Cyclic Permutations on words, Tableaux and Harmonic Polynomials

Alain Lascoux

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=Abstract. — The compatibility of tableaux with cyclic permutations provides a rank poset structure on the set of tableaux with a given evaluation, allowing the embedding of different sets of tableaux into each other. There results a decomposition of these sets into smaller sets, the cardinals of which are multiplicities of isotypic components (for the symmetric group action) either in cohomology rings of flag manifolds or equivalently in spaces of harmonic polynomials.

1. Introduction

It has been known since the work of Young that tableaux provide a way of explicitly describing the irreducible representations of the symmetric group over $\mathbb{C}$. It was soon realized by Littlewood, Robinson, Weyl and others that irreducible characters (that is to say *Schur functions*) are also given by tableaux; we prefer to express this by lifting a Schur function into the sum of all tableaux of a given shape inside the free algebra (Th.2.3).

Tableaux are now special words in the free algebra. Cyclic permutations on words induce an operation on tableaux that we call "cyclage" and give a rank-poset structure on the set of tableaux with a given evaluation, the rank being an important invariant called the *cocharge* . Cyclage connects any tableau to a row tableau which can be identified with a monomial in commuting variables (Prop.3.1). Inverting the cyclage lifts computations in the ring of polynomials to the level of tableaux, e.g. it lifts the permutation action of the symmetric group (lemma 4.1). From this one can imbed canonically each of the sets of tableaux with a given evaluation into the set of *standard* tableaux, obtaining a partition of the set of standard tableaux into disjoint sets that we call *atoms* (Th.4.2 and 4.4).

The cocharge gives also the *Kostka-Foulkes polynomials* which are the coefficients in the expansion of Hall-Littlewood polynomials in the basis of Schur functions (Th.5.1) . An interesting consequence of the construction is that the matrix of Kostka-Foulkes polynomials is the product of the incidence matrix (of the lattice of partitions) by a positive matrix, the entries of which correspond to the atoms (Cor.5.2).

Young introduced tableaux to obtain idempotents in the group algebra of the symmetric group, obtaining also multiplicities and bases of representations with the help of standard tableaux of a given shape. By identifying instead the regular representation with the cohomology ring of the flag manifold, one can further decompose the regular representation, with the elementary components corresponding to the atoms (Th.6.5). Tableaux furthermore provide explicit bases (as free modules) of the cohomology spaces of different flag manifolds (Prop.6.11).

Instead of cohomology rings, following Specht we may take spaces of harmonic polynomials. Cocharge and atoms provide multiplicities of representations for the action of the affine symmetric group (Th.7.4), and provide bases, notably Verma's basis (section 8).

Section 9 gives additional properties of atoms and shows how to generate them from two-rows tableaux.

The combinatorics of tableaux is a joint work with M.P. Schützenberger and constitutes the more substantial part of this paper, to which I have added a translation into geometrical terms.

2 . The plactic algebra

Let $\mathbb{A}^*$ be the free monoid generated by the alphabet $\mathbb{A} = \{a_1 < a_2 < \cdots\}$ and $\mathbb{Z}[\mathbb{A}^*]$ the free algebra of $\mathbb{A}$. The elements of $\mathbb{A}^*$ are *words*. The degree of a word w is denoted $|w|$, and its degree in a letter a is denoted $|w|_a$. The image in the ring of polynomials $\mathbb{Z}[\mathbb{A}]$ of a word w is called its *evaluation* and noted $ev(w)$. We shall also use the exponent notation $ev(w) = e^I$, with $I = (|w|_{a_1}\ , |w|_{a_2}\ , |w|_{a_3}\ , \ldots)$. A word w is *standard* iff there exists an integer k such that $ev(w) = a_1 a_2 \cdots a_k$.

A word is a *row* iff it is weakly increasing : $w = x_1 x_2 x_3 \cdots$, $x_1 \leq x_2 \leq x_3 \leq \cdots$, $x_i \in \mathbb{A}$. The ring $\mathbb{Z}[\mathbb{A}]$ can be imbedded into the free algebra as the free

$\mathbb{Z}$-module generated by rows (writing monomials as increasing words). A row $w = x_1 x_2 x_3 \cdots$ *dominates* another $w' = x'_1 x'_2 x'_3 \cdots$ if $|w| \leq |w'|$ and $x_1 > x'_1\ , x_2 > x'_2\ \ldots$.

Every word admits a unique factorization, called its *row-factorization*, as a product of a minimal number of rows $w = v_k \cdots v_2 v_1$. The shape $\|w\|$ of w is the sequence of degrees $|v_k|, \ldots, |v_1|$ of its rows. A word is a *tableau* iff for its row-factorization $w = v_k \cdots v_2 v_1$, one has

$$\begin{cases} \|w\| \text{ is weakly increasing} \\ v_k \text{ dominates } v_{k-1}, \ \ldots \ , v_2 \text{ dominates } v_1 \end{cases}$$

In other terms, a word is a tableau iff it is obtained by reading in the appropriate/French way a usual planar Young tableau. Thus the word $bcd\,aabd$ is a tableau that one can planarly display as $\substack{bcd \\ aabd}$.

The *Schur function* S_I of index a partition I is the sum of all tableaux of shape I in the free algebra .

We can see that the product $S_1 \cdot S_2$ is equal to the sum $S_{12} + S_3$, but that it is different from $S_2 \cdot S_1$. For, after deleting the words in common, we have on the left $\sum_{x<y}(yxx + yxy) + \sum_{x<y<z}(yxz + zxy)$, and on the right $\sum_{x<y}(xyx + yyx) + \sum_{x<y<z}(yzx + xzy)$.

To force equality, one of the principal methods is the following set of elementary relations (due to Knuth), for $x < y < z$ in $\mathbb{A}$:

$$yxx \equiv xyx \quad ; \quad xyy \equiv yyx \tag{2.1}$$

$$yxz \equiv yzx \quad ; \quad zxy \equiv xzy \tag{2.2}$$

These relations generate a congruence which is called the *plactic congruence* and denoted $\equiv$. This is the same as the algebraic formalization of "Schensted's construction" which produces, for any word, the only tableau which is congruent to it. The *plactic monoid* $\mathbb{A}^*/\equiv$ is the associated quotient of $\mathbb{A}^*$ and its algebra, the *plactic ring* $\mathbb{Z}[\mathbb{A}^*/\equiv]$.

The ring of symmetric polynomials in $\mathbb{A}$, $\mathbb{Z}[\mathbb{A}]^{\mathfrak{S}}$, is canonically imbedded into the plactic algebra, according to the following theorem (see [L& S1]) :

THEOREM 2.3 . — *The submodule of $\mathbb{Z}[\mathbb{A}^*/\equiv]$ generated by the images of the S_I is a commutative algebra isomorphic to $\mathbb{Z}[\mathbb{A}]^{\mathfrak{S}}$. In other words, one has the following diagram :*

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{A}^*] \arrow[r] & \mathbb{Z}[\mathbb{A}^*/\equiv] \arrow[r] & \mathbb{Z}[\mathbb{A}] \\
\left\{\begin{matrix} Free\,module \\ generated\,by\,the\,S_I \end{matrix}\right\} \arrow[u] \arrow[r] & \mathbb{Z}[\mathbb{A}]^{\mathfrak{S}} \arrow[u] \arrow[r, "\sim"] & \mathbb{Z}[\mathbb{A}]^{\mathfrak{S}} \arrow[u]
\end{tikzcd}$$

3 . Cyclic permutations

Let $t = xw$ be a tableau, x being its initial letter. The word wx is congruent to another tableau $t' \equiv wx$. If t has at least two rows, then $t \to t'$ is called a *cyclage*; the inverse of a cyclage, when it exists, is called a *cocyclage*. From the Schensted construction it is clear that if $t_1 \to t'$ and $t_2 \to t'$ are two cyclages leading to the same tableau, then the shapes of t_1 and t_2 are different. Thus, shape is sufficient to differentiate between all the cocyclages of a given tableau. In section 9 , we describe a more general cyclage and show how its properties are related to the Pieri decomposition of the product of a Schur function by an elementary symmetric function.

We need there only the following property, which is proved in [L& S1] :

PROPOSITION 3.1 . — *The cyclage induces a rank-poset structure on the set of tableaux with a given evaluation u ; the row u is the unique minimal element.*

We shall note the above poset $\mathcal{T}(u)$, or $\mathcal{T}(I)$, for $u = a_1^{I_1} a_2^{I_2} \cdots$. Let us call the rank of a tableau t for this structure the *cocharge* $\nu(t)$, putting $\nu(u) = 0$. An algorithm for computing directly the cocharge is given in [Mac, p.129] (in fact, what is given there is the *charge* which is the complement of the cocharge).

Example 3.2 . — The poset $\mathcal{T}(321)$, for the alphabet $\{a, b, c\}$, is

Charge		*Poset*		*Cocharge*
0		$\begin{smallmatrix}c\\bb\\aaa\end{smallmatrix}$		4
		$\downarrow$		
1	$\begin{smallmatrix}c\\b\\aaab\end{smallmatrix}$	$\begin{smallmatrix}bb\\aaac\end{smallmatrix}$	$\begin{smallmatrix}bbc\\aaa\end{smallmatrix}$	3
	$\downarrow$	$\downarrow$	$\swarrow$	
2	$\begin{smallmatrix}b\\aaabc\end{smallmatrix}$	$\begin{smallmatrix}bc\\aaab\end{smallmatrix}$		2
	$\searrow$	$\downarrow$		
3		$\begin{smallmatrix}c\\aaabb\end{smallmatrix}$		1
		$\downarrow$		
4		$aaabbc$		0

4. Canonical embeddings of tableaux

There are many linear operations on $\mathbb{Z}[\mathbb{A}]$ which can be extended to the plactic or free algebra, thanks to the cyclage connecting any tableau to a row (prop.3.1).

For example, the symmetric group $\mathfrak{S}(\mathbb{A})$ acts by permutation on $\mathbb{Z}[\mathbb{A}]$: $a, b, \ldots \longrightarrow a\mu, b\mu, \ldots$ for $\mu \in \mathfrak{S}(\mathbb{A})$ induces $a^{I_a} b^{I_b} \cdots \longrightarrow a^{I_a\mu} b^{I_b\mu} \cdots$ for $I \in \mathbb{N}^{\mathbb{A}}$.

