DOUBLE CRYSTAL GRAPHS

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Abstract

We show how to expand a non-symmetric Cauchy kernel $\prod_{i+j\leq n}(1-x_iy_j)^{-1}$ in the basis of Demazure characters for type A_n. The construction involves using the left and right structure of crystal graphs on words, and mostly reduces to properties of the jeu de taquin.

We give, without proof, the expansion of a similar kernel in the nilplactic algebra, and define characters in two sets of variables.

1 Kernels

The best characterization of Schur functions, as symmetric functions, relies on the diagonalization of the Cauchy kernel $\prod_i \prod_j (1-x_iy_j)^{-1}$.

Indeed, this kernel expands as a sum over all partitions λ :

$$\prod_i \prod_j (1-x_iy_j)^{-1} = \sum_\lambda S_\lambda(X)\, S_\lambda(Y)\ , \tag{1}$$

defining the Schur functions S_λ uniquely (choosing them to be positive).

One could as well have started from the resultant $\prod_i \prod_j (x_i-y_j)$ of the two polynomials in z, $\prod_i(z-x_i)$ and $\prod_j(z-y_j)$.

Relaxing symmetry, one still wants to "separate variables" in $\prod_{i+j\leq n}(x_i-y_j)$ and $\prod_{i+j\leq n}(1-x_iy_j)^{-1}$, or more generally, in products corresponding to the diagram of a partition.

The first function $\prod_{i+j\leq n}(x_i-y_j)$ expands as a sum of products of Schubert polynomials in X and Y separately [19]. For the second one, given a partition λ, let

$$F_\lambda := \prod_{(i,j)\in\lambda} (1-x_iy_j)^{-1}\ ,$$

product over all the boxes of the diagram of λ, noting (i,j) the coordinates of a box. We shall see that the cases $\lambda = \rho := [n,\ldots,1]$, $n \in \mathbb{N}$, allows to recover all the F_λ, and therefore we shall mostly comment the case

$$F_\rho = \prod_{i+j\leq n+1} (1 - x_i y_j)^{-1} \ ,$$

the explicit expansion given in Theorem 6 being our main new result.
A representation explanation of the decomposition of F_ρ, as we have for $\prod_i \prod_j (1 - x_i y_j)^{-1}$, seems highly desirable.

One way to understand the Cauchy kernel is to have recourse to Schur functors instead of Schur functions, and to use representations of linear groups as Schur did in his thesis [36].

One can equivalently consider spaces of sections of ample line bundles over the flag manifold $\mathcal{F}(\mathbb{C}^n)$. The non-symmetric case involves using sections over Schubert varieties, and this leads to *Demazure characters* [3]. However, the geometrical study of line bundles over Schubert varieties is more complicated than the study of characters only. Instead of representations, we shall rather use spaces of words (which can be thought as "canonical bases").

In that framework, Cauchy formula (1) is interpreted as a bijection, due mostly to Schensted, between the terms in the expansion of the left side of (1) (words in commutative biletters (x_i, y_j)), and pairs of tableaux of the same shape.

Thanks to this bijection, generating functions of monomials in commutative biletters are transformed into expressions in the free algebra, or transformed into multisets of pairs of tableaux. In particular, since F_ρ expands into a subfamily of monomials of the expansion of $\prod_i \prod_j (1 - x_i y_j)^{-1}$, we already know that F_ρ can be interpreted as a family of pairs of tableaux.

To characterize this family, we need some combinatorial tools associated to the natural two-sided structure of crystal graph on words. In fact, we do not need to prove new properties of crystal graphs to explicit F_ρ, but have mostly to gather in Theorem 4 results stated in different languages. We shall use words and tableaux as the simplest concepts in the type A_n case.

We mention, in the last two sections, the possibility of defining "Demazure characters" in two sets of variables, or inside the nilplactic algebra.

The three figures at the end provide three different realizations of the basis of a Demazure module.

2 Combinatorial Tools

Let $X = \{x_1,\, x_2, \ldots\}$ be a totally ordered alphabet (of non commuting letters).

A *row* is a weakly increasing word, a *column* is a strictly decreasing one. A pair of rows $a_1 \cdots a_\ell$, $b_1 \cdots b_k$, a_i, $b_j \in X$, is a

- *2-rows tableau* iff $\ell \le k$ & $a_i > b_i$, $i = 1, \ldots, \ell$
- *2-rows contretableau* iff $\ell \ge k$ & $a_{i+\ell-k} > b_i$, $i = 1, \ldots, k$

A sequence of rows is a *tableau* (resp. a *contretableau*) if every pair of consecutive rows is so. Both can be planarly represented, stacking rows on top of each other, and left justifying in the case of tableaux, right justifying them in the case of contretableaux. We shall identify planar objects with their row reading, and with the sequence of rows composing them.

The *plactic relations* (due to Knuth[11]) are the congruences

$$cab \equiv acb \qquad (a \le b < c)\,, \tag{2}$$
$$bac \equiv bca \qquad (a < b \le c)\,. \tag{3}$$

that can be planarly written

$$\begin{array}{|c|c} \hline c & \multicolumn{1}{c}{} \\ \hline a & \multicolumn{1}{|c|}{b} \\ \hline \end{array} \quad \equiv \quad \begin{array}{c|c|} \hline \multicolumn{1}{|c|}{a} & c \\ \hline \multicolumn{1}{c|}{} & b \\ \cline{2-2} \end{array} \qquad\qquad (a \le b < c) \tag{4}$$

$$\begin{array}{|c|c} \hline b & \multicolumn{1}{c}{} \\ \hline a & \multicolumn{1}{|c|}{c} \\ \hline \end{array} \quad \equiv \quad \begin{array}{c|c|} \hline \multicolumn{1}{|c|}{b} & c \\ \hline \multicolumn{1}{c|}{} & a \\ \cline{2-2} \end{array} \qquad\qquad (a < b \le c) \tag{5}$$

The set of words congruent modulo the plactic relations to a given word w is called the *plactic class* of w.

The following theorem is an algebraic reinterpretation [37] [22] of a construction due to Schensted [35].

Theorem 1 *In the plactic class $\mathcal{P}\ell(w)$ of a word w, there exists a tableau and only one tableau, denoted $P(w)$.*

A *right string* is the ordered string of the two-rows words in the plactic class of a 2-rows tableau (with the tableau as its origin, and the contretableau as its end, the words being ordered according to the length of their top row; edges are denoted f) :

$$\begin{smallmatrix} 3&6&&&\\ 1&2&4&5&7 \end{smallmatrix} \xrightarrow{f} \begin{smallmatrix} 1&3&6&&&\\ &2&4&5&7 \end{smallmatrix} \xrightarrow{f} \begin{smallmatrix} 1&3&4&6&\\ &&2&5&7 \end{smallmatrix} \xrightarrow{f} \begin{smallmatrix} 1&3&4&6&7\\ &&&2&5 \end{smallmatrix} \tag{6}$$

Left strings are described in terms of words on an alphabet of two letters, and it is convenient to take the two parentheses (,) as letters. Given a word w in such an alphabet, pair successive parentheses iteratively. Call the subword of paired parentheses its *paired subword* (caution: one needs to record positions). The original word w is the shuffle of it and of a subword of the type $)\cdots)(\cdots($ (i.e. closing parentheses come first).

A *left string* is a maximal sequence of words of the same degree which coincide on their paired subwords. Here is a left string, indicating below each word the positions of letters, and underlining the paired subword :

$$\begin{array}{c})\,(\,)\,)\,(\,)\,) \\ \scriptstyle 7\,\underline{6}\,\underline{5}\,4\,\underline{3}\,\underline{2}\,1 \end{array} \xrightarrow{f} \begin{array}{c})\,(\,)\,)\,(\,)\,(\\ \scriptstyle 7\,\underline{6}\,\underline{5}\,4\,\underline{3}\,\underline{2}\,1 \end{array} \xrightarrow{f} \begin{array}{c})\,(\,)\,(\,(\,)\,(\\ \scriptstyle 7\,\underline{6}\,\underline{5}\,4\,\underline{3}\,\underline{2}\,1 \end{array} \xrightarrow{f} \begin{array}{c} (\,(\,)\,(\,(\,)\,(\\ \scriptstyle 7\,\underline{6}\,\underline{5}\,4\,\underline{3}\,\underline{2}\,1 \end{array} . \qquad (7)$$

One exchanges the two notions of strings by just reading the positions of the ('s as the entries of the top rows of the elements of (6), and the)'s as the entries of the bottom rows, as can be seen on the preceding two strings. We shall later give a better interpretation in terms of Schensted bijection.

