# Yang-Baxter graphs, Jack and Macdonald polynomials

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**Abstract**

We describe properties of the affine graph underlying the recursions between the different varieties of non symmetric Macdonald and Jack polynomials. We use an arbitrary function of one variable in the definition of affine edges, and of Cherednik's elements, to unify the different theories. We describe the symmetrizing operators furnishing the symmetric polynomials from the non symmetric ones.

*Dédié à Gian-Carlo Rota, ami et maître à pensées*

## 1 Introduction

There are several families of mathematical objects related to the symmetric group which are obtained by simple recursions involving "local" operators [43].

All these constructions can be summarized as follows : one has a directed graph, with an origin $\triangleright$. Edges are labelled by elements $\theta$ of an associative algebra. One also chooses an initial element $Y_\triangleright$ in a right module over this algebra. The *evaluation* of a path from the origin $\triangleright$ to a given vertex $v$ is the product $\theta_1\theta_2\cdots\theta_k$ of the labels encountered along the path.

We shall say that a labelling is *consistent* if all the paths from the origin to any vertex $v$ give an element $Y_v := Y_\triangleright\,\theta_1\cdots\theta_k$ depending only upon $v$.

For example, Young orthogonal idempotents for a given partition $\lambda$ can be described this way. The graph is the partially ordered set of tableaux of shape the diagram of $\lambda$, with origin the "Yamanouchi tabeau", and one needs only to know the idempotent corresponding to this tableau, which will play the rôle of $Y_{\triangleright}$ (and that Young denotes $PNP$), to generate all the other idempotents, the labels of edges being of the type $c(s_i + 1/k)$, $c \in \mathbb{C}$, $k \in \mathbb{Z}$. We shall recall later that matrices representing the symmetric group can be described in the same fashion.

Other examples are associated to the *permutohedron* $\big($i.e. the *Cayley graph* of the symmetric group $\mathfrak{S}_n$, edges $v \mapsto vs_i$ corresponding to multiplication on the right by simple transposition, whenever length increases: $\ell(vs_i) = \ell(v) + 1\,\big)$.

Labelling edges by divided differences (resp. isobaric divided differences, to be defined later), and starting with

$$Y_{\triangleright} = x_1^{n-1} \cdots x_n^0 \quad \text{or} \quad Y_{\triangleright} = \prod_{i+j \le n} (x_i - y_j)$$

(resp. $Y_{\triangleright} = \prod_{i+j\le n}(1 - y_j/x_i)$) one generates polynomials $Y_v$ which are called *Schubert polynomials* [29], [34] (resp. *Grothendieck polynomials*, [29],[32]).

We shall call a data :

permutohedron + initial $Y_{\triangleright}$ + consistent labelling

a *Yang-Baxter graph* for the symmetric group, and the elements $Y_v$, $v \in \mathfrak{S}_n$, *Yang-Baxter elements.* This notion underlies Young's constructions [45], consistency arising from *Yang-Baxter relations*.

For example, writing $Y_v$ on the graph instead of $v$, here is the Yang-Baxter graph giving Schubert polynomials for $\mathfrak{S}_3$ [32], [34] :

$$\begin{tikzcd}[column sep=small]
& (x_1-y_1)(x_1-y_2)(x_2-y_1) \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_2"] & \\
(x_1-y_1)(x_2-y_1) \arrow[d, "\partial_2"'] & & (x_1-y_1)(x_1-y_2) \arrow[d, "\partial_1"] \\
x_1-y_1 \arrow[dr, "\partial_1"'] & & x_1+x_2-y_1-y_2 \arrow[dl, "\partial_2"] \\
& 1 &
\end{tikzcd}$$

Our main graph will be associated to the affine symmetric group. In fact, we shall take only "half" of the extended affine symmetric group, i.e. we shall use the monoid $\widehat{\mathfrak{S}}_n$ generated by

$$\{s_1, \ldots, s_{n-1}, \tau : 1 \le i \le n-1\}$$

satisfying relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\ ,\ \ s_i s_j = s_j s_i\ |i-j| > 1 \tag{1}$$
$$s_i\, \tau = \tau\, s_{i-1}\,,\ 2 \le i \le n-1\ ;\ \ s_1\, \tau^2 = \tau^2\, s_{n-1}\ . \tag{2}$$

The elements of $\widehat{\mathfrak{S}}_n$ can be thought as acting both on infinite vectors $v = [\ldots, v_i, \ldots]$, $i \in \mathbb{Z}$ and on an infinite set of indeterminates $x_i$, $i \in \mathbb{Z}$ : the generator $s_i$ transposes in $v$ components $v_{i+kn}, v_{i+1+kn}$ for all $k \in \mathbb{Z}$, and transposes all $x_{i+kn}, x_{i+1+kn}$ simultaneously; the generator $\tau$ acts as a shift on $v$: $(v\tau)_i = v_{i+1}$ and $x_i\,\tau = x_{i-1}$ for all $i \in \mathbb{Z}$.

Using also the inverse $\overline{\tau} := \tau^{-1}$ would give a faithful representation of the extended affine symmetric group.

Let us take the origin vector

$$\triangleright : \quad \triangleright_1 = 0 = \cdots = \triangleright_n \quad , \quad \triangleright_{i\pm n} = \triangleright_i \pm 1 \ \forall i \in \mathbb{Z} \ . \tag{3}$$

Define $\Gamma_n$ to be the following graph: vertices are the orbit of $\triangleright$ under $\widehat{\mathfrak{S}}_n$; the (oriented) edges, labelled $s_i$ or $\tau$, join pairs $v$, $v\,s_i$ or $v$, $v\tau$. The connected components obtained by erasing $\tau$-edges will be called *cells*. Since the vectors $v$ are "periodic" in the sense that $v_{i+n} = v_i + 1$, $\forall i \in \mathbb{Z}$, one will denote them by the restriction to their components $v_1, \ldots, v_n$. Thus $\triangleright$ will also be written $[0, \ldots, 0]$, and $v$ will be written $[v_1, \ldots, v_n]$.

**Definition 1** *An* affine Yang-Baxter graph *for* $\widehat{\mathfrak{S}}_n$ *is a consistent labelling of the edges of* $\Gamma_n$ *by elements of an associative algebra, together with a choice of an initial element* $Y_\triangleright$ *in a right module for this algebra. The elements* $Y_v$*,* $v \in \widehat{\mathfrak{S}}_n$*, are called* Yang-Baxter elements.

By default, we shall suppose that the algebra has a unit 1 and that $Y_\triangleright = 1$.

To find consistent labelling of edges, we shall label vertices by a second infinite vector $c(v)$, called the vector of *spectral parameters*. One chooses an initial vector $c(\triangleright)$, and one takes its orbit under $\widehat{\mathfrak{S}}_n$ : as before, the generator $s_i$ simultaneously transpose components of $c(v)$ of index $i + kn$, $i{+}1 + kn$, $\forall k \in \mathbb{Z}$, and the generator $\tau$ acts by translation : $c_i(v\tau) = c_{i+1}(v)$, $\forall i \in \mathbb{Z}$.

For example, if one takes $c(\triangleright)$ such that $c_i(\triangleright)$ is the remainder of $i{-}1$ mod $n$, then $\epsilon(v) := c(v)$ will be periodic: $\epsilon_{i+n}(v) = \epsilon_i(v) \forall i \in \mathbb{Z}$, and

$$\epsilon_1(v), \ldots, \epsilon_n(v)$$

is the *standardization* of $v_1, \ldots, v_n$ (i.e. one replaces multiple occurrence of the same value $\alpha$ in $[v_1, \ldots, v_n]$ by different letters $\alpha, \alpha', \alpha'' \ldots$, $\alpha > \alpha' > \alpha'' > \cdots$, reading from left to right; then one replaces these new letters decreasingly by $0, 1, \ldots, n{-}1$; this is an operation much used in the theory of Young tableaux, since the article of Schensted). In the case where $v_1 \geq v_2 \geq \cdots \geq v_n$, then $\epsilon(v) = [0, 1, \ldots, n{-}1]$.

The vectors $c(v)$ that we shall choose will simply be a combination of $v$ and $\epsilon(v)$.

For example, introducing extra parameters $\alpha, q, t = -t_2/t_1$, we shall take

$$c(\triangleright) = \ldots\ [\,0, \text{-}1, \ldots, \text{-}n\text{+}1\,],\ \alpha, \alpha\text{-}1, a\text{-}2, \ldots, 2\alpha;\ 2\alpha\text{-}1, \ldots$$

i.e. $c(\triangleright) = [0, \text{-}1, \ldots, \text{-}n\text{+}1]$ with periodicity $c_{i+n}(\triangleright) = \alpha + c_i(\triangleright)$. Then

$$c(v) = \alpha v - \epsilon(v) \quad \text{i.e.} \quad c_i(v) = \alpha v_i - \epsilon_i(v)\ . \tag{4}$$

Taking on the other hand $c(\triangleright) = [t^0, t^1, \ldots, t^{n-1}]$ with periodicity $c_{i+n}(\triangleright) = qc_i(\triangleright)$ entails

$$c_i(v) = q^{v_i} t^{\epsilon_i(v)}\ . \tag{5}$$

## 2 Cells and the group algebra of the symmetric group

Let us first detail Yang-Baxter graphs for the symmetric group, taking, as a first algebra to index edges, the group algebra of $\mathfrak{S}_n$ itself.

In fact, already at the beginning of the century Young gave an example of such a graph to describe representations of the symmetric group (for a modern approach, cf. [38]). Young [45] defined matrices $\sigma_i$ representing simple transpositions $s_i$. Relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ are immediate in his construction, and thus, he had essentially to check that his construction worked for $\mathfrak{S}_3$. In that case, he had three parameters $c_1, c_2, c_3$ (the *contents* of three consecutive letters in an arbitrary Young tableau). For simplicity of notation, let us specialize $c_1 = 0$, $c_2 = 2$, $c_3 = 5$. Young considered a vector space of dimension 6, with basis indexed by the permutations of $c(v^0) = [0, 2, 5]$ :

$$[0, 2, 5],\ [0, 5, 2],\ [2, 0, 5],\ [2, 5, 0],\ [5, 0, 2],\ [5, 2, 0]\ ,$$

and wrote the two matrices

$$\sigma = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & 0 & \frac{2\sqrt{6}}{5} & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{6}}{5} & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \end{bmatrix},\ \sigma' = \begin{bmatrix} -\frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & \frac{2\sqrt{6}}{5} & 0 & 0 \\ 0 & 0 & \frac{2\sqrt{6}}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

These two matrices are made of $2 \times 2$ sub-matrices,

$$\begin{bmatrix} -\rho & \sqrt{1-\rho^2} \\ \sqrt{1-\rho^2} & \rho \end{bmatrix}\ ,$$

each of which depends upon a single parameter $\rho$ $(= 1/2, 1/3$ or $1/5)$ and codes a representation of $\mathfrak{S}_2$ on a space of dimension 2. Now the matrix $\sigma$ is a patch of three elementary matrices and can be encoded by the arrows:

$$[0,2,5] \xrightarrow{s+\frac{1}{2}} [2,0,5]\ ,\ [0,5,2] \xrightarrow{s+\frac{1}{5}} [5,0,2]\ ,\ [2,5,0] \xrightarrow{s+\frac{1}{3}} [5,2,0]\ ,$$

and the matrix $\sigma'$ can similarly be encoded by:

$$[0,2,5] \xrightarrow{s'+\frac{1}{3}} [0,5,2]\ ,\ [2,0,5] \xrightarrow{s'+\frac{1}{5}} [2,5,0]\ ,\ [5,0,2] \xrightarrow{s'+\frac{1}{2}} [5,2,0]\ .$$

Thus the two matrices can be encoded at the same time by the graph

$$\begin{tikzcd}
 & {[0,2,5]} \arrow[dl, "s+\frac{1}{2}"'] \arrow[dr, "s'+\frac{1}{3}"] & \\
{[2,0,5]} \arrow[d, "s'+\frac{1}{5}"'] & & {[2,5,0]} \arrow[d, "s+\frac{1}{5}"] \\
{[0,5,2]} \arrow[dr, "s+\frac{1}{3}"'] & & {[5,0,2]} \arrow[dl, "s'+\frac{1}{2}"] \\
 & {[5,2,0]} &
\end{tikzcd}$$

which describes their action on the six-dimensional vector space with basis $[0,2,5], \ldots, [5,2,0]$.