We have defined more generally in [L & S1] an action of $\mathfrak{S}(\mathbb{A})$ on the free algebra :

$$(w \in \mathbb{A}^*, \mu \in \mathfrak{S}(\mathbb{A})) \to w^\mu \in \mathbb{A}^* \quad .$$

This action is compatible with the plactic congruence and with the cyclage, as stated in the following lemma (*cf.* [L &S1, th.4.6]) :

LEMMA 4.1 . — *Let v, w be two words, μ be a permutation. Then*

i) $\quad v \equiv w \quad \Rightarrow \quad v^\mu \equiv w^\mu \quad ,$

ii) w is a tableau $\Leftrightarrow$ w^μ is a tableau ; moreovoer, the shapes of w and w^μ are identical,

iii) $\quad w = xw', w^\mu = yw''$, *with* $x, y \in \mathbb{A} \Rightarrow (w'x)^\mu = w''y \quad .$

Given a tableau t and a permutation μ, the tableau t^μ is determined by the above lemma. For cycle the tableau to the row : $t = t_0 \to t_1 \to \cdots \to t_k = ev(t)$. We obtain t^μ by cocyclage, starting from the row $\big(ev(t)\big)^\mu$: $t^\mu \leftarrow t_1^\mu \leftarrow \cdots \leftarrow$

t^{μ}_k ; knowing t^{μ}_{j+1}, we determine t^{μ}_j among the different tableaux which cycle to t^{μ}_{j+1} by the condition that its shape equals that of t_j.

For example, the symmetric group $\mathfrak{S}(a,b,c)$ exchanges the following 6 chains of cyclages :

$$\begin{matrix}c\\bb\\aaa\end{matrix} \to \begin{matrix}bb\\aaac\end{matrix} \to \begin{matrix}bc\\aaab\end{matrix} \to \begin{matrix}c\\aaabb\end{matrix} \to aaabbc \ ; \ \begin{matrix}c\\bb\\aab\end{matrix} \to \begin{matrix}bb\\aaac\end{matrix} \to \begin{matrix}bc\\aaab\end{matrix} \to \begin{matrix}c\\aabbb\end{matrix} \to aabbbc \ ;$$

$$\begin{matrix}c\\bc\\aaa\end{matrix} \to \begin{matrix}bc\\aaac\end{matrix} \to \begin{matrix}cc\\aaab\end{matrix} \to \begin{matrix}c\\aaabc\end{matrix} \to aaabcc \ ; \ \begin{matrix}c\\bc\\aac\end{matrix} \to \begin{matrix}bc\\aacc\end{matrix} \to \begin{matrix}cc\\aabc\end{matrix} \to \begin{matrix}c\\aabcc\end{matrix} \to aabccc \ ;$$

$$\begin{matrix}c\\bc\\abb\end{matrix} \to \begin{matrix}bc\\abbc\end{matrix} \to \begin{matrix}cc\\abbb\end{matrix} \to \begin{matrix}c\\abbbc\end{matrix} \to abbbcc \ ; \ \begin{matrix}c\\bc\\abc\end{matrix} \to \begin{matrix}bc\\abcc\end{matrix} \to \begin{matrix}cc\\abbc\end{matrix} \to \begin{matrix}c\\abbcc\end{matrix} \to abbccc \ .$$

Consider the following transformation φ on words :

"change the letter a which is furthest to the right into a b; if there are no a' s, send the word to 0 " .

Let moreover $I \in \mathbb{N}^{\mathbb{A}}$ be such that $I_a = m > I_b = n$. It is clear that in this case φ transforms a tableau of evaluation I into a tableau of evaluation $I_a - 1, I_b + 1, I_c, \ldots$ of the same shape, and that φ is compatible with the cyclage. In other words, φ induces an embedding, still denoted φ, of the rank poset $\mathcal{T}(I_aI_bI_c\ldots)$ into the rank poset $\mathcal{T}(I_a - 1\ I_b + 1\ I_c\ldots)$, and the image of a cyclage is a cyclage. Combining φ with the action of the symmetric group, we obtain :

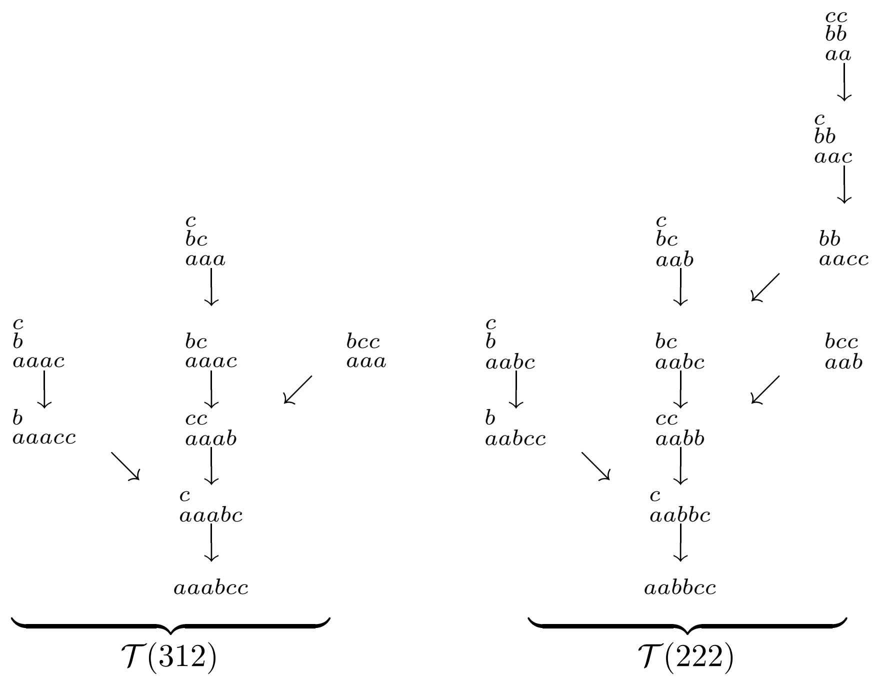
THEOREM 4.2 . — *Let I, J be two partitions : $I \le J$; $\check{I} \in \mathbb{N}^{\mathbb{A}}$ a permutation of I, $\check{J} \in \mathbb{N}^{\mathbb{A}}$ a permutation of J. Then there exists a canonical embedding of the rank poset $\mathcal{T}(\check{I})$ into the rank poset $\mathcal{T}(\check{J})$.*

The embedding of $\mathcal{T}(\check{I})$ into $\mathcal{T}(1\ldots1)$ is called the *standardization* and is denoted θ. It should be noticed that θ does not coïncide with the standardization mentionned by [Sche, th.4] obtained by numbering the successive occurrences of a from left to right, then of b, etc. To obtain the theorem, we just need to observe that the order on partitions (decreasing from left to right) is generated by the vector addition $I \to I + (0\ldots0\ -1\ 0\ldots0\ 1\ 0\ldots)$.
Therefore there exists at least one sequence of permutations $\mu, \eta, \ldots, \zeta$ which provides the required embedding by the composite

$$\mathcal{T}(\check{I}) \xrightarrow{\mu} \mathcal{T}(\check{I}^{\mu}) \overset{\varphi}{\hookrightarrow} \mathcal{T}(H) \xrightarrow{\eta} \mathcal{T}(\check{H}^{\eta}) \overset{\varphi}{\hookrightarrow} \ldots \quad \overset{\varphi}{\hookrightarrow} \mathcal{T}(\check{J}^{\zeta^{-1}}) \xrightarrow{\zeta} \mathcal{T}(\check{J}) \quad .$$

The fact that the embedding is canonical is provided by the compatibility with the cyclage of φ and of the action of $\mathfrak{S}(\mathbb{A})$: given $t \in \mathcal{T}(\check{I})$, we cycle it to the row $ev(t)$; now the image of t in $\mathcal{T}(\check{J})$ is obtained by the corresponding cocylages, starting from the row in $\mathcal{T}(\check{J})$ $\square$

For example, the poset $\mathcal{T}(321)$ described in 3.2 is isomorphic to $\mathcal{T}(312)$. Its image by φ is a subposet of $\mathcal{T}(222)$:



Definition 4.3 . — Given a partition J, the *atom* $\mathcal{A}(J)$ is the poset of standard tableaux

$$\mathcal{A}(J) := \theta\big(\mathcal{T}(J)\big) \setminus \cup_{I<J}\, \theta\big(\mathcal{T}(I)\big) \quad ,$$

i.e. is the poset of standard tableaux which are images of tableaux of evaluation e^J, but not images of tableaux of evaluation e^I with $I < J$.

The atomic decomposition of the set of standard tableaux $\mathcal{T}(1\ldots1)$ pulls back to all the other sets of tableaux and gives :

THEOREM 4.4 . — *For any partition J, the standardization θ is a bijection between the set $\mathcal{T}(J)$ and the disjoint union (on all partitions $I \leq J$) of the atoms $\mathcal{A}(I)$. The set $\theta^{-1}\big(\mathcal{A}(J)\big)$ is the complement of the canonical images of all the $\mathcal{T}(I)$, $I < J$, in $\mathcal{T}(J)$.*

For example, the 26 standard tableaux of degree 5 are distributed among the following 7 atoms (with the alphabet $\{1,\ldots,5\}$) :

Atoms

Cocharge							
10	54321						
9	43215						
8	53214 32514						
7	32145 54213 52413						
6	42513 42135 24135	54312					
5	52134 25134	43125 43512					
4	21345	53124 35124	53412				
3		31245	34125	54123			
2				41235	45123		
1						51234	
0							12345
	$\mathcal{A}(11111)$	$\mathcal{A}(1112)$	$\mathcal{A}(122)$	$\mathcal{A}(113)$	$\mathcal{A}(23)$	$\mathcal{A}(14)$	$\mathcal{A}(5)$

5 . Kostka-Foulkes polynomials

Given a tableau t, we have called cocharge of t the number of cyclages needed to obtain from it the row of the same evaluation. Any set of tableaux $\mathcal{T}$ gives rise to a cocharge polynomial $\sum_{t\in\mathcal{T}} q^{\nu(t)}$. In particular, for any $H \in \mathbb{N}^{\mathbb{A}}$, and any partition J, the *Kostka-Foulkes polynomial* $\mathcal{K}(H,J)$ is the sum $\sum q^{\nu(t)}$ taken over all tableaux of evaluation e^H and shape J , *cf* [Mac, p.126]. Kostka-Foulkes polynomials are special Kazhdan-Lusztig polynomials, see [Lus].