Given two totally ordered alphabets $X = \{x_1, x_2, \ldots\}$, $Y = \{y_1, y_2, \ldots\}$, a monomial w in the commutative variables $\binom{y}{x}$, $x \in X$, $y \in Y$, is called a *biword*.

Ordering lexicographically w, with priority to Y, one writes

$$w = \binom{y_1}{a_1} \cdots \binom{y_1}{b_1} \binom{y_2}{a_2} \cdots \binom{y_2}{b_2} \binom{y_3}{a_3} \cdots \binom{y_3}{b_3} \cdots \qquad (8)$$

where $a_1 \cdots b_1$, $a_2 \cdots b_2$, $\ldots$ are rows in the monoid X^*, and a_i, b_j belong to X.

One can erase the y_j's and interpret w as a sequence of rows :

$$w \mapsto w^x := [a_1 \cdots b_1,\ a_2 \cdots b_2,\ a_3 \cdots b_3, \ldots] \qquad (9)$$

We shall write $a_1 \cdots b_1 = row_1(w)$, $a_2 \cdots b_2 = row_2(w), \ldots$

Ordering lexicographically w with priority to X, one obtains similarly a sequence w^y of rows in Y^*.

Schensted's bijection, as reinterpreted by Knuth [11], is described by the following theorem.

Theorem 2 *The map $w \mapsto \big(P(w) := P(w^x)\,,\, Q(w) := P(w^y)\big)$ is a bijection between biwords and pairs of tableaux of the same shape.*

The algebraic interpretation of Theorem 2 is the following combinatorial version of Cauchy formula [1] :

$$\prod_{x \in X} \prod_{y \in Y} \left(1 - \binom{y}{x}\right)^{-1} = \sum\, (t, t')\ , \qquad (10)$$

sum over all pairs of tableaux $t \in X^*$, $t' \in Y^*$ of the same shape (we have written an equality instead of a bijection; the left hand side is a commutative function, and the right hand side a set of pairs of words in non commutative letters).

Strings can now be defined at the level of biwords. Given an integer i, a *right i-string* is a sequence of biwords $w_1, \ldots, w_\ell$ such that

$$[row_i(w_1),\, row_{i+1}(w_1)], [row_i(w_2),\, row_{i+1}(w_2)], \ldots, [row_i(w_\ell),\, row_{i+1}(w_\ell)]$$

is a right string, and such that $row_j(w_1) = \cdots = row_j(w_\ell)$, $\forall j \neq i, i+1$.

Similarly, a *left i-string* is a sequence of biwords $w_1, \ldots, w_\ell$ such that $w_1, \ldots, w_\ell$ coincide under the specialization $x_i = x_{i+1}$, and such that $w_1^x, \ldots, w_\ell^x$ becomes a string when one specializes x_{i+1} to '(', x_i to ')', erasing the other letters.

We could have used the symmetry in X and Y to exchange left and right strings.

Crystal bases $\mathcal{B}(\lambda)$ have been introduced by Kashiwara [9]. In the type A case, $\mathcal{B}(\lambda)$ can be identified with the set $\mathcal{T}ab(\lambda)$ of tableaux of shape λ [10]. The space $\mathcal{B}(\lambda) \otimes \mathcal{B}(\lambda)^\vee$ admits a natural structure of double crystal graph. Since the set $\cup_\lambda \big(\mathcal{T}ab(\lambda) \otimes \mathcal{T}ab(\lambda)\big)$ is, according to Theorem 2, in bijection with a set of biwords, we can describe double crystal graphs in terms of biwords. A *double crystal graph* for $\mathfrak{S}_n$ is a graph, with biwords as vertices, left and right colored oriented edges labeled $f_1, \ldots, f_{n-1}$ called *crystal operators*, such that if it contains a biword w, then it contains the full left and right i-strings, $i = 1, \ldots, n-1$, passing through w.

Crystal operators can be thought as (left or right) linear operators on sums of biwords: if $w \xrightarrow{f_i} w'$, then one puts $f_i(w) = w'$, and one puts $f_i(w) = 0$ if w is the end of the i-string passing through it.

The symmetric group $\mathfrak{S}_n$ acts by permuting the vertices of a double crystal graph [22] . The left (resp. right) action of a simple transposition s_i restricts, on a left (resp. right) i-string, to the symmetry with respect to the middle of the string [21, 27, 9]. We shall note $w \mapsto s_i(w)$ the left action.

Instead of taking symmetry with respect to middle of strings, one can define operators π_i and $\widehat{\pi}_i$, $i = 1, \ldots, n-1$, on the free algebra in X as follows [25, 8]:

- if $v \in X^*$ is the first part (or is the middle) of the left i-string containing it, then $\pi_i(v)$ is the sum of all words between v and $s_i(v)$, and $\widehat{\pi}_i(v)$ is the same sum minus v.

- the action is extended to all words by requiring that $\pi_i\big(v + s_i(v)\big) = v + s_i(v)$ and $\widehat{\pi}_i\big(v + s_i(v)\big) = 0$ (either v or $s_i(v)$ is in the first part of the string, when not in the middle).

For example, using x_1, x_2 instead of parentheses, and writing in small characters the paired letters, one gets from string (7) :

$$\pi_1\big(x_{1\,x_2x_1}x_{1\,x_2x_1}x_1\big) = x_{1\,x_2x_1}x_{1\,x_2x_1}x_1 + x_{1\,x_2x_1}x_{1\,x_2x_1}x_2 + x_{1\,x_2x_1}x_{2\,x_2x_1}x_2 + x_{2\,x_2x_1}x_{2\,x_2x_1}x_2,$$

$$\pi_1\big(x_{1\,x_2x_1}x_{1\,x_2x_1}x_2\big) = x_{1\,x_2x_1}x_{1\,x_2x_1}x_2 + x_{1\,x_2x_1}x_{2\,x_2x_1}x_2$$

and, by symmetry for the two other words,

$$\pi_1\big(x_{1\,x_2x_1}x_{2\,x_2x_1}x_2\big) = 0\ ,\ \pi_1\big(x_{2\,x_2x_1}x_{2\,x_2x_1}x_2\big) = -x_{1\,x_2x_1}x_{1\,x_2x_1}x_2 - x_{1\,x_2x_1}x_{2\,x_2x_1}x_2\ .$$

The operators π_i do not satisfy the braid relations. They induce (isobaric) divided differences on polynomials (in commutative indeterminates x_i), that we shall denote π_i^x:

$$f \xrightarrow{\pi_i^x} \frac{x_i f - x_{i+1} f^{s_i}}{x_i - x_{i+1}} \quad \text{with } s_i = \text{ transposition of } x_i, x_{i+1}\ . \tag{11}$$

The operators π_i^x, due to Demazure [3], satisfy braid relations, and therefore, for any permutation σ, there exists a well defined divided difference π_σ^x (which can be obtained by taking a product of π_i^x corresponding to a reduced decomposition of σ).

To relate biwords to permutations, I used with M.P. Schützenberger the following two notions which are in fact equivalent.

The first one consists in singling out some element $Vice(w)$ in each plactic class $\mathcal{P}\ell(w)$ of a word $w \in X^*$, called vice-tableau.

