EUCLID and ORTHOGONAL POLYNOMIALS

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Abstract

In the classical theory of elimination, or of Sturm sequences, one encounters relations which are better understood when translated in terms of orthogonal polynomials. Conversely, one may safely associate to the Tchebychef-Christoffel-Darboux kernel the names of Bézout, Sylvester, Hermite, Brioschi, and recognize that these mathematicians treated the case of a finite discrete measure, with the help of the Euclidean algorithm and of the Lagrange interpolation.

Given two generic polynomials $f(x)$, $\phi(x)$ of consecutive degrees, that I shall write in a factorized form: $f(x) = \prod_{a\in\mathbb{A}}(x-a) := R(x,\mathbb{A})$, $\phi(x) = \prod_{b\in\mathbb{B}}(x-b) := R(x,\mathbb{B})$, Bézout considered quadratic expressions in the coefficients of f, ϕ, that he called *Équations dérivées*. Following Cayley [**?**], one may reinterpret his system of equations as the quadratic form

$$\mathfrak{Bez}(\mathbb{A},\mathbb{B}) := \frac{f(x)\phi(y) - f(y)\phi(x)}{x-y} \ .$$

This form can be expanded in a unique manner :

$$\mathfrak{Bez}(\mathbb{A},\mathbb{B}) = \gamma_0\, r_0(x)r_0(y) + \gamma_1\, r_1(x)r_1(y) + \cdots + \gamma_n\, r_n(x)r_n(y) \ , \qquad (1)$$

with $f(x)$ of degree $n{+}1$, $\phi(x)$ of degree n, requiring that the polynomials $r_i(x)$, $i = 0,\ldots,n$ be monic of degree $n{-}i$, $\gamma_0,\ldots,\gamma_n$ being normalizing constants.

Let us first recognize that the $r_i(x)$ are, up to scalars, the successive remainders in the Euclidean division of f by ϕ.

Indeed, the equations defining the remainders $\mathcal{R}_i(x)$, putting $\mathcal{R}_{-1}(x) = f(x)$, $\mathcal{R}_0(x) = \phi(x)$, are

$$\mathcal{R}_{i-1}(x) = q_i(x)\mathcal{R}_i(x) - \mathcal{R}_{i+1}(x) \ , \ i = 0,\ldots,n, \ d^0(\mathcal{R}_{i+1}) \le n{-}i{-}1 \ , \qquad (2)$$

or, equivalently,

$$\mathcal{R}_{i-1}(x) \,/\, \mathcal{R}_i(x) = q_i(x) - \mathcal{R}_{i+1}(x) \,/\, \mathcal{R}_i(x) \ . \tag{3}$$

Eliminating the remainders, one can write the Euclidean division as a continued fraction:

$$\frac{\phi(x)}{f(x)} \;=\; \cfrac{1}{q_0 - \cfrac{1}{q_1 - \cfrac{1}{\ddots \cfrac{1}{q_n}}}} \tag{4}$$

Keeping on the contrary the remainders, and the top coefficients α_i of the quotients $q_i = \alpha_i x + \beta_i$, one sees that there exists normalization constants $(\star)$ such that

$$\frac{\mathcal{R}_{-1}(x)\mathcal{R}_0(y) - \mathcal{R}_{-1}(y)\mathcal{R}_0(x)}{(x-y)\mathcal{R}_0(x)\mathcal{R}_0(y)} = \alpha_0 - \frac{1}{(\star)\mathcal{R}_0(x)\mathcal{R}_0(y)} \ ,$$

and therefore, by iteration,

$$\begin{aligned} &\frac{f(x)\phi(y) - f(y)\phi(x)}{x-y} \\ &\qquad = \alpha_0\, \mathcal{R}_0(x)\mathcal{R}_0(y) + \alpha_1\, \mathcal{R}_1(x)\mathcal{R}_1(y) + \cdots + +\alpha_n\, \mathcal{R}_n(x)\mathcal{R}_n(y) \end{aligned} \tag{5}$$

is the required decomposition of the Bézoutian.