The above construction induces a decomposition of Kostka-Foulkes polynomials. For let us denote $\mathcal{R}(I,J)$ the cocharge polynomial corresponding to the subset of tableaux of shape J in the atom $\mathcal{A}(I)$. The following property is a direct corollary of theorem(4.4) :

Theorem 5.1 . — *Let $H \in \mathbb{N}^{\mathbb{A}}$, let I be the partition obtained by reordering H and J be another partition. Then the Kostka-Foulkes polynomial decompose into the following sum of smaller positive polynomials :*

$$\mathcal{K}(H,J) = \mathcal{K}(I,J) = \sum_{H\le I} \mathcal{R}(H,J)$$

Corollary 5.2 . — *i) Given any triple of partitions $I, I', J \,:\, I \le I'$, then $\sum_H \mu(H)\mathcal{K}(H,J)$, where H runs over the entire interval $[I,I']$ and μ*

is the Möbius function of the interval, is a polynomial with positive integral coefficients.

ii) The product of the Kostka matrix by the Möbius matrix is the matrix $\left[\mathcal{R}(I,J)\right]_{I,J}$.

For example, for the shape 1223 and all possible evaluations, we have the following $\mathcal{K}(*,1223)$ and $\mathcal{R}(*,1223)$ polynomials (writing only their coefficients, $122\cdots$ means $1+2q+2q^2+\cdots$) :

$$
\begin{array}{ccc}
\multicolumn{3}{c}{\overbrace{\hphantom{11222\quad 11123}}^{\textit{Evaluation}}} \\
1^8 & & \\
\downarrow & & \\
1^6 2 & & \\
\downarrow & & \\
1^4 22 & & \\
\downarrow & \searrow & \\
11222 & & 1^5 3 \\
\downarrow & \searrow & \downarrow \\
2222 & & 11123 \\
 & \searrow & \downarrow \\
 & & 1223
\end{array}
\qquad
\begin{array}{ccc}
\multicolumn{3}{c}{\overbrace{\hphantom{123467888764321}}^{\mathcal{K}(*,1223)}} \\
123467888764321 & & \\
\downarrow & & \\
1234554321 & & \\
\downarrow & & \\
123331 & & \\
\downarrow & \searrow & \\
1221 & & 11111 \\
\downarrow & \searrow & \downarrow \\
111 & & 11 \\
 & \searrow & \downarrow \\
 & & 1
\end{array}
\qquad
\begin{array}{ccc}
\multicolumn{3}{c}{\overbrace{\hphantom{000012456664321}}^{\mathcal{R}(*,1223)}} \\
000012456664321 & & \\
\downarrow & & \\
0001244321 & & \\
\downarrow & & \\
000121 & & \\
\downarrow & \searrow & \\
0011 & & 00111 \\
\downarrow & \searrow & \downarrow \\
011 & & 01 \\
 & \searrow & \downarrow \\
 & & 1
\end{array}
$$

For all partitions of weight 6, the matrix of Kostka-Foulkes polynomials can be found in [Mac, p.127] (with charge instead of cocharge). Its product by the Möbius matrix is the following, writing $\cdot$ for 0 entries (lexicographic order on decreasing partitions) :

$$
\begin{array}{ccccccccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & q & \cdot & q^2 & \cdot & \cdot & q^3 & \cdot & \cdot & q^4 & q^5 \\
\cdot & \cdot & q^2 & \cdot & \cdot & q^3 & q^4 & q^4 & \cdot & q^6+q^5 & q^8+q^7+q^6 \\
\cdot & \cdot & \cdot & q^3 & \cdot & \cdot & q^5+q^4 & \cdot & \cdot & q^7+q^6+q^5 & q^9+q^8+q^7+q^6 \\
\cdot & \cdot & \cdot & \cdot & q^3 & \cdot & \cdot & \cdot & q^5 & q^6 & q^9+q^7 \\
\cdot & \cdot & \cdot & \cdot & \cdot & q^4 & q^5 & q^5 & q^6 & q^8+2q^7+q^6 & q^{11}+2q^{10}+2q^9+2q^8+q^7 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^6 & \cdot & \cdot & q^9+q^8+q^7 & q^{11}+q^{10}+2q^9+q^8+q^7 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^6 & \cdot & q^8 & q^{12}+q^{10}+q^9 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^7 & q^9+q^8 & q^{13}+q^{12}+2q^{11}+q^{10}+q^9 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^{10} & q^{14}+q^{13}+q^{12}+q^{11} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^{15}
\end{array}
$$

In [L & S2], we gave result 5.2 *i)* only for intervals of length 2, but in fact, the same proof applies to any interval.

6 . Flag varieties

The ring $\mathbb{Z}[\mathbb{A}]$ is a free $\mathbb{Z}[\mathbb{A}]^{\mathfrak{S}}$-module. The natural scalar product on it is, for any pair P, Q of polynomials in $\mathbb{Z}[\mathbb{A}]$,

$$(P\,,\,Q)\ :=\ \sum_{\mu}\,[PQ\,/\,\Delta]^{\mu} \tag{6.1}$$

where Δ is the Vandermonde $\prod_{i<j}(a_i - a_j)$.

Many bases of $\mathbb{Z}[\mathbb{A}]$ as a free module are known ; for example, the embedding $\mathfrak{S}(n-1) \hookrightarrow \mathfrak{S}(n)$ easily gives the basis $\{e^I, I \le \rho\}$ with $\rho = n-1 \cdots 2\,1\,0$ and $n = card(\mathbb{A})$ (*cf.* [Gro]).

Schubert polynomials are another basis, which has the advantage of being self-adjoint with respect to the scalar product 6.1 (*cf.* [L& S1). Schubert polynomials correspond to the classes of Schubert varieties when one interprets $\mathbb{Z}[\mathbb{A}]$ as follows :

LEMMA 6.2 . — *Let n be a positive integer, $\mathcal{F}(\mathbb{C}^n)$ the flag variety for $Gl(n)$, $\mathbb{A}$ an alphabet of cardinal n. Then the cohomology ring $\mathcal{H}(\mathcal{F})$ of $\mathcal{F}(\mathbb{C}^n)$ is isomorphic, as a graded $\mathfrak{S}(n)$-ring, to the quotient of $\mathbb{Z}[\mathbb{A}]$ by the ideal generated by symmetric polynomials in $\mathbb{A}$ without constant terms.*

The scalar product 6.1 induces a scalar product $< \cdot | \cdot >$ on $\mathcal{H}(\mathcal{F})$: for any $\overline{P}, \overline{Q}$ of elements of $\mathcal{H}(\mathcal{F})$, let P and Q be any lifts of $\overline{P}, \overline{Q}$ in $\mathbb{Z}[\mathbb{A}]$. Then

$$< \overline{P}\ \big|\ \overline{Q} > :=\ (P, Q)^{\xi_0}$$

i.e. the scalar product of $\overline{P}, \overline{Q}$ is equal to the image of (P, Q) by the specialization $\xi_0 : \mathbb{A} \to \{0, \ldots, 0\}$.

More generally, given any partition I of weight n and a corresponding unipotent $\mathfrak{u}$ of $Gl(n)$, one can consider the subvariety $\mathcal{F}_{\mathfrak{u}}$ of $\mathcal{F}(\mathbb{C}^n)$ of flags fixed by $\mathfrak{u}$. One has the following theorem due to De Concini & Procesi [DeC-Pro] ; see proposition 6.11 below for an explicit $\mathbb{Z}$-basis and 9.10 for generators of the ideal.

THEOREM 6.3 . — *The integral cohomology ring of $\mathcal{F}_{\mathfrak{u}}$ is $\mathfrak{S}(n)$-isomorphic to the quotient $\mathcal{H}_I$ of $\mathbb{Z}[\mathbb{A}]$ by a certain explicit ideal depending only on the partition I corresponding to $\mathfrak{u}$. Moreover, $\mathcal{H}_I$ is a free $\mathbb{Z}$-module.*

The ring $\mathcal{H}_I$ inherits from $\mathbb{Z}[\mathbb{A}]$ a grading. For any partition J, the isotypic component corresponding to J is also graded, and using a formal indeterminate q, one can represent its multiplicity by the polynomial $\sum d_r(I,J)\,q^r$, where $d_r(I,J)$ is the multiplicity of the irreducible character corresponding to J in the homogeneous component of degree r of $\mathcal{H}(\mathcal{F}_{\mathfrak{u}}) \simeq \mathcal{H}_I$. The full ring corresponds to the maximal partition $I = 1 \ldots 1$.

Macdonald [Mac, II p.92 and III ex.9] states that

Theorem 6.4 . — *Given any pair of partitions I, J, the multiplicity of the character corresponding to J in the cohomology ring $\mathcal{H}_I$ is equal to the Kostka-Foulkes polynomial $\mathcal{K}(I, J)$.*

To the natural order on partitions correspond canonical $\mathfrak{S}(n)$-surjections :

$$I \geq I' \quad \Rightarrow \quad \mathcal{H}_I \mapsto \mathcal{H}_{I'}$$

(see [Hot-Spr] , [DeC-Pro]).

Translating theorem (5.1) , we get the more precise result :

Theorem 6.5 . — *Let I be a partition and $\check{\mathcal{H}}_I$ the intersection of the kernels of all the canonical surjections $\mathcal{H}_I \mapsto \mathcal{H}_{I'}$, $I > I'$. Then the multiplicity in $\check{\mathcal{H}}_I$ of the character of $\mathfrak{S}(n)$ corresponding to a partition J is equal to the polynomial $\mathcal{R}(I, J)$.*

Example 6.6 . — The atom $\mathcal{A}(1122)$ contains the tableaux 653412 (coch. 7), 534126 (coch. 6) , 346125 (coch.5) and thus $\check{\mathcal{H}}(1122)$ is concentrated in degrees 7,6,5, being an irreducible representation of $\mathfrak{S}(6)$ in each of these degrees (of respective dimensions 9, 16, 5, corresponding to the partitions 1122, 123, 33 which are the shapes of the three tableaux).

Tableaux provide not only multiplicities, but explicit bases as well of the free modules $\mathcal{H}_J$, for the following reason.

Given a partition J, let $\mathcal{Y}(J)$ be the set of all words (called *Yamanouchi* words) congruent to the tableau $\cdots 2^{J_2} 1^{J_1} 0^{J_0}$ (we have taken the Arabic alphabet $\{0, 1, 2, \ldots\}$). Let furthermore $\mathcal{E}(J)$ be the set of words which are smaller (componentwise) than at least one word in $\mathcal{Y}(J)$.

Example 6.7 . — $\mathcal{Y}(122) = \{21100, 12100, 21010, 12010, 10210\}$ and $\mathcal{E}(122) \setminus \mathcal{Y}(122) = \{20100, 02100, 11100, 20010, 21000, 02010, 11010, 12000, 00210, 10110, 10200; 01100, 20000, 01010, 02000, 11000, 00110, 00200, 10100, 10010; 10000, 01000, 00100, 00010; 00000\}$.