$Vice(w)$ can be recursively defined as follows. It is the unique word $a_1 \cdots a_\ell$ in a plactic class $\mathcal{P}\ell(w)$ such that

- a_ℓ is the suffix (i.e. the rightmost letter) of the contretableau in $\mathcal{P}\ell(w)$
- The shape of $P(a_1 \cdots a_{\ell-1})$ is maximum (with respect to the natural order on partitions) among the words in $\mathcal{P}\ell(w)$ having suffix a_ℓ.
- $a_1 \cdots a_{\ell-1}$ is a vice-tableau.

Recall that the inverse operation of "inserting a letter" into a tableau t (*Inverse-Schensted algorithm*) consists in choosing a box at the periphery of t, and finding a pair (t', x) such that t' is a tableau of shape obtained by erasing this box from the shape of t', and $t'x \equiv t$. Thus, the recursive definition of a vice-tableau implies the following algorithm, consisting of a distinguished sequence of Inverse-Schensted operations : at each step, the box which is erased is the highest which factors out on the right the suffix of the contretableau.

For example

$$\begin{array}{ccccc} 1&3&3&4&5\\ &&1&2&4\\ &&&1&2 \end{array} \equiv \begin{array}{ccccc} 1&3&3&4&5\\ &&1&2&4\\ &&&\blacksquare&1 \end{array} \cdot 2 \equiv \begin{array}{ccccc} 1&3&3&4&5\\ &&\blacksquare&2&4\\ &&&1&1 \end{array} \cdot 2 \equiv \begin{array}{ccccc} \blacksquare&1&3&3&5\\ &&1&2&4\\ &&&1&2 \end{array} \cdot 4 \tag{12}$$

shows that

$$Vice(13345\ 124\ 12) = Vice(13345\ 24\ 11) \cdot 2$$

(iterating, one finds $Vice(13345\,124\,12 = [35\,12344\,112])$.

Factoring $Vice(w)$ as a product of maximal rows, on can consider it as a biword v, with $row_1(v)\,row_2(v)\cdots = Vice(w)$. One can show that the sequence of lengths $\ell(row_1(v))$, $\ell(row_2(v)), \ldots$ is a permutation of the shape λ of the tableau $P(w)$.

Sorting the rows of $Vice(w)$ by decreasing order of length, and recording their suffixes, one gets a permutation $\sigma(w)$:

$$[35,\ 12344,\ 112,\ \emptyset,\ \emptyset] \to [1234\mathbf{4},\ 11\mathbf{2},\ 3\mathbf{5},\ \emptyset,\ \emptyset] \to \sigma(w) = [\mathbf{4}, \mathbf{2}, \mathbf{5}, 1, 3]$$

(in case some rows are of equal length, or when there are missing suffixes, i.e. empty rows, one takes the permutation of minimum length).

Using the right action of the symmetric group on a contretableau t with k rows, one generates $k!$ words $v = t^{\nu} : \nu \in \mathfrak{S}_k$. One can show by induction on the degree the following extremality property of the vice-tableau.

Lemma 3 *Given a contretableau t, then there exists a permutation ν such that $Vice(t)$ is equal to some t^{ν}. One can take for ν the permutation of maximal length among those such that the sequences of suffixes of $row_1(t^{\nu})$, $row_2(t^{\nu}), \ldots$ is the same as for t.*

For example the action of $\mathfrak{S}_3$ on the contretableau [13345 124 12], writing s and s' for the two generators of $\mathfrak{S}_3$, produces 6 words (s' acts by jeu de taquin on the two top rows, s on the bottom two) :

$$\begin{array}{ccccc} 1&3&3&4&5\\ &&1&2&4\\ &&&1&2 \end{array} \xrightarrow{s} \begin{array}{cccccc} 1&3&3&4&5&\\ &&&2&4&\\ &&&1&1&2 \end{array} \xrightarrow{s'} \begin{array}{ccccc} 3&5&&&\\ 1&2&3&4&4\\ &&1&1&2 \end{array} \xrightarrow{s} \begin{array}{ccccc} 3&5&&&\\ 2&3&4&&\\ 1&1&1&2&4 \end{array} \xrightarrow{s'} \begin{array}{cccccc} 3&3&5&&&\\ &2&4&&&\\ &1&1&1&2&4 \end{array} \xrightarrow{s} \begin{array}{ccccc} 3&3&5&&\\ 1&1&2&4&4\\ &&&1&2 \end{array} \tag{13}$$

and the vice-tableau is the third element $35\,12344\,122$, having suffixes $5, 4, 2$ and being given by a permutation of length 2.

Instead of using the action on rows of a tableau, one can similarly act on columns, still with the jeu de taquin. Given a tableau t with columns of respective lengths $\ell_1, \ldots, \ell_k$, and a permutation $\sigma \in \mathfrak{S}_k$, there exists, in the plactic class of t, a unique word $t^{col,\sigma}$ which is a product of columns of lengths $\ell_{\sigma_1}, \ldots, \ell_{\sigma_k}$. The tableau of the same shape as t such that its columns appears as a right column of some $t^{col,\sigma}$ is called the *key* of t [25, 5] . By construction, the set of columns of $\mathcal{C}\ell(t)$ (columns interpreted as sets) is a flag of sets (with respect to inclusion). Writing the letters in the order they appear in the successive columns (ordering them increasingly when they appear several at a time, and adding a complete column $[n, \ldots, 1]$), one obtains a permutation which is equal to $\sigma(t)$, i.e. to the permutation determined by the vice-tableau. Continuing with the same example, reading the objects columnwise,

$$\begin{array}{ccccc} 1 & 3 & 3 & 4 & \mathbf{5} \\ & & 1 & 2 & \mathbf{4} \\ & & & 1 & \mathbf{2} \end{array} \equiv \begin{array}{ccccc} 1 & 3 & 3 & 5 & \\ & & 2 & 4 & \mathbf{4} \\ & & 1 & 1 & \mathbf{2} \end{array} \equiv \begin{array}{ccccc} 1 & 3 & 5 & & \\ & 2 & 4 & 4 & \\ & 1 & 1 & 2 & \mathbf{4} \end{array} \;\Rightarrow\; \mathcal{C}\ell(t) = \begin{array}{ccccc} \mathbf{5} & \mathbf{5} & & & \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & & \\ \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{4} & \mathbf{4} \end{array} . \tag{14}$$

The right columns are $[\mathbf{4}]$, $[\mathbf{42}]$, $[\mathbf{542}]$ and therefore $\sigma(t) = [4\,2\,5\,1\,3]$.

3 Crystals

We restrict now our study to some distinguished subsets of tableaux and crystal graphs corresponding to Demazure modules. Given a dominant weight $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0]$, and a permutation $\sigma \in \mathfrak{S}_n$, there exists a *crystal* $\mathcal{B}(\lambda, \sigma)$, which is a collection of tableaux which can be described in many ways. It is in fact more convenient to index bases by weights, i.e. to write $\mathcal{B}(\lambda^\sigma)$ rather than $\mathcal{B}(\lambda, \sigma)$, to take into account the stabilizer of λ in $\mathfrak{S}_n$.

Define the *top component* of a crystal to be :

$$\widehat{\mathcal{B}}(\lambda^\sigma) = \mathcal{B}(\lambda^\sigma) \setminus \cup_{\nu<\sigma : \lambda^\nu \neq \lambda^\sigma} \mathcal{B}(\lambda^\nu) \ . \tag{15}$$

Moreover, instead of using tableaux only, one can choose a tableau Q of shape λ, and call *crystal* $\mathcal{B}(\lambda^\sigma, Q)$ the collection of biwords

$$w : Q(w) = Q,\ P(w) \in \mathcal{B}(\lambda^\sigma) \ .$$

Figure 1 gives $\mathcal{B}([2431], Q)$ for $Q = \begin{array}{cccc} 4 & & & \\ 3 & 4 & & \\ 2 & 3 & 3 & \\ 1 & 1 & 1 & 3 \end{array}$, the arrows being left crystal operations (increasing some i into $i+1$, $i = 1, 2, 3$); we have written w^x instead of w, separating the factors by a dot and replacing x_i by i.

