INTERPOLATION

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The original problem of interpolation is:
how to reconstruct a one-variable function knowing its values at a finite number of points?

Of course, we know that this is possible only in the case of polynomials, of degreee strictly less than the number of interpolation points. In the general case, we shall have only an *approximation* of the function.

Another way to formulate the problem is the following:

we favor a linear space V of functions. Given f, one wants a function $p(f)$ in V which is as "near" to f as possible.

The functional p should satisfy :

- Linearity: $\lambda f + \mu g \to p(\lambda f + \mu g) = \lambda p(f) + \mu p(g)$, $\lambda,\mu \in \mathbb{C}$
- Idempotence: $p^2 = p$

How do we input f ? In general, by the values $L_0(f), \ldots, L_n(f)$ of linear functionals $L_0, \ldots, L_n$ at f.

For the solution to be possible and unique, one supposes that the restriction of the L_i to V are linearly independent (which amounts to say that the L_i are a basis of the space dual to V; in particular, $n+1 = dim(V)$).

Now we can reformulate our interpolation problem this way :

Given f, find g in V such that

$$L_0(f) = L_0(g), \ldots, L_n(f) = L_n(g) \ .$$

Other problem: find a basis $\{g_i\}$ of V adapted to the L_i, for example, such that the matrix $|\,L_j(g_i)\,|$ is the identity matrix.

Symmetrically, one can start from a basis of the space V and look for the associated functionals. We shall give examples of these different points of view.

Lagrange Interpolation

The simplest spaces of functions are the space of polynomials in one variable. Take for example $V = \mathcal{P}ol(n) :=$ space of polynomials in x of degree $\leq n$.

At the same time, the simplest linear functionals are taking the value in a fixed set of points $\mathbb{A}$, with cardinal $(\mathbb{A}) = n+1$.

The looked for interpolation polynomial g is the (unique) polynomial g of degree $\leq n$ such that

$$(Lgr1) \qquad a \in \mathbb{A} \ \Rightarrow \ g(a) = f(a)$$

Denote by $R(\mathbb{A},\mathbb{B})$ the product $\prod_{a\in\mathbb{A},\, b\in\mathbb{B}}(a-b)$, and by $\mathbb{A}\setminus\mathbb{B}$ the set difference.

One remarks that for every $a\in\mathbb{A}$ the polynomial $R(x,\mathbb{A}\setminus a)$ vanishes in all the points of $\mathbb{A}$ other than a. It is clear that by linear combination of these $n+1$ polynomials of degree n, one can express any element of V, and thus these polynomials constitute the basis adapted to the functionals $L_a : f\to L_a(f) := f(a)$, modulo normalisation.

In other words, one has Lagrange formula (J. Ecole Polyt., II,p.277) :

$$(Lgr2) \qquad g(x) = \sum_{a\in\mathbb{A}} f(a)\frac{R(x,\mathbb{A}\setminus a)}{R(a,\mathbb{A}\setminus a)} \ .$$

The application $f\to g$ is a projector on the space $\mathcal{P}ol(n)$ of polynomials of degree n. This projector is no other than the "Remainder modulo $R(x,\mathbb{A})$" since it is the identity on $\mathcal{P}ol(n)$ and since it vanishes on every multiple of $R(x,\mathbb{A})$ (and thus its values on every polynomial is well determined).

Lagrange interpolation allows decomposition of rational fractions : dividing both members of $(Lgr2)$ by $R(x,\mathbb{A})$ one gets the equivalent form :

$$(Lgr3) \qquad \frac{g(x)}{R(x,\mathbb{A})} = \sum_{a\in\mathbb{A}} \frac{f(a)}{(x-a)R(a,\mathbb{A}\setminus a)} \ .$$

Indeed, in the the preceeding formula, the "variable" x plays a role symmetrical to the "interpolation points $a\in\mathbb{A}$", i.e. it can be written as a summation on the alphabet $\mathbb{A}\cup\{x\}$.

One is therefore led to replace $(Lgr3)$ by the following operator on one-variable functions :

$$f\to \sum\nolimits_{a\in\mathbb{A}} f(a)/R(a,\mathbb{A}\setminus a) \ \in \mathfrak{Sym}(\mathbb{A}) \ ,$$

where $\mathfrak{Sym}(\mathbb{A})$ is the ring of symmetric functions in $\mathbb{A}$.

This is not our last transformation of Lagrange formula. One notices that the starting space does not need to be restricted to functions of one variable, but can be taken to be $\mathfrak{Sym}(1|n)$, the space of functions of $n+1$ variables which are symmetrical in the last n ones. Put a total order on the alphabet $\mathbb{A}=\{a_1,a_2,\dots,a_{n+1}\}$. Since last century at least, one knows that functions $f(a_1;a_2,\dots,a_{n+1})$, symmetrical in $a_2,\dots,a_{n+1}$, can be expressed as functions of a_1 only, with coefficients in the ring of functions which are symmetrical in all the variables, that is $\mathfrak{Sym}(\mathbb{A})$; on another hand, the preceeding operator commutes with multiplication by every element of $\mathfrak{Sym}(\mathbb{A})$, one can thus extend it to the space $\mathfrak{Sym}(1|n)$, considered as a module over the ring $\mathfrak{Sym}(\mathbb{A})$ generated by the powers of a_1.

Thus , *Lagrange operator* is the symmetrization operator:

$$(Lgr4)\quad \mathfrak{Sym}(1|n) \ni f \to \sum f(a,\mathbb{A}\backslash a) \,/\, R(a,\mathbb{A}\backslash a) := L_{\mathbb{A}}(f) \in \mathfrak{Sym}(\mathbb{A})\ .$$

Since powers of a_1 generate $\mathfrak{Sym}(a_1|a_2,\ldots,a_{n+1})$ as a $\mathfrak{Sym}(\mathbb{A})$-module, it suffices to determine the values $L_A(a_1^k)$, $k \geq 0$ to characterize Lagrange operator (we shall see later that $\{a_1^0,\ldots,a_1^n\}$ is a basis). One can refer to Euler, who knew that

$$(Lgr5)\qquad \forall k \geq 0,\quad \sum a^k/R(a,\mathbb{A}\setminus a) = S_{k-n}(\mathbb{A})\ ,$$

where the $S_k(\mathbb{A})$ are the *complete functions* of $\mathbb{A}$, i.e. the sum of all monomials in $\mathbb{A}$ of a given degree.

Indeed, taking the generating function of the powers of a_1, that is $1/(1-za_1)$, one could compute that

$$(Lgr5')\qquad \sum_{a\in\mathbb{A}} \frac{1}{(1-za)R(a,\mathbb{A}\setminus a)} = \frac{z^n}{\prod(1-za)}\ .$$

In fact, the computation of the image by Lagrange operator of the function $1/(1-zx)$ can be avoided, by having recourse to the alphabet $\mathbb{B} = \mathbb{A}\cup\{1/z\}$. Identity Lagr5' can be written, up to the coefficient $\frac{1}{z}$:

$$(Lgr5")\qquad \frac{1}{R(1/z,\mathbb{A})} + \sum_a \frac{1}{R(a,\mathbb{A}\setminus a)\,R(a,1/z)} = \sum_{b\in\mathbb{B}} \frac{1}{R(b,\mathbb{B}\setminus b)} = 0\ ,$$

nullity being forced by the fact that the summation must be a polynomial of negative degree.

More generally, given two alphabets $\mathbb{A},\mathbb{B}$, with $\mathbb{A}$ of cardinal $n+1$, by linearity from $(Lgr5')$, one has

$$(Lgr6)\qquad L_{\mathbb{A}}\left(\frac{\prod_{b\in\mathbb{B}}(1-zb)}{1-za_1}\right) = \frac{z^n\prod(1-zb)}{\prod(1-za)}\ ,$$

i.e. , for every r, writing x for a_1,

$$(Lgr6')\qquad L_{\mathbb{A}}(S_r(x-\mathbb{B})) = S_{r-n}(\mathbb{A}-\mathbb{B})\ .$$

Lagrange operator, as given in $(Lgr4)$, sends one-variable polynomials of degree $< n$ to 0. In the general case, this is the operator which expresses the *remainder* in the usual Lagrange formula. It can also be interpreted in terms of the Euclidean division of polynomials, but this time it gives the *quotient* and not the remainder.

Indeed, let $\mathbb{B}$ be any alphabet, m an integer (which does not need to be the cardinal of $\mathbb{B}$), $\mathbb{A}$ be another alphabet of cardinal n, x a variable. Using the Lagrange operator associated to the alphabet $\{x,\mathbb{A}\}$, one can rewrite (Lgr6') as

$$(Lgr7)\ \ S_m(x-\mathbb{B})/R(x,\mathbb{A}) - \sum_a \frac{S_m(a-\mathbb{B})}{(x-a)R(a,\mathbb{A}\setminus a)} = S_{m-n}(x+\mathbb{A}-\mathbb{B}) \ ,$$

which can be considered as a division

$(Lgr7')$

$$S_m(x-\mathbb{B}) = \sum_a \frac{S_m(a-\mathbb{B})R(x,\mathbb{A}\setminus a)}{R(a,\mathbb{A}\setminus a)} + S_{m-n}(x+\mathbb{A}-\mathbb{B})\cdot R(x,\mathbb{A}) \ ,$$

showing that $S_{m-n}(x+\mathbb{A}-\mathbb{B})$ is the quotient in the Euclidean division, and that the remainder is equal to $L_{\mathbb{A}}\big(S_m(a_1-\mathbb{B})\,R(x,\mathbb{A}\backslash a_1)\big)$.

The Lagrange interpolation on an alphabet $\mathbb{A}$ of cardinal $n+1$ is given by an operator $L_{\mathbb{A}}$ from $\mathfrak{Sym}(1|n)$ to $\mathfrak{Sym}(n+1)$, characterized by the properties

- it is $\mathfrak{Sym}(\mathbb{A})$-linear
- it sends $a_1^0,\ldots,a_1^{n-1}$ to 0 and a_1^n to 1 .

Symmetric Functions

For efficient recursions, one needs to consider symmetric functions as "functors" on alphabets. This essentially means that instead of defining complete functions as coefficients in the expansion of a rational series $\prod_{a\in\mathbb{A}} 1/(1-za)$, we can take arbitrary families $\{S_k\}_{k\geq 0}$ of elements of a commutative ring, with the only restriction that $S_0 = 1$. Indeed, to any formal series $f = \sum_{k\geq 0} z^k S_k$, Littlewood [ch. 6.4] associated the infinite Hankel matrix $\mathbb{S}(f) = \ \big(S_{j-i}\big)_{j,i\geq 0}$, putting $S_i = 0$ if $i < 0$, and defined *skew Schur functions* to be the minors of this matrix.

More precisely, given $I = (i_1,\ldots,i_n) \in \mathbb{Z}^n$, $J = (j_1,\ldots,j_n) \in \mathbb{Z}^n$ he defined $S_{J/I}$ to be the minor of $\mathbb{S}(f)$ taken on rows $i_1+1, i_2+2,\ldots,i_n+n$ and columns $j_1+1,\ldots,j_n+n$ (the minor is 0 if one of these numbers is < 0). When $I = 0^n$, one writes S_J instead of $S_{J/0^n}$.

In other words,

$$(Sf1) \qquad S_{J/I} = \left| \, S_{j_k - i_h + k - h} \, \right|_{1\leq h,k\leq n} .$$

The case of a rational series $f = \prod_{b\in\mathbb{B}}(1-zb)/\prod_{a\in\mathbb{A}}(1-za)$ for two alphabets $\mathbb{A}$ and $\mathbb{B}$ corresponds by definition to the Schur functions of a difference of alphabets ("super Schur functions"), i.e. the coefficients of f are the complete functions $S_j(\mathbb{A}-\mathbb{B})$. Let us denote $\mathbb{S}(\mathbb{A}-\mathbb{B})$ the corresponding Hankel matrix. This case covers in fact the case of general formal power series, since when the cardinal of $\mathbb{A}$ or $\mathbb{B}$ is infinite, the $S_k(\mathbb{A})$ (resp. $S_k(\mathbb{B})$) are algebraically independent.

