1-q^{h_\square}} \ , \tag{127}$$

because this formula is valid for the infinity of values $a=q^n$.

---

q-specialization of monomial functions

Given a,b,q of rank 1, and a vector $v\in\mathbb{N}^n$, Lassalle [33] defines two weights $\varphi_1(v)$ and $\varphi_2(v)$ as follows :

$$\mathbb{N}^n \ni v=[v_1,\ldots,v_n] \to w := [0,\, v_1,\, v_1{+}v_2,\, v_1{+}v_2{+}v_3,\ldots,\, v_1{+}\cdots{+}v_n] \in \mathbb{N}^{n+1} \ ,$$

$$\varphi_1(v) \ = \ \frac{a^{v_1}q^{w_1}-b^{v_1}}{1-q^{w_2}}\,\frac{a^{v_2}q^{w_2}-b^{v_2}}{1-q^{w_3}}\cdots\frac{a^{v_n}q^{w_n}-b^{v_n}}{1-q^{w_{n+1}}} \tag{128}$$

$$\varphi_2(v) \ = \ \frac{a^{v_1}q^{(n-1)v_1}-b^{v_1}}{1-q^{w_2}}\,\frac{a^{v_2}q^{(n-2)v_2}-b^{v_2}}{1-q^{w_3}}\cdots\frac{a^{v_n}q^{0v_n}-b^{v_n}}{1-q^{w_{n+1}}} \tag{129}$$

(the only difference is in the powers of q in numerator).

Then Lassalle's theorem is

Theorem 42 *Let J be a partition, a,b,q be of rank 1. Then*

$$\Psi_J\left(\frac{a-b}{1-q}\right) = \sum_{H=Perm(J)} \varphi_1(H) = \sum_{H=Perm(J)} \varphi_2(H) \ , \tag{130}$$

sum over all different permutations of J.

[7] $\Psi_I((1-a)/(1-q)) = \prod_{i\in I}(1-a^i)/(1-q^i)$ has such property, and so has any symmetric polynomial, by linearity.

In fact, taking the components of the partition as indeterminates, i.e. writing $J = [x_1, \ldots, x_n]$, one sees that the preceding identities can be directly interpreted at the level of the symmetric group, and we defer their proofs to a subsequent chapter describing different statistics on the symmetric group.

```
Stat1:=proc(v) local i,j1,j2,pol; pol:=1;j1:=0;
 for i from 1 to nops(v) do
 j2:=j1+v[i];   # cumulated sums
 pol:= pol*(a∧v[i]*q∧j1 -b∧v[i])/(1-q∧j2);
 j1:=j2;
 od;
 pol
end:
Stat2:=proc(v) local n,i,j,pol; pol:=1;j:=0; n:=nops(v);
 for i from 1 to n do
 j:=j+v[i];   # cumulated sums
 pol:= pol*(a∧v[i]*q∧((n-i)*v[i]) -b∧v[i])/(1-q∧j);
 od;
 pol
end:
ACE> aa:=convert(map(Stat1,ListPerm([6,4,1])), `+`);
```

$$\frac{\left(a^6-b^6\right)\left(a^4q^6-b^4\right)\left(aq^{10}-b\right)}{\left(1-q^6\right)\left(1-q^{10}\right)\left(1-q^{11}\right)}+\frac{\left(a^6-b^6\right)\left(aq^6-b\right)\left(a^4q^7-b^4\right)}{\left(1-q^6\right)\left(1-q^7\right)\left(1-q^{11}\right)}$$
$$+\frac{\left(a^4-b^4\right)\left(a^6q^4-b^6\right)\left(aq^{10}-b\right)}{\left(1-q^4\right)\left(1-q^{10}\right)\left(1-q^{11}\right)}+\frac{\left(a^4-b^4\right)\left(aq^4-b\right)\left(a^6q^5-b^6\right)}{\left(1-q^4\right)\left(1-q^5\right)\left(1-q^{11}\right)}$$
$$+\frac{\left(a-b\right)\left(a^6q-b^6\right)\left(a^4q^7-b^4\right)}{\left(1-q\right)\left(1-q^7\right)\left(1-q^{11}\right)}+\frac{\left(a-b\right)\left(a^4q-b^4\right)\left(a^6q^5-b^6\right)}{\left(1-q\right)\left(1-q^5\right)\left(1-q^{11}\right)}$$

```
ACE> bb:=convert(map(Stat2,ListPerm([6,4,1])), `+`);
```

$$\frac{\left(a^6q^{12}-b^6\right)\left(a^4q^4-b^4\right)\left(a-b\right)}{\left(1-q^6\right)\left(1-q^{10}\right)\left(1-q^{11}\right)}+\frac{\left(a^6q^{12}-b^6\right)\left(aq-b\right)\left(a^4-b^4\right)}{\left(1-q^6\right)\left(1-q^7\right)\left(1-q^{11}\right)}$$
$$+\frac{\left(a^4q^8-b^4\right)\left(a^6q^6-b^6\right)\left(a-b\right)}{\left(1-q^4\right)\left(1-q^{10}\right)\left(1-q^{11}\right)}+\frac{\left(a^4q^8-b^4\right)\left(aq-b\right)\left(a^6-b^6\right)}{\left(1-q^4\right)\left(1-q^5\right)\left(1-q^{11}\right)}$$
$$+\frac{\left(aq^2-b\right)\left(a^6q^6-b^6\right)\left(a^4-b^4\right)}{\left(1-q\right)\left(1-q^7\right)\left(1-q^{11}\right)}+\frac{\left(aq^2-b\right)\left(a^4q^4-b^4\right)\left(a^6-b^6\right)}{\left(1-q\right)\left(1-q^5\right)\left(1-q^{11}\right)}$$

```
ACE> simplify(aa-bb), simplify(aa-SfEval(m[6,4,1],(a-b)/(1-q)));
                        0 , 0
```

Using the decomposition of monomial functions into the basis of power sums, one would get another expression of $\Psi_J((a-b)/(1-q))$ (recall that $\Psi^k((a-b)/(1-q)) = (a^k-b^k)/(1-q^k)$). For example, $J=[6,4,2]$ gives

```
ACE> map(SfEval, Top(m[6,4,1]), (a-b)/(1-q));
```

$$\frac{\left(a^6-b^6\right)\left(a^4-b^4\right)(a-b)}{(1-q^6)(1-q^4)(1-q)} - \frac{\left(a^6-b^6\right)\left(a^5-b^5\right)}{(1-q^6)(1-q^5)} - \frac{\left(a^{10}-b^{10}\right)(a-b)}{(1-q^{10})(1-q)} - \frac{\left(a^7-b^7\right)\left(a^4-b^4\right)}{(1-q^7)(1-q^4)} + \frac{2\,a^{11}-2\,b^{11}}{1-q^{11}}$$

Alphabet of square roots

We have mentionned the plethysm with a power sum Ψ^k, $k \in \mathbb{N}$. It can be seen as a transformation on alphabets

$$\mathbb{A} = \{a\} \to \Psi^k(\mathbb{A}) := \{a^k \ : a \in \mathbb{A}\} \ .$$

What about an inverse operation ?

Let us start with $k=2$. There is no reason to distinguish between $\sqrt{a}$ and $-\sqrt{a}$, and therefore, the most natural candidate to be the square root of an alphabet is

$$\mathbb{A}^{\surd} := \{\sqrt{a}, -\sqrt{a} \ : a \in \mathbb{A}\} \ .$$

At the level of power sums, it translates into

$$\Psi^{2k}(\mathbb{A}^{\surd}) = 2\psi^k(\mathbb{A}) \quad \& \quad \Psi^{2k+1}(\mathbb{A}^{\surd}) = 0\,, \forall k \in \mathbb{N}\ . \tag{131}$$

To determine the Schur functions $S_J(\mathbb{A}^{\surd})$, we shall, for a change, take their expression in terms of a determinant of powers sums, supposing $\mathbb{A}$ to be of cardinality n :

$$S_J(\mathbb{A}^{\surd}) = M_J/M_{0^n}\ , \text{ with } M = \left[\Psi^{i+j}((\mathbb{A}^{\surd})\right]_{i,j\geq 0}\,,\ J \in \mathbb{N}^{2n}\ . \tag{132}$$

The exponents in the first row of M_J

$$K = [j_1,\, j_2+1,\, \ldots,\, j_{2n}+2n-1]$$

determine the full determinant. Permuting columns in such a way as to get all even exponents first, and taking the rows in the order $1,3,\ldots,2n{-}1;\ 2,4,\ldots,2n$, one transforms M_J into a matrix with two blocks of zeros: $M_J \to \left[\begin{smallmatrix} \star & 0 \\ 0 & \star \end{smallmatrix}\right]$. If the blocks are not of size n, then it shows that $\det(M_J) = 0$. If, on the contrary, there exists $J', J'' \in \mathbb{N}^n$ such that

$$\{k_1, \ldots, k_{2n}\} = \{2j'_1, 2j'_2{+}2, \ldots, 2j'_n{+}2n{-}2\} \cup \{2j''_1{+}1, 2j''_2{+}3, \ldots, 2j''_n{+}2n{-}1\}, \tag{133}$$

then

$$\det(M_J) = \pm 2^{2n} \det(N_{J'}) \det(N_{J''}),\ N := \left[\Psi^{i+j}((\mathbb{A})\right]_{i,j\geq 0},$$

and finally,

$$S_J(\mathbb{A}^{\surd}) = \pm S_{J'}(\mathbb{A})\, S_{J''}(\mathbb{A}) . \tag{134}$$