Relation $\sigma\sigma'\sigma = \sigma'\sigma\sigma'$ becomes

$$(s_1 + \frac{1}{2})\frac{1}{\sqrt{1-\frac{1}{2^2}}}(s_2 + \frac{1}{5})\frac{1}{\sqrt{1-\frac{1}{5^2}}}(s_1 + \frac{1}{3})\frac{1}{\sqrt{1-\frac{1}{3^2}}} =$$
$$= (s_2 + \frac{1}{3})\frac{1}{\sqrt{1-\frac{1}{3^2}}}(s_1 + \frac{1}{5})\frac{1}{\sqrt{1-\frac{1}{5^2}}}(s_2 + \frac{1}{2})\frac{1}{\sqrt{1-\frac{1}{2^2}}}\ ,$$

but now, it is an identity in the group algebra of $\mathfrak{S}_3$, instead of being a matrix identity, and the graph itself, with $Y_{v^0} = 1$, is a Yang-Baxter graph for $\mathfrak{S}_3$.

Yang [44] solved the same problem as Young did, but formulated it directly in terms of group algebra instead of matrices of representation: he wanted to label the edges of the permutohedron by elements of the group algebra of the symmetric group in such a way that the product of elements from the origin to a given permutation $v$ be independent of the path. He found the solution of choosing a vector of "generic" parameters for the identity permutation (the mutual differences of these parameters are only required to be different from 0 and 1), generating vectors $c(v)$ by simple transpositions of components, and labelling the edge $[v \to v\, s_i]$ by

$$s_i + \frac{1}{c_{i+1}(v) - c_i(v)} \ . \tag{6}$$

Since the permutohedron is paved by squares $s_i s_j = s_j s_i$, $|i-j| > 1$, and hexagons $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, consistency of Yang's solution lies upon *Yang-Baxter equation*

$$(s_i + \frac{1}{\alpha})(s_{i+1} + \frac{1}{\alpha+\beta})(s_i + \frac{1}{\beta}) = (s_{i+1} + \frac{1}{\beta})(s_i + \frac{1}{\alpha+\beta})(s_{i+1} + \frac{1}{\alpha}) \ , \tag{7}$$

the commutations

$$(s_i + \frac{1}{\alpha})(s_j + \frac{1}{\beta}) = (s_j + \frac{1}{\beta})(s_i + \frac{1}{\alpha}) \ , \ |i-j| > 1 \tag{8}$$

being evident.

Thus, consistency arises from additivity :

$$c_{i+2} - c_i = (c_{i+2} - c_{i+1}) + (c_{i+1} - c_i) \ ,$$

as in the case of Young where parameters were axial distances (= contents).

Writing on the vertices of the graph $c[v]$ under $v$, and taking $c([1,2,3]) = [0, \alpha, \alpha{+}\beta]$, then a cell with edges in the group algebra of $\mathfrak{S}_3$ looks like

$$\begin{tikzcd}
 & \substack{[1,2,3] \\ [0,\alpha,\alpha+\beta]} \arrow[dl, "s+\frac{1}{\alpha}"'] \arrow[dr, "s'+\frac{1}{\beta}"] & \\
\substack{[2,1,3] \\ [\alpha,0,\alpha+\beta]} \arrow[d, "s'+\frac{1}{\alpha+\beta}"'] & & \substack{[1,3,2] \\ [0,\alpha+\beta,\alpha]} \arrow[d, "s+\frac{1}{\alpha+\beta}"] \\
\substack{[2,3,1] \\ [\alpha,\alpha+\beta,0]} \arrow[dr, "s+\frac{1}{\beta}"'] & & \substack{[3,1,2] \\ [\alpha+\beta,0,\alpha]} \arrow[dl, "s'+\frac{1}{\alpha}"] \\
 & \substack{[3,2,1] \\ [\alpha+\beta,\alpha,0]} &
\end{tikzcd}$$

# 3 Cells and the degenerate affine Hecke algebra

We have some freedom for what concerns choosing the $Y_v$ as operators on polynomials in the $x_i$'s. Indeed, any family of operators satisfying the braid relations can be used.

We can keep the relations $s_i^2 = 1$, but deform the transpositions of variables. For example, we can take

$$\sigma_i = s_i + \partial_i \ , \ 1 \leq i \leq n-1 \ , \tag{9}$$

where $\partial_i$ is the divided difference [4]

$$f \mapsto f\,\partial_i := \frac{f - f^{s_i}}{x_i - x_{i+1}} \tag{10}$$

(one could introduce an extra parameter $\gamma$ and take $\sigma_i := s_i + \gamma\,\partial_i$; this is the homogeneous version of the preceding one, and thus not more general).

The $\sigma_i$ still represent the symmetric group (they satisfy (1)) and generate an algebra with linear basis $\{\sigma_w : w \in \mathfrak{S}_n\}$ (cf. [29]).

Labelling of edges uses the spectral parameters in the same way as in the preceding section, since the $\sigma_i$'s satisfy the same relations as the $s_i$'s : the edge $[v \mapsto v\,\sigma_i]$ is labelled

$$\sigma_i + \frac{1}{c_{i+1}(v) - c_i(v)} = s_i + \partial_i + \frac{1}{c_{i+1}(v) - c_i(v)} \tag{11}$$

Relations between indeterminates $x_i$, $1 \leq i \leq n$ (considered as "multiplication by $x_i$") and the $\sigma_i$ or $\partial_i$ are given by *Leibniz rule*:

$$x_i\sigma_i = \sigma_i x_{i+1} + 1 \ , \ x_{i+1}\sigma_i = \sigma_i x_i - 1 \ , \ x_j\sigma_i = \sigma_i x_j, j \neq i, i+1 \tag{12}$$

The algebra generated by the $\sigma_i$, $1 \leq i \leq n-1$ and $x_i$, $1 \leq i \leq n$ is called the *affine degenerate Hecke algebra*.

The limit case $\sigma_i = \partial_i$ gives also rise to a Yang-Baxter graph because the $\partial_i$'s themselves satisfy the braid relations. Thus for any permutation $v$ exists a well defined divided difference $\partial_w$, as well as a $\sigma_w$.

In fact, since $1 - s_i = \partial_i\,(x_i - x_{i+1})$ the affine Hecke algebra is a subalgebra of the *algebra of divided differences* generated by the $\partial_i$, and rational functions in the $x_j$ (see [29],[30]). This last algebra is a free module over the ring of rational functions in the $x_i$'s, with basis the divided differences $\partial_w$, $w \in \mathfrak{S}_n$. Expansions of the $\sigma_w$ or $w$ in the basis of $\partial_w$ involve Schubert polynomials (cf. [27], [28], [11]).

## 4 Cells and the affine Hecke algebra

One has more general operators on polynomials satisfying the braid relations (see [29], [30]).

In particular, let $q$ be an extra parameter, and let $x_{i+n} = qx_i$, $\forall i \in \mathbb{Z}$. Then one defines *isobaric divided differences* $\overline{\pi}_i$, $i = 1, \ldots, n-1$ acting on the quotient ring $\mathbb{Z}[x_i : i \in \mathbb{Z}]/(x_{i+n} = qx_i)$ as twisted derivations:

$$fg\overline{\pi}_i = f(g\overline{\pi}_i) + f\overline{\pi}_i(gs_i)$$

satisfying the conditions

$$\begin{array}{rcl} x_j\overline{\pi}_i & = & 0 \ , \ j \not\equiv i, i{+}1 \quad \bmod (n) \ , \\ x_j\overline{\pi}_i & = & x_{j+1} \ , \ j \equiv i \ , \\ \overline{\pi}_i^2 & = & -\overline{\pi}_i \ . \end{array} \tag{13}$$

**Lemma 2** *The $\overline{\pi}_i$, $1 \le i \le n-1$, satisfy the braid relations (1).*

Indeed, the image of $\overline{\pi}_i$ under the isomorphism of the quotient ring onto the space of polynomials in $x_1, \ldots, x_n$, is the usual isobaric divided difference (that we shall denote by the same letter)

$$\overline{\pi}_i : f \mapsto (f - f^{s_i})\frac{x_{i+1}}{x_i - x_{i+1}} \ , \ 1 \le i \le n-1 \ . \tag{14}$$

These operators commute with multiplication by symmetric functions in $x_1, \ldots, x_n$ and braid relations can directly be checked on an appropriate basis of the space of polynomials (as a module over symmetric polynomials in $x_1, \ldots, x_n$).

Demazure [8] defined such operators for any root system.

Let us notice that the operators $\pi_i := \overline{\pi}_i{+}1$ also satisfy the braid relations together with $\pi_i^2 = \pi_i$.

It has been noticed in [29]( where, more generally one uses $D_i := t_1\pi_i + t_2\partial_i + t_3 s_i$) and [33] that the operators

$$T_i = T_{x_i, x_{i+1}} := \pi_i(t_1 + t_2) - t_2\, s_i \tag{15}$$

represent the Hecke algebra $\mathcal{H}_n$ of the symmetric group $\mathfrak{S}_n$. The generators $T_i$ satisfy the braid relations (1) together with

$$(T_i - t_1)\,(T_i - t_2) = 0 \ . \tag{16}$$

Reversing the order on $x_1, \ldots, x_n$ give other generators

$$\overline{T}_i := -t_1 t_2 T_i^{-1} = \overline{\pi}_i\,(t_1 + t_2) - t_2\, s_i = T_i - (t_1 + t_2) \ , \ 1 \le i \le n-1 \ , \tag{17}$$

the Hecke relation becoming in that case

$$(\overline{T}_i + t_1)\,(\overline{T}_i + t_2) = 0 \ . \tag{18}$$

We prefer keeping two parameters $t_1, t_2$, because all cases $[t_1, t_2] = [t, -1]$, $[-1, t], [t, 1/t], [1, t], [t, 1]$ occur in the literature. We shall also have to use the involutions $[t_1, t_2] \leftrightarrow [\pm t_2, \pm t_1]$.

Leibniz formula (12) implies the following commutations

$$\begin{aligned} x_i \overline{T}_i &= T_i\, x_{i+1} \ , \ 1 \le i \le n{-}1 \\ x_j T_i &= T_i\, x_j \ , \ j \ne i, i{+}1 \ , \ 1 \le j \le n \ . \end{aligned} \tag{19}$$

One still has a Yang-Baxter equation,

$$\begin{aligned} &(T_i + \frac{t_1{+}t_2}{\alpha{-}1})\,(T_{i+1} + \frac{t_1{+}t_2}{\alpha\beta{-}1})\,(T_i + \frac{t_1{+}t_2}{\beta{-}1}) = \\ &\qquad\qquad (T_{i+1} + \frac{t_1{+}t_2}{\beta{-}1})\,(T_i + \frac{t_1{+}t_2}{\alpha\beta{-}1})\,(T_{i+1} + \frac{t_1{+}t_2}{\alpha{-}1}) \ . \end{aligned} \tag{20}$$

As in the preceding section, one chooses for the identity permutation $v^0$ a vector of parameters $c(v^0) = [c_1, \ldots, c_n]$ (all different, and different from 0) and define $c(v)$, $v \in \mathfrak{S}_n$, to be $[c_{v_1}, \ldots, c_{v_n}]$.

Thanks to Yang-Baxter relation (20), consistency will be now insured by taking, on an edge $v \mapsto v\, s_i$, the label

$$T_i + \frac{t_1 + t_2}{c_{i+1}/c_i(v) - 1} \ . \tag{21}$$

In particular, given two positive integers $a, b$, taking the spectral parameters $[(-t_2)^{a+b}, t_1^a(-t_2)^b, t_1^{a+b}]$, that is $\alpha = (-t_1/t_2)^a$, $\beta = (-t_1/t_2)^b$, the left hand side of (20) becomes

$$\begin{aligned} &\Big(T_i + \frac{(-t_2)^a}{t_1^{a-1} - t_1^{a-2}t_2 + \cdots + (-t_2)^{a-1}}\Big) \\ &\qquad \Big(T_{i+1} + \frac{(-t_2)^{a+b}}{t_1^{a+b-1} - t_1^{a+b-2}t_2 + \cdots + (-t_2)^{a+b-1}}\Big) \\ &\qquad\qquad \Big(T_i + \frac{(-t_2)^b}{t_1^{b-1} - t_1^{b-2}t_2 + \cdots + (-t_2)^{b-1}}\Big) \end{aligned}$$

When dealing with reduced decompositions, it is convenient to use planar displays, Young tableaux, and jeu de taquin, to handle more easily the braid relations (cf. [31] for the nilplactic monoid). All the displays that we shall write will have the property that the words obtained by reading them columnwise, from left to right, are congruent modulo the braid relations to the words obtained by reading them row-wise (from top to bottom). For example, writing $i$ for $T_i$, then the planar object ${}^{2\,3\,4}_{1\,2\,3}$ can be read $(T_2T_3T_4)(T_1T_2T_3)$ as well as $(T_2T_1)(T_3T_2)(T_4T_3)$.

