

Generalisation of Scott permanent identity

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Abstract

Let $\mathbf{x} = \{x_1, \dots, x_r\}$, $\mathbf{y} = \{y_1, \dots, y_n\}$, $\mathbf{z} = \{z_1, \dots, z_n\}$ be three sets of indeterminates. We give the value of the determinant

$$\left| \prod_{x \in \mathbf{x}} (xy - z)^{-1} \right|_{y \in \mathbf{y}, z \in \mathbf{z}}$$

when specializing $\mathbf{y}$ and $\mathbf{z}$ to the set of roots of $y^n - 1$ and $z^n - \xi^n$ respectively.

In the case where $r = 2$ and $\mathbf{x} = \{1, 1\}$ the determinant $\left| (y - z)^{-2} \right|_{y \in \mathbf{y}, z \in \mathbf{z}}$ factorizes into the determinant of the Cauchy matrix $[(y - z)^{-1}]$ and its permanent. Scott [10, 8] found the value of this permanent when specializing $\mathbf{y}$ to the roots of $y^n - 1$ and $\mathbf{z}$ to the roots of $z^n + 1$. Han [3] described more generally the case where $\mathbf{z}$ is the set of roots of $z^n + az^k + b$ instead of $z^n + 1$.

Instead of restricting to $r = 2$ and specializing $\mathbf{x}$, we shall consider the determinant

$$\left| \prod_{x \in \mathbf{x}} (xy - z)^{-1} \right|_{y \in \mathbf{y}, z \in \mathbf{z}},$$

and obtain in Th. 2 its value when specializing $\mathbf{y}$ and $\mathbf{z}$. The remarkable feature is that this value is a product of sums of monomial functions in $\mathbf{x}$ without multiplicities, thus extending the factorized expressions of [10, 3].

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We first need a few generalities about symmetric functions [5].

Given two sets of indeterminates $\mathbf{x}, \mathbf{z}$ (we say alphabets), the complete functions $S_n(\mathbf{x} - \mathbf{z})$ are the coefficients of the generating function

$$\sum_n \gamma^n S_n(\mathbf{x} - \mathbf{z}) = \prod_{z \in \mathbf{z}} (1 - \gamma z) \prod_{x \in \mathbf{x}} (1 - \gamma x)^{-1}.$$

For any r , any $\lambda \in \mathbb{Z}^r$, $S_\lambda(\mathbf{x} - \mathbf{z}) = \det(S_{\lambda_i + j - i}(\mathbf{x} - \mathbf{z}))$.

In the case where $\mathbf{z} = 0$, and $\mathbf{x}$ of cardinality r , these functions can be obtained by symmetrisation over the symmetric group $\mathfrak{S}_r$. Let π_ω be the following operator on functions in $\mathbf{x}$:

$$f \rightarrow f\pi_\omega := \sum_{\sigma \in \mathfrak{S}_r} \left(f \prod_{1 \leq i < j \leq r} (1 - x_j/x_i)^{-1} \right)^\sigma.$$

Then, when $\lambda \geq [1-r, \dots, -1, 0]$ (i.e. $\lambda_1 \geq 1-r, \dots, \lambda_r \geq 0$), the monomial $x^\lambda = x_1^{\lambda_1} \dots x_r^{\lambda_r}$ is sent onto $S_\lambda(\mathbf{x})$ under π_ω . When λ is a partition (i.e. $\lambda_1 \leq \dots \leq \lambda_r \leq 0$), $S_\lambda(\mathbf{x})$ is the Schur function of index λ .

Let n be a positive integer, ξ and indeterminate and $\mathbf{z}$ be the set of roots of $z^n - \xi^n$. Equivalently, $e_i(\mathbf{z}) = 0$ for $1 \leq i \leq n-1$, $e_n(\mathbf{z}) = (-1)^{n-1} \xi^n$. For any integer j , any $\mathbf{x}$, one has

$$S_j(\mathbf{x} - \mathbf{z}) = S_j(\mathbf{x}) - \xi^n S_{j-n}(\mathbf{x}),$$

and more generally, from the determinantal expression of Schur functions, for any $\lambda \in \mathbb{N}^r$,

$$S_\lambda(\mathbf{x} - \mathbf{z}) = \sum_{u \in \{0, n\}^r} S_{\lambda - u}(\mathbf{x}) (-1)^{|u|/n} \xi^{|u|}.$$

In particular, when $\lambda = \underbrace{n-1, \dots, n-1}_{r-1}, p$ then the terms with negative last

component $n - p$ vanish, and the set $\{\lambda - u\}$ to consider is $\{[v, p] : v \in \{n-1, -1\}^{r-1}\}$. Reordering indices, putting $q = p - r + 1$, one rewrites the sum as

$$S_\lambda(\mathbf{x} - \mathbf{z}) = (-1)^{r-1} \sum_{u \in \{0, n\}^{r-1}} S_{q, u}(\mathbf{x}) (-1)^{(|\lambda| - |u| - q)/n} \xi^{|\lambda| - |u| - q}. \quad (1)$$