For example, for $n = 3$, $J = [0, 0, 3, 3, 5, 5]$, then $K = [\underline{0}, 1, 5, \underline{6}, 9, \underline{10}]$, and M_J, once reordered, becomes

$$\begin{vmatrix} \Psi^0(\mathbb{A}^{\surd}) & \Psi^6(\mathbb{A}^{\surd}) & \Psi^{10}(\mathbb{A}^{\surd}) & 0 & 0 & 0 \\ \Psi^2(\mathbb{A}^{\surd}) & \Psi^8(\mathbb{A}^{\surd}) & \Psi^{12}(\mathbb{A}^{\surd}) & 0 & 0 & 0 \\ \Psi^4(\mathbb{A}^{\surd}) & \Psi^{10}(\mathbb{A}^{\surd}) & \Psi^{14}(\mathbb{A}^{\surd}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \Psi^2(\mathbb{A}^{\surd}) & \Psi^6(\mathbb{A}^{\surd}) & \Psi^{10}(\mathbb{A}^{\surd}) \\ 0 & 0 & 0 & \Psi^4(\mathbb{A}^{\surd}) & \Psi^8(\mathbb{A}^{\surd}) & \Psi^{12}(\mathbb{A}^{\surd}) \\ 0 & 0 & 0 & \Psi^6(\mathbb{A}^{\surd}) & \Psi^{10}(\mathbb{A}^{\surd}) & \Psi^{14}(\mathbb{A}^{\surd}) \end{vmatrix}$$

$$= \begin{vmatrix} \Psi^0(\mathbb{A}) & \Psi^3(\mathbb{A}) & \Psi^5(\mathbb{A}) \\ \Psi^1(\mathbb{A}) & \Psi^4(\mathbb{A}) & \Psi^6(\mathbb{A}) \\ \Psi^2(\mathbb{A}) & \Psi^5(\mathbb{A}) & \Psi^7(\mathbb{A}) \end{vmatrix} \begin{vmatrix} \Psi^1(\mathbb{A}) & \Psi^3(\mathbb{A}) & \Psi^5(\mathbb{A}) \\ \Psi^2(\mathbb{A}) & \Psi^4(\mathbb{A}) & \Psi^6(\mathbb{A}) \\ \Psi^3(\mathbb{A}) & \Psi^5(\mathbb{A}) & \Psi^7(\mathbb{A}) \end{vmatrix} .$$

Taking into account signs and the value of the initial minor $S_{0^{2n}}(\mathbb{A}^{\surd})$, it proves that

$$S_{3355}(\mathbb{A}^{\surd}) = S_{023}(\mathbb{A})\, S_{012}(\mathbb{A}) .$$

The alphabet $\mathbb{A}^{\surd}$ could have been defined by the equations

$$\Psi^2\left(\Psi^{2k}(\tfrac{1}{2}\mathbb{A}^{\surd})\right) = \Psi^k(\mathbb{A}) \ ,\ \Psi^{2k+1}(\tfrac{1}{2}\mathbb{A}^{\surd}) = 0\,,\ \forall k \in \mathbb{N} . \tag{135}$$

Thus $\Psi^2(\mathbb{A}^{\surd}) \neq \mathbb{A}$ and the transformation $\mathbb{A} \to \mathbb{A}^{\surd}$ is not the inverse of $\mathbb{A} \to \Psi^2(\mathbb{A})$. However, the following lemma shows the link between the two. We now take an infinite alphabet, to be able to use without restrictions the scalar product $(\ ,\ )_{\mathbb{A}}$.

Lemma 43 *Let Φ_2 be the adjoint operation to Ψ^2. Then Φ_2 is a ring endomorphism of $\mathfrak{S}ym$, and*

$$\Phi_2(\Psi^{2k}) = 2\Psi^k \quad \& \quad \Phi_2(\Psi^{2k-1}) = 0\,,\ \forall k \geq 1 . \tag{136}$$

Proof. Φ_2 is defined by the property

$$\forall f, g \in \mathfrak{S}ym,\ \big(\Phi_2(f)\,,\, g\big) = \big(f\,,\, \Psi^2(g)\big)\ .$$

Because the scalar product is induced from the scalar product of each component of the tensor product

$$\mathfrak{S}ym \simeq \mathbb{C}[\Psi^1] \otimes \mathbb{C}[\Psi^2] \otimes \mathbb{C}[\Psi^3] \otimes \cdots\ ,$$

it is sufficient to test the lemma on each space $\mathbb{C}[\Psi^k]$. The only non-zero scalar products are

$$\big(\Psi^2(\Psi^{k^m})\,,\, \Psi^{(2k)^m}\big) = (2k)^m\, m!\,,\ m \in \mathbb{N}\ ,$$

but they are equal to

$$\big(\Psi^{k^m}\,,\, 2^m\, \Psi^{k^m}\big) = 2^m\, k^m m!$$

and this proves all the assertions of the lemma. QED

One could have used the generating function of complete function, instead of determinants in power sums. Indeed

$$\sigma_z(\mathbb{A}^{\surd}) = \prod_{a\in\mathbb{A}} \frac{1}{1 - z\sqrt{a}} \frac{1}{1 + z\sqrt{a}} = \prod_{a\in\mathbb{A}} \frac{1}{1 - z^2 a} \tag{137}$$

and therefore the alphabet $\mathbb{A}^{\surd}$ is characterized by the equations

$$S^{2k}(\mathbb{A}^{\surd}) = S^k(\mathbb{A}) \quad \& \quad S^{2k+1}(\mathbb{A}^{\surd}) = 0\,, \forall k \in \mathbb{N}\ , \tag{138}$$

or

$$\Lambda^{2k}(\mathbb{A}^{\surd}) = (-1)^k \Lambda^k(\mathbb{A}) \quad \& \quad \Lambda^{2k+1}(\mathbb{A}^{\surd}) = 0\,, \forall k \in \mathbb{N}\ . \tag{139}$$

```
# enter a symmetric function, and the basis chosen for specialization
SquareRootAlphabet:=proc(sf0,b) local sf,i,Ind,Ind0,cof,cof2;
if b='e' then sf:=Toe(sf0); cof:=1;cof2:=-1;
elif b='h' then sf:=Toh(sf0); cof:=1; cof2:=1;
elif b='p' then sf:=Top(sf0); cof:=2;cof2:=1;
fi;
Ind:=`SYMF/Sf2TableVar`(sf,b);
Ind0:=select( proc(i) evalb( type(i,even)) end, Ind);
Ind:=Ind minus Ind0;
subs(seq(cat(b,i)=0,i=Ind),seq(cat(b,i)=cof2^(i/2)*cof*cat(b,i/2),i=Ind0),sf)
end:
ACE > aa:=map(factor,[SquareRootAlphabet(s[5,5,3,3],e),
    SquareRootAlphabet(s[5,5,3,3],h),SquareRootAlphabet(s[5,5,3,3],p)]:
ACE > map(z->map(Tos,z),aa);
       [s[2, 1] s[3, 2], s[2, 1] s[3, 2], s[] s[2, 1] s[3, 2]]
```

The factorization of a Schur function $S_J(\mathbb{A}^{\surd})$, when it is different from zero, into a product of Schur functions, is also straightforward to obtain by reordering the Jacobi-Trudi determinant expressing S_J.

The preceding computations can be visualized on the diagram of J. Indeed, equation (133) can be rewritten as

$$\exists\, \sigma \in \mathfrak{S}_{2n} \text{ such that } J+\rho-\rho^{\sigma} \text{ is a vector in } 2\mathbb{N}^{2n}\ ,$$

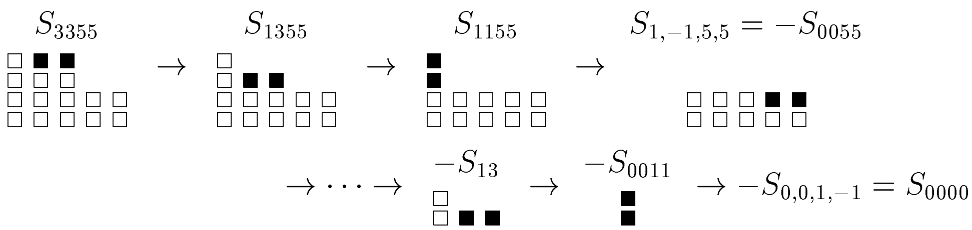
(i.e. $J+\rho-\rho^{\sigma}$ is a vector with even components).

Therefore, one can obtain the function $\pm S_{0\ldots0}$ from S_J by iterating the operation

"substract 2 to a component of the index of a non-zero Schur function"

and this amounts to build a sequence of partitions, from J to the empty partition, differing each time by a vertical or horizontal domino.

For example,



It is easy to see that the non-empty partitions from which one cannot obtain another partition by subtracting a domino, are exactly the *staircases* $[123\ldots k]$, $k \geq 1$ (they are called *2-cores*, and one could check that $S_J(\mathbb{A}^{\surd}) = 0$ iff one can obtain a 2-core from J by erasing dominos from the diagram of J.