With these conventions, a recursive application of braid relations leads to the following identities.

**Lemma 3** *Let $i, j, k$ be three positive integers, $i < j$, $i < k$. Then the following two words are congruent modulo the braid relations:*

$$\begin{array}{ccccccc}
k & \cdots & j{+}k{-}i & & k{+}j{-}i & & \\
\vdots & & \vdots & & \vdots & & \\
i & \cdots & j & & k{+}1 & & \\
 & & j{-}1 & \equiv & k & \cdots & j{+}k{-}i \\
 & & \vdots & & \vdots & & \vdots \\
 & & i & & i & \cdots & j
\end{array} . \tag{22}$$

*Moreover*

$$\begin{array}{ccc} k & \cdots & j{+}k{-}i \\ \vdots & & \vdots \\ i & \cdots & j \end{array} \overline{T}_i \cdots \overline{T}_{j-1} \equiv \overline{T}_{k+1} \cdots \overline{T}_{j+k-i} \begin{array}{ccc} k & \cdots & j{+}k{-}i \\ \vdots & & \vdots \\ i & \cdots & j \end{array} . \tag{23}$$

For example,

$$\begin{array}{ccc} 2 & 3 & 4 \\ 1 & 2 & 3 \\ & & 2 \\ & & 1 \end{array} \equiv \begin{array}{ccc} 4 & & \\ 3 & & \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{array} \qquad \text{and} \qquad {}^{2\,3\,4}_{1\,2\,3}\, \overline{T}_1 \overline{T}_2 \equiv \overline{T}_3 \overline{T}_4\, {}^{2\,3\,4}_{1\,2\,3} .$$

# 5 Symmetrizations

Each of the algebras considered in the preceding sections contains elements (that we shall call *symmetrizing operators*) sending polynomials in $x_1, \ldots, x_n$ onto symmetric polynomials, or to products of symmetric polynomials by a fixed factor of Vandermonde type.

The first, and most fundamental, of symmetrizing operators, is due to Cauchy and Jacobi:

$$f \mapsto \sum_{w \in \mathfrak{S}_n} (-1)^{\ell(w)} w \, \frac{1}{\Delta} \, , \tag{24}$$

where $\Delta := \prod_{i<j\le n}(x_i - x_j)$ is the *Vandermonde*. It sends monomials to $\pm$ a Schur function, or to 0. It can be written as a product of divided differences. In fact, it is equal to $\partial_\omega$, where $\omega := [n, \ldots, 1]$ is the permutation of maximal length in $\mathfrak{S}_n$. Its multiple $\partial_\omega / n!$ can be interpreted as the 1-dimensional idempotent corresponding to the alternating representation of the symmetric group.

In the Hecke algebra, there are two 1-dimensional (quasi)-idempotents which similarly can be expressed as a sum of linear generators, or expressed with divided differences.

Define

$$\nabla_\omega \;:=\; \sum_{w\in\mathfrak{S}_n} (-t_1)^{\ell(w\omega)} T_w \;, \tag{25}$$

$$\square_\omega \;:=\; \sum_{w\in\mathfrak{S}_n} (-t_2)^{\ell(w\omega)} T_w \;. \tag{26}$$

**Theorem 4** *1) The element $\nabla_\omega$, as an operator on polynomials, factorizes:*

$$\nabla_\omega = \partial_\omega \prod_{1\le i<j\le n} (t_2 x_i + t_1 x_j) \;.$$

*2) It factorizes inside the Hecke algebra:*

$$\nabla_\omega = \nabla_{\omega'} \big( T_{n-1}\cdots T_1 - t_2 T_{n-1}\cdots T_2 + \cdots + (-t_2)^{n-2} T_{n-1} + (-t_2)^{n-1} \big) \;,$$

*where $\omega' = [n{-}1, \ldots, 1, n]$.*

*3) It is a Yang-Baxter element $Y_\omega$, for the symmetric group $\mathfrak{S}_n$, for the initial vector of parameters $[t_1^{n-1}, -t_1^{n-2}t_2, \ldots, (-t_2)^{n-1}]$.*

*4) It is equal to another sum*

$$\nabla_\omega := \sum_{w\in\mathfrak{S}_n} t_2^{\ell(w\omega)} \overline{T}_w \;.$$

*5) The element $\square_\omega$ is obtained from $\nabla_\omega$ by interchange of $t_1$ and $t_2$ in the expressions 2,3,4. The involution $t_1 \mapsto -t_2$, $t_2 \mapsto -t_1$, $T_i \leftrightarrow \overline{T}_i, i = 1, \ldots, n{-}1$, preserves $\nabla_\omega$ and $\square_\omega$. Moreover,*

$$\square_\omega := \prod_{1\le i<j\le n} (t_1 x_i + t_2 x_j)\, \partial_\omega \;.$$

A proof of this theorem is given in [9]. Since all the operators appearing in theorem 4 commute with multiplication by symmetric functions, theorem 4 essentially reduces to check that all elements, but one, of the Schubert basis for $\mathfrak{S}_n$ are sent to 0 by $\nabla_\omega$.

The last factorization appear in geometry, $\square_\omega$ being interpreted as an Euler-Poincaré characteristic for a relative flag manifold (cf. [12], first chapter of [21], [22]).

One has similar factorizations in the degenerate affine Hecke algebra generated by the $\sigma_i = s_i + \gamma\partial_i$ , as shows the following theorem (cf. [27]):

**Theorem 5** *1) The sum of all the linear generators of the degenerate affine Hecke algebra factorizes (as operators on polynomials)*

$$\widetilde{\Box}_\omega := \sum_{w\in\mathfrak{S}_n} \sigma_w = \prod_{i<j\le n} (x_i - x_j + \gamma)\,\partial_\omega \ . \tag{27}$$

*2) It is a Yang-Baxter element for the parameters* $[1,2,\ldots,n]$*:*

$$\widetilde{\Box}_\omega = (\sigma_1+1)\Big((\sigma_2+\frac{1}{2})(\sigma_1+1)\Big)\cdots\Big((\sigma_{n-1}+\frac{1}{n-1})\cdots(\sigma_2+\frac{1}{2})(\sigma_1+1)\Big) \ .$$

# 6 Affine edges

We have shown that labelling of edges of a given cell are determined by a single vector of spectral parameters. We have only retained three algebras, the group algebra of the symmetric group, that we shall see as the specialization $\gamma = 0$ of the degenerate affine Hecke algebra, and the affine Hecke algebra.

It remains to define the labels of the affine edges connecting cells and to check the compatibility between the labels of different cells. We shall see that it amounts to introduce "periodicity" conditions on the $x_i$'s and on the initial vector $c(\triangleright)$ of spectral parameters.

To treat the different varieties of Jack and Macdonald polynomials at the same time, we shall use an arbitrary function $G := G_n$ of $x_n$, and define the label of all affine edges of the Yang-Baxter graph to be

$$\Phi := \tau\, G$$

(recall that $\tau$ is the shift: $x_i\,\tau = x_{i-1}\,\tau$).

We could even be more general and take for example $G$ to be a differential operator on functions of $x_n$. At this stage, we need only the commutation of $G$ with $x_1,\ldots,x_{n-1}$, $s_1,\ldots,s_{n-2}$.

Let us write

$$G_i := s_i \cdots s_{n-1} G_n s_{n-1}\cdots s_i \ , \ 1\le i\le n-1 \ .$$

In the case of the degenerate affine Hecke algebra with generators $\sigma_i = s_i + \gamma\partial_i$, we used differences $c_{i+1}-c_i$ and $x_{i+1}-x_i$, and consistency requires them to be invariant under $i\mapsto i+n$. For that purpose, we introduce two more parameters $\alpha,\beta$:

$$x_{i+n} = x_i + \beta \ , \ c_{i+n}(\triangleright) = c_i(\triangleright) + \alpha \ \forall i\in\mathbb{Z} \ . \tag{28}$$

The subgraph $\begin{array}{ccl} v & \rightarrow & vs_{i+1} \\ \downarrow & & \downarrow \\ v\tau & \rightarrow & v\tau s_i \ = vs_{i+1}\tau \end{array}$ give rise to the following commutative diagram (writing $c(v)$ instead of $v$, and 1,2,3 instead of $i, i+1, i+2$):

$$\begin{tikzcd}
[\ldots c_1, c_2, c_3 \ldots] \arrow[r, "s_2+\gamma\partial_2+\frac{1}{c_3-c_2}"] \arrow[d, "\Phi"'] & {[\ldots c_1, c_3, c_2 \ldots]} \arrow[d, "\Phi"'] \\
{[\ldots c_2, c_3, c_4 \ldots]} \arrow[r, "s_1+\gamma\partial_1+\frac{1}{c_3-c_2}"] & i \quad [\ldots c_3, c_2, c_4 \ldots]
\end{tikzcd} \tag{29}$$

which says nothing more than $x_2\tau = x_1$, $x_3\tau = x_2$, $\partial_2\tau = \tau\partial_1$, $s_2\tau = \tau s_1$.

The commutativity of

$$\begin{tikzcd}
{[\ldots c_{n-1}, c_n, c_{n+1}, c_{n+2} \ldots]} \arrow[r, "s_1+\gamma\partial_1+\frac{1}{c_2-c_1}"] \arrow[d, "\Phi\Phi"'] & {[\ldots c_{n-1}, c_n, c_{n+2}, c_{n+1} \ldots]} \arrow[d, "\Phi\Phi"'] \\
{[\ldots c_{n+1}, c_{n+2}, c_{n+3}, c_{n+4} \ldots]} \arrow[r, "s_{n-1}+\gamma\partial_{n-1}+\frac{1}{c_2-c_1}"] & {[\ldots c_{n+2}, c_{n+1}, c_{n+3}, c_{n+4} \ldots]}
\end{tikzcd} \tag{30}$$

is also immediate, once noticing that $\tau G_n \tau G_n = \tau^2 G_{n-1} G_n$, and that $G_{n-1}G_n$, being symmetrical in $x_{n-1}$ and $x_n$, commutes with $s_{n-1}$, as well as with $\partial_{n-1}$.

Braid relations, together with these two types of diagrams allow to deform any path $\triangleright \mapsto v$ into any other one with the same end points. Thus conditions (11) and (28) furnish a Yang-Baxter graph with edges in the degenerate Hecke algebra.

In the case of the affine Hecke algebra, we used $c_{i+1}/c_i - 1$ and $1 - x_{i+1}/x_i$ in the labels of edges of cells, and we need them to be invariant under $i \mapsto i+n$. We thus introduce two extra parameters $q, q'$ (in the following sections, we shall restrict to $q = q'$) and put

$$x_{i+n} = q x_i \ , \ c_{i+n}(\triangleright) = q' c_i(\triangleright) \ , \ \forall i \in \mathbb{Z} \tag{31}$$

Now, since $\pi_i\tau = \tau\pi_{i-1}$, $s_i\tau = \tau s_{i-1}$, $1 < i < n$, and since $G_{n-1}G_n$ commutes with $T_{n-1}$, one still has commutative diagrams (where $T_1, T_2$ stand for $T_i, T_{i+1}$, $1 \le i \le n-2$ in the first diagram) :

$$\begin{tikzcd}
{[\ldots c_1, c_2, c_3 \ldots]} \arrow[r, "T_2+\frac{t_1+t_2}{c_3/c_2-1}"] \arrow[d, "\Phi"'] & {[\ldots c_1, c_3, c_2 \ldots]} \arrow[d, "\Phi"'] \\
{[\ldots c_2, c_3, c_4 \ldots]} \arrow[r, "T_1+\frac{t_1+t_2}{c_3/c_2-1}"] & {[\ldots c_3, c_2, c_4 \ldots]}
\end{tikzcd} \tag{32}$$

and

$$\begin{array}{ccc}
[\ldots c_{n-1}, c_n, c_{n+1}, c_{n+2}\ldots] & \xrightarrow{T_1+\frac{t_1+t_2}{c_2/c_1-1}} & [\ldots c_{n-1}, c_n, c_{n+2}, c_{n+1}\ldots] \\
\Phi\Phi \Big\downarrow & & \Phi\Phi \Big\downarrow \\
[\ldots c_{n+1}, c_{n+2}, c_{n+3}, c_{n+4}\ldots] & \xrightarrow{T_{n-1}+\frac{t_1+t_2}{c_2/c_1-1}} & [\ldots c_{n+2}, c_{n+1}, c_{n+3}, c_{n+4}\ldots]
\end{array} \tag{33}$$

Conditions (21) and (31) provide a Yang-Baxter graph with edges in the Hecke algebra.