Now, addition of alphabets corresponds to multiplication of generating series or multiplication of the associated Hankel matrices, and formal substraction corresponds to division of series or multiplication by the inverse of the Hankel matrix. In other words

$$A - \mathbb{B} + \mathbb{C} \Leftrightarrow \mathbb{S}(\mathbb{A}-\mathbb{B}+\mathbb{C}) = \mathbb{S}(\mathbb{A})\,\mathbb{S}(\mathbb{B})^{-1}\,\mathbb{S}(\mathbb{C})$$

$$\Leftrightarrow \ \frac{\prod(1-zb)}{\prod(1-za)\prod(1-zc)} = \frac{1}{\prod(1-za)} \prod(1-zb) \frac{1}{\prod(1-zc)}$$

More explicitely,

$$(Sf2) \qquad S_k(\mathbb{A}-\mathbb{B}+\mathbb{C}) = \sum_{i,j,h:i+j+h=k} S_i(\mathbb{A}) S_j(-\mathbb{B}) S_h(\mathbb{C})$$

Erasing a common factor in the numerator and denominator of a rational function corresponds to writing $(\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C}) = \mathbb{A}-\mathbb{B}$.

Let us remark that, $\mathbb{A}$ being finite of cardinal n, then the $(-1)^k S_k(-\mathbb{A})$ are the *elementary symmetric functions* in $\mathbb{A}$, the $S_k(x-\mathbb{A})$, $k < n$, are the *truncations* of the polynomial $R(x,\mathbb{A})$, while $S_n(x,\mathbb{A}) = R(x,\mathbb{A})$, and for $k > n$,

$$S_k(-\mathbb{A}) = 0\ ,\ S_k(x-\mathbb{A}) = x^{k-n}\, S_n(x-\mathbb{A})\ .$$

General identities on minors imply identities on Schur functions.

For example, given two matrices M,N, with dimensions such that their product MN exists, Binet & Cauchy found a simple expression for the minors of MN (using the above conventions for indexing: $I,J \in \mathbb{N}^n$, H runs over all partitions in $\mathbb{N}^n$):

$$[MN]_{I,J} = \sum_H [M]_{I,H}\,[N]_{H,J}\ .$$

For Schur functions, this imply

$$(Sf3) \qquad S_{J/I}(\mathbb{A}\pm\mathbb{B}) = \sum_H S_{H/I}(\mathbb{A})\,S_{J/H}(\pm\mathbb{B})\ .$$

Similarly, the correspondence, due to Jacobi, between the minors of a matrix and those of its inverse reads

$$(Sf4) \qquad S_{J/I}(\mathbb{A}-\mathbb{B}) = (-1)^{|J/I|}S_{J^{\sim}/I^{\sim}}(\mathbb{B}-\mathbb{A})\ ,$$

for a pair of partitions I,J and their conjugates $J^{\sim}$, $I^{\sim}$, where $|J|$ stands for $j_1+\cdots+j_n$ and $|J/I|$ for $j_1+\cdots+j_n-i_1-\cdots-i_n$ (see the appendix or [Mcd1]).

Given a finite alphabet $\mathbb{A}$ of cardinal n, one can also have recourse to the alphabet of its inverses $\mathbb{A}^\star := \{1/a\}_{a\in\mathbb{A}}$.

The relation $S_k(-\mathbb{A}^\star) = S_{n-k}(-\mathbb{A})\,/S_n(-\mathbb{A})$ extends to

$$(Sf5) \qquad S_I(-\mathbb{A}^\star) = S_{n^r/I}(-\mathbb{A})\,S_n(-\mathbb{A})^r\ ,$$

I partition with parts $\leq n$, r sufficiently big so that $I \subseteq n^r$

We still need to enlarge the definition of a Schur function, this time allowing to play with different alphabets at the same time. In [L1], geometry imposed to take flags of alphabets (i.e. $\mathbb{A}_1 \subseteq \mathbb{A}_2 \subseteq \mathbb{A}_3 \subseteq \cdots$), but the methods used thereby extend without pain, though not using the graphical display of partitions (*Ferrers' diagrams*) rendered in [L1] the involution $\mathbb{A} \to -\mathbb{A}$ more difficult to state.

Given n, given two sets of alphabets $\{\mathbb{A}_1,\mathbb{A}_2,\ldots,\mathbb{A}_n\}$, $\{\mathbb{B}_1,\mathbb{B}_2,\ldots,\mathbb{B}_n\}$, and $I,J \in \mathbb{N}^n$, we define the *multi-Schur function*

$$(Sf6) \quad S_{J/I}(\mathbb{A}_1-\mathbb{B}_1,\ldots,\mathbb{A}_n-\mathbb{B}_n) := \left|\, S_{j_k-i_h+k-h}(\mathbb{A}_k-\mathbb{B}_k)\,\right|_{1\leq h,k\leq n}\ .$$

In the case where the alphabets are repeated, we indicate by a comma the corresponding block separation : given $H \in \mathbb{Z}^p$, $K \in \mathbb{Z}^q$, then $S_{H,K}(\mathbb{A}-\mathbb{B},\mathbb{C}-\mathbb{D})$ stands for the multi-Schur function with index the concatenation of H and K, and alphabets $\mathbb{A}_1 = \cdots = \mathbb{A}_p = \mathbb{A}$, $\mathbb{B}_1 = \cdots = \mathbb{B}_p = \mathbb{B}$, $\mathbb{A}_{p+1} = \cdots = \mathbb{A}_{p+q} = \mathbb{C}$, $\mathbb{B}_{p+1} = \cdots = \mathbb{B}_{p+q} = \mathbb{D}$.

These functions are now sufficiently general to allow easy inductions, thanks to the following transformation lemma.

Lemma $(Sf7)$. — *Let $S_J(\mathbb{A}_1-\mathbb{B}_1,\ldots,\mathbb{A}_n-\mathbb{B}_n)$ be a multi-Schur function, and $\mathbb{D}_0,\mathbb{D}_1,\ldots,\mathbb{D}_{n-1}$ be a family of finite alphabets such that $card(\mathbb{D}_i) \le i$, $0 \le i \le n-1$. Then $S_J(\mathbb{A}_1-\mathbb{B}_1,\ldots,\mathbb{A}_n-\mathbb{B}_n)$ is equal to the determinant*

$$\left| S_{j_k-i_h+k-h}(\mathbb{A}_k-\mathbb{B}_k-\mathbb{D}_{n-h}) \right|_{1\le h,k\le n}$$

Proof. — In other words, one does not change the value of a multi-Schur function S_J by replacing in row h the difference $\mathbb{A}-\mathbb{B}$ by $\mathbb{A}-\mathbb{B}-\mathbb{D}_{n-h}$. Indeed, thanks to the expansion $(Sf2)$:

$$S_j(\mathbb{A}-\mathbb{B}-\mathbb{D}_h) = S_j(\mathbb{A}-\mathbb{B}) + S_1(-\mathbb{D}_h)\, S_{j-1}(\mathbb{A}-\mathbb{B}) + \cdots + S_h(-\mathbb{D}_h)\, S_{j-h}(\mathbb{A}-\mathbb{B}),$$

the sum terminating because the $S_k(-\mathbb{D}_h)$ are null for $k > h$, we see that the determinant has been transformed by multiplication by a triangular matrix with 1's in the diagonal, and therefore has kept its value □

For example, taking $D_0 = \emptyset$, $\mathbb{D}_1 = \{x\}$, $\mathbb{D}_2 = \{y,z\}$, one has

$$\begin{pmatrix} S_i(\mathbb{A}_1-y-z) & S_{j+1}(\mathbb{A}_2-y-z) & S_{h+2}(\mathbb{A}_3-y-z) \\ S_{i-1}(\mathbb{A}_1-x) & S_j(\mathbb{A}_2-x) & S_{h+1}(\mathbb{A}_3-x) \\ S_{i-2}(\mathbb{A}_1) & S_{j-1}(\mathbb{A}_2) & S_h(\mathbb{A}_3) \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & -y-z & yz \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} S_i(\mathbb{A}_1) & S_{j+1}(\mathbb{A}_2) & S_{h+2}(\mathbb{A}_3) \\ S_{i-1}(\mathbb{A}_1) & S_j(\mathbb{A}_2) & S_{h+1}(\mathbb{A}_3) \\ S_{i-2}(\mathbb{A}_1) & S_{j-1}(\mathbb{A}_2) & S_h(\mathbb{A}_3) \end{pmatrix}$$

and the determinant of the left matrix is equal to $S_{ijh}(\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_3)$.

This lemma implies many factorization properties, e.g. for $r \ge 0$,

$$(Sf8) \qquad S_J(\mathbb{A}-\mathbb{B}-x)\, x^r = S_{J,r}(\mathbb{A}-\mathbb{B},x)$$

since taking $\mathbb{D}_1 = \mathbb{D}_2 = \cdots = \{x\}$ factorizes the determinant $S_{J,r}(\mathbb{A}-\mathbb{B},x)$.

More generally, for an alphabet $\mathbb{D}$ of cardinal $\le r$ and $J \in \mathbb{N}^r$, one has

$$(Sf9) \qquad S_I(\mathbb{A}-\mathbb{B}-\mathbb{D})\, S_J(\mathbb{D}) = S_{I,J}(\mathbb{A}-\mathbb{B},\mathbb{D}) \ .$$

Monomials themselves can be written as Schur functions. For an infinite alphabet $\mathbb{A}$, let $\mathbb{A}_n$ be the initial sub-alphabet $\mathbb{A}_n := \{a_1,\ldots,a_n\}$. Let

$I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, and write a^I for $a_1^{i_1} \cdots a_n^{i_n}$, I^ω for $(i_n, \ldots, i_1)$. Then one has

$$(Sf10) \qquad a^I = S_{I^\omega}(\mathbb{A}_n, \ldots, \mathbb{A}_1) \ ,$$

as one can see by subtracting $0, \mathbb{A}_1, \ldots, \mathbb{A}_{n-1}$ in the successive rows, from the bottom.

For example,

$$a^{742} = S_{247}(\mathbb{A}_3, \mathbb{A}_2, \mathbb{A}_1) = det \begin{vmatrix} S_2(\mathbb{A}_3) & S_5(\mathbb{A}_2) & S_9(\mathbb{A}_1) \\ S_1(\mathbb{A}_3) & S_4(\mathbb{A}_2) & S_8(\mathbb{A}_1) \\ S_0(\mathbb{A}_3) & S_3(\mathbb{A}_2) & S_7(\mathbb{A}_1) \end{vmatrix} =$$

$$= \qquad det \begin{vmatrix} a_3^2 & 0 & 0 \\ a_2 + a_3 & a_2^4 & 0 \\ 1 & S_3(a_1 + a_2) & a_1^7 \end{vmatrix} \ .$$

Any other ordering of the alphabet will lead to another determinantal expression of a monomial :

$$a^{742} = S_{742}(a_3 + a_2 + a_1, a_3 + a_2, a_3) \ .$$

From the above, we see that monomials in $\mathbb{A}_n$ can be expressed as linear combinations (with coefficients in $\mathfrak{Sym}(\mathbb{A}_n)$) of monomials $a^J : J \leq (n-1, \ldots, 1, 0)$, or monomials $a^I : I \leq (0, 1, \ldots, n-1)$. Indeed,

$$(Sf11) \quad a^H = S_H(\mathbb{A}_n, \ldots, a_n + a_{n-1}, a_n) = S_H(\mathbb{A}_n, \ldots, \mathbb{A}_n - \mathbb{A}_{n-2}, \mathbb{A}_n - \mathbb{A}_{n-1}) \ .$$

Column k is of degree 1 in the variables $a_1, \ldots, a_{k-1}$, and thus only monomials $a^J : J \leq (n-1, \ldots, 1, 0)$ appear in the expansion of the determinant. □

The resultant $R(\mathbb{A}, \mathbb{B})$ is of fundamental importance in interpolation theory, in elimination theory, in the theory of symmetric functions, in other words in premodern or postmodern algebra.

There are many determinantal expressions of it, which all are related to the fact that it is a special Schur function.