Yamanouchi words in $\mathcal{Y}(J)$ are bijectively associated to standard tableaux of shape J : taking the successive levels of $1, 2, 3, \ldots$ in a tableau t, we get a word (which has to be written from right to left) and every Yamanouchi word is obtained in this way, *e.g.* $\begin{smallmatrix} 5 & & & \\ 3 & 7 & & \\ 1 & 2 & 4 & 6 \end{smallmatrix} \leftrightarrow 1020100$ (because $1, 2, 4, 6$ are on the bottom row, $3, 7$ on the middle one, and 5 on the top row).

Lemma 6.8 . — *i) For any pair of partitions $I, J : I \leq J$, $\mathcal{E}(I) \subseteq \mathcal{E}(J)$.*

ii) For any integer $k \geq 0$, and for any I, let $\mathcal{E}(I, k)$ be the subset of words in $\mathcal{E}(I)$ beginning with k. Let $I^{\sharp k}$ be the partition obtained by reordering, if necessary, the sequence $\ldots I_{k+1}\, I_k - 1\, I_{k-1} \ldots I_0$. Then $\mathcal{E}(I^{\sharp k})$ is identical with the set obtained by erasing the first letter of each word in $\mathcal{E}(I, k)$.

Proof. i) It is sufficient to compare $\mathcal{E}(I)$ and $\mathcal{E}(J)$ for a pair of the type

$$\begin{array}{ccccccccc} I & = & (\ldots & m & n & \ldots & n & n+1 & \ldots) \\ J & = & (\ldots & m+1 & n & \ldots & n & n+1 & \ldots) \\ \begin{smallmatrix} Places\,from \\ the\,right \end{smallmatrix} & & & \uparrow h & & & & \uparrow k & \end{array}$$

with $m < n$, because such pairs generate the natural order on partitions. A word w is a Yamanouchi word iff for every factorization $w = w'w''$, $|w''|_0 \geq |w''|_1 \geq |w''|_2 \geq \cdots$ (*cf.* [L& S1, prop.3.7]). One sees that, given a word $w \in \mathcal{Y}(I)$, changing the leftmost occurrence of "$h-1$" into "h", the leftmost occurrence of "$h-2$" into "$h-1$", ..., the leftmost occurrence of "k" into "$k+1$", produces a word $v > w$ which is still a Yamanouchi word. Thus $\mathcal{Y}(I) \subset \mathcal{E}(J)$, which implies that $\mathcal{E}(I) \subseteq \mathcal{E}(J)$.

ii) Let $\begin{array}{cccccccc} I & = & (\ldots & m & n & \ldots & n & n & \ldots) \\ & & & \uparrow h & & & & \uparrow k & \end{array}$, with $m < n$.

Then $I^{\sharp k} = I^{\sharp h} = (\ldots m\ n-1 \ldots n\, n \ldots)$. Let $w \in \mathcal{E}(I^{\sharp h})$ and v be any Yamanouchi word in $\mathcal{E}(I^{\sharp h})$ such that $w \leq v$. Then the word hv is Yamanouchi, and thus belongs to $\mathcal{Y}(I)$; therefore the words hw and kw belong to $\mathcal{E}(I)$. Conversely, let kw be a word in $\mathcal{E}(I)$, hv any Yamanouchi word in $\mathcal{Y}(I)$. Then v belongs to $\mathcal{Y}(I^{\sharp h})$ and thus w belongs to $\mathcal{E}(I^{\sharp h})$ $\square$

Thus, looking back at example 6.6, one has to decompose $\mathcal{E}(122)$ into subsets of words beginning by 2, 1, 0 respectively. Erasing then this first letter, one obtains $\mathcal{E}(122^{\sharp 2}) = \{1100, 1010, 0100, 0010, 0000\} = \mathcal{E}(022)$; $\mathcal{E}(122^{\sharp 1}) = \{2100, 2010, 0210, 2000, 0200, 1100, 1010, 0110, 1000, 0100, 0010, 0000\}$
$= \mathcal{E}(112) = \mathcal{E}(122^{\sharp 0})$.

Let $\mathcal{M}(I)$ be the free module generated by the $\{e^\beta, \beta \in \mathcal{E}(I)\}$. Translating lemma 6.8 in terms of modules, and remembering that $e^{ijh\cdots}$ means $a^i b^j c^h \cdots$, we obtain

Corollary 6.9 . — *Let $k \geq 0$ be an integer, I be a partition. Then*

$$a^k \mathcal{M}(I)/a^{k+1}\mathcal{M}(I) \simeq \mathcal{M}(I^{\sharp k})$$

Now geometry provides an induction between the flags in $\mathcal{F}(\mathbb{C}^{n-1})$ and $\mathcal{F}(\mathbb{C}^n)$. More precisely, one has the following property which is the key technical point in DeConcini-Procesi's description of the cohomology of flag varieties ([DeC-Pro, lemma 4.5]) :

Lemma 6.10 . — *Let $k \geq 0$ be an integer, I a partition. Then*

$$a^k \mathcal{H}_I / a^{k+1}\mathcal{H}_I \simeq \mathcal{H}_{I^{\sharp k}}$$

Therefore the spaces $\mathcal{H}_I$ satisfy the same induction as the modules $\mathcal{M}(I)$ and $\mathcal{H}_I \simeq \mathcal{M}(I)$ (as free $\mathbb{Z}$-modules) because this is true for the starting point

$I = 0 \cdots 0$. Combining this with th.6.5, one obtains the following refinement of De Concini-Procesi's theorem :

Proposition 6.11 . — *Let J be a partition. Then $\mathcal{H}_J$ is a free $\mathbb{Z}$-module with basis $\{e^\beta, \beta \in \mathcal{E}(J)\}$; $\check{\mathcal{H}}_J$ is a free module with basis $\{e^\beta, \beta \in \mathcal{E}(J) \setminus \cup_{I<J} \mathcal{E}(I)\}$.*

From the explicit basis of $\mathcal{H}_J$, one could recover 6.2, 6.3 and 6.4 and give a basis of the ideal mentionned in 6.3. For example, since the set $\mathcal{Y}(1 \cdots 1)$ contains only the word $\rho = n-1 \cdots 1\, 0$, we recover that $\mathcal{H}_{1 \cdots 1} = \mathcal{H}(\mathcal{F})$ is generated as a free $\mathbb{Z}$-module by the monomials $\{e^\beta, \beta \leq \rho\}$.

We can now develop example 6.6 : proposition 6.11 asserts that $\check{\mathcal{H}}_{1122}$ admits the basis $\mathcal{E}(1122) \setminus \{\mathcal{E}(222) \cup \mathcal{E}(1113)\}$. In degree 7, this basis coïncide with $\mathcal{Y}(1122) = \{103210, 130210, 310210, 132010, 312010, 132100, 321010, 312100, 321100\}$; in degree 6, we have the 16 words which are not Yamanouchi : $\{102210, 120210, 122010, 122100, 103110, 130110, 310110, 311010, 131010, 131100, 311100, 103200, 130200, 310200, 312000, 132000\}$.
Finally, in degree 5, there are only 5 words which are not dominated by a word in $\mathcal{E}(222) \cup \mathcal{E}(1113)$: $\{131000, 130100, 130010, 103100, 103010\}$.

As concerns geometry, we mention that [DeC-Pro] have given another interpretation of the ring $\mathcal{H}_I$, proving a conjecture of [Kra] and showing that $\mathcal{H}_I$ is isomorphic to the coordinate ring of the scheme intersection of the closure of a nilpotent orbit with a maximal torus. A simpler description of the ideal defining $\mathcal{H}_I$, due to [Tan], is given in proposition 9.10. A stronger result has been obtained by [Wey] who dealt directly with the ideal of a nilpotent conjugacy class.

7 . Harmonic polynomials

A harmonic polynomial in $\mathbb{Q}[\mathbb{A}]$ is a polynomial which is annihilated by all symmetric derivations (*i.e.* any symmetric polynomial in the partial derivatives $\partial_a, a \in \mathbb{A}$) without constant term.

Since partial derivatives commute, any derivative of a harmonic polynomial is still a harmonic polynomial. Using Taylor's formula, one can also see that any element (μ, I) of the affine symmetric group : $\mu \in \mathfrak{S}(\mathbb{A}), I \in \mathbb{Z}^{\mathbb{A}}$, acting by

$$\{a, b, \ldots\} \longrightarrow \{a^\mu + I_a\, , b^\mu + I_b, \ldots\}$$

preserves the set of harmonic polynomials.

Let $\mathcal{V}(\mathbb{A})$ be the $\mathbb{Z}$-module of harmonic polynomials. The composite morphism

$$\mathbb{Q} \otimes \mathcal{V}(\mathbb{A}) \hookrightarrow \mathbb{Q}[\mathbb{A}] \longrightarrow \mathbb{Q} \otimes \mathcal{H}(\mathcal{F})$$

is an isomorphism (*cf.* [Bor]). The product of two harmonic polynomials is not harmonic in general, but it is congruent to one and only one harmonic polynomial modulo the ideal generated by symmetric polynomials without constant term.

There are several canonical surjections $\mathbb{Z}[\mathbb{A}] \longrightarrow \mathcal{V}(\mathbb{A})$. We shall only give two of them, one related to cohomology, the other to the Grothendieck ring.

For any finite ordered set $\mathbb{X} = \{x_1, x_2, \ldots\}$ of elements of $\mathbb{Z}[\mathbb{A}]$, let us write $\Delta(\mathbb{X})$ for the Vandermonde $\prod_{i<j}(x_i - x_j)$.

For any $I \in \mathbb{N}^{\mathbb{A}}$, denote by ∂^I the product of partial derivatives $\partial^I = \partial_a^{I_a}\partial_b^{I_b}\cdots$ and by Δ^I the image of $\frac{1}{\Delta(\rho)}\Delta(\mathbb{A})$ under ∂^I, with $\rho = (n-1, \ldots, 1, 0)$ as above (and thus, $\Delta(\rho)$ is the number $n-1!\; n-2! \cdots 1!\, 0!$).