# 7 Cherednik's Algebras

The double affine Hecke algebra $\mathbb{C}[t_1, t_2, q][T_1, \ldots, T_{n-1}, \tau^{\pm 1}, x_1^{\pm 1}, \ldots x_n^{\pm 1}]$ admits some maximal commutative subalgebra which was found by Cherednik in his study of Macdonald's polynomials. We need a more general algebra than the original Cherednik's algebra, and for that purpose, we shall use the element $G_n$ introduced in the preceding section, supposing only for the moment that it is an invertible morphism of the space of functions in $x_n$.

Let us write $\overline{G} := G^{-1}$, $\overline{\tau} = \tau^{-1}$, $\overline{T}_i := -t_1 t_2 T_i^{-1}$, $1 \le i \le n-1$. Recall that $x_i \overline{\tau} = \overline{\tau} x_{i+1}$, and that $\Phi = \tau G_n$ label the affine edges of the Yang-Baxter graph..

For $1 \le i \le n$, define the *Cherednik element* $\xi_i$ by

$$\xi_i := T_i \ldots T_{n-1}\, x_n\, \overline{G}_n\, \overline{\tau}\, \overline{T}_1 \cdots \overline{T}_{i-1}\ . \tag{34}$$

**Proposition 6** *The $\xi_i$ generate a commutative algebra and satisfy the relations*

$$\begin{gathered}
\xi_{i+1} T_i = \overline{T}_i\, \xi_i \quad \& \quad (\xi_i + \xi_{i+1}) T_i = T_i (\xi_i + \xi_{i+1})\ , 1 \le i \le n-1\ , \\
\xi_j T_i = T_i \xi_j\ , j \ne i, i+1 \\
\xi_{i+1}\, \Phi = \Phi\, \xi_i\ ,\ i = 1, \ldots, n-1\ .
\end{gathered} \tag{35}$$

*Moreover, if $G_n$ commutes with multiplication by $x_n$, then*

$$q \xi_1\, \Phi = \Phi\, \xi_n\ . \tag{36}$$

*Proof.* One checks without difficulty the relations between the $\xi_i$ and $T_j$ and the commutativity of the $\xi_i$.

Thanks to these relations, the compatibilities between $\Phi$ and the $\xi_i$ is reduced to show that $\xi_2\, \Phi = \Phi\, \xi_1$, i.e.

$$T_2 \cdots T_{n-1} x_n \overline{G}_n \overline{\tau}\, \overline{T}_1 \tau G_n = \tau G_n T_1 \cdots T_{n-2} T_{n-1} x_n \overline{G}_n\, \overline{\tau}\ .$$

Now the factor $T_1 \cdots T_{n-2}$ commutes with $G_n$ and cancels with the factor $T_2 \cdots T_{n-1}$ on the left hand side; we are left to show that

$$x_n \overline{G}_n \overline{\tau}\, \overline{T}_1 \tau G_n = \tau G_n\, T_{n-1} x_n \overline{G}_n \overline{\tau} \ .$$

Let us take the indeterminates $x_2, \ldots, x_n, x_{n+1} = q x_1$ instead of $x_1, \ldots x_n$. It allows us to use $G_{n+1} := \tau\, G_n\, \overline{\tau}$. Let us moreover write $T_n := \overline{\tau}\, T_1\, \tau = \tau\, T_{n-1}\, \overline{\tau}$; concretely $T_n$ operates on $x_n, x_{n+1}$ according to (15) and leaves $x_2, \ldots, x_{n-1}$ invariant.

Introducing factors $\overline{G}_{n+1} G_{n+1}$ and $\overline{\tau}\tau$, one has to show

$$x_n (G_{n+1} \overline{G}_{n+1})\, \overline{G}_n \overline{\tau} \overline{T}_1\, \tau G_n = \tau G_n\, (\overline{\tau}\tau) T_{n-1}\, (\overline{\tau}\tau)\, x_n (\overline{\tau}\tau)\, \overline{G}_n \overline{\tau} \ .$$

Taking into account that $\overline{G}_{n+1} \overline{G}_n$ commutes with $\overline{\tau} \bar{T}_1 \tau = T_{x_n, x_{n+1}}$, we are finally reduced to make sure that

$$x_n G_{n+1} \overline{T}_n \overline{G}_{n+1} \overline{G}_n G_n = G_{n+1} T_n x_{n+1} \overline{G}_{n+1} \ ,$$

that is

$$x_n\, \overline{T}_n = T_n x_{n+1} \ ,$$

which is no other than relation (19) for the pair $(x_n, x_{n+1})$.

One similarly checks that $T_1 \cdots T_{n-1}\, \overline{\tau}\tau\, x_n = x_1 \overline{T}_1 \cdots \overline{T}_{n-1} = \tau x_0 \overline{\tau} \overline{T}_1 \cdots \overline{T}_{n-1} = \tau x_n \xi_n / q$, that is

$$\xi_1\, \tau\, x_n = \tau\, x_n\, \xi_n / q \ . \tag{37}$$

There remains the case of $\Phi\, \xi_n = \tau G_n\, x_n G_n \overline{T}_1 \cdots \overline{T}_{n-1}$. If $G_n$ commutes with $x_n$, then

$$\Phi\, \xi_n = \tau x_n \overline{\tau}\, \overline{T}_1 \cdots \overline{T}_{n-1} = q T_1 \cdots T_{n-1}\, x_n = q \left(T_1 \cdots T_{n-1}\, x_n \overline{G}_n \overline{\tau}\right)\, (\tau G_n) \ ,$$

that is

$$\Phi\, \xi_n = q \xi_1\, \Phi \ .$$

QED

The $\xi_i$'s satisfy the same commutation relations with the $T_i$'s as the $x_i$'s do. In fact, one can interpolate between $\xi_i$ and $x_i$ without losing a commutative algebra.

**Theorem 7** *Let $\beta \in \mathbb{C}$; for $i = 1, \ldots, n-1$, let*

$$\xi_i^\beta := \xi_i + \beta x_i \ .$$

*Then the $\xi_i^\beta$'s generate a commutative algebra, and satisfy the relations*

$$\xi_{i+1}^\beta T_i = \overline{T}_i\,\xi_i^\beta \quad \& \quad (\xi_i^\beta+\xi_{i+1}^\beta)T_i = T_i(\xi_i^\beta+\xi_{i+1}^\beta)\ ,1\le i\le n-1\ , \tag{38}$$

$$\xi_j^\beta T_i = T_i\xi_j^\beta\ \ ,j\neq i,i+1 \tag{39}$$

$$\xi_{i+1}^\beta\,\Phi = \Phi\,\xi_i^\beta\ ,\ \ i=1,\ldots,n-1\ . \tag{40}$$

*Moreover, if $G_n$ commutes with multiplication by $x_n$, then*

$$q\xi_1^\beta\,\Phi = \Phi\,\xi_n^\beta\ . \tag{41}$$

*Proof.* Commutations relations (38,39,40) are a consequence of those for $\beta=0$. Given $1\le i<j\le n$, the commutator $[\xi_i^\beta,\,\xi_j^\beta]$ is equal to $\beta[x_i,\,\xi_j^\beta]-\beta[\xi_i^\beta,\,x_j]$. Explicitly,

$$x_j\xi_i^\beta = x_j\,\underbrace{T_i\cdots T_{j-2}}_{A}\,T_{j-1}\,\underbrace{T_j\cdots T_{n-1}\,x_n\,\overline{G}_n}_{B}\,\overline{\tau}\,\underbrace{\overline{T}_1\cdots\overline{T}_{i-1}}_{C}\ .$$

Now, $x_j$ commutes with $A$; $x_jT_{j-1}=\overline{T}_{j-1}x_{j-1}$; $x_{j-1}$ commutes with $B$; passing through $\overline{\tau}$, it is transformed into $x_j$ again and commutes with $C$.

Therefore

$$[\xi_i^\beta,\,x_j] = A\big(T_{j-1}-\overline{T}_{j-1}\big)B\,\overline{\tau}\,Cx_j = A(t_1+t_2)\,B\,\overline{\tau}\,Cx_j\ .$$

Similarly, starting from

$$x_i\xi_j^\beta = x_i\,B\,\overline{\tau}\,C\,\overline{T}_i\,\underbrace{\overline{T}_{i+1}\cdots\overline{T}_{j-1}}_{D}\ ,$$

one finds that

$$[\xi_j^\beta,\,x_i] = B\,\overline{\tau}\,Cx_{i+1}D(t_1+t_2)\ .$$

Finally, one remarks that $x_{i+1}D = T_{i+1}\cdots T_{j-1}\,x_j := Ex_j$, and that $A$ commutes with $B$, becomes equal to $E$ when passing through $\overline{\tau}$ and commutes with $C$ at the end. Therefore the $\xi_i^\beta$'s generate a commutative algebra. QED

One checks, for any $1\le i<n$ the commutations

$$T_i(\xi_i^\beta+\xi_{i+1}^\beta) = (\xi_i^\beta+\xi_{i+1}^\beta)T_i \quad \& \quad T_i(\xi_i^\beta\xi_{i+1}^\beta) = (\xi_i^\beta\xi_{i+1}^\beta)T_i\ , \tag{42}$$

and this ensures the commutations of each $T_i$ with the elementary symmetric functions in the Cherednik elements. Therefore one has:

**Lemma 8** *The Hecke algebra generated by the $T_i$'s commutes with the ring of symmetric functions in the $\xi_i^\beta$.*

For computations, it is convenient to write monomials in the $\xi_i$'s (without repetition) in a canonical form. Write $\theta$ for $x_n \, \overline{G}_n \, \overline{\tau}$. Then one checks, from the commutation relations (35), together with $T_i \overline{T}_i = (-t_1 t_2)$, that:

**Lemma 9** *Let $k$ be an integer, $n \geq i_1 > i_2 > \cdots > i_k \geq 1$ a decreasing sequence. Then the monomial $\xi_{i_1} \xi_{i_2} \cdots \xi_{i_k} (t_1 t_2)^{-k(k-1)}$ is equal to*

$$\begin{array}{ccccc} & & i_1 & \cdots & n{-}1 \\ & i_2 & \cdots & \cdots & n{-}2 \\ & \vdots & & & \vdots \\ i_k & \cdots & \cdots & \cdots & n{-}k \end{array} \; \theta^k \; \begin{array}{ccccc} k & \cdots & \cdots & \cdots & i_1{-}1 \\ k{-}1 & \cdots & \cdots & i_2{-}1 & \\ \vdots & \cdots & \vdots & & \\ 1 & \cdots & i_k{-}1 & & \end{array} ,$$

*where the planar display on the left must be read (columwise or rowwise) as a product of $T_j$'s, and the display on the right, as a product of $\overline{T}_j$'s.*

For example, for $n = 8$,

$$\xi_6 \xi_3 \xi_2 (t_1 t_2)^{-6} = \begin{smallmatrix} . & . & 6 & 7 \\ 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \end{smallmatrix} \; \theta^3 \; \begin{smallmatrix} 3 & 4 & 5 \\ 2 & . & . \\ 1 & . & . \end{smallmatrix} = (T_6 T_7 T_3 T_4 T_5 T_6 T_2 T_3 T_4 T_5) \; \theta^3 \; \left(\overline{T}_3 \overline{T}_4 \overline{T}_5 \overline{T}_2 \overline{T}_1\right) .$$

Exchanging left and right, one gets another commutative algebra generated by the

$$T_{i-1} \cdots T_1 \, \tau \frac{1}{x_n} G_n \, \overline{T}_{n-1} \cdots \overline{T}_i + \frac{\beta}{x_i} \; , \; 1 \leq i \leq n \; .$$

In the case of interest for vanishing Macdonald polynomials, one takes $G_n = x_n - (-t_2)^{n-1}$. The following lemma (due to Knop [18]) shows that in that case, one can choose an appropriate $\beta$ in order to get operators preserving polynomials.

