Wrońskian of symmetric functions

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Abstract

We introduce a notion of Wrońskian of symmetric functions.

Most dictionaries reduce the mathematical work of Józef Maria Hoëné Wroński to the definition of the *Wrońskian* of a family of functions of a single variable. This is far from paying justice to the many contributions of Wroński in fields like algorithms, approximation theory, continued fractions, &c.

For example, Wroński gives in different places of [13, 14] continued fractions like

$$\cfrac{1}{(1+\zeta_0 y) - \cfrac{y^2\beta_1}{(1+\zeta_1 y) - \cfrac{y^2\beta_2}{(1+\zeta_2 y) - \cfrac{y^2\beta_3}{\ddots}}}}$$

which are usually attributed to Jacobi [2], Stieltjes [10] (see the details in chapters 4 and 5 of [4]). He also gives fractions to approximate a function $F(x)$ knowing its values in $0, \xi, 2\xi, \ldots$:

$$F(x) = A_0 + \cfrac{\phi(x)}{A_1 + \cfrac{\phi(x{-}\xi)}{A_2 + \cfrac{\phi(x{-}2\xi)}{A_3 + \cfrac{\phi(x{-}3\xi)}{A_4 + \ddots}}}} \quad ,$$

with $\phi(x) = x + b_1x^2 + b_2x^3 + \ldots$. The case $\phi(x) = x$ is attributed to Thiele [11].

Wroński also claimed a *universal factorization theorem* [12], that is, he claimed to be able to factorize polynomials in a single variable over $\mathbb{C}$. This does not contradict Abel and Galois theory if one interprets his formulas as solving an approximation problem [3]. Indeed, given a polynomial in one variable which has no zero on the unit circle, Wroński provides an approximation of the factor comprising the roots of modulus less than one, or bigger than one respectively.

In this text, we shall only consider Wrońskians. Given an integer n, and n functions $f_1(x), \ldots, f_n(x)$ of a single variable x, the *Wrońskian* is the determinant $\left|f_j^{(i)}(x)\right|_{\substack{i=0\ldots n-1\\ j=1\ldots n}}$, which is of fundamental importance in the theory of linear differential equations (for the algebraic theory, see the relevant chapters [6, 7]).

However, one is not restricted to using only derivatives, one can take their discrete versions, q-derivatives or divided differences.

The symmetric group may be used to compute functions of $\mathbf{x}_n = \{x_1, \ldots, x_n\}$. *Simple transpositions* s_i act by transposing x_i, x_{i+1}, the action being denoted exponentially $f \to f^{s_i}$. The *divided difference* ∂_i, acting on its left, is the operator

$$f(\mathbf{x}_n) \to f(\mathbf{x}_n)\,\partial_i = (f - f^{s_i})\,(x_i - x_{i+1})^{-1}\,. \tag{1}$$

Notice that functions of x_1 are functions of $\mathbf{x}_n$ of degree 0 in $x_2, \ldots, x_n$, and thus products of divided differences may be used for functions of a single variable.

Divided differences furnish *discrete Wrońskians* [4, 9.3.1] of functions of x_1 :

$$W(f_1(x_1), \ldots, f_n(x_1)) = \Big|f_j(x_1)\partial_1 \ldots \partial_i\Big|_{\substack{i=0\ldots n-1\\ j=1\ldots n}}, \tag{2}$$

with the convention that a product of divided differences of degree 0 is the identity.

The following lemma shows that $W(f_1(x_1), \ldots, f_n(x_1))$ is a symmetric function of $\mathbf{x}_n$.

Lemma 1 *One has*

$$W(f_1(x_1), \ldots, f_n(x_1)) = \Big|f_i(x_j)\Big|_{\substack{i=1\ldots n\\ j=1\ldots n}} \frac{1}{\Delta(\mathbf{x}_n)}, \tag{3}$$

where $\Delta(\mathbf{x}_n) = \prod_{1\le i<j\le n}(x_j - x_i)$ *is the* Vandermonde *of* $\mathbf{x}_n$.

Proof. Newton's interpolation formula [4, 9.1.1] states that

$$f_j(x_i) = f_j(x_1) + f_j\partial_1(x_i - x_1) + \cdots + f_j\partial_1 \ldots \partial_{i-1}(x_i - x_1)\ldots(x_i - x_{i-1})\,.$$

Therefore, one can transform the determinant $|f_i(x_j)|$ into the determinant $|f_i(x_1)\partial_1 \dots \partial_{i-1}(x_i - x_1)\dots(x_i - x_{i-1})|$. Extracting the factor $\Delta(\mathbf{x}_n)$ produces the lemma. QED

When $f_1(x_1), \dots, f_n(x_1)$ are powers of x_1, the Wrońskian is equal to a Schur function. More precisely, given a partition $\lambda = [\lambda_1, \dots, \lambda_n] : \lambda_1 \geq \dots \geq \lambda_n \geq 0$, then $W\left(x_1^{\lambda_1+n-1}, x_1^{\lambda_2+n-2}, \dots, x_1^{\lambda_n}\right)$ is the *Schur function* of index λ [5].