Lemma 7.1 . — *The morphism $\phi : \mathbb{Z}[\mathbb{A}] \longrightarrow \mathcal{V}(\mathbb{A})$ defined by $\phi(e^I) = \Delta^I$ factorizes through $\mathcal{H}(\mathcal{F})$ and it induces an isomorphism $\mathcal{H}(\mathcal{F}) \simeq \mathcal{V}(\mathbb{A})$. The induced scalar product $\ll \phi(P), \phi(Q) \gg := < P \mid Q >$, with P, Q in $\mathcal{H}(\mathcal{F})$ is such that $\ll \hbar, \Delta^I \gg = \partial^I(\hbar)(0, \ldots, 0)$ for any $\hbar$ in $\mathcal{V}(\mathbb{A})$ and is such that the ∂^I are self -adjoint.*

Proof. We shall assume that $\Delta(\mathbb{A})/\Delta(\rho)$ and that its derivatives generate $\mathcal{V}(\mathbb{A})$. Thus ϕ is surjective. The action of the ring of differential operators $\mathbb{Z}[\partial_a, \partial_b, \ldots]$ on $\mathcal{V}(\mathbb{A})$ factorizes through the quotient of this ring by the ideal generated by symmetric operators without constant term; therefore ϕ factorizes through $\mathcal{H}(\mathcal{F})$ and $\mathcal{V}(\mathbb{A})$ is generated by the $\{\Delta^I : 0 \le I \le \rho\}$. Given any homogeneous polynomial P whose image in $\mathcal{H}(\mathcal{F})$ is non-zero, there exists at least one monomial e^I such that in $\mathcal{H}(\mathcal{F})$, $e^I P \equiv ce^\rho$ with $c \neq 0$. Hence the harmonic polynomial $\phi(p)$ cannot be zero, since its image by ∂^I is $c\phi(e^\rho) \neq 0$.

The operator ∂^I is the image of the operator "*multiplication by e^I*" which is clearly self-adjoint since for any P, Q in $\mathcal{H}(\mathcal{F})$, we have

$$< P \mid Q > = < PQ \mid 1 >.$$

Finally, since $\ll \hbar, \Delta^I \gg = \ll \partial^I(\hbar), \Delta^{0\cdots 0} \gg$, it only remains to compute the scalar products of $\Delta^{0\cdots 0}$ with a basis, *e.g.* $\{\Delta^I : 0 \le I \le \rho\}$. But $\ll \Delta^I, \Delta^{0\cdots 0} \gg = < e^I \mid 1 > = 0$, except $\ll 1, \Delta^{0\cdots 0} \gg = < e^\rho \mid e^{0\cdots 0} > = 1$. This implies by linearity $\ll \hbar, \Delta^{0\cdots 0} \gg = \hbar(0, \ldots, 0)$ □

The images of the different natural bases of $\mathbb{Z}[\mathbb{A}]$ are bases of harmonic polynomials, and pairs of adjoint bases give explicit decompositions in $\mathcal{V}(\mathbb{A})$:

Corollary 7.2 . — *Let $\{P_I\}$, $\{Q_I\}$ be adjoint bases of $\mathbb{Z}[\mathbb{A}]$ as a free $\mathbb{Z}[\mathbb{A}]^{\mathfrak{S}}$-module. Then every harmonic polynomial $\hbar$ admits the following decomposition :*

$$\hbar = \sum \ll \hbar\,,\, \phi(P_I) \gg \phi(Q_I) \quad .$$

For example, in the case of $\mathfrak{S}(3)$, the basis of Schubert polynomials is $\{e^{000}, e^{100}, e^{100} + e^{010}, e^{110}, e^{200}, e^{210}\}$; its adjoint is $\{-e^{012}, e^{002}, e^{011}, -e^{010} - e^{001}, -e^{001}, e^{000}\}$ (*i.e.*, it is self-adjoint up to sign and reversal of the alphabet, *cf.* [L& S3]). Corollary 7.2 implies the following decomposition, writing $\hbar^I$ for the value of $\partial^I(\hbar)$ at the point $0,0,0$:

$$\hbar = -\hbar^{000}\Delta^{012} + \hbar^{100}\Delta^{002} + (\hbar^{100}+\hbar^{010})\Delta^{011} - \hbar^{110}(\Delta^{010}+\Delta^{001}) - \hbar^{200}\Delta^{001} + \hbar^{210}\Delta^{000} \quad . \tag{7.3}$$

One could also have obtained 7.3 from Taylor formula, using relations between derivatives of an harmonic polynomial (i.e. computing modulo the ideal generated by symmetric differential operators without constant term).

Let us denote $\check{\phi} : \mathcal{H}(\mathcal{F}) \longrightarrow \mathcal{V}(\mathbb{A})$ the isomorphism defined by $\check{\phi}(e^I) = \Delta^{\rho-I}, 0 \le I \le \rho$. Let I be a partition of weight n , $J = J_1 \ge J_2 \ge \cdots \ge J_k$ its conjugate and $H \in \mathbb{N}^n$ the vector

$$\underbrace{(J_1+\cdots J_{k-1})\cdots(J_1+\cdots J_{k-1})}_{J_k} \;\ldots\; \underbrace{(J_1+J_2)\cdots(J_1+J_2)}_{J_3} \; \underbrace{J_1\cdots J_1}_{J_2} \; \underbrace{0\cdots 0}_{J_1}$$

The polynomial $\hbar_I = \Delta^H$ is a product of Vandermonde determinants. For example, $I = 5221$ leads to $J = 43111$ and $H = (4+3+1+1\,,\,4+3+1\,,\,4+3\,,\,4,4,4,0,0,0,0)$. Therefore, $\hbar_{5221} = \frac{1}{\Delta(\rho)}\,\partial^9_{a_1}\,\partial^8_{a_2}\,\partial^7_{a_3}\,(\partial_{a_4}\partial_{a_5}\partial_{a_6})^4\,(\Delta) = \frac{1}{\Delta(210)}\Delta(a_4,a_5,a_6)\cdot\frac{1}{\Delta(3210)}\Delta(a_7,a_8,a_9,a_{10})$.

The orbit of $\hbar_I$ under the symmetric group (resp. affine symmetric group) generates an irreducible (resp. reducible) $\mathbb{C}$-representation of $\mathfrak{S}$ called *Specht module* (resp. *affine Specht module*, denoted $\mathcal{V}_I$, *cf.* [Jam-Ker]). Since $\hbar_I = \check{\phi}(e^H)$, and since the space $\mathcal{H}_I$ is generated by the action of the affine symmetric group on the element e^H according to proposition 6.11, the two spaces $\mathcal{H}_I$ and $\mathcal{V}_I$ are isomorphic under $\check{\phi}$. This allows to translate theorems 6.3 and 6.4 in terms of harmonic polynomials :

Theorem 7.4 . — *Let I, J be two partitions. Then*

i) The multiplicity in the affine Specht module $\mathcal{V}_I$ of the character of the symmetric group corresponding to J is equal to the Kostka-Foulkes polynomial $\mathcal{K}(I,J)$.

ii) The multiplicity of the character J in the quotient space $\mathcal{V}_I \,/\, \cup_{I'<I} \mathcal{V}_{I'}$ is $\mathcal{R}(I,J)$.

For example, for $\mathfrak{S}(6)$, the atom $\mathcal{A}(1113)$ contains only the tableaux $6\,5\,4\,123$, $5\,46\,123$, $5\,4\,1236$, $6\,4\,1235$, $46\,1235$ and $4\,12356$, the respective cocharges of which are 6, 5, 5, 4, 4, 3. Thus the space $\mathcal{V}_{1113} \,/\, \{\mathcal{V}_{123} \cup \mathcal{V}_{114}\}$ is of dimension 10 in degree 6, of dimension 16+10 in degree 5, dimension 10+9 in degree 4 and dimension 5 in degree 3.

8 . Verma Bases

Let $\mathbb{Z}[\mathbb{A},\mathbb{A}^\vee]$ be the ring of Laurent polynomials in an alphabet $\mathbb{A}$ of cardinal n, $\mathbb{X}$ a second alphabet of the same cardinal and $\psi : \mathbb{Z}[\mathbb{A},\mathbb{A}^\vee] \longrightarrow \mathcal{V}(\mathbb{X})$ the morphism defined by $\psi(e^I) = \frac{1}{\Delta(\rho)}\Delta(\mathbb{X}+I) := \frac{1}{\Delta(\rho)}\prod_{h<k}\big((x_h+I_h)-(x_k+I_k)\big)$ (we shall denote this polynomial Δ_I). Let moreover ε_1 be the specialisation $\mathbb{A} \longrightarrow \{1,\ldots,1\}$ and $\mathcal{K}(\mathbb{A})$ be the quotient of $\mathbb{Z}[\mathbb{A},\mathbb{A}^\vee]$ by the ideal generated by the $\{(P-P^{\varepsilon_1}, P \in \mathbb{Z}[\mathbb{A},\mathbb{A}^\vee]^{\mathfrak{S}}\}$. The ring $\mathcal{K}(\mathbb{A})$ can be interpreted as the Grothendieck ring of the flag manifold $\mathcal{F}(\mathbb{C}^n)$.

The morphism ψ satisfies the same type of properties as ϕ (we shall not repeat the proof) :

LEMMA 8.1 . — *1) ψ is a surjection which factorizes through $\mathcal{K}[\mathbb{A}]$ and induces an isomorphism $\mathcal{K}[\mathbb{A}] \simeq \mathcal{V}(\mathbb{X})$.*

2) The induced scalar product $\prec\,,\succ$ defined by $\prec \psi(P),\psi(Q) \succ = (P,Q)^{\varepsilon_1}$ is such that the translation $\mathbb{X} \to \mathbb{X}+I$ is self-adjoint, and such that $\prec \hbar, \Delta_I \succ = \hbar(I)\,,\, I \in \mathbb{Z}^n$.

Adjoint bases of $\mathbb{Z}[\mathbb{A},\mathbb{A}^\vee]$ (as a free $\mathbb{Z}[\mathbb{A},\mathbb{A}^\vee]^{\mathfrak{S}}$- module) furnish explicit decompositions of harmonic polynomials. For example in the case of $\mathfrak{S}(3)$ as in 7.3, Schubert polynomials give :

$$(8.2)\qquad \hbar \;=\; -\hbar(000)\Delta_{012} + \hbar(100)\Delta_{002} + \big(\hbar(100)+\hbar(010)\big)\Delta_{011} - \hbar(110)\big(\Delta_{010}+\Delta_{001}\big) - \hbar(200)\Delta_{001} + \hbar(210)\Delta_{000} \quad .$$

Verma [Ver] has introduced another basis which is related to the cyclage structure discussed in section 3.