There are several equivalent descriptions of a crystal, or of a basis of a Demazure module, using different combinatorial objects. The following theorem summarizes, in terms of tableaux, these different descriptions for what concerns the type A_n.

Statement 1) is given by Lakshmibai and Seshadri [13], Th.9.6, Lakshmibai [12], Th.4.1, by me and M.P. Schützenberger [25], Th.4.3.

Formula 16 is given by Littelmann [27], Th.5.2, Kashiwara [8], Prop.3.2.3, [12], Th.4.1. Using simultaneously the operators π_i and $\widehat{\pi}_i$ to get the two formulas (16) and (17) is done in [25], Th.4.3.

The first part of statement 3) is Proposition 5.6 of [25]; using Lemma 3, one can refine it to get the second part describing $\widehat{W}(\mu)$.

Theorem 4 *Let n be an integer, $\lambda \in \mathbb{N}^n$ be a dominant weight, $\sigma \in \mathfrak{S}_n$ be minimum in its class modulo the stabilizer of λ, and let $\mu = \lambda^\sigma$.*

Then the crystal $\mathcal{B}(\mu)$ (resp. $\widehat{\mathcal{B}}(\mu)$) can be characterized by any of the following properties :

1) It is the set of all tableaux t of shape λ, with $\sigma(t) \le \sigma$ (resp. $\sigma(t) = \sigma$), the order being the Ehresmann-Bruhat order on permutations.

2) Given any reduced decomposition $s_i \cdots s_j = \sigma$, then

$$\begin{aligned} \mathcal{B}(\mu) &= \pi_i \cdots \pi_j \big(\cdots x_2^{\lambda_2} x_1^{\lambda_1} \big) \ , \qquad (16) \\ \widehat{\mathcal{B}}(\mu) &= \widehat{\pi}_i \cdots \widehat{\pi}_j \big(\cdots x_2^{\lambda_2} x_1^{\lambda_1} \big) \ . \qquad (17) \end{aligned}$$

3) Let $W(\mu)$ (resp. $\widehat{W}(\mu)$) be the set of all biwords (resp. biwords such that $row_1(w)\, row_2(w) \cdots$ is a vice-tableau) such that $\ell(row_1(w)) = \mu_1$, $\ell(row_2(w)) = \mu_2$, $\ldots$ and such that $\forall i$, $\forall x \in row_i(w)$, then $x \le x_{n+i-1}$.

Let moreover $Q = P(\cdots y_2^{\mu_2}\, y_1^{\mu_1})$. Then

$$W(\mu) = \mathcal{B}(\mu, Q) \quad \textit{and} \quad \widehat{W}(\mu) = \widehat{\mathcal{B}}(\mu, Q) \qquad (18)$$

The elements of a crystal are connected by left crystal operations. If a biword w belongs to a crystal $\mathcal{B}(\mu, Q)$, and i is such that w is not the origin of the left i-string passing through it, then the full i-string is contained in $\mathcal{B}(\mu, Q)$. Thus, for any i, $\mathcal{B}(\mu, Q)$ is composed of full i-strings and of origins of i-strings [8].

Demazure characters $K(\mu, X)$ and $\widehat{K}(\mu, X)$ are the images of respectively $\mathcal{B}(\mu, Q)$ and $\widehat{\mathcal{B}}(\mu, Q)$ (considered as sums) in the ring of polynomials in commutative variables $x_1, \ldots, x_n$. They depend only on μ and not on Q.

Crystals admit a dual description, using right crystal operations. We shall need only one characterization.

Given λ, σ, $\mu = \lambda^\sigma$, and any tableau P of shape λ, call *cocrystal* $\mathcal{CB}(\mu, P)$ the set of biwords w: $P(w) = P$, $Q(w) \in \mathcal{B}(\mu)$.

Theorem 5 *Let n be an integer, $\lambda \in \mathbb{N}^n$ be a dominant weight, $\sigma \in \mathfrak{S}_n$, $\mu = \lambda^\sigma$. Let P be a tableau in $\widehat{\mathcal{B}}(\mu)$. Then $\mathcal{CB}(\mu, P)$ is the set of biwords w*

$$P(w) = P \quad \& \quad \forall i, \forall x \in row_i(w) \ \textit{then}\ x \le x_{n+1-i} \ . \qquad (19)$$

Figure 1 and Figure 2 have the same underlying colored graph. In Figure 2, we are using w^x instead of a biword w, writing consecutive rows on top of each other (and using i for x_i). Crystal operations are the jeu de taquin on two consecutive rows. Vertices are all the 4-rows words in the plactic class of $\begin{array}{cccc} 4 & & & \\ 3 & 4 & & \\ 2 & 3 & 3 & \\ 1 & 1 & 1 & 3 \end{array}$ which satisfy the flag conditions

$$row_1(w) \in \{1,2,3,4\}^*,\ row_2(w) \in \{1,2,3\}^*,\ row_3(w) \in \{1,2\}^*,\ row_4(w) \in \{1\}^* .$$

One cannot add to Figure 2 any word by the jeu de taquin, without breaking the flag conditions. For example, the second lowering operator has the effect, on the two minimum elements to push a 3 on the third row :

$$\begin{array}{ccccc} 3 & 4 & 4 & & \\ 1 & 3 & 3 & 3 & \\ & & & 2 & \\ & & & 1 & 1 \end{array} \xrightarrow{f_2} \begin{array}{ccccc} 3 & 4 & 4 & & \\ 1 & 3 & 3 & & \\ & & 2 & 3 & \\ & & & 1 & 1 \end{array} \quad , \quad \begin{array}{cccccc} 4 & 4 & & & & \\ 3 & 3 & 3 & 3 & & \\ & & & 2 & & \\ & & & 1 & 1 & 1 \end{array} \xrightarrow{f_2} \begin{array}{cccccc} 4 & 4 & & & & \\ 3 & 3 & 3 & & & \\ & & 2 & 3 & & \\ & & & 1 & 1 & 1 \end{array} .$$

4 Non symmetric Cauchy kernels

The function $\prod_{i+j \le n+1} (1 - x_i y_j)^{-1}$, rewritten $F_\rho = \prod_{i+j\le n+1} \left(1 - \binom{y_j}{x_i}\right)^{-1}$, is the sum of all biwords in the commutative variables $\binom{y_j}{x_i}$, such that $i + j \le n + 1$.

However, from Theorems 4, 5, one knows that for a biword w, if $P(w) = P$, P of shape λ, $\sigma(P) = \sigma$, $\mu = \lambda^\sigma$, then $Q(w) \in \mathcal{CB}(\mu, P)$. Therefore, the conditions $i + j \le n + 1$ are equivalent to the following expansion of F_ρ, writing $\mu^\omega = [\mu_n, \ldots, \mu_1]$ when $\mu = [\mu_1, \ldots, \mu_n]$.

Theorem 6 *Let n be a positive integer. Then, under Schensted's bijection, F_ρ can be written*

$$\prod_{i+j\le n+1} \left(1 - \begin{pmatrix} y_j \\ x_i \end{pmatrix}\right)^{-1} = \sum (t, t') ,$$

sum over all weights $\mu \in \mathbb{N}^n$, all $t \in \widehat{\mathcal{B}}(\mu, X)$, $t' \in \mathcal{B}(\mu^\omega, Y)$. The commutative image of the preceding identity is

$$F_\rho = \sum_{\mu \in \mathbb{N}^n} \widehat{K}(\mu, X)\, K(\mu^\omega, Y) .$$

Let us see how expansions for staircases imply expansions for all partitions (the characterization of pairs of tableaux will be less explicit than in Theorem 6, because it probably depends on not yet known properties of the monoid generated by the π_i's and $\widehat{\pi}_j$'s. However, the image of $K(\mu, Y)$ under some π_i^y is some $K(\mu', Y)$, and the image of $\widehat{K}(\mu, X)$ under $\widehat{\pi}_i^x = \pi_i^x - 1$ is $\pm\widehat{K}(\mu', X)$; passing from Theorem 6 to Theorem 7 mostly reduces to reordering indices).