**Lemma 10** *Let*

$$\widehat{\xi}_i := T_{i-1} \cdots T_1 \, \tau \big(1 - \frac{(-t_2)^{n-1}}{x_n}\big) \overline{T}_{n-1} \cdots \overline{T}_i + \frac{(-t_1 t_2)^{n-1}}{x_i}, \; 1 \leq i \leq n \; . \quad (43)$$

*Then the $\widehat{\xi}_i$'s generate a commutative algebra which preserves polynomials in $x_1, \ldots, x_n$.*

*Proof.* Because of the commutation relations (35), we have only to check that $\widehat{\xi}_n$ preserves polynomials. Recall that $x_{i+1} T_i = \overline{T}_i x_i$, which implies, because $x_i + x_{i+1}$ commute with $T_i$, that

$$x_i T_i = T_i x_{i+1} + (t_1 + t_2) x_i \; .$$

Therefore, for $1 \le i \le n-1$, one has

$$x_i T_{n-1} \cdots T_1 = T_{n-1} \cdots T_1 x_{i+1} + (t_1+t_2) T_{n-1} \cdots T_{i+1} \overline{T}_{i-1} \cdots \overline{T}_1 \, x_1 \ ,$$

and

$$x_i \widehat{\xi}_n = T_{n-1} \cdots T_1 \, \tau (1 - \frac{(-t_2)^{n-1}}{x_n}) x_i + \frac{(-t_1 t_2)^{n-1} x_i}{x_n}$$
$$+ (t_1+t_2) T_{n-1} \cdots T_{i+1} \overline{T}_{i-1} \cdots \overline{T}_1 \, \tau \big( x_0 - \frac{(-t_2)^{n-1} x_0}{x_n} \big)$$

and this last expression is equal to $\widehat{\xi}_n x_i$ plus a polynomial (because $x_0 = x_n/q$).

On the other hand,

$$x_n \widehat{\xi}_n = \overline{T}_{n-1} \cdots \overline{T}_1 \tau \big( x_0 - \frac{(-t_2)^{n-1} x_0}{x_n} \big) + (-t_1 t_2)^{n-1} \ .$$

In all, $\widehat{\xi}_n$ commutes with polynomials, modulo addition of polynomials. Checking that

$$1 \, \widehat{\xi}_n = t_1^{n-1} \big( 1 - \frac{(-t_2)^{n-1}}{x_n} \big) + \frac{(-t_1 t_2)^{n-1}}{x_n} = t_1^{n-1} \ ,$$

one concludes that $\widehat{\xi}_n$ preserves polynomials. QED

There are also several Cherednik algebras relative to the degenerate affine Hecke algebra. We just state the analogs of the results that we described for the Hecke algebra, keeping the same notations for Cherednik's elements because they will be used in a different context.

The first Cherednik's elements are the isobaric version of Dunkl's elements ([10]).

Let $D_i$ be the isobaric derivative $f \mapsto x_i \frac{d}{dx_i}(f)$, and let $\overline{\pi}_{i,j}$ denote $\overline{\pi}_{x_i,x_j}$. Define, following [5],

$$\xi_i := \alpha D_i + \sum_{j \neq i, j=1}^{n} \overline{\pi}_{i,j} + 1 - i \ , \ 1 \le i \le n \ . \tag{44}$$

A second variety of elements involves the arbitrary function $G_n$ of $x_n$ (it can be obtained by degeneration of the $\xi_i^\beta$'s). Define

$$\widehat{\xi}_i := x_i - \sigma_{i-1} \cdots \sigma_1 \, \tau \, G_n \, \sigma_{n-1} \cdots \sigma_i \ 1 \le i \le n \ , \tag{45}$$

with $\sigma_i = s_i + \partial_i$ and $\tau$ still such that $x_i \tau = x_{i-1}$, $x_0 = x_n - \alpha$.

Define moreover

$$\Phi = s_1 \dots s_{n-1} x_n \quad , \quad \widehat{\Phi} = \tau\, G_n \ .$$

Then one has (cf. [5],[20] for the first part, [18] for the second part when $G_n = x_n + n - 1$):

**Theorem 11** *The $\xi_i$'s (resp. the $\widehat{\xi}_i$'s), $1 \le i \le n$, generate a commutative algebra.*

*They satisfy*

$$\begin{gathered}\xi_{i+1} s_i = s_i\, \xi_i - 1 \quad \& \quad (\xi_i + \xi_{i+1}) s_i = s_i(\xi_i + \xi_{i+1}) \ , 1 \le i \le n-1 \ , \\ \xi_j s_i = s_i \xi_j \ , j \ne i, i+1 \\ \xi_i\, \Phi = \Phi\, \xi_{i-1} \ , \ i = 1, \dots, n, \ \xi_0 = \xi_n - \alpha \ .\end{gathered} \tag{46}$$

$$\begin{gathered}\widehat{\xi}_{i+1} \sigma_i = \sigma_i\, \widehat{\xi}_i - 1 \quad \& \quad (\widehat{\xi}_i + \widehat{\xi}_{i+1})\, \sigma_i = \sigma_i(\widehat{\xi}_i + \widehat{\xi}_{i+1}) \ , 1 \le i \le n-1 \ , \\ \widehat{\xi}_j\, \sigma_i = \sigma_i\, \widehat{\xi}_j \ , j \ne i, i+1 \\ \widehat{\xi}_i\, \widehat{\Phi} = \widehat{\Phi}\, \widehat{\xi}_{i-1} \ , \ i = 1, \dots, n, \ \widehat{\xi}_0 = \widehat{\xi}_n - \alpha \ .\end{gathered} \tag{47}$$

# 8 Propagation of vanishing properties, and of eigenvectorness

Each edge of a cell in a Yang-Baxter graph is an operator on the ring of polynomials in only two variables, say $x, y$, as a rank 2 free-module over $\mathfrak{S}ym(x, y)$. We shall easily check, in this section, that if a vertex is an eigenvector for Cherednik elements, or vanishes for some values of the $x_i$'s, then its neighbours in a cell have similar properties.

Choosing appropriate functions $G_n$ of $x_n$, the same will be true for neighbours connected by an affine edge.

**Lemma 12** *Let $f$ be a function of $x_1, \dots, x_n$, and $i$ be an integer: $1 \le i \le n-1$. Then*

*1) if $f$ is an eigenfunction of $\xi_i^\beta, \xi_{i+1}^\beta$, of respective eigenvalues $a, b$, then $g := f\left(T_i + (t_1 + t_2)/(b/a - 1)\right)$ is an eigenvector of $\xi_i^\beta, \xi_{i+1}^\beta$ with eigenvalues $b, a$.*

*2) if $f$ is an eigenfunction of $\xi_1^\beta, \dots, \xi_n^\beta$ of respective eigenvalues $c_1, \dots, c_n$, then all elements in the linear span of $\{f,\ f\,T_i\}$ are eigenfunctions of $\xi_1^\beta, \dots, \xi_{i-1}^\beta,\ \xi_{i+2}^\beta, \dots, \xi_n^\beta$ of respective eigenvalues $c_1, \dots, c_{i-1}, c_{i+2}, \dots, c_n$.*

*Proof.*

$$f\xi_{i+1}^{\beta}T_i = bfT_i = bf(T_i + \frac{t_1+t_2}{b/a-1} - \frac{t_1+t_2}{b/a-1}) = bg - b\frac{t_1+t_2}{b/a-1}f$$

Similarly,

$$f\overline{T}_i\xi_i^{\beta} = f(T_i + \frac{t_1+t_2}{b/a-1} + \frac{b(t_1+t_2)}{a-b})\xi_i^{\beta} = g\xi_i^{\beta} + f\frac{ab(t_1+t_2)}{a-b} \ .$$

Equality $\xi_{i+1}^{\beta}T_i = \overline{T}_i\xi_i^{\beta}$ implies that

$$g\xi_i^{\beta} = g\, b \ , \ g\xi_{i+1}^{\beta} = g\, a \ .$$

This settles the first point, the second being a direct consequence of the commutation relations (38,39,40). QED

**Corollary 13** *Let $f$ be an eigenfunction of $\xi_1^{\beta}, \ldots, \xi_n^{\beta}$ of respective eigenvalues $c_1, \ldots, c_n$. Then all elements of the space $f\,\mathcal{H}_n$ (in particular the symmetric function $f\Box_\omega$) are eigenfunctions of $\prod_{i=1}^{n}(1+\gamma\xi_i)$ with eigenvalue $\prod(1+\gamma c_i)$, for any $\gamma \in \mathbb{C}$.*

**Lemma 14** *Let $f(x,y)$ be a function of two variables.*

*1) Suppose that there exist $a, b$: $a \neq b$ such that $f(a,b) = f(b,a) = 0$. Then $fT_{x,y}$, $fs_{x,y}$, $f\partial_{x,y}$ also vanish in $[x,y] = [a,b]$ and $[x,y] = [b,a]$.*

*2) Suppose that $f(a,b) = 0$, with $b = (-t_2/t_1)a$. Then $fT_{x,y}$ also vanish in $[x,y] = [a,b]$.*

*Suppose that $f(a,b) = 0$, with $b = a - \gamma$. Then $f(s_{x,y} + \gamma\partial_{x,y})$ vanish in $[x,y] = [a,b]$.*

*3) Suppose that $f(b,a) = 0$, $a \neq b$. Then*

$$f\big(T_{x,y} + \frac{t_1+t_2}{b/a-1}\big) \quad \text{and} \quad f\big(s_{x,y} + \gamma\partial_{x,y} + \frac{\gamma}{b-a}\big)$$

*vanish in $[x,y] = [a,b]$*

*Proof.* Write $f = f_1 + yf_2$, with $f_1, f_2 \in \mathfrak{S}ym(x,y)$.

If $f(a,b) = f(b,a) = 0$ and $a \neq b$, then both $f_1$ and $f_2$ vanish in $[a,b]$ and $[b,a]$, and so do $fT_{x,y}$, $fs_{x,y}$, $f\partial_{x,y}$. This proves the first point.

The expression $fT_{x,y} = t_1f_1 - t_2xf_2$ specializes, for $[x,y] = [a,b]$ into $t_1f_1(a,b) - t_2af_2(a,b) = t_1f(a,b)$, which is null in the second case. Similarly, $f(s_{x,y} + \gamma\partial_{x,y}) = f_1 + xf_2 - \gamma f_2$ specializes into $f_1(a,b) + (a-u)f_2(a,b)$, which is null when $f(a,b) = 0$.

In the third case,

$$f\big(T_{x,y}+\frac{t_1+t_2}{b/a-1}\big)=\big(t_1+\frac{t_1+t_2}{b/a-1}\big)f_1(x,y)+\big(-t_2x+\frac{t_1+t_2}{b/a-1}y\big)f_2(x,y)$$

specializes, for $[x,y]=[a,b]$, into

$$\big(\frac{t_1b/a+t_2}{b/a-1}\big)f_1(a,b)+\big(\frac{t_1b+t_2a}{b/a-1}y\big)f_2(a,b)=af(b,a)\ ,$$

that is, vanishes.

