The differential equation satisfied by a plane curve of degree n

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Abstract

Eliminating the arbitrary coefficients in the equation of a plane curve of order n by computing sufficiently many derivatives, one obtains a differential equation characteristic of these curves. This is a projective invariant. The first one, corresponding to conics, have been obtained by Monge, and Sylvester, Halphen, Cartan used invariants of higher order. The expression of these invariants is rather complicated, but become much simpler when interpreted in terms of symmetric functions.

A line in the plane can be written

$$y = a\,x + b\ ,$$

with arbitrary coefficients a, b, but it is more satisfactory to write it

$$y'' = 0\ .$$

More generally

$$y^n + (\star)y^{n-1}x + \cdots + (\star)x^n + \cdots + (\star)y^0x^0 = 0$$

is the equation of a general planar curve of order n. Writing sufficiently many derivatives of this equation, one can eliminate in their system the arbitrary coefficients $(\star)$.

However, already in the case of a conic (solved by Monge), we have to use the derivatives of order $3, 4, 5$, and the outcome is not straightforward to interpret. We need some method to perform the elimination.

It is convenient, instead of taking $\frac{d^i}{dx^i}$, to rather use the normalization

$$D^i = \frac{d^i}{i!\,dx^i}$$

With these conventions, Leibnitz formula looses its coefficients:

$$D^n(fg) = \sum_{i+j=n} D^i f\, D^j g\ .$$

We need the collection $\{D^0(y),\ D^1(y),\ D^2(y), \ldots\}$, that we can write with the help of a generating series

$$\sum_{i=0}^{\infty} z^i\, D^i(y)\ .$$

Symmetric function theory tells that we have to formally factorize this series, as we factorize the total Chern class of a vector bundle.

Thus we introduce a formal alphabet $\mathbb{A}$ and write

$$\sum_{i=0}^{\infty} \frac{d^i y}{i!\, dx^i} = \sum_{i=0}^{\infty} z^i\, D^i(y) = \prod_{a \in \mathbb{A}} (1 + za) = \sum_{i=0}^{\infty} z^i\, \Lambda^i(\mathbb{A})\ , \tag{1}$$

denoting $\Lambda^i(\mathbb{A})$ the *elementary symmetric functions* in $\mathbb{A}$, and thus identifying $D^i(y)$ to $\Lambda^i(\mathbb{A})$.

Remember that taking k copies of an alphabet (we write $k\,\mathbb{A}$) translates into taking the k-th power of the generating function :

$$\left(\sum_{i=0}^{\infty} z^i\, \Lambda^i(\mathbb{A}) \right)^k = \sum_{i=0}^{\infty} z^i\, \Lambda^i(k\,\mathbb{A})\ .$$

Adding r copies of a letter x to these k copies of $\mathbb{A}$ is written, at the level of generating series, as

$$(1 + zx)^r \left(\sum_{i=0}^{\infty} z^i\, \Lambda^i(\mathbb{A}) \right)^k = \sum_{i=0}^{\infty} z^i\, \Lambda^i(k\,\mathbb{A} + r\,x)\ . \tag{2}$$

Thus, instead of having a sum $D^n(y^2) = \sum_{i+j=n} D^i y\, D^j y$ to express the derivatives of the square of y, one can now use the more compact notation $D^n(y^2) = \Lambda^n(2\mathbb{A})$.

More generally, $D^n(y^3) = \Lambda^n(3\mathbb{A}),\ D^n(y^4) = \Lambda^n(4\mathbb{A}), \ldots$ and one has the following easy lemma resulting from Leibnitz' rule:

Lemma 1 *Given $n, k, r \in \mathbb{N}$, then*

$$D^n(x^r y^k) = x^r \Lambda^n(k\mathbb{A} + r/x)\ . \tag{3}$$

We can now easily write the derivatives of any orders of the components of the equation of a planar curve.

Let us first look at the case treated by Monge.