Indeed, let $\mathbb{A}$, $\mathbb{B}$ be of respective cardinals m, n. Then

$$(Sf12) \quad R(\mathbb{A}, \mathbb{B}) = S_{n^m}(\mathbb{A} - \mathbb{B}) = (-1)^{mn} S_{m^n}(\mathbb{B} - \mathbb{A}) = S_{n^m, 0^n}(\mathbb{A}, \mathbb{B}) \ .$$

Proof. — Take any a in $\mathbb{A}$. Then $S_{n^m}(\mathbb{A} - \mathbb{B}) = S_{n, n^{m-1}}(\mathbb{A} - \mathbb{B}, \mathbb{A} - a - \mathbb{B})$ according to $(Sf7)$, subtracting a in all the columns except the first one (instead of subtracting in rows). Now subtract $\{\mathbb{A} \backslash a\}$ in the first row of the new determinant. It factorizes into $S_n(a - \mathbb{B})\, S_{n^{m-1}}(\{\mathbb{A} \backslash a\} - \mathbb{B})$, which proves the result by induction on m. $(Sf9)$ gives the last expression where $\mathbb{A}$ and $\mathbb{B}$ have been separated. $\square$

Multi-Schur functions, being defined as determinants, are bound to technics of minors.

For example, one can use Laplace's expansion along a subfamily of rows or columns. Let $I \in \mathbb{N}^m$, $J \in \mathbb{N}^n$. Given $H \in \mathbb{N}^n$, denote $H^\omega + J := (j_1 + h_n, \ldots, j_n + h_1)$. Then

$$(Sf13) \qquad S_{I,J}(\mathbb{A},\mathbb{B}) = \sum_H (-1)^{|H|} S_{I/H^\sim}(\mathbb{A})\, S_{J+H^\omega}(\mathbb{B}) \ ,$$

sum on all partitions $H \in \mathbb{N}^n$, $H \subseteq m^n$.

In particular, for $n = 1$, one has

$$(Sf14) \qquad S_{I,j}(\mathbb{A},\mathbb{B}) = \sum_{0 \le h \le m} (-1)^h S_{I/1^h}(\mathbb{A})\, S_{j+h}(\mathbb{B}) \ .$$

The fundamental involution $\mathbb{A} \to -\mathbb{A}$, which in the case $(Sf4)$ of a Schur function $S_{J/I}(\mathbb{A})$ expresses it as a determinant of elementary functions $S_k(-\mathbb{A})$, does not allow in general to exchange the role of complete and elementary symmetric functions in a multi-Schur function. In [L1] are given conditions for having two determinantal expressions of a multi-Schur function (see also [Mcd2]). We skip this point at the moment, and shall handle it later more easily with Schubert polynomials.

Complete functions in the difference of two alphabets $\mathbb{A}$, $\mathbb{B}$, are defined by the generating function

$$\sum z^r S_r(\mathbb{A} - \mathbb{B}) := \prod_{a \in \mathbb{A},\, b \in \mathbb{B}} (1 - zb)/(1 - za) \ .$$

Multi-Schur functions are determinants in complete functions $S_r(\mathbb{A}^j - \mathbb{B}^j)$, with alphabets A^j, $\mathbb{B}^j$ fixed in each column of the determinant.

Symmetrizing operators

Divided differences ∂_μ have been introduced by Newton to solve the interpolation problem in one variable. We shall also need the *isobaric divided differences* π_μ.

In the case where μ is a simple transposition σ_i exchanging a_i and a_{i+1}, writing then ∂_i,π_i, the operators (acting on their left) are :

$$(So1) \qquad \begin{cases} \partial_i = \dfrac{1}{(a_i - a_{i+1})}(1+\sigma_i) \\ \pi_i = \dfrac{1}{(1 - a_{i+1}/a_i)}(1+\sigma_i) = a_i\partial_i \end{cases}$$

More explicitely, ∂_i and π_i are the operators on functions $f(a_1,\ldots,a_n)$ of several variables, acting only on the pair (a_i,a_{i+1}):

$$(So2) \qquad \begin{cases} f\,\partial_i = \dfrac{f(\ldots a_i,a_{i+1}\ldots) - f(\ldots a_{i+1},a_i\ldots)}{a_i - a_{i+1}} \\ f\,\pi_i = \dfrac{a_i\, f(\ldots a_i,a_{i+1}\ldots) - a_{i+1}\, f(\ldots a_{i+1},a_i\ldots)}{a_i - a_{i+1}} \end{cases}$$

One has to complete $(So1)$ by the values for $\mu =$ *identity* which are

$$\partial_{id} = 1 \ , \ \pi_{id} = 1$$

.

Both families of operators $\{\sigma_i\}$, $\{\partial_i\}$, $\{\pi_i\}$ separately satisfy the Moore/Coxeter relations (writing $\{D_i\}$ in each of these three cases) :

$$(So3) \qquad D_iD_j = D_jD_i \text{ if } |i-j| \geq 2 \quad ,$$

$$(So4) \qquad D_iD_{i+1}D_i = D_{i+1}D_iD_{i+1} \quad .$$

Relations $(So3)$ and $(So4)$ imply that ∂_μ,π_μ can be written as the product of elementary operators ∂_i, π_i corresponding to any *reduced decomposition* of μ (i.e. any decomposition $\mu = \sigma_i \cdots \sigma_j$ of minimal length gives $\partial_\mu = \partial_i \cdots \partial_j$, $\pi_\mu = \pi_i \cdots \pi_j$).

One can notice that

$$(So5) \qquad \begin{cases} \partial_i\partial_i &= & 0 \\ \pi_i\pi_i &= & \pi_i \end{cases} \quad .$$

In [**L-S ?**] , one characterizes more general operators D_i of the type

$$f \to D_i(f) = P(a_i,a_{i+1})\, f + Q(a_i,a_{i+1})\, f^{\sigma_i} \ ,$$

(where P and Q are rational functions of two variables) satisfying Moore-Coxeter relations. It happens that they must also satisfy an Hecke relation (with constants q and r)

$$(So6) \qquad D_i D_i = qD_i + r \quad .$$

In other words, these operators D_i furnish a representation of the Hecke algebra of the symmetric group as an algebra of operators on the ring of polynomials.

The most useful of these operators is, apart from the two degenerate cases ∂_i and π_i,

$$D_i := c_1\partial_i + c_2\pi_i + c_3\sigma_i \ ,$$

with constants c_1,c_2,c_3.

Let $\mathbb{A}$ be of cardinal n, and ω be the maximal permutation of $\mathfrak{S}(A)$, i.e. ω sends $a_1,\ldots,a_n$ onto $a_n,\ldots,a_1$. Then the operators ∂_ω and π_ω have the following global expression (that we shall prove at the end of the section), apart from being (in different ways) the products of the ∂_i, resp. π_i:

$$(So7 \qquad f\,\partial_\omega = \sum\nolimits_{\mu\in\mathfrak{S}(A)} (f/\Delta)^\mu \quad ,$$

$$(So8) \qquad f\,\pi_\omega = \sum\nolimits_{\mu\in\mathfrak{S}(A)} (f\rho/\Delta)^\mu \quad ,$$

where Δ is the *Vandermonde* : $\Delta = \prod_{i<j}(a_i - a_j)$ and ρ the monomial $a_1^n a_2^{n-1}\cdots a_{n+1}^0$ (which corresponds to half the sum of positive roots, in the theory of classical groups).

Divided differences satisfy a Leibniz formula, as easily seen from the definition:

$$(So9) \qquad fg\partial_i = f\,(g\partial_i) + f\partial_i\, g^{\sigma_i}$$

In particular, symmetric functions in a_i,a_{i+1} are scalars for ∂_i and π_i:

$$g = g^{\sigma_i} \Rightarrow fg\,\partial_i = f\partial_i\, g \quad \text{and} \quad fg\,\pi_i = f\pi_i\, g \ .$$

Divided differences have a simple action on complete functions: let $\mathbb{B}$ be any alphabet and $\mathbb{A}_i$ denotes $\{a_1,\ldots,a_i\}$ for any i less than the cardinal of $\mathbb{A}$. Then

$$(So10) \qquad \Big(\frac{R(1,\mathbb{B})}{R(1,\mathbb{A}_i)}\Big)\,\partial_i = \frac{R(1,\mathbb{B})}{R(1,\mathbb{A}_{i+1})} \ ,$$

$$(So11) \qquad \Big(\frac{R(1,\mathbb{B})}{R(1,\mathbb{A}_i)}\Big)\,\pi_i = \frac{R(1,\mathbb{B})}{R(1,\mathbb{A}_{i+1})}\ ,$$

because, writing $1/R(1,\mathbb{A}_i) = (1-a_{i+1})/R(1,\mathbb{A}_{i+1})$, and using the fact that $R(1,\mathbb{A}_{i+1})$ is a scalar for both ∂_i and π_i, one is reduced to check that

$$(1-a_{i+1})\,\partial_i = 1 = (1-a_{i+1})\,\pi_i\ .$$

More explicitely, $(So10)$ and $(So11)$ can be written

$$(So12) \qquad \big(S_k(\mathbb{A}_i - \mathbb{B})\big)\,\partial_i = S_{k-1}(\mathbb{A}_{i+1} - \mathbb{B})\ ,$$

and

$$(So13) \qquad \big(S_k(\mathbb{A}_i - \mathbb{B})\big)\,\pi_i = S_k(\mathbb{A}_{i+1} - \mathbb{B})$$

Divided differences have a simple action on (multi)-Schur functions, in the case that they operate only on the elements of one row or one column : take $S_{jk\ldots l}(\mathbb{A}_i - \mathbb{B}_1, \mathbb{A}_p - \mathbb{B}_2, \ldots, \mathbb{A}_r - \mathbb{B}_q)$, and suppose that all the indices of the $\mathbb{A}$, apart from, say the first, column are different from i. Then all columns are scalars for ∂_i and π_i, apart from the first one, and from $(So12)$ and $(So13)$ one gets that the images of the preceeding function under ∂_i and π_i are respectively

$$(So14) \qquad S_{j-1\,k\ldots l}(\mathbb{A}_{i+1} - \mathbb{B}_1, \mathbb{A}_p - \mathbb{B}_2, \ldots, \mathbb{A}_r - \mathbb{B}_q)$$

$$(So15) \qquad S_{jk\ldots l}(\mathbb{A}_{i+1} - \mathbb{B}_1, \mathbb{A}_p - \mathbb{B}_2, \ldots, \mathbb{A}_r - \mathbb{B}_q)\ .$$

Let us look now at the action of

$$\partial_\omega = (\partial_{n-1})(\partial_{n-2}\partial_{n-1})\cdots(\partial_1\cdots\partial_{n-1})\ .$$

Writing any monomial in $\mathbb{A}_n$ as a Schur function $S_I(\mathbb{A}_n, \ldots, \mathbb{A}_2, \mathbb{A}_1)$, we see that at each step of applying the ∂_i, only one column of the determinant is not symmetrical in a_i, a_{i+1}, and thus that the action increases alphabets and decreases degree; the final step gives a Schur function $S_J(\mathbb{A})$.

For example, $a_4^2 a_3^5 a_2^6 a_1^9 = S_{2569}(\mathbb{A}_4, \mathbb{A}_3, \mathbb{A}_2, \mathbb{A}_1)$. Our expression of ∂_ω is $\partial_{4321} = (\partial_3)\,(\partial_2\partial_3)\,(\partial_1\partial_2\partial_3)$.

Under ∂_3 the monomial becomes $S_{2469}(\mathbb{A}_4, \mathbb{A}_4, \mathbb{A}_2, \mathbb{A}_1)$, then under $\partial_2\partial_3$, $S_{2449}(\mathbb{A}_4, \mathbb{A}_4, \mathbb{A}_4, \mathbb{A}_1)$ and finaly, under ∂_1, ∂_2, ∂_3,

$$S_{2446}(\mathbb{A}_4, \mathbb{A}_4, \mathbb{A}_4, \mathbb{A}_4) = S_{2446}(\mathbb{A})\ .$$

Now, the image of a monomial under the operator $(So7)$ is an alternating sum of monomials, divided by the Vandermonde, which is Jacobi-Trudi expression of the same Schur function, and thus ∂_ω coïncides with $(So7)$, because the images of any monomial under both operators are the same. An identical proof works for π_ω, this time the indices being preserved.