Given a permutation $\mu \in \mathfrak{S}(\mathbb{A})$, considered as the word $a_1^\mu\, a_2^\mu \cdots a_n^\mu$ in $\mathbb{A}^*$, the *cocharge vector* μC is defined inductively as follows (*cf.* [Mac], p.129) : attach to a_1 the index 0; if k is attached to a_j, then attach k (resp. $k+1$) to a_{j+1} if μ has the subword $a_j a_{j+1}$ (resp. $a_{j+1}a_j$). The cocharge vector is the vector of the preceeding indices. For example, indexing $\mu = a_3a_5a_1a_7a_2a_6a_4$, one obtains $(a_3)_1(a_5)_2(a_1)_0(a_7)_3(a_2)_0(a_6)_2(a_4)_1$, and thus $\mu C = (1203021)$. The sum of the components of μC is in fact equal to the cocharge of the tableau congruent to the word μ (*cf.* [L& S1]). In the present case, the tableau is $a_7\, a_3a_5a_6\, a_1a_2a_4$ and its cocharge is $1+2+0+3+0+2+1=9$.

Verma conjectured (this was proved by Husulkar) that the constant 1 could be expressed as an integral sum of polynomials $\Delta_{\mu C} := \frac{1}{\Delta(\rho)}\Delta(\mathbb{X}+\mu C)$.

More precisely, one has (*cf.* [Hus]) :

THEOREM 8.3 . — *The set $\{\Delta_{\mu C}, \mu \in \mathfrak{S}\}$ is a $\mathbb{Z}$-basis of $\mathcal{V}$. There exist harmonic polynomials $\{\hbar_\mu, \mu \in \mathfrak{S}\}$ such that $\hbar_\mu(\eta C) = 0$, $\eta \neq \mu$, and $\hbar_\mu(\mu C) = 1$.*

The scalar product $\prec , \succ$ allows us to interpret the polynomials $\hbar_\mu$. For any $I \in \mathbb{Z}^n$, write $I\omega$ for the reverse vector $(I_n, \dots, I_1)$. It is easy to check that the matrix of scalar products $(e^{\mu C}, e^{\eta C\omega})$ is triangular, with ± 1 on the diagonal (*cf.* [Hus] for the appropriate order on $\mathfrak{S}$). Thus the set $\{e^{\mu C}\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\mathbb{A}, \mathbb{A}\check{}\,]$ as a free $\mathbb{Z}[\mathbb{A}, \mathbb{A}\check{}\,]^{\mathfrak{S}}$- module; it has a certain adjoint basis $\{P_\mu, \mu \in \mathfrak{S}\}$.

COROLLARY 8.4 . — *i) Husulkar's harmonic polynomials $\hbar_\mu$ are the image under ψ of the polynomials P_μ.*

ii) $\qquad 1 = \sum_{\mu \in \mathfrak{S}} (P_\mu)^{\varepsilon_1} \, \Delta_{\mu C}$

Proof. The condition $\hbar_\mu(\eta C) = 1$ or 0 means precisely that $\{\hbar_\mu\}$ is adjoint to $\{\Delta_{\mu C}\}$ with respect to $\prec, \succ$. This follows from the fact that the scalar product is the image of the natural one on $\mathbb{Z}[\mathbb{A}, \mathbb{A}\check{}\,]$; assertion *i)* follows.

In particular, $1 = \sum \prec 1, \hbar_\mu \succ \Delta_{\mu C}$. However, for any I in $\mathbb{Z}^n$, one has $\prec 1, \Delta_I \succ = 1 = (e^I)^{\varepsilon_1}$. Therefore, $\prec 1, \hbar_\mu \succ = (P_\mu)^{\varepsilon_1}$ $\square$

Property *ii)* explicits Verma's footnote [Ver, p.669] that the integers $\prec 1, \hbar_\mu \succ$ are the degrees of certain T(orus)-modules. Verma writes $ch(e^I)$ for the scalar product $(e^I, 1)$ and calls "trace " the specialisation morphism ε_1. His starting point was that the polynomial that calls "Weyl polynomial" : $I \longrightarrow \Delta(I)/\Delta(\rho)$ (considered as a function of $I_1, \dots, I_n$)) gives the dimension of the irreducible representation (of $Gl(\mathbb{C}^n)$) of highest weight $I - \rho$ whenever I is regular (i.e. $I_1 > I_2 > \cdots > I_n \geq 0$).

For $\mathfrak{S}(3)$, the matrix $(e^{\mu C}, e^{\eta C})$ is

e^{012}	-1	.	.	.	.	.
e^{011}	.	1	.	.	.	.
e^{001}	.	.	1	.	.	.
e^{101}	.	.	.	-1	.	.
e^{010}	.	.	.	.	-1	.
e^{000}	.	.	.	.	.	1
	e^{000}	e^{010}	e^{101}	e^{100}	e^{110}	e^{210}

Therefore, any harmonic polynomial $\hbar$ decomposes as follows :

$$\hbar \;=\; \hbar(000)\Delta_{210} - \hbar(010)\Delta_{110} - \hbar(101)\Delta_{100} + \hbar(001)\Delta_{101} + \hbar(011)\Delta_{010} - \hbar(012)\Delta_{000} \qquad . \tag{8.5}$$

For $\mathfrak{S}(4)$, apart from the diagonal, there are only 4×2 non-zero entries in the matrix of scalar products : $(e^{0101}, e^{1201}) = -1$, $(e^{0210}, e^{0102}) = 1$,

$(e^{3210}, e^{1021}) = -e^{1111} \Rightarrow \prec \Delta_{3210}, \Delta_{1021} \succ = -1$,
$(e^{2120}, e^{2012}) = e^{1111} \Rightarrow \prec \Delta_{2120}, \Delta_{2012} \succ = 1$.

Hence the adjoint basis of the polynomials
$\{\Delta_{0000}, \Delta_{0010}, \Delta_{0100}, \Delta_{1000}, \Delta_{1100}, \Delta_{0110}, \Delta_{1001}, \Delta_{1010}, \Delta_{0101}, \Delta_{1011}, \Delta_{1101}, \Delta_{1110}, \Delta_{0210}, \Delta_{2010}, \Delta_{2100}, \Delta_{2110}, \Delta_{1201}, \Delta_{1210}, \Delta_{2101}, \Delta_{1021}, \Delta_{2102}, \Delta_{2120}, \Delta_{2210}, \Delta_{3210}\}$ is
$\{(\Delta_{0123} - \Delta_{0101}), -\Delta_{0122}, (\Delta_{0212} + \Delta_{0001}), (\Delta_{0010} - \Delta_{2012}), (\Delta_{1201} - \Delta_{0112}), \Delta_{0121}, \Delta_{1012}, -(\Delta_{1021} + \Delta_{0000}), -\Delta_{0112}, -\Delta_{0012}, (\Delta_{0102} + \Delta_{0111}), (\Delta_{1011} - \Delta_{0120}), -\Delta_{0111}, \Delta_{1011}, -\Delta_{1101}, -(\Delta_{1010} + \Delta_{0011}), -\Delta_{0101}, \Delta_{0110}, \Delta_{1001}, \Delta_{0011}, -\Delta_{0001}, \Delta_{0010}, -\Delta_{0100}, \Delta_{0000}\}$.

9. Some properties of atoms

More general cyclic permutations than those used in section 3 (we shall nevertheless keep the same word *cyclage*) furnish a richer structure on the set of tableaux with a given evaluation. This structure is in fact closely related to the Pieri formulae which characterize the ring of symmetric polynomials by giving the product of a Schur function by a complete function. Through the embedding given by theorem 2.3 of $\mathbb{Z}[\mathbb{A}]^{\mathfrak{S}}$ into the plactic algebra, Pieri formulae become the following lemma (lemma (3) of Schensted [Sche] and [Schü1]).

Given a positive integer n and a partition I, we denote the set of partitions obtained from the diagram of I by adding (resp. deleting) n boxes, with no two in the same column, by $\{n \otimes I\}$ (resp. $\{I/n\}$. This set is a distributive lattice and we shall show elsewhere that its properties provide in particular a combinatorial description of the multiplication of Hall-Littlewood polynomials.

Lemma 9.1 . — *For any integer n and any partition I, we have*

$$S_n \cdot S_I \equiv \sum_{J \in \{n \otimes I\}} S_J \quad \text{and} \quad S_I \cdot S_n \equiv \sum_{J \in \{n \otimes I\}} S_J$$

Let t be a tableau of shape J, I another partition such that J/I is a single box. Then, according to 9.1, there exist two unique pairs (x, w), (w', x'), such that $x, x' \in \mathbb{A}$, that w, w' are tableaux of shape I and that $t \equiv xw$, $t \equiv w'x'$.

Definition 9.2 . — Let t be a tableau, xw and vy, $x, y \in \mathbb{A}$, two words congruent to t, and let t', t'' be the tableaux $t' \equiv wx$, $t'' \equiv yv$. Let μ be any permutation such that $|t^\mu|_{a_1} \geq |t^\mu|_{a_i}$ for any i. Then $t \to t'$ (resp. $t \to t''$) is an *I-cyclage* (resp. *I-cocyclage*) iff $\breve{x} \neq a_1$ (resp. $\breve{y} \neq a_1$) with $\breve{x}$ the initial letter of $(xw)^\mu$, $\breve{y}$ the final letter of $(vy)^\mu$, and I the shape of the tableau congruent to w (resp. v).

In other words, if a_1 is the most frequent letter in t (so that we can take μ= *identity*), then any cyclic permutation on the words congruent to t is acceptable as long as the letter which moves is different from a_1. For example, let t be the tableau $\begin{smallmatrix} c \\ bc \\ abc \end{smallmatrix}$ on the alphabet $\mathbb{A} = \{a, b, c\}$; consider the three words congruent to it :

$$t = \underbrace{c}_{x_1} \underbrace{bcabc}_{w_1} \equiv \underbrace{c}_{x_2} \underbrace{cbcabc}_{w_2} \equiv \underbrace{c}_{x_3} \underbrace{cbcab}_{w_3}$$

and their images by the permutation $\mu : \{a, b, c\} \to \{c, b, a\}$:

$$(x_1 w_1)^\mu = c\,bbaaa \equiv (x_2 w_2)^\mu = b\,cbaaa \equiv (x_3 w_3)^\mu = a\,cbbaa\ .$$

Thus, the cyclic permutations $c\,bcabc \to bcabc\,c$ and $c\,cbabc \to cbabc\,c$ are two cyclages, but not $c\,cbcab \to cbcab\,c$ since the word $(ccbcab)^\mu$ begins by a.