We start with

$$u = y^2 + c_1 xy + c_2 y + (\star)x^2 + (\star)x + (\star)$$

and take successive derivatives, starting from the third one (so that the part depending on x only (where the coefficients $(\star)$ appear) has already been eliminated).

$$\begin{aligned}
D^3 u &= \Lambda^3(2\mathbb{A}) + xc_1\Lambda^3(\mathbb{A} + 1/x) + c_2\Lambda^3(\mathbb{A}) && (4)\\
&= \Lambda^3(2\mathbb{A}) + (xc_1 + c_2)\Lambda^3(\mathbb{A}) + c_1\Lambda^2(\mathbb{A}) && (5)\\
D^4 u &= \Lambda^4(2\mathbb{A}) + (xc_1 + c_2)\Lambda^4(\mathbb{A}) + c_1\Lambda^3(\mathbb{A}) && (6)\\
D^5 u &= \Lambda^5(2\mathbb{A}) + (xc_1 + c_2)\Lambda^5(\mathbb{A}) + c_1\Lambda^4(\mathbb{A}) && (7)
\end{aligned}$$

Elimination of the coefficients among these three equations gives the vanishing :

$$\begin{vmatrix} \Lambda^2(\mathbb{A}) & \Lambda^3(\mathbb{A}) & \Lambda^3(2\mathbb{A}) \\ \Lambda^3(\mathbb{A}) & \Lambda^4(\mathbb{A}) & \Lambda^4(2\mathbb{A}) \\ \Lambda^4(\mathbb{A}) & \Lambda^5(\mathbb{A}) & \Lambda^5(2\mathbb{A}) \end{vmatrix} = \begin{vmatrix} \Lambda^2 & \Lambda^3 & 2\Lambda^{30} + 2\Lambda^{21} \\ \Lambda^3 & \Lambda^4 & 2\Lambda^{40} + 2\Lambda^{31} + \Lambda^{22} \\ \Lambda^4 & \Lambda^5 & 2\Lambda^{50} + 2\Lambda^{41} + 2\Lambda^{32} \end{vmatrix} , \qquad (8)$$

writing Λ^i for $\Lambda^i(\mathbb{A})$, $\Lambda^{ij\cdot\cdot}$ for $\Lambda^i(\mathbb{A})\Lambda^j(\mathbb{A}), \ldots$

The last determinant can be simplified and becomes

$$\begin{vmatrix} \Lambda^2 & \Lambda^3 & 0 \\ \Lambda^3 & \Lambda^4 & \Lambda^{22} \\ \Lambda^4 & \Lambda^5 & 2\Lambda^{32} \end{vmatrix} = \begin{vmatrix} y''/2 & y'''/6 & 0 \\ y'''/6 & y^{iv}/24 & (y''/2)^2 \\ y^{iv}/24 & y^v/120 & y''y'''/6 \end{vmatrix} , \qquad (9)$$

which is Monge's equation, after suppressing the extra factor $y''/2$:

$$D^2yD^2yD^5y - 3D^2yD^3yD^4y + 2D^3yD^3yD^3y = 0 \; . \qquad (10)$$

The general case takes only a few lines more to be written down.

From Eq.2, one has

$$\Lambda^n(\mathbb{A} + rx) = \Lambda^n\mathbb{A} + rx\,\Lambda^{n-1}\mathbb{A} + \binom{r}{2}x^2\,\Lambda^{n-2}\mathbb{A} + \cdots + \binom{r}{n}x^n\,\Lambda^0\mathbb{A} \; .$$

Let u be a polynomial in x, y of total degree n with leading term y^n. The equations