Similarly

$$f\big(s_{x,y}+\gamma\partial_{x,y}+\frac{\gamma}{b-a}\big)=f_1\big(1+\frac{\gamma}{b-a}\big)+f_2\big(x-\gamma+\frac{\gamma y}{b-a}\big)$$

specializes, for $[x,y]=[a,b]$, into $af(b,a)$. QED

Propagation of vanishing properties and of eigenvectors along affine edges are insured by the choice of appropriate $\Phi$'s, as shows the following lemma :

**Lemma 15** *1) Let $f$ be an eigenfunction of the Cherednik elements $\xi_1,\ldots,\xi_n$, with eigenvalues $c_1,\ldots,c_n$. Let $\Phi=\tau G_n$, with $G_n$ commuting with multiplication by $x_n$. Then $f\Phi$ is an eigenfunction of $\xi_1,\ldots,\xi_n$, with eigenvalues $c_2,\ldots,c_n,qc_1$ (in the case of the Hecke algebra) or $c_2,\ldots,c_n,c_1+\alpha$ (in the case of the degenerate Hecke algebra).*

*2) Let $k$ be an integer, and let $f$ be a function vanishing in all $c(u)$, $|u|\le k$. Let $\Phi=\tau G_n$, with $G_n((-t_2)^{n-1})=0$ (in the case of the Hecke algebra) and $G_n(1-n)=0$ (in the case of the degenerate Hecke algebra). Then $f\,\Phi$ vanishes in all $c(u)$, $|u|\le k+1$.*

*Proof.* Commutation rules of Cherednik's elements with $\Phi$ insure the first point. If $f$ vanishes in $c(u)$, then $f\tau$ vanishes in $c(u\tau)$. The vectors $u$ : $|u|\le k+1$ which are not in the image of $\tau$ are such that $u_n=0$, in which case $c_n(u)=(-t_2)^{n-1}$ or $1-n$, depending on the algebra. In that case, the vanishing of $G_n$ implies the vanishing of $f\,\Phi$. QED

# 9 Macdonald polynomials

Take Yang-Baxter graph with

$$\sigma_i:=T_i\ ,\ 1\le i\le n-1\ ,\ c_i(\triangleright)=t_1^{n-i}(-t_2)^{i-1}\ ,\ 1\le i\le n\ ,G_n=x_n\ ,$$
$$x_{i+n}=qx_i\ ,\ c_{i+n}(\triangleright)=qc_i(\triangleright)\ ,\ \forall i\in\mathbb{Z} \tag{48}$$

**Definition 16** *The* Macdonald polynomial $M_v$*,* $v \in \mathbb{N}^n$*, is the image of the polynomial* $1$ *under the element* $Y_v$ *associated to the Yang-Baxter graph satisfying (48). In other words, one has:*

$$M_v := 1\, Y_v \ , \ \forall v \in \mathbb{N}^n \ .$$

The easiest characterization of these polynomials (cf. [36], [7]) is that they are eigenfunctions of Cheredenik's elements $\xi_i$ (for the same $G$), that is for

$$\xi_i := T_i \ldots T_{n-1}\, \overline{\tau}\, \overline{T}_1 \cdots \overline{T}_{i-1} \ .$$

**Theorem 17** *Let* $v \in \mathbb{N}^n$*. Then the Macdonald polynomial* $M_v$ *is a simultaneous eigenfunction of the* $\xi_i$*'s, i.e. one has*

$$M_v\, \xi_j = M_v\, c_j(v) \ , \ j : 1 \le j \le n \ .$$

*Proof.* Relations (35,36) ensure that $M_v$ is an eigenfunction, with eigenvalue $c(v)$, if it is true for the origin $v = \triangleright$. Indeed, $1\, T_i = t_1$, $1\, \overline{T}_i = -t_2$ implies that $1\, \xi_i = t_1^{n-i}(-t_2)^{i-1} = c_i(\triangleright)$. QED

Mimachi and Noumi give in [37] a reproducing kernel for the Macdonald polynomials.

The labelling of the affine edges of the Yang-Baxter graph by $\Phi = \tau\, x_n$ is due to Sahi [40] and Knop [18]. They call $\Phi$ a "creation operator". Baker and Forrester [2] similarly define a *lowering operator* $\widehat{\Phi}$ :

$$\widehat{\Phi} = \big(T_{n-1} \cdots T_1\, \tau - t_1^{n-1}\big) \frac{1}{x_n}\, \overline{T}_{n-1} \cdots \overline{T}_1 \ . \tag{49}$$

**Lemma 18** $\widehat{\Phi}$ *preserves polynomials and sends the Macdonald polynomial* $M_v$ *onto a multiple of* $M_{v\overline{\tau}}$ *if* $v_n > 0$*, and onto 0 otherwise.*

*Proof.* Let $\overline{\xi}_n := T_{n-1} \cdots T_1\, \tau$. Then $\overline{\xi}_n\, \xi_n = (-t_1 t_2)^{n-1}$ and $\overline{\xi}_n - t_1^{n-1}$ annihilates all $M_v$, with $v_n = 0$ (because in that case, $c_n(v) = (-t_2)^{n-1}$). Therefore $\widehat{\Phi}$ annihilates those Macdonald polynomials.

Since $\Phi$ has zero kernel, we shall evaluate $\Phi\widehat{\Phi}$ to determine the action of $\widehat{\Phi}$ on the other Macdonald polynomials.

$$\begin{aligned}
\Phi\widehat{\Phi} &= \tau\, x_n\, T_{n-1} \cdots T_1 \tau \frac{1}{x_n}\, \overline{T}_{n-1} \cdots \overline{T}_1 - t_1^{n-1} \tau\, x_n \frac{1}{x_n}\, \overline{T}_{n-1} \cdots \overline{T}_1 \\
&= \tau\, \overline{T}_{n-1} \cdots \overline{T}_1\, (x_1 \tau \frac{1}{x_n})\, \overline{T}_{n-1} \cdots \overline{T}_1 - t_1^{n-1} \tau\, \overline{T}_{n-1} \cdots \overline{T}_1 \\
&= \frac{1}{q}\, \overline{\xi}_1^2 - t_1^{n-1} \overline{\xi}_1 \ ,
\end{aligned}$$

with $\overline{\xi}_1 := \tau\,\overline{T}_{n-1}\cdots\overline{T}_1$. Therefore $\widehat{\Phi}$ preserves polynomials, and sends $M_v$ on a non-zero multiple of $M_{v\overline{\tau}}$ if $v_n \neq 0$. QED

A simpler lowering operator is

$$\widehat{\widehat{\Phi}} := \left(T_{n-1}\cdots T_1\,\tau - t_1^{n-1}\right)\frac{1}{x_n}\,\overline{\tau}\ . \tag{50}$$

Since the eigenvalues of $M_v$ under the $\xi_j$'s are permuted of those of $M_{vs_i}$, any linear combination of $M_v$, where the $v$'s are in the orbit under $\mathfrak{S}_n$, is an eigenfunction of any symmetric function in the $\xi_j$. In particular, let $\Xi := \xi_1 + \cdots + \xi_n$. One has the following theorem ([36], [7]).

**Theorem 19** *For each partition $\lambda \in \mathbb{N}^n$, there exists a unique polynomial $M_\lambda^{\mathfrak{S}}$, symmetrical in $x_1,\ldots,x_n$, whose expansion in the basis of Schur functions is of the type $M_\lambda^{\mathfrak{S}} = S_\lambda + \sum_{\mu<\lambda} c_\lambda^\mu S_\mu$ and which is an eigenfunction of $\Xi$ (the sum being over all partitions smaller than $\lambda$ with respect to the dominance order).*

*Given any $v$ in the $\mathfrak{S}_n$-orbit of $\lambda$, then up to multiplication by a scalar (we write $\doteq$), one has*

$$M_\lambda^{\mathfrak{S}} \doteq M_v \sum_{w\in\mathfrak{S}_n} (-t_2)^{\binom{n}{2}-\ell(w)} T_w = M_v \prod_{1\le i<j<\le n} (t_1x_i + t_2x_j)\,\partial_\omega = M_v\,\Box_\omega\ .$$

*Proof.* Since $\Xi$ commutes with $\Box_\omega$, $M_v\,\Box_\omega$ is a symmetric eigenfunction of $\Xi$ with eigenvalue $q^{\lambda_1}t_1^{n-1} + \cdots + q^{\lambda_n}(-t_2)^{n-1}$ which characterizes the partition $\lambda$. One can check that the $\xi_i$'s have a triangular action on polynomials by putting an appropriate order on monomials, related to the Bruhat order on the affine symmetric group. We refer for that point to [20] and [17]. Thus $M_\lambda^{\mathfrak{S}}$ is not only characterized by an eigenvalue, but also by a leading term. QED

The first polynomials, in the basis of Schur functions are, for two variables

$$M_{0,0}^{\mathfrak{S}} = 1,\ M_{1,0}^{\mathfrak{S}} = S_1,\ M_{2,0}^{\mathfrak{S}} \doteq (qt_1+t_2)S_2 + (qt_2+t_1)S_{1,1},\ M_{1,1}^{\mathfrak{S}} = S_{1,1}$$
$$M_{3,0}^{S} \doteq (t_2+q^2t_1)S_3 - (q+1)(t_1+qt_2)S_{2,1},$$

$$\begin{aligned}M_{4,0}^{\mathfrak{S}} \doteq{}& (t_2+q^3t_1)(t_2+q^2t_1)S_4\\ &+ (1+q+q^2)(t_2+q^2t_1)(t_1+qt_2)S_{3,1} + (q+q^3)(t_1+t_2)(t_1+qt_2)S_{2,2}\end{aligned}$$

From the expression of the $\xi_i$'s one sees that all $\Box_\omega\,\xi_i\,\Box_\omega$, and thus also $\Box_\omega\,\Xi\,\Box_\omega$, $\Box_\omega\,\Xi$, are proportional to $\Box_\omega\,\overline{\tau}\,\Box_\omega$ . But now, $\Box_\omega$ acts as a scalar on symmetric function, and therefore the action of $\Xi(1\,\Box_\omega)$ on symmetric functions is just

$$f \mapsto f\,\tau\,\Box_\omega = f\,\tau \prod_{1\le i<j<\le n} (t_1x_i + t_2x_j)\,\partial_\omega$$
$$= f(x_2,\ldots,x_n,qx_1) \prod_{1\le i<j<\le n} (t_1x_i + t_2x_j)\,\partial_\omega\ , \quad (51)$$

which is the usual expression of the Debiard-Sekiguchi-Macdonald operator.

Creation (or raising) operators on symmetric Macdonald polynomials are similar operators ([23])

$$B_k := \big(t_1\tau - (-t_2)\big)\cdots\big(t_1^k\tau - (-t_2)^k\big)\,x_1\cdots x_k\,\Box_\omega\ .$$

The image of $M_\lambda^{\mathfrak{S}}$, when $\ell(\lambda)\le k$, is proportional to $M_{\lambda+1^k}^{\mathfrak{S}}$. Other creation operators are given in [26], [14], [15], [16].

## 10 Vanishing Macdonald polynomials

Take Yang-Baxter graph with

$$\sigma_i := T_i\ ,\ 1\le i\le n-1\ ,\ c_i(\triangleright) = t_1^{n-i}(-t_2)^{i-1}\ ,\ 1\le i\le n\ , G_n = x_n - (-t_2)^{n-1}\ ,$$
$$x_{i+n} = qx_i\ ,\ c_{i+n}(\triangleright) = qc_i(\triangleright)\ ,\ \forall i\in\mathbb{Z} \quad (52)$$

**Definition 20** *The* vanishing Macdonald polynomial $\widehat{M}_v$, $v\in\mathbb{N}^n$, *is the image of* 1 *under the element* $Y_v$ *associated to the Yang-Baxter graph satisfying (52):*

$$\widehat{M}_v := 1\,Y_v\ .$$

The appropriate Cherednik's elements are [18]

$$\widehat{\xi}_i := T_{i-1}\cdots T_1\,\tau\big(1 - \frac{(-t_2)^{n-1}}{x_n}\big)\overline{T}_{n-1}\cdots\overline{T}_i + \frac{(-t_1t_2)^{n-1}}{x_i},\ 1\le i\le n$$

**Theorem 21** *Let* $v\in N^n$, *and* $k = |v|$. *Let* $\mathcal{P}ol(v)$ *be the subspace of polynomials* $f$ *of degree* $\le |v|$ *such that*

$$\forall u\in\mathbb{N}^n,\ |u|\le|v|,\ u\ne v \Rightarrow f(c(u)) = 0\ .$$

*Then* $\mathcal{P}ol(v)$ *is 1-dimensional and contains* $\widehat{M}_v$.

*The polynomial* $\widehat{M}_v$ *is an eigenfunction of the* $\widehat{\xi}_i$*'s with eigenvalues*

$$(-t_1t_2)^{n-1}/c_1(v),\ldots,(-t_1t_2)^{n-1}/c_n(v)\ .$$

*The component of top degree of* $\widehat{M}_v$ *is* $M_v$.

The proof is the same as for Macdonald polynomials for what concerns eigenfunctions. One checks that 1 is an eigenfunction of the $\widehat{\xi}_i$'s with eigenvalues $(-t_2)^{n-1}, (-t_2)^{n-2}t_1, \ldots, t_1^{n-1}$, because $1T_i = t_1$, $1\overline{T}_i = -t_2$. On the other hand, it is not difficult to see that $\mathcal{P}ol(v)$ is 1-dimensional. The choice of $G_n = x_n - (-t_2)^{n-1}$ ensures that vanishing propagates along an affine edge.

Restricting $G_n$ to its top-degree component $x_n$ gives $M_v$ instead of $\widehat{M}_v$.