A simpler proof of the validity of $(So7)$ consists first in noticing that both operators (∂_ω and the summation) are $\mathfrak{Sym}(\mathbb{A})$-linear. Therefore, we need only to check their action on a generating set, which from $(Sf11)$ can be taken as the monomials $\{a^J : J \le \rho\}$. Now all these monomials are sent to 0 by both operators, except for a^ρ which is sent to 1 by $f \longrightarrow \sum \pm f^\mu/\Delta(\mathbb{A})$, and to $S_{0\ldots 0} = 1$ by ∂_ω □

Multi-Schur functions can be obtained with the help of ∂_ω. For example, let $\{\mathbb{B}^1,\mathbb{B}^2,\ldots,\mathbb{B}^n\}$ be n finite alphabets of respective cardinals $i_1,i_2,\ldots,i_n$; we denote this sequence by I, and put $J = i_1 - 0, i_2 - 1, \ldots, i_n - (n-1)$. Then

$$(So16) \qquad R(a_n,\mathbb{B}^1)\cdots R(a_1,\mathbb{B}^n)\,\partial_\omega = S_J(\mathbb{A}_n - \mathbb{B}^1, \ldots, \mathbb{A}_n - \mathbb{B}^n)\ .$$

Proof. — Write $R(a_n,\mathbb{B}^1)\cdots R(a_1,\mathbb{B}^n)$ as $S_I(\mathbb{A}_n - \mathbb{B}^1, \ldots, \mathbb{A}_1 - \mathbb{B}^n)$, according to $(Sf7)$: the determinant becomes triangular after subtraction of $\{0,\mathbb{A}_1,\ldots,\mathbb{A}_{n-1}\}$, because in the upper part we are left with elementary symmetric functions of degree strictly higher than the cardinals of the alphabets involved. Now, the action of $(\partial_{n-1})(\partial_{n-2}\partial_{n-1})\cdots(\partial_1\cdots\partial_{n-1})$ consists, thanks to $(So14)$, in transforming all the $\mathbb{A}_i$ into $\mathbb{A}_n$ while decreasing the indices □

The Schur function $S_I(\mathbb{A} - \mathbb{B})$ can be obtained by a double sommation on $\mathfrak{S}(\mathbb{A})$ and $\mathfrak{S}(\mathbb{B})$ as stated by Sergeev-Pragacz formula [].

For every permutation $\mu \in \mathfrak{S}(\mathbb{A})$, there exist a divided difference ∂_μ and an isobaric divided difference π_μ. The maximum divided difference ∂_ω is the operator

$$f \longrightarrow \sum \pm f^\mu/\Delta(\mathbb{A})\ .$$

The ring $\mathfrak{Sym}(1\,|\,n)$

We already used the fact that the ring

$$\mathfrak{Sym}(1|n) = \mathfrak{Sym}(a_1) \otimes_{\mathbb{Z}} \mathfrak{Sym}(a_2,\ldots,a_{n+1})$$

with coefficients in $\mathbb{Z}$ is a $\mathbb{Z}$-free module with basis $\{S_I(\mathbb{A}-a_1)\, a_1^k, I \text{ partition} \in \mathbb{N}^n, k \in \mathbb{N}\}$. One of the essential properties of Lagrange operator is its compatibility with Schur functions as shown by the following lemma :

LEMMA $(Pr1)$. — *Let $\mathbb{A}$ be of cardinal $n+1$ and $L_{\mathbb{A}}$ the corresponding Lagrange operator. Then*

1) $L_{\mathbb{A}}$ is equal to the product $\partial_1 \cdots \partial_n$ (acting on its left).

2) $\forall I \in \mathbb{Z}^n, I \geq -(0\,1 \cdots n-1)$, $\forall k \geq 0$,

$$L_{\mathbb{A}}\big(S_I(\mathbb{A}-a_1)\, a_1^k\big) = S_{I\ k-n}(\mathbb{A}) \ .$$

Proof. — Writing the product $S_I(\mathbb{A}-a_1)a_1^k$ as the Schur function $S_{I,k}(\mathbb{A},a_1)$, we are reduced to compute the image of powers of a_1 under $L_{\mathbb{A}}$, and thus, thanks to $(Lgr5)$ and $(So12)$, we obtain the required determinant of complete functions of $\mathbb{A}$.

Now, since $S_{I,k}(\mathbb{A},a_1)$ is a $\mathbb{Z}$-basis, the powers of a_1 are a generating set for $\mathfrak{Sym}(1|n)$ as a $\mathfrak{Sym}(\mathbb{A})$-module. The formulas $(Lgr6')$ and $(So12)$ show that the $\mathfrak{Sym}(\mathbb{A})$-linear operators $L_{\mathbb{A}}$ and $\partial_1 \cdots \partial_n$ coincide $\square$

The interest of taking $I \geq -(0\,1 \cdots n-1)$ instead of $I \in \mathbb{N}^n$ will appear later.

It was already clear in the preceeding proof that an essential property of $\mathfrak{Sym}$(1|n) is to be a $\mathfrak{Sym}(\mathbb{A})$-module. Indeed, the $\mathbb{Z}$-basis $S_I(\mathbb{A}-a_1)a_1^k = S_{I,k}(\mathbb{A},a_1)$ can be written in terms of powers of a_1 only (with coefficients in $\mathfrak{Sym}(\mathbb{A})$); moreover, because the elementary symmetric functions of the alphabet $\{\mathbb{A}\backslash a_1\}$ vanish in degree $> n$, one has

$$a_1^k = S_k(a_1) = S_k\big(\mathbb{A} + (a_1 - \mathbb{A})\big) = \sum_0^n S_{k-i}(\mathbb{A})\, S_i(a_1 - \mathbb{A}) \ .$$

Thus, $\{a_1^0, \ldots, a^n\}$ is a generating system for our ring as a module over $\mathfrak{Sym}(\mathbb{A})$. To prove that it is a basis, i.e. that every element of $\mathfrak{Sym}(1|n)$ can be uniquely written

$$f = c_0 a_1^0 + \cdots + c_n a_1^n \ ,$$

with coefficients c_i in $\mathfrak{Sym}(\mathbb{A})$, (in other words, the module is free), we shall first define on it a quadratic form with values in $\mathfrak{Sym}(\mathbb{A})$.

Definition$(Pr2)$. — $\forall P,Q \in \mathfrak{Sym}(1|n)$, set $(P\,|\,Q) := L_{\mathbb{A}}(PQ)$.

Proposition ($Pr3$). —

1)$\mathfrak{Sym}(1|n)$ *is a* $\mathfrak{Sym}(\mathbb{A})$*-free module of basis* $\{1,a_1,\ldots,a_1^n\}$*, as well as basis* $\{S_n(a_1-\mathbb{A}),\ldots,S_0(a_1-\mathbb{A})\}$.

2) The quadratic form is non-degenerate, i.e there does not exist $P\neq 0$ *such that for all* Q *in* $\mathfrak{Sym}(\mathbb{A})$ (P,Q)*=0. The two above bases are adjoint, i.e., for* $0\leq h,k\leq n$, $\big(a_1^k,S_{n-h}(a_1-\mathbb{A})\big)=1$ *or* 0 *according that* $k=h$ *or not.*

3) Every element f *of* $\mathfrak{Sym}(1|n)$ *can be written*

$$f=\sum_0^n L_{\mathbb{A}}\big(f S_k(a_1-\mathbb{A})\big)\, a_1^{n-k}\ ,$$

$$f=\sum_0^n L_{\mathbb{A}}\big(f a_1^{n-k}\big)\, S_k(a_1-\mathbb{A})\ .$$

Proof. — We have to write

$$S_h(a_1-\mathbb{A})\, a_1^k=(-1)^h S_{1^h}(\mathbb{A}-a_1)\, a_1^k=(-1)^h S_{1^h,k}(\mathbb{A},a_1)\ .$$

Now the image under $L_{\mathbb{A}}=\partial_1\cdots\partial_n$ is $(-1)^h S_{1^h,k-n}(\mathbb{A},\mathbb{A})$.

The indices in the first row of this determinant are $1,2,\ldots,h,k-n+h$. They are all different and non negative only when $k-n+h=0$, i.e. only when the determinant is obtained from the determinant expressing $S_{0^{h+1}}(\mathbb{A})=1$ by permuting columns.

Therefore the square matrix with entries the $L_{\mathbb{A}}\big(S_i(a_1-\mathbb{A})a_1^k\big)$ is the identity matrix, which proves that the quadratic form is non-degenerate and that $a_1^0,\ldots,a_1^n$ are linearly independent, with adjoint basis $S_n(a_1-\mathbb{A}),\ldots,S_0(a_1-\mathbb{A})$. This pair of adjoint bases allows now to expand in two different ways any element of the module by taking scalar products □

The ring $\mathfrak{Sym}(1|n)$ can be identified with the equivariant cohomology ring of a relative projective space of dimension n, and the operator $L_{\mathbb{A}}$ to the *Gysin morphism* from $\mathfrak{Sym}(1|n)$ to the cohomology of the base (identified with $\mathfrak{Sym}(\mathbb{A})$). The variable a_1 is the *first Chern class* of the *tautological line bundle*, $(1+a_2)\cdots(1+a_{n+1})$ is the total Chern class of the tautological kernel (see [Fu1])

The ring $\mathfrak{Sym}(1|n)$ is a free module over $\mathfrak{Sym}(\mathbb{A})$, with basis $\{1,a_1,\ldots a_1^n\}$, and an explicit adjoint basis with respect to the scalar product

$$(f,g):=L_{\mathbb{A}}(fg)\ .$$

The ring $\mathfrak{Sym}(r \mid m)$

Lagrange operator can be expressed as a summation on subsets of cardinal 1 of a given alphabet $\mathbb{A}$. Its generalization is straightforward : sum on subsets of cardinal r.

Let r,m be two posiive integers,$\mathbb{A}$ an alphabet of card. $n = r + m$. Then the operator

$$(Gr1) \quad \mathfrak{Sym}(r) \otimes \mathfrak{Sym}(m) \ni f \to \sum_{\mathbb{A}', card(\mathbb{A}')=r} f(\mathbb{A}', \mathbb{A} \setminus \mathbb{A}')/R(\mathbb{A}', \mathbb{A} \setminus \mathbb{A}')$$

may be called the *Sylvester operator* of order r. Indeed, Sylvester expressed the discriminant of a polynomial, or the greatest common divisor of two polynomials using this operator.

The alphabet being totally ordered, Sylvester operator can also be expressed through divided differences. Denote $\partial_{r|m}$ the product

$$(Gr2) \qquad \partial_{r|m} := (\partial_r \cdots \partial_{n-1})(\partial_{r-1} \cdots \partial_{n-2}) \cdots (\partial_1 \cdots \partial_{m-1})$$

It is the divided difference indexed by the permutation

$$\mu = (r{+}1) \cdots n, 1 \cdots r \ .$$

Let $\mathfrak{Sym}(r|m)$ denote the ring of functions in $r + m$ variables, which are symmetrical in the first r ones, and symmetrical separately in the last m ones. When identifying the variables to the elements of $\mathbb{A}_r := \{a_1, \cdots, a_r\}$ and $\mathbb{A} \setminus \mathbb{A}_r$, we shall also write $\mathfrak{Sym}_{r|m}(\mathbb{A})$.

Proposition $(Gr3)$. —

1) Sylvester operator is equal to $\partial_{r|m}$.

2) Sylvester operator is $\mathfrak{Sym}(\mathbb{A})$-linear and is the only $\mathfrak{Sym}(\mathbb{A})$-linear operator sending the $S_J(\mathbb{A}_r)$, J partition such that $J \subset m^r$, to zero and $S_{m^r}(\mathbb{A}_r)$ to 1.

3) It sends products of Schur functions to Schur functions, i.e. for every $I \in \mathbb{N}^m$, $J \in \mathbb{N}^r$, one has

$$\big(S_I(\mathbb{A} - \mathbb{A}_r) \cdot S_J(\mathbb{A}_r)\big)\partial_{r|m} = S_{(IJ)-(0^m m^r)}(\mathbb{A}) \ ,$$

where $IJ - 0^m m^r$ means $I_1 - 0, \ldots, I_m - 0, J_1 - m, \ldots, J_r - m$.