To understand why obtaining a 2-core from a partition is independent of the order in which one erases dominos, it is better to consider the more general case of a p-core, $p \in \mathbb{N}$, $p \geq 2$. This is what we shall do in the next section.

---

p-cores and p-quotients

To describe conveniently the transformation that we have effected on $S_J(\mathbb{A}^{\surd})$, and to generalize it to roots of any order, it is convenient to introduce another combinatorial object used in modular representations of the symmetric group [35], [22].

Let p be an integer, $p \geq 2$. One numbers consecutively the integral points of the plane, of x-coordinates $0, -1, \ldots, 1-p$, as follows (the y-axis is pointing downwards, to stick to the most frequent conventions) :

$$
\begin{array}{cccc|c}
 & & & & \textit{level} \\
\vdots & & \vdots & \vdots & \vdots \\
1-2p & \cdots & -p-1 & -p & -1 \\
1-p & \cdots & -1 & 0 & 0 \\
\hline
1 & \cdots & p-1 & p & 1 \\
1+p & \cdots & 2p-1 & 2p & 2 \\
\vdots & & \vdots & \vdots & \vdots
\end{array} \; .
$$

Let μ be a partition, *considered as an infinite decreasing vector, with only a finite number of non-zero components.* Thus, we reorder usual partitions and concatanate an infinite string of 0's. Using the conventions of physicists, (for whom these vectors are bases of Fock spaces), let $|\emptyset\rangle$ be the *vacuum vector* $|\emptyset\rangle := [0,\ -1,\ -2,\ -3, \ldots]$, and let

$$|\mu\rangle := [\mu_1,\ \mu_2-1,\ \mu_3-2, \ldots] \; .$$

To $|\mu\rangle$, associate the *p-abacus* of beads placed at the points $\mu_1,\ \mu_2-1,\ \mu_3-2, \ldots$. Each column contains the components of $|\mu\rangle$ having same residue modulo p.

Consider now each column of the p-abacus separately. Some beads have been pushed down, and their displacement is recorded by a partition (as we did for minors of a matrix) : given any bead, count how many empty spaces lie above it. For example

$$
\begin{array}{cc}
\vdots & \vdots \\
\bigcirc & 0 \\
\bigcirc & 0 \\
\bullet & \cdot \\
\bigcirc & 1 \\
\bullet & \cdot \\
\bullet & \cdot \\
\bigcirc & 3
\end{array}
\qquad \text{gives the partition} \qquad [3, 1, 0, 0, \ldots]
$$

The p-uple of such partitions read from the p-abacus of μ is called the *p-quotient* of μ. Packing the beads upwards, one gets the p-abacus of another partition, which is called the *p-core* of μ.

Of course, from the p-core and the p-quotient, one reconstructs the p-abacus of a partition, and thus, for any integer $p \geq 2$, there exists a bijection

$$(p\text{-core},\ p\text{-quotient}) \leftrightarrow \text{partition} \; .$$

Moving one bead upwards by one step (when it is possible) corresponds to subtracting p to one of the component of $|\mu\rangle$, and to erasing a ribbon of length p from the edge of the diagram of μ. It implies, in particular, that p-cores are those partitions from which one cannot peel off a p-ribbon.

One can code p-cores differently, by listing the levels of the bottom beads in each column. Because the number of beads in the lower plane is equal to the number of holes in the upper plane, p-cores are in bijection with vectors $v \in \mathbb{Z}^p$, of null sum $|v| := v_1 + \cdots + v_p = 0$ (called 0-weights).

Let ν be the p-core of μ. Evaluate $|\nu\rangle$ and $|\mu\rangle$ modulo p (i.e. replace each component by their residue modulo p). Then these two new vectors differ by a (minimal) permutation, the sign of which is called the *p-sign* of the partition μ.

When the p-core is null, the p-sign of μ counts the minimal number of transpositions to reorder the residue of $|\mu\rangle$ into the vector $[0, p\text{–}1, \ldots, 1, 0, p\text{–}1, \ldots, 1, 0, \ldots]$.

```
Part2Abacus:=proc(pa,p) local n,i,j,v,part,ma,x;
part := [op(pa),0$(p-irem(nops(pa),p) )];
                 # length must be a multiple of p
n := nops(part);
v := [seq(part[i]-i+1, i=1..n)]; lprint([seq(modp(v[i],p),i=1..n)]);
ma:=matrix(n/p+ ceil(pa[1]/p),p);
for i from 1 to rowdim(ma) do
for j from 1 to p do x:=-n+1+p*(i-1)+j-1;
if member(x,v) then ma[i,j]:=x
else ma[i,j]:=`.`;
fi;
od;
od;
eval(ma)
end:
ACE> Part2PCore([8,8,5,4,4,4,4,2,1,1,1],3); # p-core= first component
                  [[4, 2], [1], [4, 2, 1], [2, 2]]
ACE> Part2Abacus([8,8,5,4,4,4,4,2,1,1,1],3), Part2Abacus([4,2,0$8], 3);
[2,1,0,1,0,2,1,1,2,1,0,1] , [1,1,1,0,2,1,0,2,1,0,2,1,0,2,1]
```

$$\left[\begin{array}{ccc} -11 & . & -9 \\ -8 & -7 & . \\ -5 & . & . \\ -2 & -1 & 0 \\ \hline 1 & . & 3 \\ . & . & . \\ 7 & 8 & . \end{array}\right] , \left[\begin{array}{ccc} -11 & -10 & -9 \\ -8 & -7 & -6 \\ -5 & -4 & -3 \\ -2 & . & . \\ \hline 1 & . & . \\ 4 & . & . \\ . & . & . \end{array}\right]$$

The 3-core of $[8,8,5,4,4,4,4,2,1,1,1]$ is $[4,2]$, which corresponds to the vector $[2, -1, -1]$. Its 3-quotient is $[[1],[4,2,1],[2,2]]$, and its 3-sign is $+1$.

Alphabet of p-th roots

It is now easy to adapt our analysis of square roots to p-th roots, $p \geq 2$. Let ζ be a primitive root of unity, and $\Omega := \{\zeta^0, \zeta, \ldots, \zeta^{p-1}\}$ be the alphabet of p-th roots of unity. Define

$$\mathbb{A}^{p\surd} := \Omega\, \mathbb{A} = \cup_{a\in\mathbb{A}}\{a^{1/p}, \zeta a^{1/p}, \ldots, \zeta^{p-1}a^{1/p}\} \ . \tag{140}$$

Then

$$\sigma_z(\mathbb{A}^{p\surd}) = \sigma_z(\Omega\mathbb{A}) = \prod_{a\in\mathbb{A}} \frac{1}{1-z^p a} = \exp\left(\sum_{i=1}^{\infty} z^{ip} p\Psi^i(\mathbb{A})/ip\right) \ , \tag{141}$$

and therefore

$$S^{pk}(\mathbb{A}^{p\surd}) = S^k(\mathbb{A}) \quad \& \quad S^k(\mathbb{A}^{p\surd}) = 0 \text{ if } k \not\equiv 0 \mod p \ , \tag{142}$$

$$\Psi^{pk}(\mathbb{A}^{p\surd}) = p\Psi^k(\mathbb{A}) \quad \& \quad \Psi^k(\mathbb{A}^{p\surd}) = 0 \text{ if } k \not\equiv 0 \mod p \ . \tag{143}$$

Given n and $J \in \mathbb{N}^{pn}$, then by reordering the rows and columns of the Jacobi-Trudi matrix expressing S_J, one obtains a determinant which is non zero iff J has no p-core, the diagonal blocks being the Jacobi-Trudi determinants of $S_{J^0}(\mathbb{A}), \ldots, S_{J^{p-1}}(\mathbb{A})$, $[J^0, \ldots, J^{p-1}]$ being the p-quotient of J. Indeed, the reordering of columns exactly corresponds to grouping the numbers j_1, $j_2{+}1$, $\ldots$, $j_{pn}{+}pn{-}1$, according to their residues modulo p.

Let Φ_p be the transformation $\mathbb{A} \to \mathbb{A}^{p\surd}$ (defined as a endomorphism of $\mathfrak{S}ym$ by equations (142) or (143)). Then the same proof as for the case $p = 2$ gives the following characterization of Φ_p.

Lemma 44 *The adjoint of Ψ^p with respect to the canonical scalar product on $\mathfrak{S}ym$ is Φ_p.*

We can now state the following proposition, due to Littlewood [35].