In the case $n = 2$, the first polynomials, and the vectors $c(v)$ are

$$\begin{array}{ccccccc} [0,0] & & [0,1] & & [1,0] & & [0,2] \\ [t_1,-t_2] & \xrightarrow{\Phi} & [-t_2, qt_1] & \longrightarrow & [qt_1, -t_2] & \xrightarrow{\Phi} & [-t_2, q^2 t_1] \\ 1 & & x_2 + t_2 & & -\frac{(t_1+t_2)t_2(x_2+t_2)}{qt_1+t_2} - t_2(x_1 - t_1) & & t_2(x_2+t_2)\left(t_1 - \frac{x_2}{q} - \frac{t_1+t_2}{qt_1+t_2}(x_1+t_2)\right) \end{array}$$

Given any $v \in \mathbb{N}^n$, there is a unique symmetric polynomial (up to normalization) in the linear span of $\widehat{M}_{v\,w}$, $w \in \mathfrak{S}_n$. This polynomial is obtained as the image of $\widehat{M}_v$ under

$$\square_\omega = \sum_{w \in \mathfrak{S}_n} (-t_2)^{\binom{n}{2} - \ell(w)} T_w = \prod_{1 \le i < j < \le n} (t_1 x_i + t_2 x_j)\, \partial_\omega$$

and clearly inherits from $\widehat{M}_v$ vanishing and eigenfunction properties, which are stated in the next theorem.

**Theorem 22** *Let $n$ be an integer, $\lambda \in \mathbb{N}^n$ be a partition, $v$ any vector in its orbit under $S_n$. Then*

$$\widehat{M}_v^{\mathfrak{S}} := \widehat{M}_v \prod_{1 \le i < j < \le n} (t_1 x_i + t_2 x_j)\, \partial_\omega$$

*is, up to a scalar, the unique symmetric polynomial vanishing in all $c(\mu)$, $\mu$ partition $\in \mathbb{N}^n$, $|\mu| \le |\lambda|$, $\mu \ne \lambda$.*

*It is also the unique eigenfunction of $\widehat{\xi}_1 + \cdots + \widehat{\xi}_n$ with eigenvalue*

$$(-t_2)^{n-1} q^{-\lambda_1} + (-t_2)^{n-2} t_1 q^{-\lambda_2} + \cdots + t_1^{n-1} q^{-\lambda_n} \ .$$

For $n = 2$, one has the following polynomials, in terms of Schur functions:

$$\widehat{M}_{0,0}^{\mathfrak{S}} = 1 \ , \ \widehat{M}_{1,0}^{\mathfrak{S}} = S_1 + t_2 - t_1 \ , \ \widehat{M}_{1,1}^{\mathfrak{S}} = S_{1,1} + t_2 S_1 + t_2^2 \ ,$$

$$\widehat{M}_{2,0}^{\mathfrak{S}} \doteq (qt_1 + t_2) S_2 + (qt_2 + t_1) S_{1,1} + (q+1)(t_2^2 - qt_1^2) S_1 + q(t_1 - t_2)(qt_1^2 - t_2^2) \ .$$

## 11 Jack polynomials

Take Yang-Baxter graph with

$$\sigma_i := s_i \ , \ 1 \le i \le n-1 \ , \ c_i(\triangleright) = 1-i \ , \ 1 \le i \le n \, , \, G_n = x_n \ ,$$
$$x_{i+n} = x_i \ , \ c_{i+n}(\triangleright) = c_i(\triangleright) + \alpha \ , \ \forall i \in \mathbb{Z} \tag{53}$$

**Definition 23** *The* non symmetric Jack polynomial $J_v$, $v \in \mathbb{N}^n$, *is the image of* 1 *under the Yang-Baxter element* $Y_v$ *given by the graph satisfying (53):*

$$J_v := 1\, Y_v \ .$$

For example, for $n = 2$, one has the following sequence of polynomials, writing $v, c(v), J_v$ on top of each other as before, and writing $\sigma$ for $\sigma_1 = s_1$:

$$\begin{matrix} [0,0] & & [0,1] & & [1,0] & & [0,2] & & [2,0] & & \\ [0,-1] & \xrightarrow{\Phi} & [-1,\alpha] & \xrightarrow{\sigma} & [\alpha,-1] & \xrightarrow{\Phi} & [-1,2\alpha] & \xrightarrow{\sigma} & [2\alpha,-1] & \xrightarrow{\Phi} & \cdots \\ 1 & & x_2 & & x_1 + \frac{x_2}{1+\alpha} & & x_2(x_2 + \frac{x_1}{1+\alpha}) & & x_1^2 + \frac{2x_1x_2+x_2^2}{1+2\alpha} & & \end{matrix}$$

The symmetric polynomials are obtained, up to normalization, as images of non-symmetrical ones under $\Delta(X)\partial_\omega = \sum_{w\in\mathfrak{S}_n} w$, i.e. for any partition $\lambda \in \mathbb{N}$, any $v$ in the orbit of $\lambda$ under $\mathfrak{S}_n$, one has

$$J_\lambda^{\mathfrak{S}} = J_v\, \Delta(X)\, \partial_\omega = \sum_{w\in\mathfrak{S}_n} \big(J_v\big)^w \ .$$

One checks that 1 is an eigenvector of the Cherednik elements (defined in (44))

$$\xi_i := \alpha D_i + \sum_{j\neq i, j=1}^{n} \overline{\pi}_{i,j} + 1 - i \ , \ 1 \le i \le n \ ,$$

with eigenvalues $0, -1, \ldots, -n+1$. These values are our initial spectral parameters, and therefore the commutation relations (46) imply the following theorem due to Opdam ( [39], [20]).

**Theorem 24** *For any* $v \in \mathbb{N}^n$, *the polynomial* $J_v$ *is a simultaneous eigenfunction of* $\xi_1, \ldots, \xi_n$, *with eigenvalues* $c_1(v), \ldots, c_n(v)$.

*For any partition* $\lambda \in \mathbb{N}^n$, *and any symmetric function* $f \in \mathfrak{S}ym(n)$, *the Jack polynomial* $J_\lambda^{\mathfrak{S}}$ *is an eigenvector of* $f(\xi_1, \ldots, \xi_n)$ *with eigenvalue* $f(c_1(\lambda), \ldots, c_n(\lambda))$.

Baker and Forrester [1] give another symmetrization, using Sahi's scalar product. Introduce another set of variables $Y = \{y_1, \dots, y_n\}$. Then, according to Sahi [41], the kernel

$$\Omega(X,Y) := \prod_{i=1}^{n} (1 - x_i y_i)^{-1} \prod_{i=1}^{n} \prod_{j=1}^{n} (1 - x_i y_j)^{-1/\alpha}$$

diagonalizes in the basis of Jack polynomials: there exist non-zero constants $m_v$ such that

$$\Omega(X,Y) = \sum_{v \in \mathbb{N}^n} m_v J_v(X) J_v(Y) \ .$$

Recall that, thanks to the Cauchy determinant,

$$\prod_{i=1}^{n} (1 - x_i y_i)^{-1} \partial_\omega^X = \Delta(Y) \prod_{i,j} (1 - x_i y_j)^{-1} \ ;$$

this last function is sent, under $\partial_\omega^Y$ onto $n! \prod_{i,j} (1 - x_i y_j)^{-1}$.

Therefore, the image of $\Omega(X,Y)$ under $\partial_\omega^X \partial_\omega^Y$ is $n! \prod_{i,j} (1 - x_i y_j)^{-1-1/\alpha}$. This last function diagonalizes in the basis of symmetric Jack polynomials, but this time for the parameter $1/(1 + 1/\alpha) = \alpha/(1+\alpha)$.

Therefore, all $J_{vw} \partial_\omega$, for a fixed $v \in \mathbb{N}^n$, coincide, up to non zero-scalars, and are proportional to the same symmetric Jack polynomial ( one has to check that the polynomial of index a permutation of $[\lambda_1{+}n{-}1, \dots, \lambda_n{+}0]$ is sent to the polynomial of index the (decreasing) partition $\lambda$. As for normalization, one can check that, for a weakly increasing $u \in \mathbb{N}^n$,

$$J_{u_1, u_2+1, \dots, u_n+n-1} \partial_\omega = {\sum_w}' (J_u)^w \ ,$$

where the sum on the right is over cosets representatives of $\mathfrak{S}_n$ modulo the stabilizer of $u$, and the polynomial $J_u$ is for the parameter $\alpha/(1+\alpha)$. For example,

$$J_{014;\,\alpha} \partial_{321} = J_{002;\,\alpha/(1+\alpha)} (1 + s_2 + s_2 s_1) \ .$$

Knop and Sahi [20] show that symmetric Jack polynomials can be obtained through still another symmetrization :

**Lemma 25** *Let $\lambda$ be a partition of length $k \le n$, and $v$ be any permutation of $[\lambda_1 - 1, \dots, \lambda_k - 1, 0^{n-k}]$. Then*

$$J_\lambda^{\mathfrak{S}} = \sum_{w \in \mathfrak{S}_n} \left( J_v \, x_1 \cdots x_k \right)^w \ .$$

*Proof.*

$J_{0^{n-k},\lambda_1,\ldots,\lambda_k} = J_{\lambda_1-1,\ldots,\lambda_k-1,0^{n-k}}\,\tau^k\, x_n \cdots x_{n-k+1} = J_{\lambda_1-1,\ldots,\lambda_k-1,0^{n-k}}\, x_k \cdots x_1\, \tau^k.$

The image of the left hand-side under $\sum w$ is $J_\lambda^{\mathfrak{S}}$. However,

$$\sum_{w\in\mathfrak{S}_k\times\mathfrak{S}_{n-k}} (J_{\lambda_1-1,\ldots,\lambda_k-1,0^{n-k}})^w = \sum_{w\in\mathfrak{S}_k\times\mathfrak{S}_{n-k}} (J_v)^w$$

and this allows to replace $J_{\lambda_1-1,\ldots,\lambda_k-1,0^{n-k}}$ by $J_v$ in the summation on $\mathfrak{S}_n$. QED

There is an abundant literature about symmetric Jack polynomials (cf. the references given in [35]). They can be obtained through "creation operators" [25]. Their simplest explicit expression seems to be a determinant given in [24]. They are eigenfunctions of the *Sekiguchi operator*

$$\big(D_1 + (n-1)\beta + t\big)\big(D_2 + (n-2)\beta + t\big) \cdots \big(D_n + 0\beta + t\big)\, x_1^{n-1} \cdots x_n^0\, \partial_\omega \ ,$$

$D_i$ still being $f \mapsto x_i \frac{d}{dx_i}(f)$, and $\beta = 1/\alpha$.

## 12 Vanishing Jack polynomials

Take Yang-Baxter graph with

$$\sigma_i := s_i + \partial_i \ ,\ 1 \le i \le n-1 \ ,\ c_i(\triangleright) = 1 - i \ ,\ 1 \le i \le n \ , G_n = x_n + n - 1 \ ,$$
$$x_{i+n} = x_i + \alpha \ ,\ c_{i+n}(\triangleright) = c_i(\triangleright) + \alpha \ ,\ \forall i \in \mathbb{Z} \tag{54}$$

and let the Cherednik algebra be generated by the

$$\widehat{\xi}_i := x_i - \sigma_{i-1} \cdots \sigma_1\, \tau\, (x_n + n - 1)\, \sigma_{n-1} \cdots \sigma_i \ ,\ 1 \le i \le n \ . \tag{55}$$

**Definition 26** *The* vanishing Jack polynomial $\widehat{J}_v$, $v \in \mathbb{N}^n$, *is the image of* 1 *under the Yang-Baxter element* $Y_v$ *associated to graph satisfying (54):*

$$\widehat{J}_v := 1\, Y_v \ .$$

For example, for $n = 2$, one has the following sequence of polynomials :

$$\begin{array}{ccccccccc} [0,0] & & [0,1] & & [1,0] & & [0,2] & & [2,0] \\ [0,-1] & \xrightarrow{\Phi} & [-1,\alpha] & \xrightarrow{\sigma} & [\alpha,-1] & \xrightarrow{\Phi} & [-1,2\alpha] & \xrightarrow{\sigma} & [2\alpha,-1] \\ 1 & & x_2+1 & & x_1+\frac{x_2+1}{1+\alpha} & & (x_2+1)(x_2-\alpha+\frac{x_1+1}{1+\alpha}) & & x_1(x_1-\alpha)+\frac{(x_2+1)(2x_1+x_2+1-3\alpha)}{1+2\alpha} \end{array}$$

Knop and Sahi [19] have shown that these polynomials can be characterized by vanishing conditions. Let $\mathcal{P}ol_k$ be the space of polynomials in $x_1, \ldots, x_n$ of degree $\le k$.