Starting instead with Newton's polynomials $x^{<k>} = x(x{-}1)(x{-}2)\dots(x{-}k{+}1)$, then $W\left(x_1^{<\lambda_1+n-1>}, \dots, x_1^{<\lambda_n>}\right)$ is the *factorial Schur function* of index λ [9]. Putting now $x^{<k>} = (x{-}1)(x{-}q)\dots(x{-}q^{k-1})$, then $W\left(x_1^{<\lambda_1+n-1>}, \dots, x_1^{<\lambda_n>}\right)$ is the *q-factorial Schur function* of index λ. More generally, $x^{<k>} = (x - y_1)(x{-}y_2)\dots(x{-}y_k)$ leads to *Graßmannian Schubert polynomials*, more details will come later.

One may also use the q-derivative: $D_q : f \to \big(f(x) - f(qx)\big)(x{-}qx)^{-1}$, but this is a limit case of divided differences for the alphabet $\{x, qx, q^2x, \dots\}$. Indeed, writing $[k]$ for the q-integer $1{+}q{+}\cdots{+}q^{k-1}$, one sees that $D_q^k/[1]\dots[k]$ is the operator which sends x^r onto $x^{r-k}\frac{[r]\dots[r-k+1]}{[1]\dots[k]}$. Since $\partial_1\dots\partial_{k-1}$ sends x_1^r onto the sum of all monomials in $x_1, \dots, x_k$ of total degree $r-k$, the operator $\partial_1\dots\partial_{k-1}$, acting on functions of x_1, indeed specializes into $D_q^k/[1]\dots[k]$.

If a function is symmetrical in x_i, x_{i+1}, then it is sent to 0 by ∂_i. Starting with symmetric functions in $\mathbf{x}_n$, one can however use ∂_n, and more generally, products of divided differences beginning by ∂_n. To denote the products that we shall need, we use a graphical display.

Given n, a partition $\mu = [\mu_1, \dots, \mu_r]$, then ∂_n^μ is the product

$$
\begin{array}{ccccc}
\partial_n & \dots & \dots & \dots & \partial_{n+\mu_1-1} \\
\partial_{n-1} & \dots & \dots & \partial_{n+\mu_2-2} & \\
\vdots & & \vdots & & \\
\partial_{n+1-r} & \dots & \partial_{n+\mu_r-r} & &
\end{array}
$$

that one reads by successive rows (reading by column would give the same operator, thanks to the braid relations $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$).

We can now extend the Wrońskian to symmetric functions. Given n, a partition $\lambda \in \mathbb{N}^n$, and N symmetric functions $f_i(\mathbf{x}_n)$, the integer N being the number of partitions contained in λ, then the Wrońskian is

$$
W_\lambda(f_1(\mathbf{x}_n), \dots, f_N(\mathbf{x}_n)) = \left| f_i(\mathbf{x}_n)\, \partial_n^\mu \right|_{\substack{i=1\dots N \\ \mu\subseteq\lambda}} . \tag{4}
$$

For example, for $\lambda = [2, 2]$, the set $\{\mu \subseteq [2, 2]\}$ is $\{[0, 0], [1, 0], [2, 0], [1, 1],$

$[2,1],[2,2]\}$, and the Wrońskian is equal to

$$\Big| f_i,\ f_i\partial_2,\ f_i\partial_2\partial_1,\ f_i\partial_2\partial_3,\ f_i\partial_2\partial_3\partial_1,\ f_i\partial_2\partial_3\partial_1\partial_2 \Big|_{i=1\ldots 6}\ .$$