Given a partition λ, let $\rho(\lambda) = [k, k-1, \ldots, 1]$ be the biggest staircase, the diagram of which is contained in the one of λ.

Take any box in the staircase $[k+1, \ldots, 1]$ which does not belong to λ. The diagonal passing through this box cuts the diagram of $\lambda/\rho(\lambda)$ into two pieces that we call respectively the *North-West part* and the *South-East part* of λ/ρ. Fill now each box of row r, $r = 2, \ldots$ of the North-West part with the number $r-1$. Similarly, fill each box of column c of the South-East part with the number $c-1$.

Reading the North-West part by columns, from right to left, and interpreting r as the simple transposition s_r, gives a reduced decomposition of a certain permutation $\sigma(\lambda, NW)$; similarly, reading rows, from right to left, and from top to bottom, of the South-East part, gives a permutation $\sigma(\lambda, SE)$ (these two permutations depend on the choice of the box cutting the diagram λ/ρ, there can be several ways to get F_λ from F_ρ).

Theorem 7 *Let λ be a partition, $\rho(\lambda) = [k, \ldots, 1]$ the maximal staircase contained in the diagram of λ, $\sigma(\lambda, NW)$, $\sigma(\lambda, SE)$ be the two permutations obtained by cutting the diagram of λ/ρ as explained above. Then*

$$F_\lambda(X,Y) = \prod_{(i,j)\in\lambda} (1-x_i y_j)^{-1} = \sum\nolimits_{\mu\in\mathbb{N}^k} \pi^x_{\sigma(\lambda,NW)}\Big(\widehat{K}(\mu,X)\Big)\, \pi^y_{\sigma(\lambda,SE)}\Big(K(\mu^\omega,Y)\Big)\,,$$

sum over all weights $\mu \in \mathbb{N}^k$.

Proof. Divided differences allow to increase the number of poles in a rational function. Indeed, if F is symmetrical in x_k, x_{k+1}, then the divided difference π^x_k (acting on x_k, x_{k+1}) has the effect

$$(1-x_k y)^{-1}\, F \xrightarrow{\pi^x_k} (1-x_k y)^{-1}(1-x_{k+1}y)^{-1}\, F$$

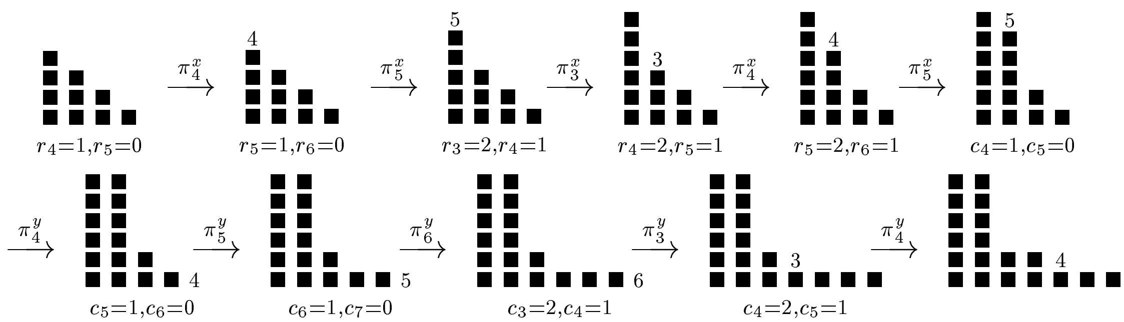
because $\pi^x_k(x_{k+1}) = 0$ and π^x_k commutes with multiplication by F. Now, a function F_λ is symmetrical in x_k, x_{k+1} if rows k and $k+1$ of the diagram of λ have equal length; it is symmetrical in y_j and y_{j+1} if columns j and $j+1$ have equal length. Therefore, when the difference of lengths of row k and $k+1$ is 1, the image of F_λ under π^x_k is F_{λ^+}, where λ^+ is obtained by adding a box to row $k+1$ of the diagram of λ.

It is easy to see that the chosen sequence of π^x_k or π^y_j has the effect of producing a sequence F_ρ, $F_{\nu^1}, \ldots,$ F_{ν^ℓ}, F_λ, with partitions differing by one box at a time. QED

For example, when $\lambda = [7,5,2,2,2,2]$, one has $\rho = [4,3,2,1]$,

$$\begin{array}{lllllll} 5 & 5 & & & & & \\ 4 & 4 & & & & & \\ \blacksquare & 3 & & & & & \\ \blacksquare & \blacksquare & & & & & \\ \blacksquare & \blacksquare & \blacksquare & 3 & 4 & & \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & 4 & 5 & 6 \end{array} \quad \Longrightarrow \quad \sigma(\lambda, NW) = s_5 s_4 s_3\, s_5 s_4 \;\&\; \sigma(\lambda, SE) = s_4 s_3\, s_6 s_5 s_4$$

Representing F_λ by the diagram of λ (filled with ■, except for the box just added), writing below it the lengths of rows $i, i+1$ (when using π_i^x), or columns $j, j+1$ (when using π_j^y), Theorem 7 provides the following sequence to pass from F_{4321} to F_{752222} :



5 Nilplactic kernel

The two sets of indeterminates X and Y played a symmetrical role in the preceding sections. We present now an example where it is not the case and where tableaux are given another interpretation.

The x_i's are commutative variables, and commute with the y_j's. The y_j's obey the nilplactic relations [24], [4] :

$$y_i^2 = 0 \quad , \quad y_i y_{i+1} y_i = y_{i+1} y_i y_{i+1} \tag{20}$$

$$y_j y_i y_k = y_j y_k y_i \quad , \quad y_i y_k y_j = y_k y_i y_j \quad (i < j < k) \tag{21}$$

One says that a word in the y_i's is *reduced* if its image under $y_i \to s_i$, s_i simple transposition, is reduced. Two reduced words equivalent under the nilplactic relations project onto the same permutation. The *nilplactic algebra* is the algebra generated by the y_j's, modulo the ideal generated by non reduced words.

Schensted's construction can be adapted to the nilplactic case. The basic objects are now sequences of columns $w = [u_1, \ldots, u_\ell]$, and one has the following theorem ([24], [4]):

Theorem 8 *There exists a bijection between ℓ-uples of columns (=strictly decreasing words) $w = [u_1, \ldots, u_\ell]$, such that $u_1 \cdots u_\ell$ is reduced, and pairs of tableaux $t = P^{nil}(w)$, $t' = Q^{nil}(w) \in \{x_1, \ldots, x_\ell\}^*$ of conjugate shape, t being reduced.*

The tableau $P^{nil}(w)$ is the unique tableau in the nilplactic class of $u_1 \cdots u_\ell$. Multiplying $P^{nil}(u_1 \cdots u_{\ell-1})$ by the column u_ℓ increase the shape by a vertical strip (as for the multiplication of a Schur function by an elementary symmetric function, cf. [32]), which is recorded as the set of boxes occupied by the letter x_ℓ in the tableau $Q^{nil}(w)$.

For example, the product $\begin{smallmatrix}5\\1\end{smallmatrix} \cdot \begin{smallmatrix}6\\5\\2\\1\end{smallmatrix} \cdot \begin{smallmatrix}4\\3\\2\end{smallmatrix}$ produces the following sequence of tableaux and transposed shapes differing by a strip

$$
\begin{array}{lccccc}
\textit{tableau} & \begin{smallmatrix}5\\1\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}6&\\5&\\2&6\\1&2\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}6&&\\5&&\\4&6&\\2&3&\\1&2&3\end{smallmatrix} \\
\textit{transposed shape} & \begin{smallmatrix}1&1\end{smallmatrix} & & \begin{smallmatrix}2&2&&\\1&1&2&2\end{smallmatrix} & & \begin{smallmatrix}3&&&&\\2&2&3&&\\1&1&2&2&3\end{smallmatrix} = Q^{nil}
\end{array} .
$$

As before, instead of taking all possible reduced words, one wants to characterize the possible pairs of tableaux which correspond to the expansion of some kernel.