Theorem 45 *Let p, n be two positive integers, $\mathbb{A}$ be an alphabet, J a partition in $\mathbb{N}^{pn}$.*

If J has a non-empty p-core, then $S_J(\mathbb{A}^{p\surd}) = 0$. Else, let $[J^0, \ldots, J^{p-1}]$ be the p-quotient of J, $\epsilon_p(J)$ be its p-sign. Then

$$\Phi_p(S_J)(\mathbb{A}) = S_J(\mathbb{A}^{p\surd}) = \epsilon_p(J)\, S_{J^0}(\mathbb{A}) \cdots S_{J^{p-1}}(\mathbb{A}) \ . \tag{144}$$

```
# choose the basis b='p' or 'h' in which to expand sf0
PHI:=proc(sf0,n,b) local sf,i,lp,lpn,val;
if b='p' then sf:=Top(sf0); val:=n
elif b='h' then sf:=Toh(sf0); val:=1
fi;
lp:=`SYMF/Sf2TableVar`(sf,b);
lpn:=select(proc(i,n) evalb(modp(i,n)=0) end, lp,n);
subs(seq(cat(b,i)=val*cat(b,i/n),i=lpn),
subs(seq(cat(b,i)=0,i=lp minus lpn),sf))
end:
ACE> map(Tos,factor(PHI(s[9, 5, 4, 3, 2, 2, 2],3,h)));
                     - s[] s[1] s[2, 1] s[3, 1, 1]
ACE> Part2PCore([9, 5, 4, 3, 2, 2, 2], 3);
                      [[], [2, 1], [1], [3, 1, 1]]
```

Alphabet of p-th roots of 1

Given a positive integer p, we have met the alphabet $\Omega = \Omega^{(p)}$ of p-th roots of 1. Let us complete the description of the specialization of symmetric functions in Ω.

From the preceding section, putting $\mathbb{A} = \{1\}$, one already knows :

Lemma 46 *The only non-zero specializations in the alphabet Ω of complete, elementary functions and power sums are*

$$\Lambda^p(\Omega) = (-1)^{p-1}\,,\ S^{(pk)}(\Omega) = 1\,,\ \Psi^{(pk)}(\Omega) = p\,. \tag{145}$$

If J is a partition in $\mathbb{N}^p$ with empty p-core, then $S_J(\Omega) = \epsilon_p(J)$. Otherwise $S_J(\Omega) = 0$.

There essentially remains to determine the values of monomial functions in Ω. We shall follow A. Forsyth [10] and Thrall [51].

Let us take another alphabet $\mathbb{A}$, and let ζ be a primitive p-th root of unity. Then, according to Cauchy's formula

$$\prod_{a\in\mathbb{A}}(1-(-a)^p) = \prod_{i=0}^{p-1}\lambda_{\zeta^i}(\mathbb{A}) = \lambda_1(\Omega\mathbb{A}) = \sum_I \Psi_I(\Omega)\Lambda^I(\mathbb{A})\ . \tag{146}$$

Decomposing

$$\begin{aligned}\lambda_\zeta(\mathbb{A}) = \left(1+\Lambda^p(\mathbb{A})+\Lambda^{2p}(\mathbb{A})+\cdots\right)+\zeta\left(\Lambda^{p+1}(\mathbb{A})+\Lambda^{2p+1}(\mathbb{A})+\cdots\right)+\\ \cdots+\zeta^{p-1}\left(\Lambda^{p-1}(\mathbb{A})+\Lambda^{2p-1}(\mathbb{A})+\cdots\right)\\ =\theta_0+\zeta\theta_1+\cdots+\zeta^{p-1}\theta_{p-1}\ ,\end{aligned} \tag{147}$$

i.e. collecting the $\Lambda^k(\mathbb{A})$ according to the residue of k modulo p, one also has

$$\lambda_1(\Omega\mathbb{A}) = \prod_{i=0}^{p-1} \left(\theta_0 + \zeta^i\theta_1 + \cdots + \zeta^{(p-1)i}\theta_{p-1}\right) . \tag{148}$$

However, this last product, putting $\theta_{i\pm p} = \theta_i$, is equal to the determinant (cf. ex ?)

$$\begin{vmatrix} \theta_0 & \theta_{p-1} & \cdots & \theta_1 \\ \theta_1 & \theta_0 & & \theta_2 \\ \vdots & \vdots & & \vdots \\ \theta_{p-1} & \theta_{p-2} & \cdots & \theta_0 \end{vmatrix} = \left|\theta_{i-j}\right|_{1\le i,j\le p} \tag{149}$$

The expansion of such determinant is determined by the case where $\mathbb{A}$ is of cardinality p–1, but implies the expansion of $\lambda_1(\Omega\mathbb{A})$ for any $\mathbb{A}$. For example, for $p=3$, $card(\mathbb{A})=2$, one has

$$\lambda_1(\Omega\mathbb{A}) = \Lambda^{000}(\mathbb{A}) + \Lambda^{111}(\mathbb{A}) + \Lambda^{222}(\mathbb{A}) - 3\Lambda^{012}(\mathbb{A})$$

and this implies, for a general $\mathbb{A}$,

$$\lambda_1(\Omega\mathbb{A}) = \sum \Psi_I(\Omega)\Lambda^I(\mathbb{A}) = \theta_0^3+\theta_1^3+\theta_2^3-3\theta_0\theta_1\theta_2 = \begin{vmatrix} 1+\Lambda^3+\cdots & \Lambda^2+\Lambda^5+\cdots & \Lambda^1+\Lambda^4+\cdots \\ \Lambda^1+\Lambda^4+\cdots & 1+\Lambda^3+\cdots & \Lambda^2+\Lambda^5+\cdots \\ \Lambda^2+\Lambda^5+\cdots & \Lambda^1+\Lambda^4+\cdots & 1+\Lambda^3+\cdots \end{vmatrix} .$$

Instead of taking $\lambda_1(\Omega\mathbb{A})$ as generating function of the coefficients $\lambda_1(\Omega\mathbb{A}) = \Psi_I(\Omega)$, one can use :

$$\sigma_1(\Omega\mathbb{A}) = \prod_{a\in\mathbb{A}} (1-a^p)^{-1} = \Psi^p(\sigma_1(\mathbb{A})) . \tag{150}$$

It implies, for any $k \in \mathbb{N}$,

$$S^{(pk)}(\Omega\mathbb{A}) = \Psi^p(S^k(\mathbb{A})) = \sum_{I:|I|=pk} \Psi_I(\Omega) S^I(\mathbb{A}) , \tag{151}$$

$$\Psi_I(\Omega) = \left(S^{pk}(\Omega\mathbb{A}) \,,\, \Psi_I(\mathbb{A})\right)_{\mathbb{A}} = \left(\Psi^p(S^k(\mathbb{A})) \,,\, \Psi_I(\mathbb{A})\right)_{\mathbb{A}} \\ = \left(S^k(\mathbb{A}) \,,\, \Phi_p(\Psi_I(\mathbb{A}))\right)_{\mathbb{A}} \tag{152}$$

In other words, because the scalar products (S^k, Ψ^J), $|J| = k$, are all equal to 1, one can obtain $\Psi_I(\Omega)$ from the expansion of $\Psi_I(\mathbb{A})$ in the basis of power sums, as follows. Let $\hbar(J) = p^\ell(J)$ if all parts of J are divisible by p, and $\hbar(J) = 0$ otherwise. Then

$$\Psi_I = \sum_J c_J \Psi^J \quad \text{implies} \quad \Psi_I(\Omega) = \sum_J \hbar(J)\, c_J . \tag{153}$$

For example, for $p = 4$,

$$\Psi_{246} = \Psi^6\Psi^2\Psi^4 - (\Psi^6)^2 - \Psi^{10}\Psi^2 - \Psi^8\Psi^4 + 2\Psi^{12}$$

gives $\Psi_{246}(\Omega) = 0 + 0 + 0 - 4^2 + 2 \times 4 = -8$.

Before closing the subject, let us remark that in the case where I is such that $|I| = p$, then equation (152) shows that the value of $\Psi_I(\Omega)$, $I = 1^{m_1}2^{m_2}\cdots$, is given by Waring's formula :

$$\Psi_I(\Omega) = p(-1)^{\ell(I)-1}\frac{(\ell(I)-1)!}{m_1!\, m_2! \cdots}\,. \tag{154}$$

Ex. 70 Let k be a positive integer, J be a partition $J \in \mathbb{N}^k : J \supseteq [2^{k-1}, 3]$, $|J|$ odd. Show that the the forgotten symmetric function of index J vanishes on the Bernoulli alphabet $\mathbb{B}$. For example, in weight 7, one has vanishing for $J \in \{[7], [2,5], [3,4], [2,2,3]\}$.

Corr.Ex.70. The expansion of the forgotten function involves product of power sums of degrees sums of parts of J. For the choosen partitions, there is no occurence of Ψ_1 and always a Ψ_{2i+1} in each product. For the same reason, the monomial functions $\Psi_J(\mathbb{B})$ also vanish.

Ex. 71 To the $n \times (n-1)$ matrix with entries $[i,j]$ equal to $2^{i-j}/(i-j+1)!$, $i < j$, entries $[i,i]$ equal to 1, the others being 0, add a first column $[1/1!, 1/2!, \ldots, 1/n!]$ and show, after Hernandez (MuirV, p.260), that the determinant is 0 for n odd.

Corr.Ex.71. Once more, one has to recognize the Bernoulli alphabet, the determinant being proportional to the Bernoulli number of order n.

Ex. 72 Let n, m be two postive integers. Write

$$1^m + 2^m + \cdots + n^m = c_1 n^{m+1} + c_2 n^m + \cdots + c_{m+1} n \ ,$$

and show, after Ferrari (Muir V, P.262) that

$$(m+1)m \cdots (m-k+2)\, c_k$$

is equal to the determinant of order $k-1$ with entries $[i,j] = \binom{m+2-j}{i+2-j}$, for $j \leq i+1$, other entries being 0.