$$D^{n+1}u = 0 = \cdots = D^{n(n+3)/2}u$$

are

$$\begin{aligned}
0 &= \Lambda^{n+1}(n\mathbb{A}) + c_{1,n-1}\Lambda^{n}((n-1)\mathbb{A}) + c_{2,n-1}\Lambda^{n+1}((n-1)\mathbb{A}) + \cdots \\
&\qquad\qquad\qquad + c_{1,1}\Lambda^{2}(\mathbb{A}) + c_{2,1}\Lambda^{3}(\mathbb{A}) + \cdots + \Lambda^{n+1}(\mathbb{A}) \\
0 &= \Lambda^{n+2}(n\mathbb{A}) + c_{1,n-1}\Lambda^{n+1}((n-1)\mathbb{A}) + c_{2,n-1}\Lambda^{n+2}((n-1)\mathbb{A}) + \cdots \\
&\qquad\qquad\qquad + c_{1,1}\Lambda^{3}(\mathbb{A}) + c_{2,1}\Lambda^{4}(\mathbb{A}) + \cdots + \Lambda^{n+2}(\mathbb{A}) \\
& \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
0 &= \Lambda^{n(n+3)/2}(n\mathbb{A}) + c_{1,n-1}\Lambda^{n(n+3)/2-1}((n-1)\mathbb{A}) + c_{2,n-1}\Lambda^{n(n+3)/2}((n-1)\mathbb{A}) + \cdots + \\
&\qquad c_{1,1}\Lambda^{n(n+1)/2+1}(\mathbb{A}) + c_{2,1}\Lambda^{n(n+1)/2+2}(\mathbb{A}) + \cdots + \Lambda^{n(n+3)/2}(\mathbb{A}) \ ,
\end{aligned}$$

where the coefficients $c_{i,j}$ are polynomials in x only.

Eliminating these coefficients, one obtains the vanishing of the following determinant (we have written the columns in a different order):

$$\begin{vmatrix}
\Lambda^2(\mathbb{A}) & \Lambda^3(\mathbb{A}) & \cdots & \Lambda^{n+1}(\mathbb{A}) & \cdots & \Lambda^{n}((n-1)\mathbb{A}) & \Lambda^{n+1}((n-1)\mathbb{A}) & \Lambda^{n+1}(n\mathbb{A}) \\
\Lambda^3(\mathbb{A}) & \Lambda^4(\mathbb{A}) & \cdots & \Lambda^{n+2}(\mathbb{A}) & \cdots & \Lambda^{n+1}((n-1)\mathbb{A}) & \Lambda^{n+2}((n-1)\mathbb{A}) & \Lambda^{n+2}(n\mathbb{A}) \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
 & & \cdots & \Lambda^{N}(\mathbb{A}) & \cdots & \Lambda^{N-1}((n-1)\mathbb{A}) & \Lambda^{N}((n-1)\mathbb{A}) & \Lambda^{N}(n\mathbb{A})
\end{vmatrix} \tag{11}$$

with $N = n(n+3)/2$, which is the differential equation satisfied by a planar curve of order n.

This determinant has a simple structure, with blocks of $n, n{-}1, \ldots, 1$ columns involving the elementary symmetric functions in $\mathbb{A}$, $2\mathbb{A}, \ldots, n\mathbb{A}$.

One can simplify it a little. Because the image of a curve of degree n under the transformation $y \to \alpha y + \beta$ is still a curve of the same degree, the value of the determinant is independent of $y = \Lambda^0(\mathbb{A})$ and $y' = \Lambda^1(\mathbb{A})$, that one can put both equal to 0.

Therefore, instead of the generating series (2), one can now take a second alphabet $\mathcal{D}$ such that

$$\Lambda^i(\mathcal{D}) = \Lambda^{i+2}(\mathbb{A}) \ , \ i = 0, 1, \ldots$$

(as usual $\Lambda^i = 0$ for $i < 0$).

In other words, $\sum_i z^i\, \Lambda^i(\mathcal{D}) = \Lambda^2(\mathbb{A}) + z\Lambda^3(\mathbb{A}) + \cdots$, and for what concerns its powers, one has that

$$\Lambda^i(k\mathcal{D}) = \Lambda^{i+2k}(k\mathbb{A}) \quad , \quad i, k \in \mathbb{N} \ .$$