Since $x^v \pi_\omega = S_v(\mathbf{x})$, for any $v \geq [1-r, \dots, -1, 0]$, one can rewrite (1) as a symmetrisation of monomial functions in $\mathbf{x} - x_1 = \{x_2, \dots, x_r\}$:

$$S_\lambda(\mathbf{x} - \mathbf{z}) = (-1)^{r-1} \sum_{j=0}^{r-1} x_1^q m_{nj}(\mathbf{x} - x_1) (-1)^{r-1-j} \xi^{|\lambda| - jn - q} \pi_\omega. \quad (2)$$

From the identity

$$m_{nj}(\mathbf{x} - x_1) = m_{nj}(\mathbf{x}) - x_1^n m_{n,j-1}(\mathbf{x}) + x_1^{2n} m_{n,j-2}(\mathbf{x}) + \cdots + (-x_1^n)^j,$$

one sees that $x_1^q m_{nj}(\mathbf{x} - x_1) \pi_\omega$ is equal to

$$S_q(\mathbf{x}) m_{nj}(\mathbf{x}) - S_{q+n}(\mathbf{x}) m_{n,j-1}(\mathbf{x}) + \cdots + (-1)^j S_{q+jn}(\mathbf{x}).$$

Since on the other hand $S_{(n-1)r-1p}(\mathbf{x} - \mathbf{z})$ belongs to the linear span of Schur functions, or monomial functions, indexed by partitions μ such that $\mu_1 \leq n-1$, one can restrict this last sum to the term $(-1)^j S_{q+jn}(\mathbf{x})$.

In summary, one has the following expression for the specialisation of the Schur function that we are considering.

Proposition 1 *Let $\mathbf{x}$ be an alphabet of cardinality r , $\mathbf{z}$ be the set of roots $z^n - \xi^n = 0$, $p \leq n-1$, $N = (n-1)(r-1)$. Then*

$$S_{(n-1)r-1p}(\mathbf{x} - \mathbf{z}) = \sum_{\mu} m_{\mu}(\mathbf{x}) \xi^{N+p-|\mu|}, \quad (3)$$

sum over all partitions $\mu \in \mathbb{N}^r$, $\mu_1 \leq n-1$.

For example, for $n = 4$, $r = 2$, one has

$$\begin{aligned} S_{30}(\mathbf{x} - \mathbf{z}) &= m_3(\mathbf{x}) + m_{21}(\mathbf{x}), \quad S_{31}(\mathbf{x} - \mathbf{z}) = m_{31}(\mathbf{x}) + m_{22}(\mathbf{x}) + \xi^4, \\ S_{32}(\mathbf{x} - \mathbf{z}) &= m_{32}(\mathbf{x}) + \xi^4 m_1(\mathbf{x}), \quad S_{33}(\mathbf{x} - \mathbf{z}) = m_{33}(\mathbf{x}) + \xi^4 (m_2(\mathbf{x}) + m_{11}(\mathbf{x})). \end{aligned}$$

Let

$$D(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \left| \prod_{x \in \mathbf{x}} (xy - z)^{-1} \right|_{y \in \mathbf{y}, z \in \mathbf{z}}.$$

In the case $r = 2$, this determinant has been obtained by Izergin and Korepin [4] as the partition function of the Heisenberg XXZ-antiferromagnetic model. Gaudin [2] had previously described the partition function of some other model as the determinant $|(x-y)^{-1}(x-y+\gamma)^{-1}|$ for some parameter γ .

The Izergin-Korepin determinant is used in the enumeration of alternating sign matrices [1]. In that case, one first specializes $\mathbf{x} = \{e^{2i\pi/3}, e^{4i\pi/3}\}$. Okada [9] evaluates more general partition functions corresponding to similar determinants or Pfaffians, and to other roots of unity (see also [6, Th. 7.2]).

We shall take another point of view, keep $\mathbf{x}$ generic, but specialize instead $\mathbf{y}$ and $\mathbf{z}$. In [7, Formula 4], it is shown that the function

$$G(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{D(\mathbf{x}, \mathbf{y}, \mathbf{z})}{\Delta(\mathbf{z})} \prod_{x \in \mathbf{x}} \prod_{y \in \mathbf{y}} \prod_{z \in \mathbf{z}} (xy - z)$$

is equal to the determinant of the matrix

$$\left[S_{\square j}(y_i \mathbf{x} - \mathbf{z}) \right]_{j=0 \dots n-1, i=1 \dots n}, \quad (4)$$

where $\square = (n-1)^{r-1}$, and $\Delta(\mathbf{z}) = \prod_{1 \leq i < j \leq n} (z_i - z_j)$.