To recover not only the remainders, but also the *convergents* of the continued fraction, one uses the remark that the Bézoutian be identical with its remainder modulo $f(x), f(y)$ (because it is of degree at most n in x, resp. in y). However

$$\mathcal{R}_i(x) \equiv S_{i^i}(\mathbb{A}{-}\mathbb{B} - x)\, \phi(x) \quad \mod \ f(x) \ ,$$

the Schur functions[1] $S_{i^i}(\mathbb{A}{-}\mathbb{B} - x)$ being Euler's multiplicators[2] for the remainders.

[1]Recall that, given two alphabets $\mathbb{A}$, $\mathbb{B}$, the *complete functions* $S_i(\mathbb{A}{-}\mathbb{B})$ are defined by the generating function:

$$\sum_i z^i S_i(\mathbb{A}{-}\mathbb{B}) = \frac{\prod_{b\in\mathbb{B}} 1 - zb)}{\prod_{a\in\mathbb{A}} 1 - za} \ .$$

Schur functions are determinants in complete functions, but it is appropriate to generalize the usual definition and allow different alphabets in the successive columns. Given $\lambda \in \mathbb{N}^n$, given $2n$ sets of variables $\mathbb{A}_1, \ldots, \mathbb{A}_n$, $\mathbb{B}_1, \ldots, \mathbb{B}_n$, then the Schur function $S_\lambda(\mathbb{A}_1{-}\mathbb{B}_1, \ldots, \mathbb{A}_n{-}\mathbb{B}_n)$ is equal to $\left|S_{\lambda_j+j-i}(\mathbb{A}_j{-}\mathbb{B}_j)\right|_{1\le i,j\le n}$.
When alphabets are the same in consecutive columns, one does not repeat them. For example, one writes $S_\lambda(\mathbb{A}{-}\mathbb{B})$ when the argument is $\mathbb{A}{-}\mathbb{B}$ in all columns. Adding a letter x to an alphabet $\mathbb{A}$ is denoted $\mathbb{A} \to \mathbb{A}{+}x$, and $\mathbb{A} - \mathbb{B}{-}x$ stands for $\mathbb{A} - (\mathbb{B}{+}x)$. More informations may be found in [**?**].

[2]Euler writes a remainder in the division of $R(x,\mathbb{B})$ by $R(x,\mathbb{A})$ as a combination

Consequently, one can write

$$\begin{aligned}\mathfrak{Bez}(\mathbb{A},\mathbb{B}) = \phi(x)\phi(y)\Big((\star)S_0S_0 &+ (\star)S_1(\mathbb{A}-\mathbb{B}-x)S_1(\mathbb{A}-\mathbb{B}-y)\\ &+ (\star)S_{22}(\mathbb{A}-\mathbb{B}-x)S_{22}(\mathbb{A}-\mathbb{B}-y) + \cdots\\ &\qquad + (\star)S_{n^n}(\mathbb{A}-\mathbb{B}-x)S_{n^n}(\mathbb{A}-\mathbb{B}-y)\Big)\,, \qquad (6)\end{aligned}$$

with some normalizing constants denoted $(\star)$.

We are now in the world of orthogonal polynomials, the polynomials $1, S_1(\mathbb{A}-\mathbb{B}-x), \ldots, S_{n^n}(\mathbb{A}-\mathbb{B}-x)$ being the unique[3] orthogonal basis $P_k(x)$, $k = 0, \ldots, n$, $d^0(P_k) = k$, with respect to the functional

$$g(x) \longrightarrow \sum_{a\in\mathbb{A}} g(a)\frac{\phi(a)}{R(a, \mathbb{A}-a)}\ .$$

Indeed, this functional is an avatar of the Lagrange interpolation in the points $a \in \mathbb{A}$. It is characterized by the fact that it sends each x^i, $i \in \mathbb{N}$, onto the complete function $S^i(\mathbb{A}-\mathbb{B})$ (which should be considered as the *i-th moment*).