Instead of taking a general partition, we restrict to $\rho = [n, \ldots, 1]$ for simplicity, and define the *NilCauchy kernel* to be

$$
NilC(n) = \Big((1+x_1y_n)\cdots(1+x_1y_1)\Big) \\
\Big((1+x_2y_n)\cdots(1+x_2y_2)\Big)\cdots\cdots\Big((1+x_ny_n)\Big) . \quad (22)
$$

One notices that each block expands into a sum of columns :

$$
(1+xy_n)\cdots(1+xy_k) = \sum\nolimits_u x^{\ell(u)}u , \quad (23)
$$

sum over all possible columns u in the letters $y_n, \ldots, y_k$.

Thus, $NilC(n)$ can be studied using Theorem 8. All the tools presented in the first sections involving right actions can be adapted to the nilplactic case. For example, there are two symmetric groups acting on a tableau, permuting either rows, or columns. One can choose to record the columns appearing on the left, the columns appearing on the right, the rows appearing upwards or the rows appearing downwards. One has indeed four possible keys in the nilplactic case. To describe $NilC(n)$, one needs the *bottom key*, which is illustrated on the following example, which will be used in Figure 3 (read now objects as sequences of rows in the y_i (noted $y_i = i$). Transformations are now given by the nil-jeu de taquin exchanging a two-rows tableau with the two-rows contretableau in its nilplactic class.

$$
\begin{array}{l}5\\46\\235\\1234\end{array} \mapsto
\begin{array}{l}5\\46\\1235\\\;\;124\end{array} \mapsto
\begin{array}{r}5\;\;\;\;\;\;\\1246\\35\\124\end{array} \mapsto
\begin{array}{r}5\;\;\;\;\;\;\\1246\\135\\24\end{array} \mapsto
\begin{array}{r}1256\\4\\135\\24\end{array} \mapsto
\begin{array}{r}1256\\145\\3\\24\end{array} \mapsto
\begin{array}{r}1256\\145\\34\\2\end{array}
$$

In that case, the bottom key is the set of bottom rows (with the same multiplicities as the lengths of rows of the tableau)

$$\{\,[2],\ [2,4],\ [1,2,4],\ [1,2,3,4]\,\}\ , \tag{24}$$

with *weight* $\mu = [2,4,1,3]$ (number of occurrences of $1,2,3,4$ respectively).

We shall just state, but not prove, the following property which shows another use of tableaux and the jeu de taquin.

Property 9 *The NilCauchy kernel expands, in the nilplactic algebra, as*

$$NilC(n) \overset{Nil}{\equiv} \sum_t K(\mu(t),X)\ t\ , \tag{25}$$

sum over all tableaux (read by columns, and which are reduced words) that appear as subwords of

$$(y_n\cdots y_1)\,(y_n\cdots y_2)\cdots(y_n)\ ,$$

$\mu(t)$ being the weight of the bottom key of t.

For example,

$$NilC(2) \overset{Nil}{\equiv} x_1^2x_2 \begin{array}{|c|c|}\cline{1-1} y_2 \\ \hline y_1 & y_2 \\ \hline\end{array} + x_1^2 \begin{array}{|c|}\hline y_2 \\ \hline y_1 \\ \hline\end{array} + x_1x_2 \begin{array}{|c|c|}\hline y_1 & y_2 \\ \hline\end{array} + x_1 \begin{array}{|c|}\hline y_1 \\ \hline\end{array} + (x_1{+}x_2)\begin{array}{|c|}\hline y_2 \\ \hline\end{array} + 1$$

(the term y_2y_2 has disappeared, because it is not a reduced word).

Figure 3 shows all the terms in $NilC(6)$ belonging to the nilplactic class of $t = \begin{smallmatrix} 5 & & & \\ 4 & 6 & & \\ 2 & 3 & 5 & \\ 1 & 2 & 3 & 4 \end{smallmatrix}$ (writing i instead of y_i). They constitute a crystal, the crystal operations being the nil-jeu de taquin on two consecutive columns. Each vertex must be read as a product of four columns in the y_i, of lengths $\ell_1,\ldots,\ \ell_4$, appearing as a subword of

$$\Big((1{+}x_1y_6)\cdots(1{+}x_1y_1)\Big)\Big((1{+}x_2y_6)\cdots(1{+}x_2y_2)\Big)\Big((1{+}x_3y_6)\cdots(1{+}x_3y_3)\Big)$$
$$\Big((1{+}x_4y_6)(1{+}x_4y_5)(1{+}x_4y_4)\Big)\Big(\ \Big)\Big(\ \Big)\Big(\ \Big)\ ,$$

its factor being $x_1^{\ell_1}x_2^{\ell_2}x_3^{\ell_3}x_4^{\ell_4}$.

For example, $\begin{smallmatrix} 2 & 5 & & \\ 1 & 4 & 6 & \\ & 3 & 5 & \\ & 2 & 3 & 4 \end{smallmatrix}$ stands for

$$(x_1y_2x_1y_1)(x_2y_5x_2y_4x_2y_3x_2y_2)(x_3y_6x_3y_5x_3y_3)(x_4y_4) = x_1^2x_2^4x_3^3x_4^1\,y_2y_1\,y_5y_4y_3y_2\,y_6y_5y_3\,y_4.$$

Because the weight of the bottom key is $\mu = [2,4,1,3]$, then, according to Property 9, the sum of all factors $x_1^{\ell_1}x_2^{\ell_2}x_3^{\ell_3}x_4^{\ell_4}$ is the Demazure character $K([2413]) = x_1^4x_2^3x_3^2x_4^1 + \cdots + x_1^2x_2^4x_3^1x_4^3$.

Thus the study of $NilC(n)$ essentially reduces to recognize that the intersection of a nilplactic class with the set of reduced words appearing in the expansion of $NilC(n)$ is isomorphic to some crystal.

Property 9 implies the expansion of Schubert polynomials in terms of Demazure characters given in [26].

Projecting expression (25) on the NilCoxeter algebra with generators y_i satisfying :

$$y_i^2 \overset{Cox}{=} 0 \quad , \quad y_iy_{i+1}y_i \overset{Cox}{=} y_{i+1}y_iy_{i+1} \quad , \quad y_iy_k \overset{Cox}{=} y_ky_i \, , \, |k-i| \neq 1 \tag{26}$$

one gets Fomin-Kirillov expansion [7] of the nilCoxeter kernel involving Schubert polynomials (see also [20], [6])

$$NilC(n) \overset{Cox}{=} \sum_{\sigma \in \mathfrak{S}(n+1)} Schub_\sigma(X)\, \sigma \ . \tag{27}$$

For example

$$NilC(2) \overset{Cox}{=} x_1^2x_2\, s_2s_1s_2 + x_1^2\, s_2s_1 + x_1x_2\, s_1s_2 + x_1s_1 + (x_1+x_2)\, s_2 + 1 \ .$$

6 Double characters

Having a combinatorial interpretation of characters in terms of tableaux and crystal graphs, it is easy to modify their definition to handle functions in two sets of indeterminates. Let w be a word in **non commuting letters** $\binom{j}{i}$, and let w_x be the word obtained by sending $\binom{j}{i}$ to x_i, $\forall j$. We have seen that a left crystal operator f_i sends a word w_x to 0, or change in it a single precise occurrence of the letter x_i into a x_{i+1}.

Define accordingly the action of f_i on w to be

$$\left\{ \begin{array}{ccc} f_i(w_x) = 0 & \Rightarrow & f_i(w) = 0 \\ f_i(w'_x\, x_i\, w''_x) = w'_x\, x_{i+1}\, w''_x & \Rightarrow & f_i\big(w'\, \binom{j}{i}\, w''\big) = w'\, \binom{j+1}{i+1}\, w'' \end{array} \right. \tag{28}$$

One extends similarly the action of π_i from words in the x_j's to words in the $\binom{j}{i}$'s.