Proposition 3.1 and theorem 4.2 extend to this more general cyclage (*cf.* [L& S1]) :

THEOREM 9.3 . — *i) Let I be a partition. The cyclage induces a rank-poset structure on the set of tableaux with evaluation e^I (this set is still denoted $\mathcal{T}(I)$); the row e^I is the unique minimal element.*

ii) Let I, J be two partitions : $I \leq J$; $\check{I} \in \mathbb{N}^{\mathbb{A}}$ a permutation of I, $\check{J} \in \mathbb{N}^{\mathbb{A}}$ a permutation of J. Then there exists a canonical embedding of the rank poset $\mathcal{T}(\check{I})$ into the rank poset $\mathcal{T}(\check{J})$.

For the example 3.2, one has only to complete the poset with the extra cyclage

$$\begin{smallmatrix} c \\ bb \\ aaa \end{smallmatrix} \longrightarrow \begin{smallmatrix} c \\ b \\ aaab \end{smallmatrix} \quad .$$

We need moreover the special composition of cyclages $t = wu \longrightarrow uw \equiv t'$, with u the bottom row of t. We denote by $K(t)$ the tableau t' congruent to uw and call *katabolism* the morphisme $t \to K(t)$. Katabolism provide an invariant of tableaux. For applying repeatedly Pieri formulae 9.1 (which imply that katabolism is a bijection), one obtains :

THEOREM 9.4 . — *i) The sequence of shapes $\{\|K^i(t)\|\,, i \geq 0\}$ characterizes t among all the tableaux of the same evaluation.*

ii) For any t, the tableau $\theta(t)$ is the unique standard tableau s with identical sequence $\{\|K^i(s)\|\,, i \geq 0\} = \{\|K^i(t)\|\,, i \geq 0\}$.

Example 9.5 . — The image by θ of the sequence of katabolisms

$$\begin{smallmatrix} ce \\ abbcdde \end{smallmatrix} \longrightarrow \begin{smallmatrix} d \\ abbcddee \end{smallmatrix} \longrightarrow \begin{smallmatrix} e \\ abbcddde \end{smallmatrix} \longrightarrow abbcdddee$$

is the sequence (taking now as alphabet $\{1, 2, \ldots\}$)

$$\begin{smallmatrix} 79 \\ 1234568 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 8 \\ 12345679 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 9 \\ 12345678 \end{smallmatrix} \longrightarrow 123456789$$

Another invariant can also be read on the sequence $K^i(t)$. We shall consider here only the case of standard tableaux. The bottom row of a standard tableau s contains a maximal subword of consecutive integers starting with 1, the degree of which is denoted $f(s)$.

Lemma 9.6 . — *For any standard tableau s, the sequence $f(s), f(K(s) - f(s), f(K^2(s) - f(K(s), \ldots$ is a decreasing partition I. Furthermore s belongs to the atom $\mathcal{A}(I)$.*

Going back to example 9.5, we obtain for the standard tableau 79 1234568 the sequence $6, 7-6, 8-7, 9-8, 9-9, \ldots$, *i.e.* the partition 6111. Therefore this tableau belongs to $\mathcal{A}(6111)$.

We can also characterize the tableaux t in $\mathcal{T}(I)$, I a partition, such that $\theta(t)$ belongs to the atom $\mathcal{A}(I)$ $\big($ a, b are the first two letters of $\mathbb{A}$ and $\mathbb{A}_p$ is the alphabet of the first p letters$\big)$:

Lemma 9.7 . — *i) Let $t \in \mathcal{T}(I)$. Then $\theta(t) \in \mathcal{A}(I)$ iff for every permutation μ, the restriction of t^μ to $\{a, b\}^*$ is congruent to $b^{I_{b\mu}} a^{I_{a\mu}}$.*

ii) Let I be a partition, J a permutation of I, p an integer, I' the partition obtained by reordering $(J_1, \ldots, J_p)$. Let t be a tableau in $\mathcal{T}(J)$, t' its restriction to $\mathbb{A}_p^*$. Then $\theta(t) \in \mathcal{A}(I)$ implies that $\theta(t') \in \mathcal{A}(I')$.

iii) The set of tableaux $t \in \mathcal{T}(I)$ such that $\theta(t) \in \mathcal{A}(I)$ is closed by cocyclage.

iv) The minimum elements in $\mathcal{A}(I)$ are two-row tableaux.

Proof. *i)* According to 4.2, t is such that $\theta(t) \in \mathcal{A}(I)$ iff there exists no partition $J < I$ such that t belongs to the image of $\mathcal{T}(J)$ in $\mathcal{T}(I)$. But from the description of the canonical embedding in terms of the morphism φ in 4.2, t belongs to some $\mathcal{T}(J)$ iff there exists a permutation μ such that $|t^\mu|_a \geq |t^\mu|_b$ and the tableau t^μ has some occurrence of b in its bottom row. This settles the case where $|t^\mu|_a \geq |t^\mu|_b$; otherwise, we take the product of μ by the transposition σ of a, b and notice that the image by σ of the word $b^j a^i$, $j \geq i$ is the word $a^{j-i}\, b^i\, a^i \;\equiv b^i\, a^j$.

ii) Follows from *i)*, since we have to use only those permutations μ which preserve globally $\mathbb{A}_p$.

iii) Let $t \to t'$ be an H-cyclage of tableaux in $\mathcal{T}(I)$. Let μ be any permutation such that $|t^\mu|_a \geq |t^\mu|_b$. Then $t^\mu \to t'^\mu$ is an H-cyclage : $t^\mu \equiv xw$, $t'^\mu \equiv wx$, where w is a tableau of shape I and $x \neq a$. Thus if t'^μ has no b in its bottom row, then $x \neq b$ and the restriction of t^μ and t'^μ to $\{a, b\}^*$ are identical. According to criterium *i)*, this ensures that $\theta(t) \in \mathcal{A}(I)$ if $\theta(t') \in \mathcal{A}(I)$.

iv) Let t be a tableau of shape J and $t \to t'$ be an $H - cyclage$ such that $J_2 = H_2$. Then for any μ, $t^\mu = xw \to wx \equiv t'^\mu$ is such that $x \neq a, b$. Therefore the restrictions of t^μ and t'^μ to $\{a, b\}^*$ coïncide and thus $\theta(t) \in \mathcal{A}(I)$ implies $\theta(t') \in \mathcal{A}(I)$. If t is a two-row tableau of shape J, then t admits only the

(J_2-1, J_1)-cyclage ; if $t = b^m ua^n v$, $m > 0$, then the cyclage $t \to t' \equiv b^{m-1}ua^n vb$ is such that t' as a b in its bottom row, and thus $\theta(t) \in \mathcal{A}(I) \Rightarrow \theta(t') \notin \mathcal{A}(I)$ $\square$

Property *iii)* shows that to describe atoms, we need only to enumerate the two-row tableaux. Moreover, given a partition I, if we start from the maximal tableau $t = \cdots w_3 w_2 w_1$ with $w_k = (I_1 + I_2 + \cdots I_{k-1} + 1) \cdots (I_1 + I_2 + \cdots I_k)$ and cycle the left letter of each successive tableau till we get a two-row tableau, then this final tableau (which we shall call *canonical*) is $w_2 w_4 \cdots w_{2k}\ w_1 w_3 \cdots$.

The following further simplification shows that we can restrict ourselves to the case of partitions with no parts repeated more than twice, and that in the case that no part is repeated, the atom has only one two-row tableau.

LEMMA 9.8 . — *i) Let I be a partition, $I = 1^{m_1} 2^{m_2} \cdots k^{m_k}$. Let $\varepsilon_i = inf(2, m_i)$, $n_i = m_i - \varepsilon_i, \forall i \geq 1$ and $I' = 0^{p-q} 1^{\varepsilon_1} 2^{\varepsilon_2} \cdots k^{\varepsilon_k}$, with$p = |I'| = \varepsilon_1 + 2\varepsilon_2 + \cdots + k\varepsilon_k$, $q = \varepsilon_1 + \cdots + \varepsilon_k$. Let moreover $J = 1^{n_1} 2^{n_2} \cdots k^{n_k}\ J_p \cdots J_1$, with $J_p + \cdots J_1 = p$. Let finally $t \in \mathcal{T}(J)$. Then $\theta(t) \in \mathcal{A}(I)$ iff $\theta(t') \in \mathcal{A}(I')$, where t' denotes the restriction of t to the first p letters of $\mathbb{A}$.*

ii) if no part of I is repeated, then $\mathcal{A}(I)$ has only one two-row tableau, namely its canonical tableau.

Proof. *i)* Let $t \in \mathcal{T}(J)$ be such that $\theta(t') \in \mathcal{A}(I')$. Then t' is the image of a tableau $t'_1 \in \mathcal{T}(I')$; denote by t_1 the tableau obtained from t by changing the subtableau t' into t'_1. According to 9.7*i)* , to determine to which atom $\theta(t_1)$ belongs, we have to test the restriction to $\{a, b\}^*$ of all $t_1{}^{\mu}, \mu \in \mathfrak{S}(\mathbb{A})$. In fact we need only to test a subfamily of permutations η such that each pair of parts of I occurs at least once in the set $\{\big(|t_1^{\eta}|_a, |t_1^{\eta}|_b\big)\}$. Because of the definition of the multiplicities m_i, we can suppose that η fixes the letters outside $\mathbb{A}_p$. In other words, η acts only on the subtableau t'_1, the restriction of t_1^{η} and t'^{η}_1 to $\{a, b\}^*$ coïncide and $\theta(t'_1) \in \mathcal{A}(I')$ implies that $\theta(t_1) \in \mathcal{A}(I)$.

Conversely, property 9.7*ii)* shows that $\theta(t) \in \mathcal{A}(I)$ implies that $\theta(t') \in \mathcal{A}(I')$.

ii) Let $t \in \mathcal{T}(I)$ be a two-row tableau such that $\theta(t) \in \mathcal{A}(I)$. We know from proposition 9.7 that for any k the

restriction t' of t to $\mathbb{A}_k$ is such that $\theta(t') \in \mathcal{A}(I')$, with $I' = (I_1, \ldots, I_k)$. Let k be the smallest integer such that $\theta(t')$ is not canonical. It is equivalent to require that there is at least one occurrence of a_k on the bottom (resp. top) row when k is even (resp. odd). Since $|t|_{a_k} < |t|_{a_{k-1}}$, there is also one misplaced a_k in the tableau $K(t')$. Therefore $\theta\big(K(t')\big)$ also is not canonical. Since it belongs to the atom $\mathcal{A}(I'')$, with $I'' = (I_1+I_2, I_3, \ldots, I_k)$, the assertion follows by induction on the number of parts $\square$

To sum up, atoms are totally characterized by the enumeration of all the non-canonical two-row tableaux for partitions having no part repeated more than twice.