For example, for $k = 5$,

$$(m+1)\cdots(m-3)\, c_5 = \begin{vmatrix} \binom{m+1}{2} & m & 0 & 0 \\ \binom{m+1}{3} & \binom{m}{2} & m-1 & 0 \\ \binom{m+1}{4} & \binom{m}{3} & \binom{m-1}{2} & m-2 \\ \binom{m+1}{5} & \binom{m}{4} & \binom{m-1}{3} & \binom{m-2}{2} \end{vmatrix} .$$

Corr.Ex.72. Write each binomial $\binom{n}{k}$ as $n!/k!(n-k)!$. Factorizing out the common factorials in rows and columns, one is left with $\Lambda_{1^{k-1}}(\mathbb{A})$, with $\mathbb{A}$ the Bernoulli alphabet : $\Lambda^i(\mathbb{A}) = 1/(i+1)!$. Therefore, Ferrari's result is another way of writing that power sums of $1, \ldots, n$ are given by Bernoulli polynomials.

Ex. 73 Let n be an integer. Take the lower triangular matrix of order n with entries $[i,j] = 1/(2i-2j+2)!$, $i \le j$. Border it with a first column $[1/1!, 1/3!, \ldots, 1/(2n-1)!]$, and with a bottom row $[1/(2n-2)!, 1/(2n-1)!, \ldots, 1/3!, 1/1!]$. Show that the determinant of such matrix is null. For example, for $n = 4$,

$$\begin{vmatrix} 1 & 1/2 & 0 & 0 & 0 \\ 1/6 & 1/24 & 1/2 & 0 & 0 \\ 1/120 & 1/720 & 1/24 & 1/2 & 0 \\ 1/5040 & 1/40320 & 1/720 & 1/24 & 1/2 \\ 1/720 & 1/5040 & 1/120 & 1/6 & 1 \end{vmatrix} = 0\ .$$

Corr.Ex.73. We have to permute the first and second column, as well as row n and row $n+1$ to get monotonous sequences. Now, we recognize

$$\Lambda_{[1,\ldots,n-2,\,n-1,\,n-1,\,n-1]/[0001\ldots n-2]}(\mathbb{A})$$

with $\mathbb{A}$ such that $\Lambda^i(\mathbb{A}) = 1/(i+1)!$, that is $\mathbb{A}$ is the Bernoulli alphabet. In terms of the $S_i(\mathbb{A})$, the Schur function is $S_{[34\ldots n+1]/[01\ldots n-2]}(\mathbb{A})$, which is null, because its first row is composed of multiples of Bernoulli numbers of odd index.

Ex. 74 Let n be an integer. Show that the skew multi-Schur function

$$S_{[2;\ldots;n+2;\,n]/[0,0,1,\ldots,n]}(2n{+}3;\ 2n{+}1, \ldots,\ 3;\ 2)$$

is null.

Corr.Ex.74. (MuirV, p. 268: Williams Am.Math. Monthly 23 (1916) 263-264). Extracting common factorials in rows and columns, one gets, apart from the first column, a determinant of inverses of factorials, which shows a connection with the Bernoulli alphabet and the nullity is a consequence of the vanishing of Bernoulli numbers of odd index.

Ex. 75 Following Pascal (MuirV, p.254) show that the following determinant of order $n{+}1$ vanishes :

$$\begin{vmatrix} 1/1! & 2 & 0 & \cdots & 0 \\ 1/3! & 1/2! & 2 & \cdots & 0 \\ 1/5! & 1/4! & 1/2! & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1/(2n-1)! & 1/(2n-2)! & 1/(2n-4)! & \cdots & 2 \\ 1/(2n)! & 1/(2n-1)! & 1/(2n-3)! & \cdots & 1/1! \end{vmatrix} = 0$$

Corr.Ex.75. Transposing the determinant, and multiplying the $n-1$ first columns by $1/2$, one can recognize $S_{12\ldots nn/0012\ldots n-1}(\mathbb{A})$, for $\mathbb{A}$ such that $S_i(\mathbb{A}) = 1/2i!$, $i = 1, 2, \ldots$. Now, this alphabet is the "Genocchi alphabet", defined by

$$\lambda_z(\mathbb{A}) = \frac{z}{1 + exp(z)} = z + \sum_{n \geq 1} z^{2n}(-1)^n G_{2n}/(2n)! \ ,$$

where $G_{2n} = 2(1 - 2^{2n})\, Bernoulli(2n)$ is a Genocchi number. This alphabet is such that $\Lambda^{2i}(\mathbb{A}) = 0$, $i = 1, 2, \ldots$. Since the current Schur function is equal to $\Lambda_{23\ldots n+1/01\ldots n-1}(\mathbb{A})$, which has a first row composed of even elementary symmetric functions, it vanishes.

Ex. 76 Compute the following determinant of binomial coefficients, and generalize it to any order :

$$\begin{vmatrix} \binom{5}{1} & \binom{5}{3} & \binom{5}{5} & 0 \\ \binom{5}{0} & \binom{5}{2} & \binom{5}{4} & 0 \\ 0 & \binom{5}{1} & \binom{5}{3} & \binom{5}{5} \\ 0 & \binom{5}{0} & \binom{5}{2} & \binom{5}{4} \end{vmatrix} .$$

Corr.Ex.76. This is $S_{1234}(5)$, and therefore, it is equal to 2^{10}. The general case is $S_{12\ldots n}(n+1)$, which is equal to $2^{n(n-1)/2}$.

Ex. 77 Let $\mathbb{A}$ be defined by $\Lambda^k(\mathbb{A}) = 1/k!$, $k \geq 1$. Compute all the monomial functions $\Psi_J(\mathbb{A})$.

Corr.Ex.77. Use a second alphabet $\mathbb{B}$. Then, according to Cauchy :

$$\prod\nolimits_{a \in \mathbb{A}, b \in \mathbb{B}} (1 + ab) = \sum \Lambda^k(\mathbb{A}\mathbb{B}) = \sum \Psi_J(\mathbb{A})\Lambda^J(\mathbb{B}) \ .$$

But $\prod_a (1 + ab)$ specializes into $\exp(b)$, and therefore $\prod_{a,b}(1 + ab)$ specializes into $\sum (\Lambda^1(\mathbb{B}))^k / k!$. All monomial functions of $\mathbb{A}$ are null, except for $J = 1^k$, $k \geq 0$.

Cayley (Am.J.M. **7** (1885)47-56) uses this property to control his tables of expansion of the monomial functions in the basis of elementary symmetric functions.

Ex. 78 Generalize the preceding exercise. Take a rank-1 element x, define $\mathbb{A}$ by $S^k(\mathbb{A}) = x^k/k!$. Given any homogeneous symmetric function of degree n, show that the scalar product $(f, (S_1)^n)$ (which is the dimension of the virtual representation of $\mathfrak{S}_n$ associated to f) is equal to $n!\, f(\mathbb{A})\, x^{-n}$.

Corr.Ex.78. Since $\sigma_z(\mathbb{A}) = exp(zx)$, then $\Psi_1(\mathbb{A}) = x$, $\Psi_i(\mathbb{A}) = 0$, $i > 1$. One concludes, using

$$f(\mathbb{A}) = \sum (f, \Psi^J)\, \Psi^J(\mathbb{A})/(\Psi^J, \Psi^J) = (f, (\Psi_1)^n)\, x^n/n!\ .$$

Ex. 79 Given two integers n, m, take the two matrices $M^{\pm}$ with entries $M^{\pm}[i,j] = \binom{2i+1-2j}{i-j}$, $1 \le j \le i$, $M^{\pm}[i,i{+}1] = \pm 2i$, $1 \le i < n$, $M^{\pm}[n,j] = \binom{2n+m-2-2j}{n-j}$, $1 \le j \le n$.

Compute their determinant, after Muir (1918) (MuirV, p.273).

Corr.Ex.79. One recognizes the determinantal expression of $n!\,\Lambda^n$ or $n!\,S^n$ in terms of power sums, after dividing rows by 2, apart from the last row. Having an arbitrary last row is treated in Ex.(35).

Thus, we are essentially asked to compute the complete and elementary symmetric functions of the alphabet $\mathbb{A}$ such that

$$\Psi^i(\mathbb{A}) = \frac{1}{2}\binom{2i-1}{i-1}\ ,$$

which is equal to half of the Catalan alphabet $\mathbb{B}$: $\Psi^i(\mathbb{B}) = \binom{2i-1}{i-1}$.