Proof. — Thanks to $(Sf9)$ one writes $S_J(\mathbb{A}_r)S_I(\mathbb{A} - \mathbb{A}_r)$ as $S_{J,I}(\mathbb{A}, \mathbb{A} - \mathbb{A}_r)$. The expansion of this determinant is a linear combination of $S_H(-\mathbb{A}_r)$, H partition : $H \leq r^m$.

Therefore, when taking coefficients in $\mathfrak{Sym}(\mathbb{A})$, one can express any element of $\mathfrak{Sym}(r|m)$ as a combination of the $S_J(\mathbb{A}_r)$, with partitions J contained in m^r.

Now one remarks that Sylvester operator and $\partial_{r|m}$ are both $\mathfrak{Sym}(\mathbb{A})$-linear, and that they decrease degree by rm. Thus they send all $S_J(\mathbb{A}_r)$, J partition : $J \subset m^r$, to zero, and they send $S_{m^r}(\mathbb{A}_r)$ to constants. We check below that the image of Schur functions in $\mathbb{A}_r$ by $\partial_{r|m}$, and we get in this case $S_{0\ldots0} = 1$. As for Sylvester summation, we can as well compute the sum

$$\sum_{\mathbb{A}'} \frac{R(\mathbb{A}',\mathbb{A} \setminus \mathbb{A}_r)}{R(\mathbb{A}',\mathbb{A} \setminus \mathbb{A}')}$$

because $R(\mathbb{A}',\mathbb{A} \setminus \mathbb{A}_r)$ has leading term $S_{m^r}(\mathbb{A}')$. Now, clearly in this summation there is only the term $\mathbb{A}' = \mathbb{A}_r$ which is non zero (and equal to 1).

The two operators, coïnciding on a generating set, are equal, and this proves 1) and 2). We shall check later that the generating set is linearly independent, and thus that $\mathfrak{Sym}(r|m)$ is a free module of dimension rm over $\mathfrak{Sym}(\mathbb{A})$.

As for the last point, one writes (once more!) $S_I(\mathbb{A} - \mathbb{A}_r)S_J(A_r) = S_{I,J}(\mathbb{A},\mathbb{A}_r)$. However, the action of the divided difference ∂_r on this Schur function is not straightforward, because ∂_r acts non trivially on the r last columns of the determinant, and one would have to use Leibniz formula. Fortunately, one can overcome this by taking other products of divided differences. Let us take numerical values to follow more easily the computation, say $r = 3$, $m = 2$, $IJ = 12467$. Instead of starting with $S_{12,\,467}(\mathbb{A},\mathbb{A}_3)$, we take $S_{12479}(\mathbb{A},\mathbb{A},\mathbb{A}_3,\mathbb{A}_2,\mathbb{A}_1)$ which gives under $\partial_2\partial_1\partial_2$ the preceeding function.

Now we have to compute the image of $S_{12479}(\mathbb{A},\mathbb{A},\mathbb{A}_3,\mathbb{A}_2,\mathbb{A}_1)$ under $(\partial_2\partial_1\partial_2)\,(\partial_3\partial_4\partial_2\partial_3\partial_1\partial_2)$, acting on the left. But, thanks to the braid relations, this operator can be written $(\partial_3\partial_4)(\partial_2\partial_3\partial_4)(\partial_1\partial_2\partial_3\partial_4)$.

This sequence of operators is such that at each step of the computation of the image of $S_{12479}(\mathbb{A},\mathbb{A},\mathbb{A}_3,\mathbb{A}_2,\mathbb{A}_1)$, there is only one column which is transformed (in which case, the transformation amounts to $S_k(\mathbb{A}_i) \longrightarrow S_{k-1}(A_{i+1)})$ and finally we get the Schur function $S_{12233}(\mathbb{A},\mathbb{A},\mathbb{A},\mathbb{A},\mathbb{A})$ as required $\square$

Of course, we could have proved the preceeding proposition by mere expansions of determinants (as we did in [L1], [L2]), but using divided differences avoids the subtilities of determinantal calculus which have been lost since the time of Sylvester.

We did not give full details here, because with Schubert polynomials, we shall not even need determinants at all.

Having seen that we have to use the structure of $\mathfrak{Sym}(r|m)$ as a free $\mathfrak{Sym}(\mathbb{A})$-module, we define on it a quadratic form with the help of Sylvester operator :

$$(Gr4) \qquad \mathfrak{Sym}(r|m) \ni f,g \Rightarrow (f,g) := fg\,\partial_{r|m} \in \mathfrak{Sym}(\mathbb{A}) \ .$$

Proposition $(Gr5)$. —
1) $\mathfrak{Sym}(r|m)$ *is a free* $\mathfrak{Sym}(\mathbb{A})$*-module with the two basis* $\{S_I(\mathbb{A}_r)\}$ *and* $\{S_I(\mathbb{A}_r - \mathbb{A}\}$*, where* I *runs over all partitions* $I : 0 \subseteq I \subseteq m^r$*.*

2) The quadratic form is non degenerate, and the two bases $\{S_I(\mathbb{A}_r)\}$ *and* $\{S_I(\mathbb{A}_r - \mathbb{A}\}$ *are adjoint with respect to it. More precisely, one has* $\big(S_I(\mathbb{A}_r - \mathbb{A}) \cdot S_J(\mathbb{A}_r)\big)\,\partial_{r|m} = 1$ *or* 0 *according that* $I = (m - J_r \ldots m - J_1)$ *or not.*

3) Every element f *of* $\mathfrak{Sym}(r|m)$ *can be written*

$$f = \sum_{I \subseteq m^r} f\, S_I(\mathbb{A} - \mathbb{A}_r)\partial_{r|m}\, S_I(\mathbb{A}_r) \ .$$

Proof. — The proof is the same as in the case of $\mathfrak{Sym}(1|m)$. We have to compute all the $S_K(\mathbb{A} - \mathbb{A}_r)\, S_J(\mathbb{A}_r)\,\partial_{r|m}$.

Now, proposition $(Gr3)$ gives them as a determinant of complete functions in $\mathbb{A}$, with respective indices in the first row

$$K_1, K_2 + 1, \ldots, K_m + m - 1, J_1, J_2 + 1, \ldots, J_r + r - 1 \ .$$

Using the hypothesis that $0 \subseteq K \subseteq r^m$ and $0 \subseteq J \subseteq m^r$, we see that these n integers belongs to the set $\{0,1,\ldots n-1\}$. Therefore, for every partition J, there exists one and only one partition K such that the determinant has all its columns distinct, and no column with all indices negative. In this case, the determinant is obtained from $S_{0\ldots 0} = 1$ by a permutation of columns. The conjugate of the partition K gives us the partition I specified by the theorem (we have taken $S_K(\mathbb{A} - \mathbb{A}_r)$ instead of $S_I(\mathbb{A}_r - \mathbb{A})$).

Thus the matrix $\big|S_I(\mathbb{A}_r - \mathbb{A})S_J(\mathbb{A}_r)\,\partial_{r|m}\big|$ is the identity matrix, which proves that the quadratic form is non degenerate and that the two sets are bases of the module.

Having two adjoint bases, we can now expand any element of $\mathfrak{Sym}(r|m)$ by computing scalar products $\square$

We have seen in the preceeding section that computing in the ring $\mathfrak{Sym}(1|m)$ was related to finding the first remainder in the Euclidean division. Similarly, we can use $\mathfrak{Sym}(r|m)$ to express the remainder of degree $r < n-1$. This gives some determinantal expressions of the remainder, as well as expressions using Sylvester operator as a summation on subsets of given cardinal.

The ring $\mathfrak{Sym}(r|m)$ can be identified with the (equivariant) cohomology ring of a relative Grassmann variety and $\mathfrak{Sym}(\mathbb{A})$ to the cohomology ring of the base. The variables $a_1, \ldots, a_r$ are the *Chern roots* of the *tautological quotient bundle* . The operator $\partial_{r|m}$ is the *Gysin morphism*. The Schur functions $S_I(\mathbb{A}_r)$ are representatives of the Schubert subvarieties of the Grassamanniann. When computing modulo the ideal generated by symmetric polynomials without constant term, the quadratic form $\partial_{r|m}(f, g)$ becomes the *intersection form*. It was already well known at the end of the 19th century that Schubert varieties in complementary degree intersect in 0 point or 1 point, and that the matrix representing the intersection form is the identity matrix.

The ring $\mathfrak{Sym}(r|m)$ is a free module over $\mathfrak{Sym}(\mathbb{A})$, with basis $S_I(a_1, \ldots, a_r)$, and adjoint basis $S_{I^\sim}(a_{r+1}, \ldots, a_{r+m})$, where the partition I runs over all partitions $I \le r^m$.

Rational interpolation

The resultant $R(\mathbb{A},\mathbb{B})$ plays a fundamental role in the summations generalising Lagrange interpolation, as well as in invariant theory or elimination theory. It can be itself expressed through summations implying other resultants, and we shall only present the following "exchange lemma".

LEMMA *Rt*1. — *Let* $\mathbb{A},\mathbb{B},\mathbb{C}$ *be three alphabets of respective cardinals* $m+n$, n, m. *Then one has the following identity*

$$\sum_{\mathbb{A}'+\mathbb{A}''=\mathbb{A}} \frac{R(\mathbb{A}',\mathbb{B}')}{R(\mathbb{A}',\mathbb{C})\,R(\mathbb{A}',\mathbb{A}'')} = \frac{R(\mathbb{B},\mathbb{C})}{R(\mathbb{A},\mathbb{C})}\ ,$$

sum on all disjoint decompositions of $\mathbb{A}$ *in* $\mathbb{A}'+\mathbb{A}''$, $card(\mathbb{A}')=m$, $card(\mathbb{A}'')=n$, *as well as the identity*

$$\sum_{\mathbb{A}'+\mathbb{A}''=\mathbb{A}} \frac{R(\mathbb{A}',\mathbb{B}')\,R(\mathbb{A}",\mathbb{C})}{R(\mathbb{A}',\mathbb{A}'')} = R(\mathbb{B},\mathbb{C})\ .$$

Proof. — The two identities are equivalent up to multiplication by $R(\mathbb{A},\mathbb{C}) = R(\mathbb{A}',\mathbb{C})R(\mathbb{A}'',\mathbb{C})$.