Compact table of all atoms for $\mathfrak{S}(n)$, $n \leq 12$

alphabet $\{1,\ldots,9,0,\flat,\natural\}$

Atom	*Non−canonical Tableaux*	*Cocharge*
$\mathcal{A}(1223)$	45 123678	5
$\mathcal{A}(1224)$	56 1234789	5
$\mathcal{A}(1225)$	67 12345890	5
$\mathcal{A}(1334)$	567 1234890♭	7
$\mathcal{A}(1226)$	78 12345690♭	5
$\mathcal{A}(12234)$	5670♭♮ 123489	11
$\mathcal{A}(2334)$	567♮ 1234890♭	9
	567 1234890♭♮	8
$\mathcal{A}(1335)$	678 1234590♭♮	7
$\mathcal{A}(1227)$	89 12345670♭♮	5

Let us see how to use the table by a concrete example. Take $I = 11333$; reducing the multiplicities of parts to 2 at most, we obtain the partition 1133 which is not in the table. Thus $\mathcal{A}(1133)$ has only one two-row tableau, namely 456 8 123 7. Cocycling, we obtain that $\mathcal{A}(1133)$ is equal to

$$\begin{array}{l}4568\\1237\end{array} \longleftarrow \begin{array}{l}7\\456\\1238\end{array} \longleftarrow \begin{array}{l}8\\7\\456\\123\end{array} \quad .$$

Now according to 9.8, the pull-back of $\mathcal{A}(11333)$ in $\mathcal{T}(3\,1^8)$ is exactly the set of all tableaux obtained from the three preceeding ones by shuffling three occurrences of 9. We give below each of these tableaux (written as words) together with their respective images by θ, and their cocharges, taking as alphabet $\{1,\ldots,9,0,\natural\}$:

Cocharge

$$\begin{array}{lll} \textit{image by}\,\theta \longrightarrow & \natural0789456123 & \\ \textit{tableau} \longrightarrow & 98799456123 & (16) \end{array}$$

$$\begin{array}{l}0789456123\natural\\87994561239\end{array}\qquad\begin{array}{l}\natural0894561237\\98794561239\end{array}\tag{15}$$

$$\begin{array}{l}789456\natural1230\\79945691238\end{array}\quad\begin{array}{l}089456\natural1237\\97945691238\end{array}\quad\begin{array}{l}0894561237\natural\\87945612399\end{array}\quad\begin{array}{l}089\natural4561237\\97994561238\end{array}\quad\begin{array}{l}\natural0945612378\\98745612399\end{array}\tag{14}$$

$$\begin{array}{l}89\natural45601237\\99945681237\end{array}\ \begin{array}{l}89456\natural12370\\79456912389\end{array}\ \begin{array}{l}09456\natural12378\\97456912389\end{array}\ \begin{array}{l}09\natural45612378\\97945612389\end{array}\ \begin{array}{l}89\natural45612370\\79945612389\end{array}\ \begin{array}{l}0945612378\natural\\87456123999\end{array}\tag{13}$$

$$\begin{array}{l}9\natural456012378\\99456812379\end{array}\quad\begin{array}{l}9456\natural123780\\74569123899\end{array}\quad\begin{array}{l}9\natural456123780\\79456123899\end{array}\quad\begin{array}{l}\natural9456123780\\97456123899\end{array}\tag{12}$$

$$\begin{array}{l}\natural4560123789\\94568123799\end{array}\qquad\begin{array}{l}9456123780\natural\\74561238999\end{array}\tag{11}$$

$$\begin{array}{l}4560123789\natural\\45681237999\end{array}\tag{10}$$

We shall now give a property which allows us to generate atoms by induction on the size of the alphabet.

Let $I = (I_1 \geq I_2 \geq \cdots)$ be a partition and r an integer such that $r \leq I$, let k be the integer such that $I_k \geq r > I_{k+1}$ and denote by $I \setminus r$ the partition $(I_1, \ldots, I_{k-1},\, I_k + I_{k+1} - r\,, I_{k+2}\,, \ldots)$.

Proposition 9.9 . — *Let I be a partition, r, n two integers, t a tableau such that $\theta(t) \in \mathcal{A}(I)$ and such that $|ev(t)|_{a_m} = 0$ for $m > n$, $|ev(t)|_{a_n} = r$. Then the tableau t' obtained from t by erasing all occurrences of a_n is such that $\theta(t')$ belongs to the atom $\mathcal{A}(I \setminus r)$.*

We will not give the proof, which reduces by cyclage to a detailed study of two-row tableaux. Note however that in the case where r is a part of I, this property is a corollary of lemma 9.7.

For example, the pull-back of $\mathcal{A}(4221)$ in $\mathcal{T}(1^6 3)$ is the set

$$\left\{\begin{array}{l}7\\67\\45\\1237\end{array},\begin{array}{l}7\\6\\45\\12377\end{array},\begin{array}{l}67\\45\\12377\end{array},\begin{array}{l}6\\45\\123777\end{array},\begin{array}{l}77\\457\\1236\end{array},\begin{array}{l}77\\45\\12367\end{array},\begin{array}{l}7\\457\\12367\end{array},\begin{array}{l}7\\45\\123677\end{array},\begin{array}{l}457\\123677\end{array},\begin{array}{l}45\\1236777\end{array}\right\}$$

Erasing the "7" , we get the two tableaux $\begin{array}{l}6\\45\\123\end{array}$, $\begin{array}{l}45\\1236\end{array}$ which belong to the atom $\mathcal{A}(321)$, in accordance with the fact that $4221 \setminus 3 = (4{+}2{-}3)\; 2\,1 = 321$.

In section 6 we saw that tableaux provide multiplicities of representations, as well as bases, in the cohomology ring $\mathcal{H}_I$ of the space of flags corresponding to a partition I.

Generators of the ideal defining $\mathcal{H}_I$ can also be read from tableaux.

For a triple of integers $k, r, n\, :\, k \geq r, k + r \leq n$, let $\tau_{k\,r\,n}$ be the tableau $\begin{array}{l}k{+}1\cdots k{+}r\\1\,\cdots\,k\quad k{+}r{+}1\cdots\,n\end{array}$. Using criterium 9.6, we see that $\tau_{k\,r\,n}$ belongs to the atom $\mathcal{A}\big(P(\tau_{k\,r\,n})\big)$, where $P(\tau_{k\,r\,n})$ is the following partition of n : $P(\tau_{k\,r\,n}) := \big(k, r, \ldots, r, n - k - [\frac{n-k}{r}]\big)$ and $[\frac{n-k}{r}]$ is the integral part of $(n - k)/r$. Moreover $\tau_{k\,r\,n}$ has cocharge $n - k$.

To the tableau $\tau_{k\,r\,n}$ we associate the complete function $S(\tau_{k\,r\,n}) := S_{n-k}(\mathbb{A}_r)$, with $\mathbb{A}_r = \{a_1, \ldots, a_r\}$ as before. Given a partition I of n, let $\mathfrak{Tab}(I)$ be the set of tableaux $\tau_{k\,r\,n}$ such that $P(\tau_{k\,r\,n}) \not\leq I$, and let $\mathfrak{Ideal}(I)$ be the ideal of $\mathbb{Z}[\mathbb{A}]$ generated (as a $\mathfrak{S}(\mathbb{A})-$ideal) by the $\{S(\tau), \tau \in \mathfrak{Tab}(I)\}$. Recall that the order on partitions is defined by the componentwise order on the associated vectors : $I \to$ *conjugate partition* $J = (J_1 \geq J_2 \geq \cdots) \to J^{\Sigma} := (J_1, J_1 + J_2 - 1, J_1 + J_2 + J_3 - 2, \ldots)$.

On the other hand, given an integral vector $H = (H_1, \ldots, H_p)$, let $\mathfrak{Tan}(H)$ (due to Tanisaki [Tan]) be the $\mathfrak{S}(\mathbb{A})-$ideal in $\mathbb{Z}[\mathbb{A}]$ generated by $\{S_{k_1}(\mathbb{A}_1), Sk_2(\mathbb{A}_2), \ldots, S_{k_p}(\mathbb{A}_p)\,,\, k_1 \geq H_1, \ldots, k_p \geq H_p\}$.

Proposition 9.10 . — *Let I and J be conjugate partitions. Then the two ideals $\mathfrak{Ideal}(I)$ and $\mathfrak{Tan}(J^{\Sigma})$ coïncide.*

Proof. Since order is defined by the vectors J^{Σ}, it is clear that $\mathfrak{Ideal}(I)$ can be described by the components of J^{Σ} rather than by the partitions $P(\tau_{k\,r\,n})$. We shall just check by an example that the two sets of generators coïncide.

Let $I = 864421$ and thus $J = 65442211$, $J^{\Sigma} = 6\ 10\ 13\ 16\ 17\ 18\ 18\ 18$. The tableau $\tau_{12,3,25}$ has cocharge 13 and belongs to the atom $\mathcal{A}(12\ 3\,3\,3\,3\,1)$; the conjugate of this last partition is $6\,5\,5\,1^9$ which gives the vector $6\,11\,14\,14\ldots$ not comparable to J^{Σ}. Thus, $P(\tau_{12,3,25}) \not\leq I$. On the other hand, the tableau $\tau_{13,3,25}$ has cocharge 12 and belongs to the atom $\mathcal{A}(13\ 3\,3\,3\,3)$; the corresponding conjugate partition is $5\,5\,5\,1^{10}$ giving the vector $5\,9\,13\,13\,13\ \ldots$ which is less than J^{Σ}. Therefore, $\mathfrak{Ideal}(I)$ contains $S_{13}(\mathbb{A}_3)$, but not $S_{12}(\mathbb{A}_3)$, in accordance with the fact that the third component of J^{Σ} is 13 □

Tanisaki [Tan] uses elementary symmetric functions e_j as generators of the ideal, but since $e_j(\mathbb{A} \setminus \mathbb{B}) = e_j(\mathbb{A}) - S_1(\mathbb{B})e_{j-1} + S_2(\mathbb{B})e_{j-2} - \cdots + (-1)^j S_j(\mathbb{B})$ for any subalphabet $\mathbb{B}$ of $\mathbb{A}$, one has $e_j(a_1, \ldots, a_p) \equiv (-1)^j S_j(a_{p+1}, \ldots, a_n)$ modulo the ideal generated by symmetric polynomials in $\{a_1, \ldots, a_n\}$ without constant terms : elementary symmetric functions and complete functions are interchangeable.

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