Using the generating function of Catalan numbers

$$\sigma_z(\mathbb{B}) = (1 - \sqrt{1-4z})/2\ ,$$

we find that

$$\sigma_z(\mathbb{A}) = \sqrt{\frac{1-\sqrt{1-4z}}{2z}}\ ,$$

$$S^n(\mathbb{A}) = \frac{(2n+1)(2n+3)\cdots(4n-1)}{n!\,2^n}\ \&\ \Lambda^n(\mathbb{A}) = (-1)^{n-1}\frac{(2n-1)(2n+1)\cdots(4n-3)}{n!\,2^n}$$

and

$$\begin{aligned} det(M^+) &= (-1)^{n-1}(2n+2m-5)(2n+2m-3)\cdots(4n+2m-9) \\ det(M^-) &= (2n+2m-3)(2n+2m-1)\cdots(4n+2m-7)\ . \end{aligned}$$

```
SpecDemiCatalan:=proc(sf0) local i,sf; sf:=Top(sf0);
subs(seq(cat(p,i)=binomial(2*i-1,i-1)/2, i=Sf2TableVar(sf,p)), sf)
end:
MatrixMuir:=proc(n,m,sgn) local i,j;
```

```
matrix([
seq([seq(binomial(2*i+1-2*j,i-j),j=1..i),2*i*sgn,0$(n-i-1)],i=1..n-1),
[seq(binomial(2*n-2-2*j+m,n-j),j=1..n)] ])
end:
```

$ACE >$ `seq(ifactor(2∧(2*i-1)*SpecDemiCatalan(cat(h,i))),i=1..6);`

```
1, (7), (2)(3)(11), (5)(11)(13), (2)(13)(17)(19), (2)(7)(17)(19)(23)
```

$ACE >$ `seq(ifactor(2∧(2*i-1)*SpecDemiCatalan(cat(e,i))),i=1..6);`

```
1,-(5), (2)(3)(7),-(3)(11)(13), (2)(11)(13)(17), -(2)(7)(13)(17)(19)
```

$ACE >$ `factor(expand(det( MatrixMuir(5,m,-1) )));`

```
(2 m + 13) (2 m + 11) (2 m + 9) (2 m + 7)
```

Ex. 80 Let k be a positive integer. Define an alphabet $\mathbb{A}$ by the equations

$$S^i(\mathbb{A}) = S^{i-1}(k\mathbb{A}) \ , \ i \geq 1 \ .$$

Compute the complete, elementary functions and power sums of $\mathbb{A}$.

Corr.Ex.80. The functional equation for $\sigma_z(\mathbb{A})$ is

$$\sigma_z(\mathbb{A}) = 1 + z\,\sigma_z(k\mathbb{A}) == 1 + z\,\big(\sigma_z(\mathbb{A})\big)^k \ ,$$

from which one deduces functional equations for $\lambda_z(\mathbb{A})$ and for $\log\big(\sigma_z(\mathbb{A})\big)$.

One solves them by induction, or with the help of ACE, obtaining

$$S^i(\mathbb{A}) = \frac{1}{ki{+}1}\binom{ki{+}1}{i} \ , \ \Lambda^i(\mathbb{A}) = (-1)^{i-1}\frac{k{-}1}{ki{-}1}\binom{ki{-}1}{i{-}1} \ , \ \Psi^i(\mathbb{A}) = \frac{ki{-}i{+}1}{k(ki{+}1)}\binom{ki{+}1}{i} \ .$$

In the case $k = 2$, the $S^i(\mathbb{A})$ are the Catalan numbers. U. Tamm (*Some aspects of Hankel matrices in Coding Theory and Combinatorics*, Electronic J. 8 (2001) # A1) gives the continued fraction expansion of $\sigma_z(\mathbb{A})$ in the case $k = 3$.

Ex. 81 Let Φ be the adjoint of the algebra automorphism of $\mathfrak{S}ym$ induced by $\Psi_i \to \Psi_i + \Psi_{i+1}$, $i \geq 1$. Show that Φ is an algebra morphism and determines it.

Gives the image of $\sigma_z = \sum z^i S^i$ under Φ.

Corr.Ex.81. It is easy to see that the equations

$$(\Psi_i, \Phi(\Psi_j)) = (\Psi_i + \Psi_{i+1}\,,\, \Psi_j) = j \text{ if } i \in \{j, j{-}1\} \text{ or } 0 \text{ otherwise},$$

have the solution

$$\Phi(\Psi_1) = \Psi_1\ , \quad \Phi(\Psi_i) = \Psi_i + \frac{i}{i-1}\Psi_{i-1}\, i > 1\ .$$

With these explicit values, one checks that Φ is compatible with product. The image of $\sigma_z = \exp(\sum_{i>0} z^i \Psi_i/i)$ is

$$\exp(\sum_{i>0} z^i \Psi_i/i)\ \exp(\sum_{i>1} z^i \Psi_{i-1}/(i-1))\ .$$

Ex. 82 Let m, n be two positive integers. Compute after Mignosi (1907), MuirV p.257, the determinant of order n, with entries $[i,j] = m,\ i \le j$, $[i, i{+}1] = -i$, and 0 elsewhere.

Corr.Ex.82. This is the expression of $n!\, S_n(\mathbb{A})$ as a determinant of power sums, with $\Psi^i(\mathbb{A}) = m$, i.e. with $\mathbb{A} = m$ (which could be a complex number). Therefore the determinant is equal to

$$m(m+1)\cdots(m+n-1)\ .$$

Ex. 83 Let i, j, k, n be non negative integers ($j > 0$). Show that

$$\frac{(n{+}i)!}{n!\, i!}\, \frac{(n{+}i{+}j)!}{(n{+}1)!\,(i{+}j)}\, j\, \frac{(n{+}i{+}2j)!}{(n{+}2)!\,(i{+}2j)}\, 2j^2 \cdots \frac{(n{+}i{+}kj)!}{(n{+}k)!\,(i{+}kj)}\, kj^k$$

is an integer. Find the q-analogue of this property.

Corr.Ex.83. The above expression is equal to $S_I(n)$, with

$$I = [i,\ i{+}j{-}1,\ i{+}2j{-}2, \ldots,\ i{+}kj{-}k]\ .$$

Of course, its q-analogue is $S_I\big((1{-}q^n)/(1{-}q)\big)$.

Ex. 84 Let p, r be positive integers. Show that

$$\sum_{k\geq 0} q^{k(p+r)} \frac{(1-q^{-p})\cdots(1-q^{k-1-p})}{(1-q^{1+r})\cdots(1-q^{k+r})(1-q^{-1})\cdots(1-q^{-k})} = \frac{1}{(1-q^{1+r})\cdots(1-q^{p+r})}$$

Corr.Ex.84. Up to a power of q, this is the expansion

$$S^p\left(\frac{1-1/a}{1-1/q} + \frac{1}{a-a/q}\right) = S^p(\frac{1}{1-1/q})$$

Ex. 85 Let m, n be two integers. Write $G(m;n)$ for the Gauss polynomial $(1-q)\cdots(1-q^{m-n+1})/(1-q)\cdots(1-q^n)$. Compute the value of

$$\det\left(1/G(m+i+j-2\,;\,j)\right)_{1\leq i,j\leq n} \ .$$

Composto (1916; MuirV, p.349) gave a similar determinant, replacing Gauss polynomials by binomial coefficients $\big($ which corresponds to taking the limit, for $q \to 1$, of a multiple by an appropriate power of $(1-q)\big)$.

Corr.Ex.85. Getting rid of numerators by factoring them out, and completing denominators in q-factorials, one tranforms the determinant into

$$\det\left(1/(1-q)\cdots(1-q^{m+i+j-2})\right) \ ,$$

that is, in $S_{(m+n-1)^n}(1/(1-q)$ (whose value is given in (eq:Hook3)). In summary, one finds, up to sign and a power of q, a product of q-integers and inverses of q-integers.

For example, for $m=5$, $n=3$, one finds

$$\begin{vmatrix} \frac{1}{G(5,1)} & \frac{1}{G(6,2)} & \frac{1}{G(7,3)} \\ \frac{1}{G(6,1)} & \frac{1}{G(7,2)} & \frac{1}{G(8,3)} \\ \frac{1}{G(7,1)} & \frac{1}{G(8,2)} & \frac{1}{G(9,3)} \end{vmatrix} = -q^{21}[2]^3\,[3]\,[5]^{-1}\,[6]^{-2}\,[7]^{-3}\,[8]^{-2}\,[9]^{-1} \ .$$

Ex. 86 Show that

$$(1-q)(1-q^3)\cdots(1-q^{2m-1}) = \sum_0^{2m}(-1)^i\begin{bmatrix}2m\\ i\end{bmatrix}$$

(sum of Gauss polynomials).

Corr.Ex.86. Dividing by $(1-q)(1-q^2)\cdots(1-q^{2m})$, one transforms the identity into $S_m(1/(1-q^2)) = \sum(-1)^i S_i(1/(1-q))\, S_{2m-i}(1/(1-q))$. However, the right member is equal to

$$\sum_{i=0}^{m} S_{i,2m-i}\left(\frac{1}{1-q}\right) = S_m\left(\Psi^2(\frac{1}{1-q})\right) .$$

This identity is due to Gauss.