The product of the two resultants can be written as the Schur function:

$$R(\mathbb{A}',\mathbb{B})\,R(\mathbb{A}'',\mathbb{C}) = S_{m^n,n^m}(\mathbb{A}-\mathbb{C},\mathbb{A}'-\mathbb{B})\ .$$

Using the factorisation of Sylvester operator $f \longrightarrow \sum f(\mathbb{A}',\mathbb{A}'')/R(\mathbb{A}',\mathbb{A}'')$ as a product of divided differences, one obtains

$$\sum R(\mathbb{A}',\mathbb{B})\,R(\mathbb{A}'',\mathbb{C})/R(\mathbb{A}',\mathbb{A}'') = S_{m^n,0^m}(\mathbb{A}-\mathbb{C},\mathbb{A}-\mathbb{B})$$

Writing $\mathbb{A}-\mathbb{C} = (\mathbb{A}-\mathbb{B}) + (\mathbb{B}-\mathbb{C})$, one develops

$$S_{m^n0^m}(\mathbb{A}-\mathbb{C},\mathbb{A}-\mathbb{B}) = \sum\nolimits_I S_{I\,0^m}(\mathbb{A}-\mathbb{B})\,S_{m^n/I}(\mathbb{B}-\mathbb{C})\ .$$

All the S_{I0^m}, $I \subseteq m^n$, are null (having at least two identical columns), except for $I=0^n$; therefore

$$\sum R(\mathbb{A}',\mathbb{B})\,R(\mathbb{A}'',\mathbb{C})/R(\mathbb{A}',\mathbb{A}'') = S_{m^n}(\mathbb{B}-\mathbb{C}) = R(\mathbb{B},\mathbb{C}) \qquad \square$$

A consequence of this exchange property is the determination of the numerator and denominator of a rational function knowing sufficiently many of its values (Cauchy(1841) , Rosenhain (1846)):

Theorem *Rt*2. — *Let P et Q be two unitary polynomials of respective degrees n,m, and $f = P/Q$. Let $\mathbb{A}$ be an alphabet of cardinal $m+n+1$. Then, if $\mathbb{A}$ is generic enough,*

$$f(x) = \frac{\sum_{\mathbb{A}',card=n} R(x,\mathbb{A}') \prod_{a'\in\mathbb{A}'} f(a') \,/\, R(\mathbb{A}',\mathbb{A}\backslash\mathbb{A}')}{\sum_{\mathbb{A}',card=m} R(x,\mathbb{A}') \prod_{a'\in\mathbb{A}'} f(a') \,/\, R(\mathbb{A}',\mathbb{A}\backslash\mathbb{A}')}$$

Proof. — Write $P(x) = R(x,\mathbb{B}), card(\mathbb{B}) = n, Q(x) = R(x,\mathbb{C}), card(\mathbb{C}) = m$.
The rational function f is therefore equal to

$$f = R(x,\mathbb{B})/R(x,\mathbb{C}) = (-1)^{mn} R(x+\mathbb{C},\mathbb{B})/R(x+\mathbb{B},\mathbb{C})$$

Using the preceeding lemma for the pair of alphabets $(x+\mathbb{B},\mathbb{C})$, one has

$$\frac{R(x+\mathbb{B},\mathbb{C})}{R(\mathbb{A},\mathbb{C})} = \sum_{card(\mathbb{A}')=m,\mathbb{A}''=\mathbb{A}-\mathbb{A}'} \frac{R(\mathbb{A}',x+\mathbb{B})}{R(\mathbb{A}',\mathbb{C})R(\mathbb{A}',\mathbb{A}'')} =$$

$$= \sum_{\mathbb{A}'} \frac{R(\mathbb{A}',x)}{R(\mathbb{A}',\mathbb{A}'')} \frac{R(\mathbb{A}',\mathbb{B})}{R(\mathbb{A}',\mathbb{C})} = \sum_{\mathbb{A}'} \prod_{a'} f(a') \frac{R(\mathbb{A}',x)}{R(\mathbb{A}',\mathbb{A}'')}$$

and similarly for the numerator.

Since we have introduced the extra factor $R(\mathbb{B},\mathbb{C})$, we must have a genericity condition which ensures that it is non zero $\square$

One can also give a determinantal form to the preceeding computation (with $z = 1/x$) :

Proposition *Rt*3. — *Let $f(z) = P(z)/Q(z)$ be a rational function, $d^o(P) \le n$, $d^o(Q) \le m$, and $\mathbb{A}$ be an alphabet of cardinal $m+n+1$. Then one has the identity*

$$\begin{array}{c} \mathbb{A} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ \end{array} \begin{vmatrix} \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 1 & a & \cdots & a^n & a^m f(a) & \cdots & af(a) & f(a) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ Q(z) & zQ(z) & \cdots & z^n Q(z) & z^m P(z) & \cdots & zP(z) & P(z) \end{vmatrix} = 0$$

Proof. — Multiply the row by the corresponding factor $Q(a)$. The new determinant is an alternating function in $a_1,\ldots,a_{m+n+1},z$ of degree $\le n+m$ in each letter. Being divisible by the Vandermonde in $\mathbb{A}+z$, it must be null $\square$

The expansion of the preceeding determinant is of the type

For example, for $f(z) = (1+2z)(1+3z)/(1+z)$ and $\mathbb{A} = \{0,1,2,3\}$, one has the determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 6 & 6 \\ 1 & 2 & 4 & 70/3 & 35/3 \\ 1 & 3 & 9 & 105/2 & 35/2 \\ Q & zQ & z^2Q & zP & P \end{vmatrix}$$

the nullity of which is equivalent to

$$(1+z)P = (1+2z)(1+3z)Q \ .$$

One can take a different view of rational interpolation: instead of finding the numerator and denominator of a rational function from sufficiently many of its values, one looks for a rational approximation of an analytic function

$$f(z) = 1 + zf_1 + z^2 f_2 + z^3 f_3 + \cdots \ ,$$

i.e. one looks for a rational function $g(z) = \prod(1-zb)/\prod(1-za)$, whose development in the neighbourhood of $z=0$ coincide with the development of f up to a certain power of z.

In other words, the first coefficients of f are equal to the complete functions $S_j(\mathbb{A}-\mathbb{B})$, and one has to find back the rational function $g(z)$ from its development in the neighbourhood of $z=0$.

Puting $x = 1/z$, $card(\mathbb{A}) = m$, $card(\mathbb{B}) = n$, one can write $g(z) = x^{m-n} R(x,\mathbb{B})/R(x,\mathbb{A})$, and transform this expression by multiplying both numerator and denominator by the factor $R(\mathbb{A},\mathbb{B})$:

$$g(1/x) = \pm x^{m-n} R(\mathbb{A}+x,\mathbb{B}) \,/\, R(\mathbb{B}+x,\mathbb{A})$$

In other terms, the rational function is the quotient of two resultants. This is fortunate, because resultants can be written as special Schur functions and one has many determinantal expressions of them.

This gives without computation many expressions that one could find in the flourishing 19th century litterature.

For example Sylvester gives

$$(Rt4) \qquad \pm x^{n-m} g(1/x) = S_{n^{m+1}}(\mathbb{A}+x-\mathbb{B})/S_{m^{n+1}}(\mathbb{A}-\mathbb{B}-x) \ ,$$

while Jacobi proposes

$$(Rt5) \qquad \pm x^{n-m} g(1/x) = S_{n,n^m}(\mathbb{A}+x-\mathbb{B},\mathbb{A}-\mathbb{B})/S_{m^{n+1},0}(\mathbb{A}-\mathbb{B},x)$$

One could also exchange the rôles of $\mathbb{A}$ and $\mathbb{B}$, or transforms the determinants by linear combination of rows and columns.

For example, for $m = 3$, $n = 2$, writing $\mathbb{C}$ for $\mathbb{A} - \mathbb{B}$, one has

$$-g(x)/x = \begin{vmatrix} S_2(x+\mathbb{C}) & S_3(x+\mathbb{C}) & S_4(x+\mathbb{C}) & S_5(x+\mathbb{C}) \\ S_1(x+\mathbb{C}) & S_2(x+\mathbb{C}) & S_3(x+\mathbb{C}) & S_4(x+\mathbb{C}) \\ S_0(x+\mathbb{C}) & S_1(x+\mathbb{C}) & S_2(x+\mathbb{C}) & S_3(x+\mathbb{C}) \\ 0 & S_0(x+\mathbb{C}) & S_1(x+\mathbb{C}) & S_2(x+\mathbb{C}) \end{vmatrix} /$$

$$/ \begin{vmatrix} S_3(\mathbb{C}-x) & S_4(\mathbb{C}-x) & S_5(\mathbb{C}-x) \\ S_2(\mathbb{C}-x) & S_3(\mathbb{C}-x) & S_4(\mathbb{C}-x) \\ S_1(\mathbb{C}-x) & S_2(\mathbb{C}-x) & S_3(\mathbb{C}-x) \end{vmatrix}$$

$$= \begin{vmatrix} S_2(x+\mathbb{C}) & S_3(\mathbb{C}) & S_4(\mathbb{C}) & S_5(\mathbb{C}) \\ S_1(x+\mathbb{C}) & S_2(\mathbb{C}) & S_3(\mathbb{C}) & S_4(\mathbb{C}) \\ S_0(x+\mathbb{C}) & S_1(\mathbb{C}) & S_2(\mathbb{C}) & S_3(\mathbb{C}) \\ 0 & S_0(\mathbb{C}) & S_1(\mathbb{C}) & S_2(\mathbb{C}) \end{vmatrix} / \begin{vmatrix} S_3(\mathbb{C}) & S_4(\mathbb{C}) & S_5(\mathbb{C}) & x^3 \\ S_2(\mathbb{C}) & S_3(\mathbb{C}) & S_4(\mathbb{C}) & x^2 \\ S_1(\mathbb{C}) & S_2(\mathbb{C}) & S_3(\mathbb{C}) & x^1 \\ S_0(\mathbb{C}) & S_1(\mathbb{C}) & S_2(\mathbb{C}) & x^0 \end{vmatrix}$$

The above expressions degenerate when the extra factor $R(\mathbb{A},\mathbb{B}) = S_{n^m}(\mathbb{A} - \mathbb{B})$ is null, i.e. when the determinant

$$\begin{vmatrix} f_n & \cdots & f_{m+n-1} \\ \vdots & & \vdots \\ f_{m-n+1} & \cdots & f_m \end{vmatrix}$$

is null.

Qestions of convergence and algorithms to compute by recursion the different determinants corresponding to variable m and n have been studied by Padé in his thesis, and have generated a numerous litterature which has not much relations with our algebraic manipulations.

Rational interpolation mostly reduces to using different determinantal expressions of the resultant $R(\mathbb{A},\mathbb{B})$ coming from the fact that it is a special Schur function.

Newton interpolation

Newton's approach to interpolation is different from Lagrange's. Interpolation points are totally ordered, and adding a new interpolation point has the effect of adding an extra term to the interpolation polynomial. As for Lagrange interpolation, we have to interpret Newton's interpolation as an identity on polynomials.

Given an infinite totally ordered alphabet $\mathbb{A}$, recall that $\mathbb{A}_n$ is the subinitial alphabet $\mathbb{A}_n := \{a_1, \ldots, a_n\}$. Because $\{1,\, R(x,\mathbb{A}_1),\, R(x,\mathbb{A}_2), \ldots\}$ is a linear basis of polynomials (triangulary equivalent to $\{1,x,x^2,\ldots\}$), we can expand any polynomial $f(x)$ in this basis :

$$(Ni1) \quad f(x) = f(a_1) + f^{\partial} R(x,\mathbb{A}_1) + f^{\partial\partial} R(x,\mathbb{A}_2) + f^{\partial\partial\partial} R(x,\mathbb{A}_3) + \cdots \ ,$$

the coefficients $f,f^{\partial},f^{\partial\partial},\ldots$ being uniquely determined.

Taking $x = a_1$, $x = a_2$, $x = a_3$, ..., we can easily identify the first of them :

$$f^{\partial} = (f(a_1) - f(a_2))/(a_1 - a_2) \ ,$$

$$f^{\partial\partial} = (f(a_1)-f(a_2))/(a_1-a_2)(a_2-a_3) - (f(a_1)-f(a_3))/(a_1-a_3)(a_2-a_3), \ldots$$

and infer that $f^{\partial\ldots\partial}$ is the image of $f(a_1)$ under a product of successive divided differences $\partial_1,\partial_2,\partial_3,\ldots$ acting on their left.

Thus one is led to Newtons's formula :

$(Ni2)$

$$f(x) = f(a_1) + f\partial_1\, R(x,\mathbb{A}_1) + f\partial_1\partial_2\, R(x,\mathbb{A}_2) + f\partial_1\partial_2\partial_3\, R(x,\mathbb{A}_3) + \cdots$$

It is sufficient to prove this formula for a linear basis of polynomials, for example the polynomials $1, R(x,\mathbb{B}_1), R(x - \mathbb{B}_2), \ldots$ associated to another infinite alphabet $\mathbb{B}$. Because the action of divided differences on complete functions is easy, remembering that $R(x,\mathbb{B}_k)$ can be written $S_k(x - \mathbb{B}_k)$, we have to prove all the identities, for $k = 0,1,2\ldots$

$$(Ni3) \qquad S_k(x - \mathbb{B}_k) = S_k(\mathbb{A}_1 - \mathbb{B}_k) + S_{k-1}(\mathbb{A}_2 - \mathbb{B}_k)\, R(x,\mathbb{A}_1)$$

$$+ S_{k-2}(\mathbb{A}_3 - \mathbb{B}_k)\, R(x,\mathbb{A}_2) + \cdots + S_0(\mathbb{A}_k - \mathbb{B}_k) R(x,\mathbb{A}_k)$$

But now, we could have taken any $\mathbb{B}$ to have a linear basis of polynomials, for example $\mathbb{B} = \mathbb{A}$.