For any $k \in \mathbb{N}$, let φ_k be the sum of all monomial functions $m_\mu(\mathbf{x})$ of degree k , with $\mu_1 \leq n-1$ (notice that $\varphi_k = 0$ when $k > (n-1)r$). From (3), one has that $S_{\square j}(y_i \mathbf{x} - \mathbf{z})$ specializes, when $\mathbf{z}$ is the set of roots of $z^n - \xi^n$, into

$$S_{\square j}(y_i \mathbf{x} - \mathbf{z}) = y_i^{N+j} \varphi_{N+j} + \xi^n y_i^{N+j-n} \varphi_{N+j-n} + \xi^{2n} y_i^{N+j-2n} \varphi_{N+j-2n} + \dots \quad (5)$$

Specializing further $\mathbf{y}$ into the roots of $y^n - 1$, one sees that the matrix (4) factorizes into the product of the matrix $\left[y_i^{(N+j)} \right]$, where $(k) = k \pmod n$, and the diagonal matrix

$$\text{diag} \left((\varphi_N + \xi^n \varphi_{N-n} + \xi^{2n} \varphi_{N-2n} + \dots), (\varphi_{N+1} + \xi^n \varphi_{N+1-n} + \xi^{2n} \varphi_{N+1-2n} + \dots), \dots, (\varphi_{N+n-1} + \xi^n \varphi_{N+n-1-n} + \xi^{2n} \varphi_{N+n-1-2n} + \dots) \right).$$

For example, for $n = 3, r = 3$,

$$S_{220}(y_i \mathbf{x} - \mathbf{z}) = y_i^4 \varphi_4 + y_i \varphi_1 \xi^3, S_{221}(y_i \mathbf{x} - \mathbf{z}) = y_i^5 \varphi_5 + y_i^2 \varphi_2 \xi^3, \\ S_{222}(y_i \mathbf{x} - \mathbf{z}) = y_i^6 \varphi_6 + y_i^3 \varphi_3 \xi^3 + \xi^6,$$

and the matrix factorizes into

$$\begin{bmatrix} y_1 & y_1^2 & 1 \\ y_2 & y_2^2 & 1 \\ y_3 & y_3^2 & 1 \end{bmatrix} \begin{bmatrix} \varphi_4 + \varphi_1 \xi^3 & 0 & 0 \\ 0 & \varphi_5 + \varphi_2 \xi^3 & 0 \\ 0 & 0 & \varphi_6 + \varphi_3 \xi^3 + \xi^6 \end{bmatrix}$$

Taking into account that $\prod_{x,y,z} (xy - z)$ specializes into $\prod_{y,x} (x^n y^n - \xi^n) = \prod_{x \in \mathbf{x}} (x^n - \xi^n)^n$, and that the determinant of powers of the $y \in \mathbf{y}$ is a permutation of the Vandermonde in $\mathbf{y}$, one obtains the following theorem.

Theorem 2 *Let n, r be two positive integers, $N = (n-1)(r-1)$. Let $\mathbf{x}$ be an alphabet of cardinality r , $\mathbf{y}$ be the set of roots of $y^n - 1$, $\mathbf{z}$ be the set of roots of $z^n - \xi^n$. Then*

$$\Delta(\mathbf{y}) \Delta(\mathbf{z}) \left| \prod_{k=1}^r (x_k y_i - z_j)^{-1} \right|_{i,j=1 \dots n} = \frac{(-1)^{(n-1)(n/2+r-1)}}{\prod_{x \in \mathbf{x}} (x^n - \xi^n)^n} \prod_{i=0}^{n-1} \left(\sum_{j=0}^{\infty} \varphi_{N+i-nj} \xi^{nj} \right).$$

For $\mathbf{x} = \{1, 1\}$, this theorem is due to Han[3]. In that case, $\varphi_i = i + 1$ and $\varphi_{n-1+i} = n - i$ for $i = 0, \dots, n-1$, and the product appearing in the theorem is

$$n(n-1+\xi^n)(n-2+2\xi^n)\cdots(1+(n-1)\xi^n).$$

For $r = 5$, $n = 3$, as a further example, the theorem furnishes the expression

$$\prod_{k=1}^5 (x_k^3 - \xi^3)^{-3} (\varphi_8 + \varphi_5 \xi^3 + \varphi_2 \xi^6) (\varphi_9 + \varphi_6 \xi^3 + \varphi_3 \xi^6 + \xi^9) \\ (\varphi_{10} + \varphi_7 \xi^3 + \varphi_4 \xi^6 + \varphi_1 \xi^9),$$

which specializes, for $\mathbf{x} = \{1, 1, 1, 1, 1\}$, into

$$(1 - \xi^3)^{-15} (15 + 51\xi^3 + 15\xi^6) (5 + 45\xi^3 + 30\xi^6 + \xi^9) (1 + 30\xi^3 + 45\xi^6 + 5\xi^9).$$

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