As in the case of a single variable, the Wrońskian on a set of functions can be written in terms of the images of the initial functions under permutations of the variables. To make precise these statements, we introduce some more notations. First, a polynomial $f(x_1,\ldots,x_n)$ must be considered as a polynomial in $\mathbf{x} = \{x_1,\ldots,x_n,\ldots,x_\infty\}$ of degree 0 in x_i, $i > n$. We shall write it $f(\mathbf{x})$. Similarly, a permutation in the symmetric group $\mathfrak{S}_n$ may be considered as a permutation in the infinite symmetric group by addition of the fixed points $n{+}1, n{+}2, \ldots$. Given a partition $\mu \in \mathbb{N}^n$, let $\sigma(\mu)$ denote the permutation

$$\sigma(\mu) = [1+\mu_n,\ldots,n+\mu_1,1,\ldots,\widehat{1{+}\mu_n},\ldots,\widehat{n{+}\mu_1},n{+}\mu_1{+}1,n{+}\mu_1{+}2,\ldots]\,,$$

and let $f(\mathbf{x}^{\sigma(\mu)})$ denote $f(x_{\sigma_1},\ldots,x_{\sigma_n})$. Finally, the *inversion polynomial* $⋒(\sigma)$ of a permutation σ is equal to

$$⋒(\sigma) = \prod_{i<j,\,\sigma_i>\sigma_j} (x_{\sigma_i} - x_{\sigma_j})\,.$$

Proposition 2 *Given a partition $\lambda \in \mathbb{N}^n$, one has*

$$W_\lambda(f_1(\mathbf{x}_n),\ldots,f_N(\mathbf{x}_n)) = \frac{1}{\prod_{\mu\subseteq\lambda} ⋒(\sigma(\mu))} \Big| f_i(\mathbf{x}^{\sigma(\mu)}) \Big|_{\substack{\mu\subseteq\lambda \\ i=1\ldots N}}\ . \tag{5}$$

Proof. Newton's formula extends to polynomials of several variables. In the case of symmetric functions [4, 9.7], one has

$$f(\mathbf{x}_n^{\sigma(\mu)}) = \sum_{\nu\not\subseteq\mu} f(\mathbf{x})\partial_n^\nu P_\mu^\nu + f(\mathbf{x})\partial_n^\mu\ ⋒\,(\sigma(\mu))\,,$$

where the coefficients P_μ^ν are some universal polynomials (in fact, they are specializations of Schubert polynomials [4, 9.6.3]). This formula shows that the determinant $\Big| f_i(\mathbf{x}^{\sigma(\mu)}) \Big|$ is obtained from the Wrońskian by multiplication by a triangular matrix having the inversion polynomials $⋒(\sigma(\mu))$ in its diagonal. QED

For example, for $n = 2$, $\lambda = [3,1]$, then $N = 7$, and the Wrońskian $W_{31}(f_1(x_1,x_2),\ldots,f_7(x_1,x_2))$ is equal to

$$\frac{1}{\Delta(\mathbf{x}_5)(x_2{-}x_1)^2(x_4{-}x_3)(x_5{-}x_3)(x_5{-}x_4)} \Big| f_i(x_\alpha,x_\beta) \Big|_{\substack{\alpha<\beta,\,\alpha=1,2,\beta=2,3,4,5 \\ i=1,\ldots,7}}\,,$$

the permutations $\sigma(\mu)$, $\mu \subseteq [3,1]$ being $[1,2,3,4,5], [1,3,2,4,5], [1,4,2,3,5]$, $[1,5,2,3,4], [2,3,1,4,5], [2,4,1,3,5], [2,5,1,3,4]$.

In the special case where λ is a rectangular partition (i.e. all the parts of λ are equal to some integer r), then the right-hand side of (5) shows that $W_{r^n}(f_1,\ldots,f_N)$ is symmetrical in $x_1,\ldots,x_{n+r}$ and that the product $\prod \Cap(\sigma(\mu)$ is a power of the Vandermonde. For example, for $\lambda = [3,3]$, one has $N = 10$ and

$$W_{33}\big(f_1(x_1,x_2),\ldots,f_{10}(x_1,x_2)\big) = \frac{1}{\Delta(\mathbf{x}_5)^4}\left| f_i(x_\alpha,x_\beta)\right|_{1\le\alpha<\beta\le5,\,1\le i\le10}.$$

The Wrońskian of symmetric functions still possesses a factorization property with respect to multiplication by a single symmetric function. Indeed, if $g = g(\mathbf{x}_n)$ is symmetrical, then the expression (5) shows that

$$W_\lambda(gf_1(\mathbf{x}_n),\ldots,gf_N(\mathbf{x}_n)) = \prod_{\mu\subseteq\lambda} g\left(\mathbf{x}_n^{\sigma(\mu)}\right)\, W_\lambda(f_1(\mathbf{x}_n),\ldots,f_N(\mathbf{x}_n))\,. \qquad (6)$$

This property can also be seen starting with the definition (4) using repeatedly Leibniz's formula for a product:

$$f(\mathbf{x}_n)g(\mathbf{x}_n)\partial_i = f(x_n)(g(\mathbf{x}_n)\partial_i) + f(\mathbf{x}_n)\partial_i\, g(\mathbf{x}_n^{s_i})\,.$$