**Theorem 27** *Let $v \in N^n$, and $k = |v|$. Let $\mathcal{P}ol(v)$ be the subspace of polynomials $f$ of degree $\leq |v|$ such that*

$$\forall u \in \mathbb{N}^n,\ |u| \leq |v|,\ u \neq v \Rightarrow f(c(u)) = 0\ .$$

*Then $\mathcal{P}ol(v)$ is 1-dimensional and contains $\widehat{J}_v$.*

*The polynomial $\widehat{J}_v$ is an eigenfunction of the $\widehat{\xi}_i$'s with eigenvalues $c(v)$.*

*The restriction of $\widehat{J}_v$ to its component of top degree ( $= |v|$) is equal to $J_v$.*

*Proof.* The dimension of $\mathcal{P}ol_k$ is equal to the number of equations $+1$, this accounts for the existence of a vanishing polynomial, unicity follows by putting an appropriate order on monomials. Moreover, restricting $s_i + \partial_i$ and $G_n = x_n + n - 1$ to their terms of highest degree shows that the leading term of $\widehat{J}_v$ is $J_v$.

The labels on the edges of cells have been chosen to propagate vanishing conditions, at the same time as eigenvector properties. The choice of $G = x_n + n - 1$ insures that for any $f$, $f\Phi$ vanishes in $c(v)$ for any $v$ such that $v_n = 0$ (in which case $c_n(v) = 1 - n$).

Because of the commutation relations (47), it is sufficient to test that 1 is an eigenfunction for $\widehat{\xi}_n$, with the right eigenvalue, to get that all $\widehat{\xi}_i$ diagonalize. Indeed

$$1\,\widehat{\xi}_n = x_n - 1\sigma_{n-1}\cdots\sigma_1\,\tau\,(x_n + n - 1) = -n + 1\ .$$

QED

Given any $v \in \mathbb{N}^n$, the linear span of $\widehat{J}_{vw}$, $w \in \mathfrak{S}_n$ contains, up to normalization, a unique symmetric polynomial (27)

$$\widehat{J}_v^{\mathfrak{S}} := \sum_{w\in\mathfrak{S}_n} \widehat{J}_v\,\sigma_w = \widehat{J}_v \prod_{1\leq i,j\leq n} (1 + x_i - x_j)\,\partial_\omega$$

(Sahi and Knop give another symmetrization), and this symmetric polynomial inherits from $\widehat{J}_v$ the properties stated by the following theorem :

**Theorem 28** *Let $n$ be an integer, $\lambda \in \mathbb{N}^n$ be a partition, $v$ any vector in its orbit under $\mathfrak{S}_n$ and let $\gamma$ be an indeterminate. Then*

$$\widehat{J}_v^{\mathfrak{S}} := \widehat{J}_v \prod_{1\leq i<j<\leq n} (1 + x_i - x_j)\,\partial_\omega$$

*is, up to a scalar, the unique symmetric polynomial vanishing in all $c(u)$, $u \in \mathbb{N}^n$, $|u| \leq |v|$, $u$ not a permutation of $v$.*

*It is also the unique eigenfunction of $(\gamma + \widehat{\xi}_1)\cdots(\gamma + \widehat{\xi}_n)$ with eigenvalue*

$$(\gamma + \alpha\lambda_1 - 0)(\gamma + \alpha\lambda_2 - 1)\cdots(\gamma + \alpha\lambda_n - n{+}1)\ .$$

For $n = 2$ the first polynomials, in the basis of Schur functions, are

$$\widehat{J}^{\mathfrak{S}}_{0,0} = 1 \ , \widehat{J}^{\mathfrak{S}}_{1,0} = S_1 + 1 \ , \ \widehat{J}^{\mathfrak{S}}_{1,1} = S_{1,1} + S_1 + 1 \ ,$$

$$\widehat{J}^{\mathfrak{S}}_{2,0} = (\alpha+1)S_2 + (1-\alpha)S_{1,1} + ((1-\alpha)(2+\alpha)S_1 - (\alpha^2+2\alpha-1)$$

$$\widehat{J}^{\mathfrak{S}}_{3,0} = (2\alpha+1)S_3 - 2(\alpha-1)S_{2,1} - 3(\alpha+1)(2\alpha-1)S_2 + 3(2\alpha-1)(\alpha-1)S_{1,1} \\ + (-12\alpha + 4\alpha^3 + 2\alpha^2 + 3)S_1 + (2\alpha-1)(2\alpha^2+5\alpha-1) \ .$$

The symmetric Jack polynomials specialize, for $\alpha = 1$, into Schur functions. Similarly, vanishing symmetric Jack polynomials specialize into some Schubert polynomials. Indeed, Schubert polynomials are polynomials in two sets of variables, indexed by permutations, or by integral vectors (which are the codes of the permutations), which have vanishing properties related to the Bruhat order [32]. In the special case that we need now, let $\lambda \in \mathbb{N}^n$ be a partition and $X_\lambda$ be the Schubert polynomial in $x_1, \ldots, x_n, y_1, \ldots, y_\infty$, of index the increasing reordering of $\lambda$ (it is a *Grassmannian Schubert polynomial* [34]). Then $X_\lambda$ vanishes in every partition $\mu \in \mathbb{N}$: $|\mu| \le |\lambda|$, $\mu \ne \lambda$, i.e $X_\lambda$, evaluated in

$$x_1 = \mu_1 + n - 1, \ldots, x_n = \mu_n + 0 \quad , \quad y_1 = 0, y_2 = 1, y_3 = 2, \ldots, y_i = i{-}1, \ldots$$

is null. Therefore, one has the following lemma (stated by Knop and Sahi [19] in terms of *factorial Schur functions*; in the literature, $X_\lambda$ is rather denoted $\mathbb{Y}_{\lambda_n,\ldots,\lambda_1}$).

**Lemma 29** *For any partition $\lambda \in \mathbb{N}^n$, the specialization $\alpha$=1 of $\widehat{J}^{\mathfrak{S}}_\lambda$ is equal to the Schubert polynomial $X_\lambda$ evaluated in $\{x_1 + n - 1, \ldots, x_n + n - 1\}$, $\{0, 1, 2, \ldots\}$.*

# References


[1] T. H. Baker and P. J. Forrester. Symmetric Jack polynomials from non-symmetric theory. *Annals of Combinatorics*, **3** (1999) 159–170.

[2] T. H. Baker and P. J. Forrester. A $q$-analogue of the type $A$ Dunkl operator and integral kernel. *Int. Math. Res. Not.*, **14**(1997) 667–686.

[3] T. H. Baker and P. J. Forrester. Isomorphisms of type $A$ affine Hecke algebras and multivariate orthogonal polynomials. *Pacific J.M.*, **194** (2000) 19–41.

[4] I.N. Bernstein, I.M. Gelfand, S. Gelfand. Schubert cells and cohomology of the spaces $G/P$ , Russ. Math. Survey, **28** (1973) 1–26.

[5] I. Cherednik. A unification of the Knizhnik–Zamolodchikov and Dunkl operators via affine Hecke algebras. *Inv. Math.*, **106**(1991) 411–432.

[6] I. Cherednik. Integration of quantum many–body problems by affine Knizhnik–Zamolodchikov equations. *Commun. Math. Phys.*, **106** (1994) 65–95.

[7] I. Cherednik. Non-symmetric Macdonald polynomials. *Int. Math. Res. Not.*, **10** (1995) 483–515.

[8] M. Demazure. Une formule des caractères , *Bull. Sc. Math.*, **98** (1974) 163–172.

[9] G. Duchamp, D. Krob, A. Lascoux, B. Leclerc, T. Scharf, J.Y. Thibon. Euler-Poincaré characteristics and polynomial representations of Iwahori-Hecke algebras *Pub. RIMS (Kyoto)*, **31** (1995) 179–201.

[10] C. F. Dunkl. Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.*, **311** (1989) 167–183.

[11] S. Fomin and A. Kirillov. The Yang-Baxter equation, symmetric functions and Schubert polynomials, *Discrete Math.*, **153** (1996) 123–143.

[12] F. Hirzebruch. *Gesammelte Abhandlungen*, Springer (1987).

[13] J. Kaneko. $q$-Selberg integrals and Macdonald polynomials. *Ann. Sci. Éc. Norm. Sup.* $4^e$ *série*, **29** (1996) 1086–1110.

[14] Y. Kajihara and M. Noumi Raising operators of row type for Macdonald polynomials *Compositio Math.*, **120** (2000) 119–136.

[15] A. Kirillov and M. Noumi. Affine Hecke algebra and raising operators for Macdonald polynomials, *Duke Math. J.*, **93** (1998) 1–39.

[16] A. Kirillov and M. Noumi. q-difference raising operators for Macdonald polynomials and the integrality of transition coefficients, *CRM Proceedings and Lect. Notes*, **22** (1999) 227–243.

[17] F. Knop. Integrality of two variable Kostka functions. *J. Reine Ang. Math.*, **482** (1997) 177–189.

[18] F. Knop. Symmetric and non-symmetric quantum Capelli polynomials. *Comment. Math. Helvet.*, **72** (1997) 84–100.

[19] F. Knop and S. Sahi. Difference equations and symmetric polynomials defined by their zeros. *Int. Math. Res. Not.*, **10** (1996) 473–486.

[20] F. Knop and S. Sahi. A recursion and combinatorial formula for Jack polynomials. *Inv. Math.*, **128** (1997) 9–22.

[21] A. Lascoux. Polynômes symétriques, Foncteurs de Schur et Grassmanniennes. *Thèse, Université Paris 7* (1977).

[22] A. Lascoux. About the "$y$" in the $\chi_y$-characteristic of Hirzebruch. *Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), Contemp. Math.*, **241**(1999) 285–296.

[23] L. Lapointe, A. Lascoux, J. Morse. Determinantal expressions for Macdonald polynomials *Int. Math. Res. Notices*, **18** (1998) 957–978.

[24] L. Lapointe, A. Lascoux, J. Morse. Determinantal expressions for Jack polynomials, *Electronic Journal of Combinatorics*, **7** 1n (2000).

[25] L. Lapointe, L. Vinet. A Rodrigues formula for the Jack polynomials and Stanley conjecture, *Int. Math. Res. Notices* **9** (1995) 419–424.

[26] L. Lapointe, L. Vinet. Creation operators for the Macdonald and Jack polynomials *Letters Math. Phys.*, **40** (1997) 269–286.

[27] A. Lascoux, B. Leclerc and J.Y. Thibon. Twisted Action of the symmetric group on the cohomology of a flag manifold, *Banach Center Publications*, **36** (1996) 111–124.

[28] A. Lascoux, B. Leclerc and J.Y. Thibon. Flag Varieties and the Yang-Baxter Equation , *Letters in Math. Phys.*, **40** (1997) 75–90.

[29] A. Lascoux & M.P. Schützenberger. Symmetry and Flag manifolds, *Invariant Theory, Springer L.N.*, **996** (1983) 118–144.

[30] A. Lascoux & M.P. Schützenberger. Symmetrization operators on polynomial rings, *Funct.Anal.*, **21** (1987) 77–78.

[31] A. Lascoux & M.P. Schützenberger. Fonctorialité des polynômes de Schubert *in Invariant Theory, Contemporary Maths*, **88** (1989) 585–598.

[32] A. Lascoux & M.P. Schützenberger. Algèbre des différences divisées, *Discrete Maths*, **99** (1992) 165–179.

[33] G. Lusztig. Affine Hecke algebras and their graded version. *J. Amer. Math. Soc.*, **2** (1989) 599-685.

[34] I. G. Macdonald. *Notes on Schubert polynomials* LACIM, Publi. Université Montréal, (1991).

[35] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford University Press, Oxford, 2nd edition (1995).

[36] I. G. Macdonald. Affine Hecke algebras and orthogonal polynomials. *Séminaire Bourbaki, 47ème année*, **797** (1995).

[37] K. Mimachi and M. Noumi. A reproducing kernel for nonsymmetric Macdonald polynomials. q-alg/9610014.

[38] A. Okounkov, A. Vershik, A new approach to representation theory of symmetric groups, *Selecta Math.*, **2** (1996) 581–605.

[39] E. M. Opdam. Harmonic analysis for certain representations of graded Hecke algebras. *Acta Math.*, **175** (1995) 75–121.

[40] S. Sahi. Interpolation, integrality and a generalization of Macdonald's polynomials. *Int. Math. Res. Not.*, **10** (1996) 457–471.

[41] S. Sahi. A new scalar product for nonsymmetric Jack polynomials. *Int. Math. Res. Not.*, **20** (1996) 997–1004.

[42] S. Sahi. The binomial formula for nonsymmetric Macdonald polynomials. q-alg/9703024.

[43] A. Vershik. Local algebras and a new version of Young's orthonormal form *Topics in Algebra, Banach Center Pub.*, **26** (1990) 467–473.

[44] C.N. Yang. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, Phys. Rev. Lett., **19** (1967), 1312-1315.

[45] A. Young. *The Collected Papers of Alfred Young*, University of Toronto Press (1977).