The equation of a planar curve can now be rewritten

$$\begin{vmatrix}
\Lambda^0(\mathcal{D}) & \cdots & \Lambda^{n-1}(\mathcal{D}) & \Lambda^{-1}(2\mathcal{D}) & \cdots & \Lambda^{n-3}(2\mathcal{D}) & \cdots & \Lambda^{1-n}(n\mathcal{D}) \\
\Lambda^1(\mathcal{D}) & \cdots & \Lambda^{n}(\mathcal{D}) & \Lambda^{0}(2\mathcal{D}) & \cdots & \Lambda^{n-2}(2\mathcal{D}) & \cdots & \Lambda^{2-n}(n\mathcal{D}) \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
 & \cdots & \Lambda^{N}(\mathcal{D}) & & \cdots & \Lambda^{N-2}(2\mathcal{D}) & & \Lambda^{N-2n+2}(n\mathcal{D})
\end{vmatrix} \tag{12}$$

with $N = (n-1)(n+4)/2$, and, apart from notations, this is the equation given by Sylvester.

For the conic, this equation is the determinant (9) that we have seen above :

$$\begin{vmatrix} \Lambda^0\mathcal{D} & \Lambda^1\mathcal{D} & \Lambda^{-1}(2\mathcal{D}) \\ \Lambda^1\mathcal{D} & \Lambda^2\mathcal{D} & \Lambda^0(2\mathcal{D}) \\ \Lambda^2\mathcal{D} & \Lambda^3\mathcal{D} & \Lambda^1(2\mathcal{D}) \end{vmatrix} = \begin{vmatrix} \Lambda^0\mathcal{D} & \Lambda^1\mathcal{D} & 0 \\ \Lambda^1\mathcal{D} & \Lambda^2\mathcal{D} & \Lambda^0\mathcal{D}\,\Lambda^0\mathcal{D} \\ \Lambda^2\mathcal{D} & \Lambda^3\mathcal{D} & 2\Lambda^1\mathcal{D}\,\Lambda^0\mathcal{D} \end{vmatrix} \tag{13}$$

The equation of a planar cubic is :

$$\begin{vmatrix} \Lambda^0\mathcal{D} & \Lambda^1\mathcal{D} & \Lambda^2\mathcal{D} & \Lambda^{-1}(2\mathcal{D}) & \Lambda^0(2\mathcal{D}) & \Lambda^{-2}(3\mathcal{D}) \\ \Lambda^1\mathcal{D} & \Lambda^2\mathcal{D} & \Lambda^3\mathcal{D} & \Lambda^0(2\mathcal{D}) & \Lambda^1(2\mathcal{D}) & \Lambda^{-1}(3\mathcal{D}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda^5\mathcal{D} & \Lambda^6\mathcal{D} & \Lambda^7\mathcal{D} & \Lambda^4(2\mathcal{D}) & \Lambda^5(2\mathcal{D}) & \Lambda^3(3\mathcal{D}) \end{vmatrix} = 0\ . \tag{14}$$

In terms of $\Lambda^i(\mathcal{D})$ only, written Λ_i, and putting $\Lambda_0 = 1$, the determinant reads

$$\begin{vmatrix} 1 & \Lambda_1 & \Lambda_2 & 0 & 1 & 0 \\ \Lambda_1 & \Lambda_2 & \Lambda_3 & 1 & 2\Lambda_1 & 0 \\ \Lambda_2 & \Lambda_3 & \Lambda_4 & 2\Lambda_1 & \Lambda_1^2 + 2\Lambda_2 & 1 \\ \Lambda_3 & \Lambda_4 & \Lambda_5 & \Lambda_1^2 + 2\Lambda_2 & 2\Lambda_1\Lambda_2 + 2\Lambda_3 & 3\Lambda_1 \\ \Lambda_4 & \Lambda_5 & \Lambda_6 & 2\Lambda_1\Lambda_2 + 2\Lambda_3 & \Lambda_2^2 + 2\Lambda_1\Lambda_3 + 2\Lambda_4 & 3\Lambda_1^2 + 3\Lambda_2 \\ \Lambda_5 & \Lambda_6 & \Lambda_7 & \Lambda_2^2 + 2\Lambda_1\Lambda_3 + 2\Lambda_4 & 2\Lambda_2\Lambda_3 + 2\Lambda_1\Lambda_4 + 2\Lambda_5 & \Lambda_1^3 + 6\Lambda_1\Lambda_2 + 3\Lambda_3 \end{vmatrix}$$