Ex. 87 Let $p \in \mathbb{N}$, and let ζ be a p-primitive root of unity. Define $\binom{n}{k}_\zeta$ by the recursion

$$\binom{n}{k}_\zeta = \zeta^k \binom{n-1}{k}_\zeta + \binom{n-1}{k-1}_\zeta , \quad \binom{n}{0}_\zeta = 1 .$$

Show that, for $m, n, j, k \in \mathbb{N}$, $n = mp + r$, $k = jp + h$, $0 \le r, h \le p{-}1$, one has

$$\binom{n}{k}_\zeta = \binom{m}{j}\binom{r}{h}_\zeta .$$

Corr.Ex.86. (Kalyuzhnyj, Vest. Kharkov Un. **230**(1982) 73-82). The binomial type coefficient $\binom{n}{k}_\zeta$ is the specialization $q = \zeta$ of the Gauss polynomial $\begin{bmatrix} n \\ k \end{bmatrix}$, the recursion being identified with

$$q^k S^k\Big(\frac{1-q^{n-k}}{1-q}\Big) = S^k\Big(\frac{1-q^{n+1-k}}{1-q} - 1\Big) = S^k\Big(\frac{1-q^{n+1-k}}{1-q}\Big) - S^{k-1}\Big(\frac{1-q^{n+1-k}}{1-q}\Big) .$$

Suppressing equal factors and using that $(1-q^{jp})/(1-q^p)$ specializes into j, one gets the required formula.

For example, for $p = 3$, $n = 8$, $k = 4$,

$$\frac{(1-\zeta^8)(1-\zeta^7)(1-\zeta^6)(1-\zeta^5)}{(1-\zeta)(1-\zeta^2)(1-\zeta^3)(1-\zeta^4)} = \frac{(1-\zeta^2)(1-\zeta^6)}{(1-\zeta)(1-\zeta^2)} = 2\frac{1-\zeta^2}{1-\zeta} = \binom{2}{1}\binom{2}{1}_\zeta .$$

Ex. 88 Let k be a positive integer. Show that

$$\frac{a^k q^{k(k-1)}}{(1{-}a)\cdots(1{-}aq^{k-1})} = \frac{1}{(1{-}a)\cdots(1{-}aq^{k-1})} + \sum_{i\ge 1} (-1)^i q^{\binom{i}{2}} \frac{(1-q^k)\cdots(1-q^{k-i-1})}{(1{-}q)\cdots(1{-}q^i)(1{-}a)\cdots(1{-}aq^{k-1-i})}$$

For example, for $k = 2$, one has

$$\frac{a^2q^2}{(1-a)(1-q)} = \frac{1}{(1-a)(1-q)} - \frac{1+q}{1-a} + q .$$

Corr.Ex.88. First, it suffices to show the identity for $a = q^{p+1}$, $p \in \mathbb{N}$. In that case, dividing both members by $(1-q)\cdots(1-q^p)$, one recognizes

$$S^{p+k}\left(\frac{q^k}{1-q}\right) = \sum_i (-1)^i \Lambda^i \left(\frac{1-q^k}{1-q}\right) S^{p+k-i}\left(\frac{1}{1-q}\right) .$$

Ex. 89 Show that

$$\sum_{i\geq 0} z^i \frac{(1+q)(1+q^3)\cdots(1+q^{2i-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2i})} = \prod_{i\geq 0} \frac{1+zq^{2i+1}}{1-zq^{2i}} .$$

Corr.Ex.89. Change q into q^2, a in aq in the q-binomial identity. Specialize now a into -1.

Ex. 90 Prove the identity in the preceding exercise by identifying the right hand side with $\lambda_z(q/(1-q^2))\,\sigma_z(1/(1-q^2))$, with q of rank 1.

Corr.Ex.90. One has

$$S^i\left(\frac{1}{1-q^2}\right)\Lambda^j\left(\frac{q}{1-q^2}\right) = S^k\left(\frac{1}{1-q^2}\right)\Lambda^j\left(\frac{q-q^{2k+1}}{1-q^2}\right) ,$$

with $k = i+j$, and therefore the coefficient of z^k in $\lambda_z(q/(1-q^2))\,\sigma_z(1/(1-q^2))$ is

$$S^k\left(\frac{1}{1-q^2}\right)\sum_{j\geq 0}\Lambda^j\left(\frac{q-q^{2k+1}}{1-q^2}\right) = S^k\left(\frac{1}{1-q^2}\right)(1+q)(1+q^3)\cdots(1+q^{2k-1}) .$$

Ex. 91 Show, after Vilenkin, that, for n, k, r positive integers, the binomial coefficient $\binom{nk}{r}$ is equal to

$$\binom{nk}{r} = \sum_I \binom{n}{1}^{m_1}\binom{n}{2}^{m_2}\cdots \frac{1}{m_1!\,m_2!\cdots}\,\frac{k!}{(k-\ell(I))!} ,$$

sum over all partitions $I = 1^{m_1}2^{m_2}\cdots$ of weight r.

Corr.Ex.91. (Vilenkin, Math Notes **24**(1978)658-660). This is a specialization of the Cauchy identity :

$$\Lambda^r(nk) = \sum_{I:\,|I|=r} \Lambda^I(n)\,\Psi_I(k) .$$

Ex. 92 Let a, b be two elements of binomial type. Let $\mathbb{A}$, $\mathbb{A}'$, $\mathbb{B}$ be the alphabets such that

$$S_i(\mathbb{A}) = \left(i! S_i(a)\right)^{-1} \ , \ S_i(\mathbb{A}') = 1/S_i(a) \ , \ S_i(\mathbb{B}) = S_i(b)/S_i(a) \, , \ i = 1, 2, \ldots$$

Show that for any positive integer n, any partition $J \in \mathbb{N}^n : J \supseteq (n{-}1)^n$, the Schur functions $S_J(\mathbb{A})$, $S_J(\mathbb{A}')$ and $S_J(\mathbb{B})$ factorize into simple factors.

Corr.Ex.92. Some help can be obtained from ACE.

```
ACE Det_ab:= proc(pa)  local i,j,n; n:=nops(pa);
 factor(det( array([seq(map(proc(k) if (k>0) then
    SfAExpand(h[k](b))/SfAExpand(h[k](a))
      elif(k=0)then 1 else 0 fi end,[seq(pa[i]-i+j,j=1..n)]),i=1..n)])));
end:
ACE Det_ab([3,2,2]);
                                           2                          2
                                           b  (b + 1) (b + 2) (b - a)  (b - a - 1)
                                        6 ---------------------------------------
                                             3        3        2
                                            a  (a + 1)  (a + 2)  (a + 3) (a + 4)
```

Ex. 93 Let B, C be variables, α, β be integers. Show that for any positive integer n, any partition $J \in \mathbb{N}^n : J \supseteq (n{-}1)^n$, the Schur function $S_J(\mathbb{A})$ specializes into simple factors when $\mathbb{A}$ is such that, for $i = 1, 2, \ldots$,

$$S_i(\mathbb{A}) = (B{-}q^{\alpha})(B{-}q^{\alpha+1}) \cdots (B{-}q^{\alpha+i-1})/(C{-}q^{\beta})(C{-}q^{\beta+1}) \cdots (C{-}q^{\beta+i-1}) \, .$$

Corr.Ex.93.

One can use another manner than in the preceding exercise to specialize Schur functions.

```
ACE > SpecialSf_BC:= proc(sf,a,b) local i,j,sf2;
sf2:=Toh(sf);
factor( subs(seq(cat(h,i)= convert([seq((B-q∧(a+j))/(C-q∧(b+j)),
j=0..i-1)],‘*‘), i=Sf2TableVar(sf2,’h’)), sf2));
end:
ACE > SpecialSf_BC(s[3,2,2], 5,4);
```

$$\frac{(B-q^5)^2(B-q^6)(B-q^7)q^{15}(q+1)(q^2+q+1)(q-1)^3(C-B)(qC-B)^2}{(C-q^4)^3(C-q^5)^3(C-q^6)^2(C-q^7)(C-q^8)}$$

Ex. 94 Let $n \in \mathbb{N}$, $\beta \in \mathbb{R}$, $\mathbb{A}$ of cardinality $\le n{-}1$, $\mathbb{B}$ the set of n-th roots of β.

Show that $\prod_{a\in\mathbb{A}} \prod_{b\in\mathbb{B}} (1+ab)$ is equal to the determinant of order n

$$\begin{vmatrix} \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \cdots & \Lambda^{n-2}(\mathbb{A}) & \Lambda^{n-1}(\mathbb{A}) \\ \beta\Lambda^{n-1}(\mathbb{A}) & \Lambda^0(\mathbb{A}) & \cdots & \Lambda^{n-3}(\mathbb{A}) & \Lambda^{n-2}(\mathbb{A}) \\ \vdots & \vdots & & \vdots & \vdots \\ \beta\Lambda^1(\mathbb{A}) & \beta\Lambda^2(\mathbb{A}) & \cdots & \beta\Lambda^{n-1}(\mathbb{A}) & \Lambda^0(\mathbb{A}) \end{vmatrix} .$$

Corr.Ex.94. (Kapteyn(1881); MuirIV, p.315). Let $\mathbb{B}^\vee := \{1/b\}_{b\in\mathbb{B}}$. Since $\beta\Lambda^k(\mathbb{A}+\mathbb{B}^\vee) = \beta\Lambda^k(\mathbb{A}) + (-1)^{n-1}\Lambda^{k-n}(\mathbb{A})$, the determinant is equal to $\beta^n\Lambda_{n^n}(\mathbb{A}+\mathbb{B}^\vee)$, up to the exchange of β and $(-1)^{n-1}\beta = \Lambda^n(\mathbb{B})$. Developping $\mathbb{A}+\mathbb{B}$, one gets that

$$\beta^n \sum_I \Lambda_I(\mathbb{A})\Lambda_{n^n/I}(\mathbb{B}^\vee) = \sum \Lambda_I(\mathbb{A})\Lambda_I(\mathbb{B}) \ .$$