In this case every $S_{k-j+1}(\mathbb{A}_j - \mathbb{A}_k)$, $j < k$, is null, because it is, up to a sign, the $k-j+1$ elementary symmetric function of an alphabet ($= \mathbb{A}_k \setminus \mathbb{A}_j$) of cardinal $k-j$. The above summation reduces to $R(x - \mathbb{A}_k) = S_0() R(x - \mathbb{A}_k)$

$\square$

Newton's formula is closely connected with a discrete analog of the Wronskian, and could be checked using it.

Indeed, given n one variable-functions f_i, we define their $\mathbb{A}$-Wronskian to be

$$(Ni4) \qquad W_{\mathbb{A}}(f_1,\ldots,f_n) := det\,|\,f_i\partial_1\cdots\partial_{j-1}\,|_{1\le i,j\le n}$$

Lemma $(Ni5)$. — *The $\mathbb{A}$-Wronskian of n functions $f_1,\ldots,f_n$ is equal to the quotient $det\,|f_i(a_j)|\;/\;\Delta(\mathbb{A})$.*

It is also equal to the determinant

$$det\,|f_i\,\partial_1\cdots\partial_{j-1}\pi_j\cdots\pi_{n-1}|\ .$$

Proof. — Replace each $f(a_j)$ by its Newton's expansion: $f(a_j) = f(a_1) + \cdots + f\partial_1\cdots\partial_{j-1}\,R(a_j,\mathbb{A}_{j-1})$. Then we see that the determinant has rows equal to linear combinations of the rows of the Wronskian,

$$row_j = row_1 + row_2\,R(a_j,\mathbb{A}_1) + \cdots + row_j\,R(a_j,\mathbb{A}_{j-1})\ .$$

Therefore, the two determinants coincide, up to the factor

$$R(a_2,\mathbb{A}_1)R(a_3,\mathbb{A}_2)\cdots R(a_n,\mathbb{A}_{n-1}) = \Delta(\mathbb{A})\ .$$

Because $det|f_i(a_j)|/\Delta(\mathbb{A})$ is a symmetric function, one does not change the value of the Wronskian by taking its image under

$$\pi_\omega = (\pi_{n-1})(\pi_{n-2}\pi_{n-1})\cdots(\pi_1\cdots\pi_{n-1})\ .$$

Now, the Wronskian has been written as a determinant such that the entries in the successive rows are symmetric functions of $\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_n$ respectively. The operator π_{n-1} operates non trivially only on row $n-1$, then the operator $\pi_{n-2}\pi_{n-1}$ operates on row $n-2$ of the new determinant, and finally the image of the Wronskian under π_ω is the determinant where each of the operators (π_{n-1}), $(\pi_{n-2}\pi_{n-1})$, $\ldots(\pi_1\cdots\pi_{n-1})$ have been applied to the appropriate row. The entries of the final determinant are now symmetric functions in $\mathbb{A}$ $\square$

In the special case where all the functions f_j are powers of a variable, we see that the Wronskian interpolates between the expression of a Schur function as a determinant of complete functions, and as a quotient of determinant of powers by the Vandermonde. For example, with $\mathbb{A}$ of cardinal 3, one has

$$W_{\mathbb{A}}(x^2,x^5,x^9) = \begin{vmatrix} S_2(\mathbb{A}_1) & S_5(\mathbb{A}_1) & S_9(\mathbb{A}_1) \\ S_1(\mathbb{A}_2) & S_4(\mathbb{A}_2) & S_8(\mathbb{A}_2) \\ S_0(\mathbb{A}_3) & S_3(\mathbb{A}_3) & S_7(\mathbb{A}_3) \end{vmatrix} = \begin{vmatrix} a_1^2 & a_1^5 & a_1^9 \\ a_2^2 & a_2^5 & a_2^9 \\ a_3^2 & a_3^5 & a_3^9 \end{vmatrix} \;/\; \Delta(\mathbb{A}) =$$

$$\begin{vmatrix} a_1^2\pi_1\pi_2 & a_1^5\pi_1\pi_2 & a_1^9\pi_1\pi_2 \\ S_1(\mathbb{A}_2)\pi_2 & S_4(\mathbb{A}_2)\pi_2 & S_8(\mathbb{A}_2)\pi_2 \\ S_0(\mathbb{A}_3) & S_3(\mathbb{A}_3) & S_7(\mathbb{A}_3) \end{vmatrix} = \begin{vmatrix} S_2(\mathbb{A}) & S_5(\mathbb{A}) & S_9(\mathbb{A}) \\ S_1(\mathbb{A}) & S_4(\mathbb{A}) & S_8(\mathbb{A}) \\ S_0(\mathbb{A}) & S_3(\mathbb{A}) & S_7(\mathbb{A}) \end{vmatrix}$$

Given $\{a_1, a_2, \ldots\}$, Newton interpolation develops any polynomial $f(x)$ in the basis $\{(x-a_1)\cdots(x-a_k)\}$; the coefficients are obtained by applying divided differences on f.

Newton interpolation in several variables

A linear basis of the ring of polynomials in an alphabet $\mathbb{A}$ allows to expand any polynomial in $\mathbb{A}$, as we did for polynomials in one variable. It is reasonable to ask that this basis contains Newton's polynomials $R(a_1,\mathbb{B}_k)$, and has simple vanishing properties.

One may also start from the images of a polynomial in $\mathbb{A}$ under divided differences, evaluated in a second alphabet $\mathbb{B}$. It is easy to see that the divided differences ∂_μ, for all permutations μ, are linearly independent, and therefore there exist universal coefficients in $\mathbb{A}$,$\mathbb{B}$ which generalize Newton's polynomials, and allow the expansion of any polynomial f in terms of its images under all the divided differences.

We shall define recursively these polynomials. Let $\mathbb{N}^\infty$ denote the linear space of infinite integral vectors with only a finite numbers of non zero components. To each $I \in \mathbb{N}^\infty$, M.P. Schützenberger and I have attached a *Schubert polynomial* Y_I.

We first define *dominant Schubert polynomials* corresponding to dominant weights. For $K \in \mathbb{N}^\infty$, K weakly decreasing, i.e. $K = k_1 \geq k_2 \geq k_3 \geq \cdots 0\, 0 \cdots$, let Y_K be

$$(Nis1) \qquad Y_K := \prod\nolimits_{(i,j)} (a_i - b_j) \ ,$$

product on all points (i,j) of the Ferrers' diagram of K.

For example

$$Y_{221100\ldots} = \begin{matrix} (a_1 - b_2) & (a_2 - b_2) & & \\ (a_1 - b_1) & (a_2 - b_1) & (a_3 - b_1) & (a_4 - b_1) \end{matrix} \ ,$$

where the above planar display stands for the product of its entries.

Definition. — *Schubert polynomials* are all the non-zero images of dominant Schubert polynomials under divided differences.

We index Schubert polynomials according to the following reordering rules:

given $p \geq 0$ and $I \in \mathbb{N}^\infty$, write i,j for the p and $p+1$-th components of I. Then,

$$i > j \ \Rightarrow \ Y_{\ldots i\, j \ldots}\, \partial_p = Y_{\ldots j\, i-1 \ldots}$$

On the other hand, the nullity ${\partial_p}^2 = 0$ implies that

$$i \leq j \ \Rightarrow \ Y_{\ldots i\, j \ldots}\, \partial_p = 0 \ .$$

Note that the same Y_I can be obtained in several manners from dominants Y_K. We shall skip for the moment the proof that braid relations imply that the polynomials Y_I are uniquely determined by the above recursions (see also exercice ?).

When K is decreasing, the dominant Schubert polynomial Y_K can be written as a Schur function. Indeed, thanks to (Sf7), for $K = k_1 \geq k_2 \geq \cdots \geq k_n \geq 00\ldots$, writing $K^\omega = k_n \ldots k_1$, one has

$$(Nis2) \qquad Y_K = S_{K^\omega}(\mathbb{A}_n - \mathbb{B}_{k_n}, \ldots, \mathbb{A}_1 - \mathbb{B}_{k_1}) \ .$$

For example,

$$Y_{22110\ldots} = S_{1122}(\mathbb{A}_4 - \mathbb{B}_1, \mathbb{A}_3 - \mathbb{B}_1, \mathbb{A}_2 - \mathbb{B}_2, \mathbb{A}_1 - \mathbb{B}_2) =$$

$$= \begin{vmatrix} S_1(\mathbb{A}_4 - \mathbb{B}_1) & S_2(\mathbb{A}_3 - \mathbb{B}_1) & S_4(\mathbb{A}_2 - \mathbb{B}_2) & S_5(\mathbb{A}_1 - \mathbb{B}_2) \\ S_0(\mathbb{A}_4 - \mathbb{B}_1) & S_1(\mathbb{A}_3 - \mathbb{B}_1) & S_3(\mathbb{A}_2 - \mathbb{B}_2) & S_4(\mathbb{A}_1 - \mathbb{B}_2) \\ 0 & S_0(\mathbb{A}_3 - \mathbb{B}_1) & S_2(\mathbb{A}_2 - \mathbb{B}_2) & S_3(\mathbb{A}_1 - \mathbb{B}_2) \\ 0 & 0 & S_1(\mathbb{A}_2 - \mathbb{B}_2) & S_2(\mathbb{A}_1 - \mathbb{B}_2) \end{vmatrix} \ .$$

In particular, $Y_{00\ldots} = 1$.

It is clear that all specializations $Y_K(\mathbb{A},\mathbb{A})$, $K \neq 0$ decreasing, are null. We shall need more generally that all $Y_I(\mathbb{A},\mathbb{A})$ are null, apart from $Y_{00\ldots}$. Stronger vanishing properties will be proved in the next sections (see also exercice ?).

We also need that the Y_I are linearly independent, this coming from the fact that their leading monomial (for the lexicographic order from the right) is equal to $a^I = a_1^{I_1}\, a_2^{I_2}\, a_3^{I_3} \cdots$

For $I \in \mathbb{N}^n$, write $I = Jk$, $J \in \mathbb{N}^{n-1}$, $k = I_n$, and define recursively $\partial_{[I]}$ as the product

$$\partial_n \cdots \partial_{n+k-1}\, \partial_{[J]} \ .$$

It is clear by induction on n that $\partial_{[I]}$ sends $Y_{I0\ldots}$ to $Y_{00\ldots}$ and all other Schubert polynomials to 0 or another Schubert polynomial $\neq Y_{00\ldots}$, because the action of a simple divided difference consists into reordering the subscript or annihilating the polynomial.

For example,

$$I = 412 \Rightarrow \partial_{[412]} = \partial_3\partial_4\, \partial_{[41]} = (\partial_3\partial_4)(\partial_2)(\partial_1\partial_2\partial_3\partial_4) \ .$$

$$Y_{412}\, \partial_{[412]} = Y_{412}\, \partial_3\partial_4\partial_{[41]} = Y_{41000}\, \partial_{[41]} = Y_{00\ldots}$$

and more generally,

$$Y_{ijh}\partial_{[412]} = Y_{ijh}\, \partial_3\partial_4\partial_{[41]} = Y_{ij00\,h-2}\, \partial_{[41]} \text{ or } 0 \text{ if } h < 2,$$

$$= Y_{i\,0\,j-1\,0\,h-2}\, \partial_1\partial_2\partial_3\partial_4 \ \text{ or } 0 \text{ if } j = 0$$

$$Y_{0\,i-1\,j-1\,0\,h-2}\, \partial_2\partial_3\partial_4 = Y_{0\,j-1\,i-2\,0\,h-2}\, \partial_3\partial_4 \text{ or } 0 \text{ if } i \leq j \text{ or } i < 2$$

$= Y_{0\,j-1\,0\,i-3\,h-2}\,\partial_4 = Y_{0\,j-1\,0\,h-2\,i-4}$ or 0 if $i < 4$ or $i-3 \leq h-2$.

Finally, the image of Y_{ijh} is a Schubert polynomial iff

$$i \geq 4, j \geq 1, h \geq 2, i \geq h+2, i \geq j+1 \text{ and } 0 \text{ otherwise} .$$

It is remarkable that the above simple properties are sufficient to obtain the multi-variables generalisation of Newton's formula.