which expands into the rather less attractive expression : $6\Lambda_1^7\Lambda_2\Lambda_3^2 - 30\Lambda_2^4\Lambda_1^2\Lambda_5 - 10\Lambda_3^3\Lambda_1\Lambda_5 + 5\Lambda_2\Lambda_7\Lambda_3^2 - 2\Lambda_5\Lambda_7\Lambda_1^3 + 5\Lambda_7\Lambda_1^2\Lambda_2^3 + 30\Lambda_4\Lambda_2^5\Lambda_1 + 6\Lambda_5\Lambda_2^2\Lambda_3^2 + 6\Lambda_1^5\Lambda_6\Lambda_4 - 6\Lambda_1^6\Lambda_6\Lambda_3 + 3\Lambda_6\Lambda_1\Lambda_4^2 - 3\Lambda_6^2\Lambda_1\Lambda_2 - 20\Lambda_1^4\Lambda_4^2\Lambda_3 - 60\Lambda_4\Lambda_1^3\Lambda_2^4 + 5\Lambda_2\Lambda_5\Lambda_4^2 + 10\Lambda_5\Lambda_4\Lambda_3^2 - 3\Lambda_1^8\Lambda_5\Lambda_2 + 20\Lambda_2^2\Lambda_4^2\Lambda_3 - 3\Lambda_4\Lambda_1\Lambda_5^2 + 10\Lambda_2^2\Lambda_4^2\Lambda_1^3 - 15\Lambda_7\Lambda_1^4\Lambda_2^2 + 2\Lambda_6^2\Lambda_1^3 + 4\Lambda_7\Lambda_2^4 + 5\Lambda_2^6\Lambda_3 - 8\Lambda_2\Lambda_4\Lambda_3\Lambda_1\Lambda_5 - 5\Lambda_2\Lambda_6\Lambda_1^2\Lambda_5 - 10\Lambda_3^2\Lambda_2^2\Lambda_1\Lambda_4 + 5\Lambda_1^3\Lambda_4^3 + 12\Lambda_1^7\Lambda_5\Lambda_3 - 6\Lambda_2\Lambda_1^5\Lambda_4^2 + 5\Lambda_1^2\Lambda_4\Lambda_3^3 + 5\Lambda_1^2\Lambda_3\Lambda_5^2 - 30\Lambda_1^4\Lambda_2\Lambda_3^3 - 50\Lambda_2^4\Lambda_3^2\Lambda_1 - 12\Lambda_5\Lambda_1^6\Lambda_2^2 + 6\Lambda_2^2\Lambda_4\Lambda_1^7 - 10\Lambda_4\Lambda_2^4\Lambda_3 - 10\Lambda_4\Lambda_2^3\Lambda_5 - 10\Lambda_1^6\Lambda_3^3 + 10\Lambda_2\Lambda_4\Lambda_1^4\Lambda_5 - 7\Lambda_1^2\Lambda_6\Lambda_4\Lambda_3 + 12\Lambda_2\Lambda_7\Lambda_1^6 + 10\Lambda_2^3\Lambda_3^3 + \Lambda_1^2\Lambda_5\Lambda_4^2 + 3\Lambda_3\Lambda_6\Lambda_1\Lambda_5 - 3\Lambda_3\Lambda_7\Lambda_1\Lambda_4 - 2\Lambda_4\Lambda_6\Lambda_5 - 15\Lambda_2\Lambda_6\Lambda_4\Lambda_1^3 + 5\Lambda_2\Lambda_7\Lambda_1^2\Lambda_4 - 2\Lambda_2\Lambda_6\Lambda_4\Lambda_3 + 10\Lambda_3^2\Lambda_1\Lambda_4^2 + 6\Lambda_1^7\Lambda_4^2 - 10\Lambda_2\Lambda_1\Lambda_4^3 - 10\Lambda_1\Lambda_2\Lambda_3^4 + 3\Lambda_5\Lambda_7\Lambda_1\Lambda_2 - 5\Lambda_1^3\Lambda_6\Lambda_3^2 + 10\Lambda_1^4\Lambda_5\Lambda_3^2 - 3\Lambda_1^8\Lambda_4\Lambda_3 + 30\Lambda_1^5\Lambda_4\Lambda_3^2 - 6\Lambda_1^6\Lambda_4\Lambda_5 - 4\Lambda_1^5\Lambda_7\Lambda_3 - 2\Lambda_1^5\Lambda_5^2 - 10\Lambda_6\Lambda_2^4\Lambda_1 - 6\Lambda_6\Lambda_2^3\Lambda_3 - 4\Lambda_7\Lambda_2^2\Lambda_4 + 4\Lambda_6\Lambda_2^2\Lambda_5 + 30\Lambda_6\Lambda_2^3\Lambda_1^3 - 3\Lambda_1^8\Lambda_7 + 54\Lambda_2^2\Lambda_3^2\Lambda_1^5 + 50\Lambda_2^2\Lambda_3^3\Lambda_1^2 + 24\Lambda_4\Lambda_1^5\Lambda_2^3 - 5\Lambda_6\Lambda_3^3 - 30\Lambda_2^3\Lambda_1^3\Lambda_3^2 + 45\Lambda_2^3\Lambda_1^4\Lambda_5 + 60\Lambda_2^5\Lambda_1^2\Lambda_3 - 45\Lambda_2^4\Lambda_1^4\Lambda_3 - 10\Lambda_2^3\Lambda_1\Lambda_4^2 - 10\Lambda_2^3\Lambda_1^6\Lambda_3 - 5\Lambda_3\Lambda_4^3 + 3\Lambda_2^5\Lambda_1^5 + 10\Lambda_2\Lambda_1^2\Lambda_4^2\Lambda_3 - 14\Lambda_7\Lambda_1\Lambda_2^2\Lambda_3 + 10\Lambda_2^6\Lambda_1^3 + 5\Lambda_2\Lambda_1^3\Lambda_5^2 - 27\Lambda_6\Lambda_1^5\Lambda_2^2 - 15\Lambda_2^7\Lambda_1 + 40\Lambda_2^3\Lambda_3\Lambda_1\Lambda_5 + 6\Lambda_2\Lambda_6\Lambda_1^7 - 8\Lambda_2\Lambda_3\Lambda_5^2 + 6\Lambda_3^5 + \Lambda_5^3 + \Lambda_1^9\Lambda_6 + \Lambda_6^2\Lambda_3 - 20\Lambda_2\Lambda_4\Lambda_3^3 + 10\Lambda_4\Lambda_1^2\Lambda_2^3\Lambda_3 + 45\Lambda_4\Lambda_1^4\Lambda_3\Lambda_2^2 - 40\Lambda_5\Lambda_1^3\Lambda_3\Lambda_2^2 - 15\Lambda_6\Lambda_1^2\Lambda_3\Lambda_2^2 + 14\Lambda_6\Lambda_4\Lambda_1\Lambda_2^2 + 10\Lambda_2\Lambda_7\Lambda_3\Lambda_1^3 + 18\Lambda_2\Lambda_6\Lambda_1\Lambda_3^2 - 5\Lambda_2\Lambda_5\Lambda_1^2\Lambda_3^2 - 42\Lambda_2\Lambda_4\Lambda_1^6\Lambda_3 - 20\Lambda_2\Lambda_4\Lambda_1^3\Lambda_3^2 - 12\Lambda_2\Lambda_5\Lambda_1^5\Lambda_3 +$