For example, for $n=2$, $\lambda=[2,1]$, one has, for $f = f_1(x_1,x_2),\ldots,f_5(x_1,x_2)$, $g = g(x_1,x_2)$, all functions being symmetrical,

$$\begin{aligned}
fg\partial_2 &= f(g\partial_2) + f\partial_2 g^{s_2}\\
fg\partial_2\partial_3 &= f(g\partial_2\partial_3) + f\partial_2(g^{s_2}\partial_3) + f\partial_2\partial_3(g^{s_2s_3})\\
fg\partial_2\partial_1 &= f(g\partial_2\partial_1) + f\partial_2(g^{s_2}\partial_1) + f\partial_2\partial_1(g^{s_2s_1})\\
fg\partial_2\partial_1\partial_3 &= f(g\partial_2\partial_1\partial_3) + f\partial_2(g^{s_2}\partial_1\partial_3) + f\partial_2\partial_1(g^{s_2s_1}\partial_3)\\
&\quad + f\partial_2\partial_3\Big((g^{s_2}\partial_1)^{s_3}\Big) + f\partial_2\partial_1\partial_3(g^{s_2s_1s_3})\,.
\end{aligned}$$

Therefore, by linear combination of columns, one reduces the determinant to

$$\left| f,\ f\partial_2 g^{s_2},\ f\partial_2\partial_3 g^{s_2s_3},\ f\partial_2\partial_1 g^{s_2s_1},\ f\partial_2\partial_1\partial_3 g^{s_2s_1s_3}\right|_{f=f_1,\ldots,f_5}.$$

Like in the case of a single variable, Wrońskians reveal their interest when applied to specific problems or evaluated in specific functions. Since the fundamental symmetric functions are Schur functions, and since images of Schur functions are equal to some Schubert polynomials, Wrońskians of Schur functions involve properties of Schubert polynomials that we cannot develop here, referring to [4]. We shall content ourselves with two examples. Let us first recall the definition of Schubert polynomials.

The *Schubert polynomials* $Y_v(\mathbf{x},\mathbf{y}) : v \in \mathbb{N}^n$ constitute a linear basis of the space of polynomials in $x_1,\ldots,x_n$, with coefficients in $\mathbf{y} = \{y_1, y_2, \ldots, y_\infty\}$. Those indexed by a partition $\lambda : \lambda_1 \geq \cdots \geq \lambda_n$ are called *dominant* and are equal to the product $Y_\lambda(\mathbf{x},\mathbf{y}) = \prod_{i,j=1\ldots\lambda_i}(x_i - y_j)$. The other polynomials are all their non-zero images under products of divided differences. More precisely, they are defined recursively by

$$Y_v(\mathbf{x},\mathbf{y})\partial_i = Y_{\ldots v_{i-1},v_{i+1},v_i-1,v_{i+2},\ldots}(\mathbf{x},\mathbf{y}) \quad \text{when } v_i > v_{i+1} \tag{7}$$

Notice that when $v_i \leq v_{i+1}$, the polynomial $Y_v(\mathbf{x},\mathbf{y})$ is the image of another polynomial under ∂_i. Therefore it is symmetrical in x_i, x_{i+1}, and one has in that case $Y_v(\mathbf{x},\mathbf{y})\partial_i = 0$. The polynomials are invariant under concatanation of 0 to their index: $Y_v(\mathbf{x},\mathbf{y}) = Y_{v,0}(\mathbf{x},\mathbf{y})$, so that one may use elements of $\mathbb{N}^\infty$ with a finite number of non-zero components as the indexing set of Schubert polynomials.

When v is (weakly) increasing, the Schubert polynomial is symmetrical in $x_1,\ldots,x_n$ and called *Graßmannian Schubert polynomial.* When moreover $\mathbf{y}$ is specialized to $0,0,\ldots$, a Graßmannian Schubert polynomial specializes to a Schur function.

Starting with a Graßmannian Schubert polynomial $Y_v(\mathbf{x},\mathbf{y})$, the hypothesis $v_i > v_{i+1}$ for the recursive rule (7) implies that $v_{i+1} = 0$. In other words, at the level of indices, the action of divided differences is always of the type $Y_{\ldots,k,0,\ldots}(\mathbf{x},\mathbf{y}) \to Y_{\ldots,0,k-1,\ldots}(\mathbf{x},\mathbf{y})$.