Let $\lambda \in \mathbb{N}$ be a dominant weight, $\sigma \in \mathfrak{S}_n$, and $\mu = \lambda^\sigma$. Let now

$$w^\lambda := \begin{array}{lllll} \binom{1}{n} & \cdots & \binom{\lambda_n}{n} & & \\ \binom{1}{n-1} & \cdots & \cdots & \binom{\lambda_{n-1}}{n-1} & \\ \vdots & & & \vdots & \\ \binom{1}{1} & \cdots & \cdots & \cdots & \binom{\lambda_1}{1} \end{array} \qquad \text{(read by rows)}$$

Let $s_i \cdots s_j = \sigma$ be a reduced decomposition of σ (σ can be supposed to be minimum in its class modulo the stabilizer of λ).

Then one defines the crystal $\mathcal{BB}(\mu)$ to be

$$\mathcal{BB}(\mu) = \pi_i \cdots \pi_j \left(w^\lambda\right) \tag{29}$$

This sum of words does not depend on the choice of the reduced decomposition of σ, as shows the characterization given in the next theorem.

Given a tableau $t \in X^*$, let $\varphi(t)$ be the tableau in the $\binom{j}{i}$'s obtained by changing an entry x_i in row r, column c, into the letter $\binom{i+c-r}{i}$, and extends φ by linearity to sums of tableaux. Considering $\mathcal{B}(\mu, X)$ to be a sum of tableaux rather than a set, one has :

Theorem 10 *Let $\lambda \in \mathbb{N}$ be a dominant weight, μ be a permutation of λ. then*

$$\varphi\Big(\mathcal{B}(\mu, X)\Big) = \mathcal{BB}(\mu) \tag{30}$$

The proof consists just in noticing that the action of any f_i preserves the differences between subscripts and superscripts: $\binom{j}{i} \mapsto \binom{j+1}{i+1}$. Thus, recording the initial superscripts in w^λ, one can as well erase them and operate on tableaux in the x_i's, reintroducing the superscripts at the last step.

Now the entries of the initial tableau w^λ are of the type $\binom{i+c-r}{i}$, for a letter in column c, row r. QED

For example,

$$\begin{aligned}
\mathcal{B}([1,0,2], X) \; &= \; \begin{array}{ll} \boxed{x_2} & \\ \boxed{x_1} & \boxed{x_1} \end{array} + \begin{array}{ll} \boxed{x_2} & \\ \boxed{x_1} & \boxed{x_2} \end{array} + \begin{array}{ll} \boxed{x_3} & \\ \boxed{x_1} & \boxed{x_1} \end{array} + \begin{array}{ll} \boxed{x_2} & \\ \boxed{x_1} & \boxed{x_3} \end{array} + \begin{array}{ll} \boxed{x_3} & \\ \boxed{x_1} & \boxed{x_3} \end{array} \quad \text{implies} \\
\mathcal{BB}([1,0,2]) \; &= \; \begin{array}{ll} \boxed{\binom{1}{2}} & \\ \boxed{\binom{1}{1}} & \boxed{\binom{2}{1}} \end{array} + \begin{array}{ll} \boxed{\binom{1}{2}} & \\ \boxed{\binom{1}{1}} & \boxed{\binom{3}{2}} \end{array} + \begin{array}{ll} \boxed{\binom{2}{3}} & \\ \boxed{\binom{1}{1}} & \boxed{\binom{2}{1}} \end{array} + \begin{array}{ll} \boxed{\binom{1}{2}} & \\ \boxed{\binom{1}{1}} & \boxed{\binom{4}{3}} \end{array} + \begin{array}{ll} \boxed{\binom{2}{3}} & \\ \boxed{\binom{1}{1}} & \boxed{\binom{4}{3}} \end{array}
\end{aligned}$$

The characters $K(\mu, X)$ are obtained by taking the sum of the commutative images of the tableaux in $\mathcal{B}(\mu, X)$. Similarly, one defines a *double character* $\mathbb{K}(\mu, X, Y)$ in two sets of commutative indeterminates by taking the image of a word under the morphism

$$\binom{j}{i} \mapsto (x_i - y_j) \ .$$

By construction, one recovers $K(\mu, X)$ by specializing the y_j's to 0 :

$$K(\mu, X) = \mathbb{K}(\mu, X, 0) \ .$$

But now, the polynomials $\mathbb{K}(\mu, X, Y)$ have some specific properties (e.g. vanishing properties) that make them easier to characterize than the polynomials $K(\mu, X)$.

For example, if μ is the *code* of a *vexillary* permutation σ (cf. [30]), then one can show (but we shall not) that $\mathbb{K}(\mu, X, Y)$ coincides with the Schubert polynomial indexed by σ.

A special case, when μ is of the type

$$\mu = [\mu_1 \le \mu_2 \le \cdots \le \mu_r, 0, 0, \ldots]$$

is known since long. Indeed, specializing $y_1 = 0$, $y_2 = 1$, $y_3 = 2$, ..., one recognizes that $\mathbb{K}(\mu, X, \{0, 1, 2, \ldots\})$ is a *factorial Schur function*, which was defined by Biedenharn and Louck [2] through the enumeration of the same tableaux as in Theorem 10. The more general case were the y_j's are not specialized has been obtained by several authors ([31], [33], [34]), but was already used, following Giambelli, in [15] to describe Schubert cycles for relative Grassmann varieties.

Some historical comments. I shall end this text evoking why Schur's work was personally important for me, even before I was aware of it, restricting to my work in 1974.

In [14] I considered symmetric functions as operators on isobaric determinants, Schur functions (not named) being operators transforming Vandermonde determinants by increasing exponents. Shortly after [15], multi-Schur functions appeared as Schubert cycles. In [16] Schur functions were "Schur-Littlewood" functions, sums of all the tableaux of a given shape [29]. In [17], Schur functions became elements of the plactic algebra, this interpretation being needed to explain their products. Finally Schur functors are used in my thesis [18]. I could add that much of my further work with M.P. Schützenberger can be interpreted as reflections about what remains of Schur functions when allowing non-symmetry [23] and non-commutativity [26].

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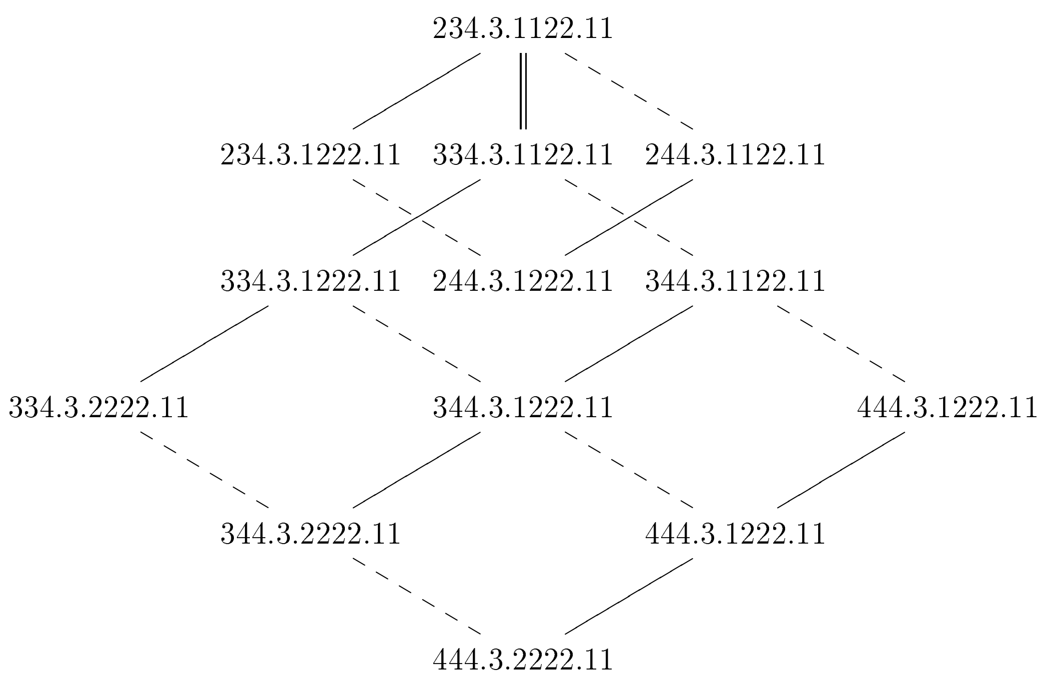