Therefore, it sends $S_{k^k}(\mathbb{A}-\mathbb{B}-x)x^i = S_{k^k;\,i}(\mathbb{A}-\mathbb{B};\ x)$ onto $S_{k^k;\,i}(\mathbb{A}-\mathbb{B};\ \mathbb{A}-\mathbb{B})$. When $i < k$, this last determinant vanishes, having two identical columns. In other words, the polynomial $P_k = S_{k^k}(\mathbb{A}-\mathbb{B}-x)$ is orthogonal to $x^0, \ldots, x^{k-1}$.

Thus, the decomposition of the Bézoutian becomes

$$\mathfrak{Bez}(\mathbb{A},\mathbb{B}) = \phi(x)\phi(y)\Big(1 + (\star)P_1(x)P_1(y) + \cdots + (\star)P_n(x)P_n(y)\Big)\,, \qquad (7)$$

and inside the parenthesis, one recognizes the Christoffel-Darboux reproducing kernel[4].

It is well known, in fact, that the denominators $\mathcal{D}_i(x)$ of the convergents of the continued fraction (**??**):

$$\frac{N_i(x)}{D_i(x)} := \cfrac{1}{q_0 - \cfrac{1}{q_1 - \cfrac{1}{\ddots \cfrac{1}{q_i}}}}\quad,\quad i \le n\,, \qquad (8)$$

$A(x)\,R(x,\mathbb{A}) + B(x)\,R(x,\mathbb{B})$, the coefficients $A(x)$, $B(x)$ being polynomials of the smallest possible degree. Explicitely, for $|\mathbb{A}| = \alpha$, $|\mathbb{B}| = \beta$, the k-th remainder can be written

$$\mathcal{R}_k = (-1)^{\alpha-k+1}S_{(\beta-\alpha+k)^{k-1}}(\mathbb{A}-\mathbb{B}-x)\,R(x,\mathbb{B}) + S_{k^{\beta-\alpha+k-1}}(\mathbb{B}-\mathbb{A}-x)\,R(x,\mathbb{A})\,.$$

[3] up to normalization

[4] the image of $g(x)K(x,y)$ under the functional is $g(y)$, if g is of degree $\le n$, and $K(x,y)$ is the kernel.

may be interpreted as orthogonal polynomials.

One recovers this fact by noticing that the denominators satisfy the recursions

$$\mathcal{D}_{i-1} = q_i\mathcal{D}_i - \mathcal{D}_{i+1}\ ,\ \ i = 0, \dots, n\ , \tag{9}$$

with initial conditions $\mathcal{D}_0 = 1$, $\mathcal{D}_1 = q_1$ (or equivalently $\mathcal{D}_{-1} = 0$, $\mathcal{D}_0 = 1$). These are the same equations as for the remainders, the initial conditions for the remainders being multiplied by $\phi(x)$, compared to those for the $\mathcal{D}_i$'s. This time, recursions provide a sequence of polynomials of increasing degrees, instead of decreasing degrees in the case of remainders.

In final, one has verified again that

$$\mathcal{R}_i(x) \equiv \phi(x)\mathcal{D}_i(x) \mod f(x),\ i = 0, \dots, n\ . \tag{10}$$

Let us now trace back related formulas in the classical literature.

For example, in his study of *Sturm sequences* (this is the case where $n\phi(x)$ is the derivative of $f(x)$), Sylvester stated without proof the vanishing of the following determinant :

$$\Big| D_i(a_j)^2 \Big|_{0\le i,j\le n}\ ,$$

where $a_0, \dots, a_n$ are the roots of $f(x)$ (and thus are the letters of the alphabet $\mathbb{A}$), and $D_0 = 1$.