On the other hand, according to Cauchy,

$$\prod(1+ab) = \sum \Lambda_I(\mathbb{A})\Lambda_{I^\sim}(\mathbb{B}) \ ,$$

but one also has that $\Lambda_{I^\sim}(\mathbb{B}) = \pm\Lambda_I(\mathbb{B})$, when $I \subseteq n^n$, the n-sign of I and $I^\sim$ being not necessarily the same. However, we had changed β into $(-1)^{n-1}\beta$, and there remains to control signs, to be able to conclude. One could also have factorized Kapteyn's matrix into the product

$$\begin{bmatrix} \Lambda^0(\mathbb{A}) & \cdots & \Lambda^{2n-1}(\mathbb{A}) \\ \vdots & & \vdots \\ \Lambda^{-n+1}(\mathbb{A}) & \cdots & \Lambda^n(\mathbb{A}) \end{bmatrix} \begin{bmatrix} \mathbf{1}_{n\times n} \\ \boldsymbol{\beta}_{n\times n} \end{bmatrix} ,$$

but this amounts factorizing

$$\beta^n\Lambda_{n^n}(\mathbb{A}+\mathbb{B}^\vee)$$

into a matrix with entries $\Lambda^k(\mathbb{A})$ and another, with entries $\Lambda^k(\mathbb{B})$.

Ex. 95 Let p be a positive integer, $\mathbb{A}$ a finite alphabet. Find the minimal polynomial, with coefficients in $\mathfrak{S}ym(\mathbb{A})$, having the root $\sum_{a\in\mathbb{A}} a^{1/p}$.

Corr.Ex.95. Let ζ be a p-primitive root of unity, $\mathbb{B} = \{b\}$ be any alphabet such that $\Psi^p(\mathbb{B}) = \mathbb{A}$. Taking into account the arbitrariness in the choice of $\mathbb{B}$, one sees that the required polynomial has roots

$$\sum_{0 \le i_b \le p-1,\, b \in \mathbb{B}} \zeta^{i_b} b \ .$$

Let $\mathbb{D}$ be the alphabet of such roots (it is of cardinality p^n, with $n = card(\mathbb{A})$). Then $\Psi^k(\mathbb{D}) = 0$ if $k \neq\equiv 0 \mod p$. Expanding each $(\sum \zeta^{i_b} b)^{kp}$, and keeping teh terms independent of ζ, one sees that

$$\Psi^{kp)}(\mathbb{D}) = p^n \sum_{I:|I|=k} \frac{(kp)!}{(pm_1)!(pm_2)!\cdots} \Psi_I(\mathbb{A}) \ ,$$

writing $I = 1^{m_1} 2^{m_2} \cdots$. One finishes the computation by using the expression of the elementary symmetric functions in terms of power sums.

MacMahon (Collected Papers I, p.58-60) essentially considers the case of the product of all elements of $\mathbb{D}$.

Ex. 96 Let ζ be a $(2m+1)$-primitive root of unity, and $\mathbb{A} = \{1, \zeta, \ldots, \zeta^{2m-1}\}$. Show that

$$\Lambda^i(\mathbb{A}) = (-\zeta)^{-i} \ , \ i = 0, \ldots, 2m \ .$$

Use exercice 86 to prove the following identity due to Gauss :

$$\sum_{j=0}^{2m} \zeta^{j^2} = (\zeta - \zeta^{-1})(\zeta^3 - \zeta^{-3}) \cdots (\zeta^{2m-1} - \zeta^{1-2m}) \ .$$

Corr.Ex.96. Let $x = \zeta^{2m}$. Then $\mathbb{A} + x$ is the alphabet Ω of $2m+1$)-roots of unity, and

$$\Lambda^i(\mathbb{A}) = \Lambda^i(\Omega) - x\Lambda^{i-1}(\Omega) + \cdots + (-x)^i$$

is equal to $(-x)^i = (-\zeta)^{-i}$.

To deduce Gauss formula, one has to notice that the sets of numbers $\{0^2, 1^2, \ldots, (2m)^2\}$ and $\{m^2 - \binom{0}{2} - 0, \ldots, m^2 - \binom{i}{2} - i, \ldots, m^2 - \binom{2m}{2} - 2m\}$ coincide modulo $2m+1$.

Ex. 97 Let $n \in \mathbb{N}$, $\mathbb{A}$ be the set of n-roots of -1. Show that

$$\sum_{a \in \mathbb{A}} (1-a)^{-1} = n/2 \ , \ \sum_{a \in \mathbb{A}} (1-a)^{-2} = n(2-n)/4 \ .$$

Corr.Ex.97. (D. Svrtan, *Proof of Scott's conjecture*, Proc. AMS **87** (1983) 203-207). Take a new alphabet $\mathbb{B} := \{b := (1-a)^{-1}\}_{a \in \mathbb{A}}$. Then $\mathbb{B}$ is the set of roots of the polynomial $(x-1)^n + x^n$. Waring's formula gives

$$\sum (1-a)^{-1} = \Psi_1(\mathbb{B}) = n/2 \, , \ \Psi_2(\mathbb{B}) = n(2-n)/4 \, , \ldots$$

Ex. 98 (Graeffe Method for localizing roots). Let $\mathbb{A}$ be a finite alphabet, $P(x) = P^0(x) := \prod_{a\in\mathbb{A}}(x-a)$. For any positive integer k, define $P^k(x) = P^{k-1}(\sqrt{x})P^{k-1}(-\sqrt{x})$. Compute the coefficients of $P^k(x)$ in terms of those of $P^0(x)$.

Corr.Ex.98.

$$P^1(x) = \prod(\sqrt{x}-a)(-\sqrt{x}-a) = \prod(a^2-x)$$

and therefore

$$P^k(x) = (-1)^k \prod(x-a^{2^k}) \ .$$

The coefficients are, up to sign, the monomial functions $\Psi_{rj}(\mathbb{A})$, $r=2^k$, $0\le j\le card(\mathbb{A})$.

We have seen that the expansion of Ψ_{rj} is the sum (with signs) of all Schur functions without r-core, such that their r-quotient is of the type $[1^{i_1},\ldots,1^{i_r}]$, $i_1+\cdots+i_r=j$. Therefore, it is the sum $\pm\Lambda_J$, J partition without r-core, wtih r-quotient of the type $[i_r,\ldots,i_1]$. One has to expand these Schur functions in terms of the $\Lambda^i(\mathbb{A})$.

For example, for $k=2=j$, one obtains 10 partitions: 6 such that their 4-quotient is a permutation of $[1,1,0,0]$, 4 such that it is a permutation of $[2,0,0,0]$.

```
ACE> sf:=SfOmega(Tos(m[4,4])); # Schur functions indexed by conjugate
partitions
sf:=-s[7,1] +s[4,4] +s[8] -s[5,1,1,1] +s[2,2,2,2] -s[4,3,1] +s[4,2,1,1]
    +s[6,1,1] +s[3,3,2] -s[3,2,2,1]
ACE> tt:=Sf2Table(sf,'s'):  map(Part2PCore,map(op,[indices(tt)]),4);
[[[],[],[],[1],[1]], [[],[1],[],[],[1]], [[],[],[1],[1],[]],
[[],[2],[],[],[]], [[],[],[2],[],[]], [[],[],[],[2],[]],
[[],[],[1],[],[1]],[[],[],[],[],[2]],[[],[1],[1],[],[]],[[],[1],[],[1],[]]]
```

Ex. 99 Let $m\in\mathbb{N}$, θ an irational number. For every $k\ge 0$, define

$$\begin{Bmatrix} m \\ k \end{Bmatrix} = \frac{\sin m\theta\ \sin(m-1)\theta\cdots(m-k+1)\theta}{\sin\theta\ \sin 2\theta\cdots\sin k\theta} \ .$$

Let $\zeta := \exp(\theta\sqrt{-1})$, and

$$\mathbb{A} = \{\zeta^{-m+1},\, \zeta^{-m+3},\ldots,\, \zeta^{m-3},\, \zeta^{m-1}\} \ .$$

Show that, for all $k \geq 0$, $p \geq 1$, one has

$$\begin{aligned}
\Lambda^k \mathbb{A} &= \begin{Bmatrix} m \\ k \end{Bmatrix} , \\
\Lambda_{k^p}(\mathbb{A}) &= \begin{Bmatrix} m \\ k \end{Bmatrix} \cdots \begin{Bmatrix} m+p-1 \\ k \end{Bmatrix} \Big/ \begin{Bmatrix} k \\ k \end{Bmatrix} \cdots \begin{Bmatrix} k+p-1 \\ k \end{Bmatrix}
\end{aligned}$$

Corr.Ex.99. (S. Karlin, *Total Positivity*, Stanford Un. Press (1968) 396–399).

$$\prod_{a \in \mathbb{A}} (1 + az\zeta^{m-1}) = (1 - z\zeta^0)(1 - z\zeta^2) \cdots (1 - z\zeta^{2m-2}) \ ,$$

and therefore, up to powers of ζ, any Schur function $S_I(\mathbb{A})$ is equal to $S_I\big((1-q^m)/(1-q)\big)$, with $q = \zeta^2$.

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