Figure 1

Crystal $\mathcal{B}([2413], Q)$ with $Q = \begin{array}{llll} 4 & & & \\ 3 & 4 & & \\ 2 & 3 & 3 & \\ 1 & 1 & 1 & 3 \end{array}$

Vertices must be read as products of four rows, crystal operations are the jeu de taquin on consecutive rows

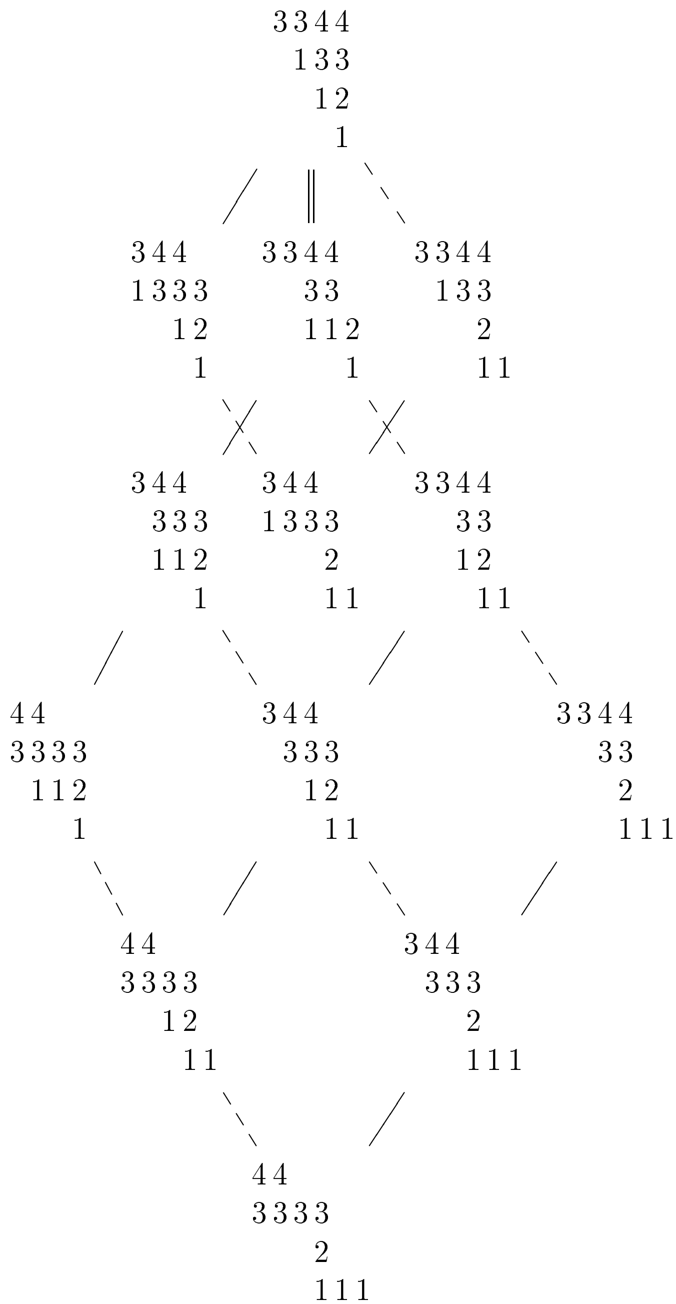

Figure 2: Cocrystal $\mathcal{CB}([2413])$

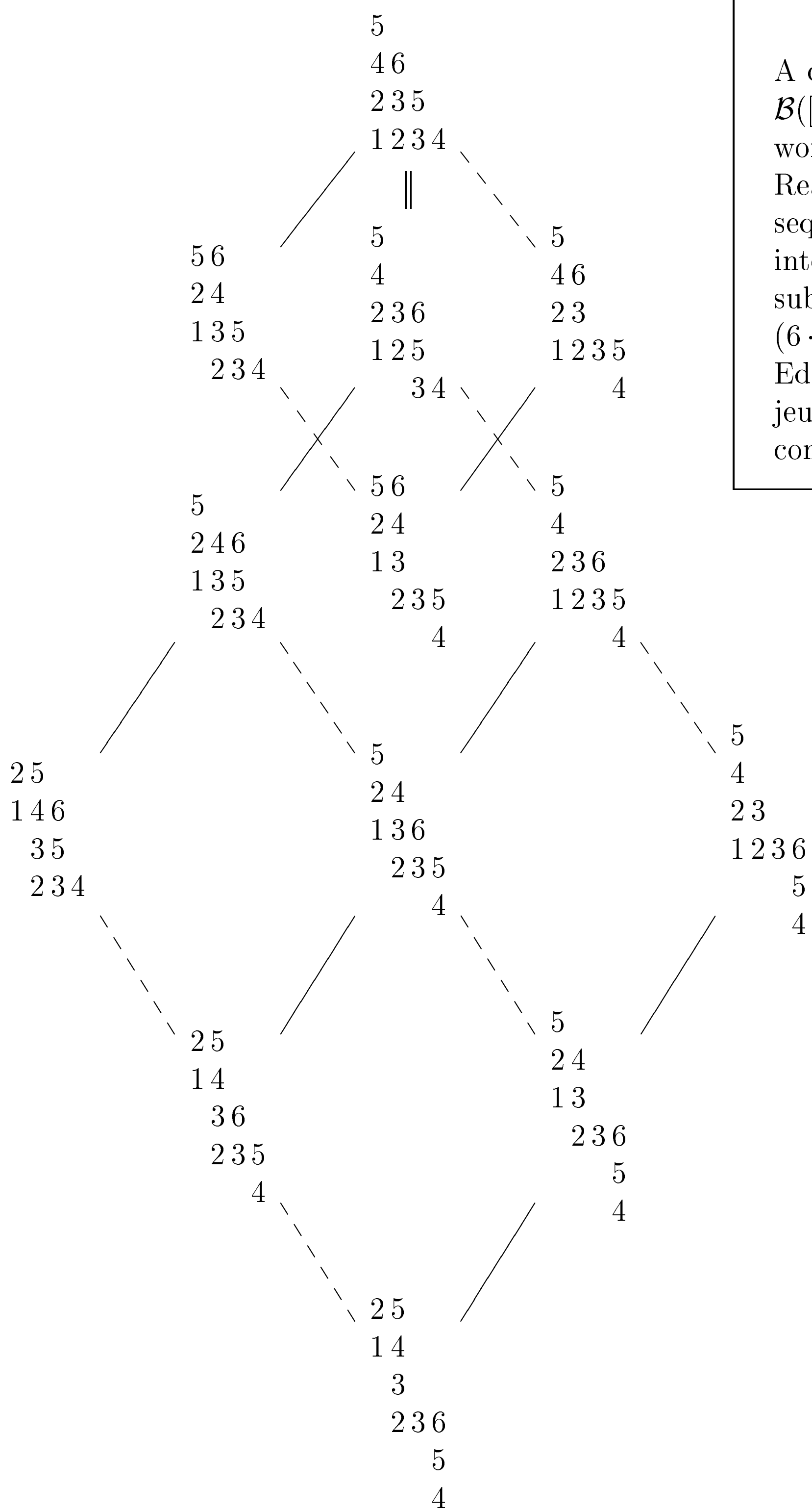

Figure 3

A copy of the crystal $\mathcal{B}([2,4,1,3])$, with reduced words as vertices. Read each vertex as a sequence of four columns, interpreting it as a reduced subword of $(6\cdots1)(6\cdots2)(6\cdots3)(6\cdots4)$. Edges are given by the (nil) jeu de taquin on pairs of consecutive columns