In other words, Sylvester stated that there exists constants c_i such that the $n{+}1$ following equations are satisfied :

$$c_0 + c_1 D_1(a)^2 + \cdots + D_n(a)^2 = 0 \quad \forall a \in \mathbb{A}\ . \tag{11}$$

But this comes from the expression of the specializations $x = y = a \in \mathbb{A}$ of the Bezoutian, divided by $\phi(a)^2 = f'(a)^2/n^2$:

$$\begin{aligned} \mathfrak{Bez}(\mathbb{A},\mathbb{B})\Big|_{x=y=a} &= f'(a)\phi(a)/\phi(a)^2 = n \\ &= (\star) + (\star)D_1(a)^2 + \cdots + (\star)D_n(a)^2 \end{aligned} \tag{12}$$

Brioschi generalized Sylvester's relation to an arbitrary $\phi(x)$ of degree $n{-}1$. One still has the vanishing of the determinant

$$\Big| D_i(a_j)^2 \Big|_{0\le i,j\le n}\ ,$$

where $a_0, \dots, a_n$ are still the roots of $f(x)$, and $D_0(x)$ is now equal to

$$D_0(x) = 1 - \frac{f'(x)}{\phi(x)}\ .$$

Orthogonality properties have also been obtained by Brioschi, though he states them only for two consecutive remainders in [**?**].

Proposition 1 *Let $r_0, r_1, \ldots, r_{n-1}$ be the successive remainders in the division of $f(x) = S^n(x-\mathbb{A})$ by $\phi(x) = S^{n-1}(x-\mathbb{B})$. Then, for any $k, j : 0 \le k <\le j \le n-1$, one has the nullities*

$$\sum_{a\in\mathbb{A}} r_k(a) r_j(a) \frac{1}{\phi(a) R(a, \mathbb{A}-a)} = 0 \ . \tag{13}$$

Indeed, writing $r_k(a) = \phi(a) D_k(a)$, one recognizes the scalar product of the two orthogonal polynomials $D_k(x), D_j(x)$:

$$\sum_{a\in\mathbb{A}} D_k(a) D_j(a) \frac{\phi(a)}{R(a, \mathbb{A}-a)} \ . \tag{14}$$

However, in the article [**?**], in the case where $\phi(x) = f'(x)/n$, Brioschi gives the following nullities :

$$\sum_{a\in\mathbb{A}} x^k D_j(a) = 0 \quad 0 \le k < j \le n-1 \tag{15}$$

The functional looks different, because the term $\phi(a)/R(a, \mathbb{A}-a) = \phi(a)/f'(a) = 1/n$ can be erased.

Notice that

$$f'(x)/f(x) = \sum_{a\in\mathbb{A}} \frac{1}{x-a} \ ,$$

so that the case treated by Sylvester and Brioschi is the case of orthogonal polynomials for a discrete measure, in relation with Lagrange interpolation. As a matter of fact, it was also the starting point of Tchebychef. Taking a more general $\phi(x)$ of degree $n-1$, one has similarly

$$\frac{\phi(x)}{f(x)} = \sum_{a\in\mathbb{A}} \phi(a) \, \frac{R(x, \mathbb{A}-a)}{R(a, \mathbb{A}-a)} \ ,$$

thanks to Lagrange interpolation, and this time the weights are no more the "Lagrange weights" $1/R(a, \mathbb{A}-a)$, but $\phi(a)/R(a, \mathbb{A}-a)$, and still concentrated in the points of $\mathbb{A}$.

Starting with two polynomials $f(x)$, $\phi(x)$ of the same degree n, one may homogeneize them, introducing an extra variable y, and use the Jacobian of f and ϕ.

Sylvester obtained that there exists constants $c_1, \ldots, c_n$ such that :

$$\frac{1}{n} Jacobian(f, \phi)\Big|_{y=x} = \frac{\mathcal{R}_1(x)^2}{c_1} + \cdots + \frac{\mathcal{R}_n(x)^2}{c_1 \cdots c_n} \ , \tag{16}$$

where $\mathcal{R}_1 = \phi(x), \ldots, \mathcal{R}_n$ are the successive remainders in the division of $f(x)$ by $\phi(x)$.

Once more, this is a decomposition of the Christoffel-Darboux kernel for $x = y$. We are back to our starting point, which is also the one of Hermite [**?**], the decomposition of quadratic forms.

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