Given two integers n, m, and two sets of indeterminates $\mathbf{x}_{n+m}$, $\mathbf{y}_{n+m}$, the determinant $\left|R(A,B)\right|_{A,B}$, where A runs over all subsets of cardinality n of $\mathbf{x}_{n+m}$, B runs over all subsets of cardinality m of $\mathbf{y}_{n+m}$, and $R(A,B) = \prod_{a\in A, b\in B}(a-b)$ is the resultant of A, B, can be encountered in the literature. Of course, it is clear that it is divisible by some power of the Vandermonde in $\mathbf{x}_{n+m}$, and some power of the Vandermonde in $\mathbf{y}_{n+m}$, the question is to determine the power and to find the remaining factor. Instead of proceeding with successive transformations of the determinant, one first recognizes that $R(\mathbf{x}_n, \mathbf{y}_m) = Y_{m^n}(\mathbf{x},\mathbf{y})$. Using (5), one reduces the problem to the computation of the Wrońskian $W_{m^n}(Y_{m^n}(\mathbf{x},\mathbf{y})\partial_n^\mu)$. By symmetry between $\mathbf{x}$ and $\mathbf{y}$, one can also use the divided differences δ_i relative to $y_1,\ldots,y_{n+m}$, and further reduce the Wrońskian to the determinant $\left|Y_{m^n}(\mathbf{x},\mathbf{y})\partial_n^\mu \delta_m^\nu\right|_{\mu\subseteq m^n,\, \nu\subseteq n^m}$. In fact, the image of a Schubert polynomial under $-\delta_i$ is either 0 or another Schubert polynomial [4], denoting $|\nu| = \nu_1 + \nu_2 + \ldots$:

$$(-1)^{|\nu|} Y_{m^n}(\mathbf{x},\mathbf{y})\delta_m^\nu = (-1)^{|\nu|} Y_{m-\nu_n,\ldots,m-\nu_1}(\mathbf{x},\mathbf{y})\,.$$

Ordering conveniently the partitions indexing the two families of divided differences, one obtains a triangular matrix with ± 1 on the diagonal. In the end, taking into account the two products of inversion polynomials in $\mathbf{x}$ and $\mathbf{y}$ figuring in (5), which must be symmetrical in $x_1, \ldots, x_{n+m}$, and also symmetrical in $y_1, \ldots, y_{n+m}$, one obtains

$$\Big| R(A,B) \Big|_{A,B} = \pm \Big(\Delta(\mathbf{x}_{n+m}) \Delta(\mathbf{y}_{m+n}) \Big)^{\binom{n+m-2}{n-1}} .$$

We evaluated the preceding determinant of resultants by recognizing in it a Wrońskian of Schubert polynomials. As a second example, let us take a more general determinant and compute the Wrońskian

$$W_{22}\big(Y_{00},\ Y_{01},\ Y_{11},\ Y_{03},\ Y_{24},\ Y_{35}\big) = \begin{array}{|cccccc|c} 1 & Y_{01} & Y_{11} & Y_{03} & Y_{24} & Y_{36} & \\ 0 & 1 & Y_{1} & Y_{002} & Y_{203} & Y_{305} & \partial_2 \\ 0 & 0 & 1 & 0 & Y_{013} & Y_{025} & \partial_2\partial_1 \\ 0 & 0 & 0 & Y_{0001} & Y_{2002} & Y_{3004} & \partial_2\partial_3 \\ 0 & 0 & 0 & 0 & Y_{0102} & Y_{0204} & \partial_2\partial_3\partial_1 \\ 0 & 0 & 0 & 0 & Y_{0002} & Y_{0014} & \partial_2\partial_3\partial_1\partial_2 \end{array} ,$$

where Y_v stands for $Y_v(\mathbf{x}, \mathbf{y})$, indicating which divided differences are used on the right of the determinant.

Evaluating the 2×2 remaining block requires some familiarity with Schubert calculus. In final one finds that the Wrońskian, which expands in fact as a sum of $4 \times 10 \times 70 = 2800$ terms, is equal to

$$-Y_{0001} Y_{0002} Y_{0114} .$$

Even when specializing $\mathbf{y}$ to $0, 0, \ldots$, it is not straightforward that the following determinant of Schur functions, with A running over all pairs in $x_1, \ldots, x_4$,

$$\Big| s_0(A),\ s_1(A),\ s_{11}(A),\ s_3(A),\ s_{42}(A),\ s_{53}(A) \Big|$$

admits three Schur functions as factors. Some determinants of Schur functions may be found in the literature, see for example the recent papers [1, 8]. However, most published cases deal only with determinants equal to a power of the Vandermonde.

A systematic study of determinants of Schur functions, or of Schubert polynomials, like the one above, having factors other than $x_i - x_j$, is certainly of great interest.

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