$20\Lambda_2\Lambda_6\Lambda_1^4\Lambda_3 + \Lambda_7\Lambda_4^2 + \Lambda_1^2\Lambda_7\Lambda_3^2 - \Lambda_5\Lambda_7\Lambda_3$ Recall that $\Lambda_0 = y''/2!$, $\Lambda_1 = y'''/6!, \ldots, \Lambda_7 = y^{vii}/7!$, so that the equation of a cubic is a differential polynomial of order 15, and degree 10.

Invariance under the projective group

We did not yet use that the image of a curve under a projective transformation of the plane is still a curve of the same degree, and therefore that *the differential equation is a projective invariant.*

Let us introduce another alphabet $\mathcal{E}$:

$$\Lambda^i(\mathcal{E}) = \Lambda^i(\mathcal{D})/i! = \frac{d^{i+2}y}{i!\,(i\,+2)!\,dx^{i+2}}\ .$$

Expressed in terms of $\mathcal{D}$, invariance under some subgroup of the projective group amounts to belonging to the kernel of

$$\nabla_{\mathcal{D}} := \frac{d}{d_{\Lambda^1}} + 2\Lambda^1(\mathcal{D})\frac{d}{d_{\Lambda^2}} + 3\Lambda^2(\mathcal{D})\frac{d}{d_{\Lambda^3}} + 4\Lambda^3(\mathcal{D})\frac{d}{d_{\Lambda^4}} + \cdots$$

This operator appears in the theory of *binary forms* in x, y, and simply expresses the invariance under the translation $x \to x+1$. Elements of this kernel are called *semi-invariants* in the theory of binary forms.

To simplify the operator $\nabla_{\mathcal{D}}$, let us introduce another alphabet $\mathcal{E}$:

$$\Lambda^i(\mathcal{E}) = \Lambda^i(\mathcal{D})/i! = \frac{d^{i+2}y}{i!\,(i\,+2)!\,dx^{i+2}}\ .$$

Under the change of alphabet, our projective invariants belong to the kernel of

$$\nabla_{\mathcal{E}} := \frac{d}{d_{\Lambda^1}} + \Lambda^1(\mathcal{E})\frac{d}{d_{\Lambda^2}} + \Lambda^2(\mathcal{E})\frac{d}{d_{\Lambda^3}} + \Lambda^3(\mathcal{E})\frac{d}{d_{\Lambda^4}} + \cdots$$

But this kernel is very easy to determine. Indeed, $\nabla_{\mathcal{E}}$ sends $\psi_1(\mathcal{E})$ onto 1, and all the other power sums $\psi_i(\mathcal{E})$, $2 \le i$, onto 0. Therefore a semi-invariant is a polynomial in $\psi_2(\mathcal{E})$, $\psi_3, \ldots$

Monge's element is of degree 3 in $\mathcal{E}$. Therefore it must be proportional to $\psi_3(\mathcal{E})$. In final, the equation of a conic is

$$\psi_3(\mathcal{E}) = 0\ , \tag{15}$$

or in terms of elementary symmetric functions,

$$\psi_3(\mathcal{E}) = 3\Lambda^3(\mathcal{E}) - 3\Lambda^{12}(\mathcal{E}) + \Lambda^{111}(\mathcal{E}) = 0 \tag{16}$$

which rewrites, with $\Lambda^1(\mathcal{E}) = \frac{d^3y}{1!\,3!!\,dx^3}$, $\Lambda^2(\mathcal{E}) = \frac{d^4y}{2!\,4!!\,dx^4}$, $\Lambda^3(\mathcal{E}) = \frac{d^5y}{1!\,3!!\,dx^5}$, into the equation that we have already seen:

$$3\left(\frac{y''}{2!}\right)^2 \frac{y^v}{3!5!} - 3\frac{y''}{2!}\frac{y'''}{1!3!}\frac{y^{iv}}{2!4!} + \left(\frac{y'''}{1!3!}\right)^3 = 0\ .$$

The equations of curves of higher degree can be similarly treated, using the usual theory of binary forms, and involve generalizations of the Hessian.

The next projective invariant, after the invariant of Monge, has been found by Halphen. Cartan takes it as the projective analogue of the curvature. The equation of the cubic is a polynomial in it and Monge's invariant. Its expression, in terms of power sums in $\mathcal{E}$, is

$$48\psi_5\psi_3 - 20\psi_3^2\psi_2 - \psi_2^4 + 12\psi_2^2\psi_4 - 36\psi_4^2 \tag{17}$$

and is easier to handle that the determinant of order 6 displayed in Eq. 14 !

References

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