THEOREM ($Nis3$). — *Let $\mathbb{A}$ and $\mathbb{B}$ be two infinite alphabets. Then for any polynomial $f(\mathbb{A})$ in $\mathcal{P}ol(\mathbb{A})$, one has*

$$f(\mathbb{A}) = \sum\nolimits_I \; f\partial_{[I]}(\mathbb{B}) \; Y_I(\mathbb{A},\mathbb{B}) \; ,$$

where $\partial_{[I]}$ is the divided difference (acting only on $\mathbb{A}$) which sends Y_I onto $Y_{00\ldots} = 1$, and where the sum is over all $I \in \mathbb{N}^\infty$.

Proof. — It is sufficient to check the theorem on a linear basis of the polynomials in $\mathbb{A}$, which can involve the elements of $\mathbb{B}$ which are scalars for the divided differences in $\mathbb{A}$. Thus we choose the $Y_J(\mathbb{A},\mathbb{B})$ as a linear basis. Now images of Y_I under divided differences are either 0 or other Schubert polynomials, which specialize to 0 when $\mathbb{A} = \mathbb{B}$, except for $Y_{00\ldots}$. Thus, Newton's formula, in the case of the polynomial $f(\mathbb{A}) = Y_J$ reduces to the irrefutable identity

$$Y_J(\mathbb{A},\mathbb{B}) = Y_{00\ldots}\; Y_J(\mathbb{A},\mathbb{B}) \qquad \square$$

For example, writing $f\partial\cdots\partial$ for $f\partial\cdots\partial\,(\mathbb{B})$, and Y_I for $Y_I(\mathbb{A},\mathbb{B})$, the interpolation of $f := a_1^3 a_2$ is

$$f = f\,Y_{0,0,0,0} + f\partial_2\,Y_{0,1,0,0} + f\partial_1\,Y_{1,0,0,0} + f\partial_1\partial_2\,Y_{2,0,0,0} + f\partial_2\partial_1\,Y_{1,1,0,0} \\ + f\partial_1\partial_2\partial_3\,Y_{3,0,0,0} + f\partial_2\partial_1\partial_2\,Y_{2,1,0,0} f\partial_2\partial_1\partial_2\partial_3\,Y_{3,1,0,0} \; ,$$

that is

$$a_1^3 a_2 = b_1^3 b_2\,Y_{0,0,0,0} + b_1^3\,Y_{0,1,0,0} + (b_1^2 + b_1 b_2 + b_2^2)\,Y_{1,0,0,0} + (b_1^2 + b_1 b_2 + b_1 b_3)\,Y_{2,0,0,0} \\ + (b_1^2 + b_1 b_2 + b_2^2)\,Y_{1,1,0,0} + b_1\,Y_{3,0,0,0} + (b_1 + b_2 + b_3)\,Y_{2,1,0,0} + Y_{3,1,0,0} \\ .$$

The generalization of Lagrange interpolation to functions of several variables is no more complicated. Fix n sufficiently big (the "order of interpolation"), and let X_ω be the polynomial

$$X_\omega(\mathbb{A},\mathbb{B}) := Y_{n-1cdots210} = \prod_{i+j \leq n} (a_i - b_j) \; .$$

The vanishing properties that we need are this time :

LEMMA (*Nis*4). — *For any* $\mu \in \mathfrak{S}(n)$, *all the* $X_\omega(\mathbb{A},\mathbb{A}^\mu)$ *are null, except for* $\mu = \omega = (n,\ldots,1)$, *in which case*

$$X_\omega(\mathbb{A},\mathbb{A}^\omega) = \Delta(\mathbb{A}) := \prod_{i<j\leq n} (a_i - a_j) \ .$$

Proof. — If $X_\omega(\mathbb{A},\mathbb{A}^\mu) \neq 0$, then b_1 has to be specialized to a_n; this leaves only the value a_{n-1} for b_2, &c. ∎

THEOREM (*Nis*5). — *Let* $\mathbb{A}$ *be an infinite alphabet,* $f(a_1,a_2,\ldots)$ *a polynomial in* $\mathbb{A}$, *and* $n \in \mathbb{N}$ *be such that all the exponents* I *of the monomials appearing in* f *are (componentwise) majorized by* $(n-1,\ldots,1,0)$. *Let* $\mathbb{B}$ *be an alphabet of cardinal* n. *Then*

$$f(\mathbb{A}) = \sum_{\mu\in\mathfrak{S}(\mathbb{B})} f(\mathbb{B}^\mu)\, X_\omega(\mathbb{A},\mathbb{B}^{\omega\mu})/\Delta(\mathbb{B}^\mu) \ .$$

Proof. — For any permutation $\nu \in \mathfrak{S}(\mathbb{B})$, the specialization $\{a_1,\ldots,a_n\} = \mathbb{B}^\nu$ reduces the theorem to the identity

$$f(\mathbb{B}^\nu) = f(\mathbb{B}^\nu)\, X_\omega(\mathbb{B}^\nu, \mathbb{B}^{\omega\nu}\ /\Delta(\mathbb{B}^\nu) \ .$$

These $n!$ equations on f determine f and force the equality for general $\mathbb{A}$ ∎

For example, for $n = 3$, one has for all polynomials of the type $f = \sum_I c_I a^I$, with $I \leq (2,1,0)$

$$\begin{aligned} f(a_1,a_2,a_3) = \frac{1}{(b_1-b_2)(b_1-b_3)(b_2-b_3)} \Big(&-f(b_1,b_2,b_3)(b_2-a_1)(b_3-a_2)(b_3-a_1) \\ &+f(b_1,b_3,b_2)(b_3-a_1)(b_2-a_2)(b_2-a_1) - f(b_3,b_1,b_2)(b_1-a_1)(b_2-a_2)(b_2-a_1) \\ &+f(b_2,b_1,b_3)(b_1-a_1)(b_3-a_2)(b_3-a_1) - f(b_2,b_3,b_1)(b_3-a_1)(b_1-a_2)(b_1-a_1) \\ &+f(b_3,b_2,b_1)(b_2-a_1)(b_1-a_2)(b_1-a_1)\Big) \end{aligned}$$

Dominant polynomials are certain products of the type $\prod(a_i - b_j)$. Their images $Y_I(\mathbb{A},\mathbb{B})$, $I \in \mathbb{N}^\infty$, are called *Schubert polynomials.* The vanishing properties $Y_I(\mathbb{A},\mathbb{A}) = 0$, for $I \neq 0$, imply that any polynomial $f(\mathbb{A})$ decomposes onto the Schubert basis, the coefficients being the images of f under divided differences.

Interpolation of symmetric functions

One can use Newton's formula to interpolate symmetric functions. Let $f(\mathbb{A}_n)$ be a function symmetrical in $\{a_1,\ldots,a_n\}$. According to the preceeding theorem, one can write this function

$$(Isf1) \qquad f(\mathbb{A}_n) = \sum\nolimits_I c_I Y_I \ .$$

I claim that all the I corresponding to non zero coefficients c_I are of the type

$$I = 0 \le i_1 \le i_2 \cdots \le i_n \, 0\, 0 \ldots$$

Indeed, $f(\mathbb{A}_n)$ depending only on $a_1,\ldots,a_n$ is annihilated by all ∂_k, with $k > n$. Similarly, being symmetrical in $a_1,\ldots,a_n$, it is also annihilated by $\partial_1,\ldots,\partial_{n-1}$.

Therefore, all the Schubert polynomials appearing on the right side must be annihilated by the same divided differences, because otherwise we would have a sum of Schubert polynomials equal to 0. This is equivalent to say that I can have only the decrease $\ldots i_n \, 0 \ldots$ $\square$

Let us take for example a symmetric function of two variables, say $Y_{25}(\mathbb{A},\mathbb{C})$, where $\mathbb{C}$ is another infinite alphabet. Write $Y_I Y_J$ for the product $Y_I(\mathbb{B},\mathbb{C}) Y_J(\mathbb{A},\mathbb{B})$. Then formula $(Isf1)$ reads

$$\begin{aligned} Y_{25}(\mathbb{A},\mathbb{C}) &= Y_{25}Y_{00} + Y_{204}Y_{01} + Y_{014}Y_{11} + Y_{2003}Y_{02} \\ &+ Y_{0103}Y_{12} + Y_{20002}Y_{03} + Y_{0003}Y_{22} + Y_{01002}Y_{13} + Y_{200001}Y_{04} \\ &+ Y_{00002}Y_{23} + Y_{010001}Y_{14} + Y_{200000}Y_{05} + Y_{000001}Y_{24} + Y_{010000}Y_{15} + Y_{000000}Y_{25} . \end{aligned}$$

In the case $\mathbb{B} = 0$, the above expansion expresses the Schubert polynomial $Y_{25}(\mathbb{A},\mathbb{C})$ as a linear combination of all the Schur functions $S_I(a_1 + a_2) = Y_I(\mathbb{A},0)$, $I \subseteq 25$.

The symmetrical Schubert polynomials $Y_I(\mathbb{A},\mathbb{B})$ are a natural generalization of Schur functions : the leading term of $Y_I(\mathbb{A},\mathbb{B})$ (i.e. term of higher degree in $\mathbb{A}$) is the Schur function $S_I(\mathbb{A}_n) = Y_I(\mathbb{A},0)$, and can be written as determinants.

Indeed, let $I = i_1 \le \cdots \le i_n$, and $K = I^\omega + \rho := i_n + n - 1, \ldots, i_2 + 1, i_1 + 0$. Then, still writing $\mathbb{B}_k$ for $\{b_1,\ldots,b_k\}$, one has, according to (Sf11)

$$Y_K = R(a_n,\mathbb{B}_{k_1}) \cdots R(a_1,\mathbb{B}_{k_n}) = S_{K^\omega}(\mathbb{A}_n - \mathbb{B}_{k_1},\ldots,\mathbb{A}_1 - \mathbb{B}_{k_n}) \ .$$

The image of Y_K under ∂_ω, with $\omega = n \cdots 1$, is by definition Y_I, but thanks to (So16), it is also equal to

$$(Isf2) \qquad Y_I = S_I(\mathbb{A}_n - \mathbb{B}_{k_1},\ldots,\mathbb{A}_n - \mathbb{B}_{k_n}) \ ,$$

the leading term being evidently the Schur function $S_I(\mathbb{A}_n)$. It is a special case of the determinant used in [L1].

Expression (So7) shows that $\partial_\omega(Y_K)$ can be written as the quotient

$$(Isf3) \qquad Y_I = \left| R(a,\mathbb{B}_{k_1}),\,\ldots,R(a,\mathbb{B}_{k_n})\right|_{a\in\mathbb{A}_n} \,/\, \Delta(\mathbb{A}_n)\ ,$$

i.e.

$$(Isf4) \qquad Y_I = \sum\nolimits_\mu (-1)^{\ell(\mu)} \Big(R(a_1,\mathbb{B}_{k_1})\cdots R(a_n,\mathbb{B}_{k_n})\Big)^\mu \,/\, \Delta(\mathbb{A}_n)\ .$$

If I has repeated components, one can reduce the preceeding summation to a subset of the symmetric group.

For example, for any $m \geq 0$, let $r = m+n-1$; then

$$Y_{0^{n-1}m} = Y_{r\,00\ldots}\partial_1\cdots\partial_{n-1} = L_{\mathbb{A}_n}\big(R(a_1,\mathbb{B}_r)\big)\ .$$

Now Leibniz formula expresses the image of $R(a_1,\mathbb{B}_r)$ as a summation of products $\prod(a_i-b_j)$.

This can be visualized as follows. A string of boxes, some of them occupied by a $\bullet$, means a product of linear factors according to the following recepee : " in an empty box at position j, put the factor $a_i - b_j$, where $i-1$ is equal to the number of dots on the left of the box. Finally erase the $\bullet$'s."

For example,

$$\square\square\bullet\bullet\square\bullet\,\square = (a_1-b_1)(a_1-b_2)\ (a_3-b_5)\ (a_4-b_7)\ .$$

Now Y_r is equal to a string of r boxes. Its image under ∂_1 is equal to the sum of all strings with one dot,

$$Y_{0\,r-1} = \sum \square\ldots\square\bullet\square\ldots\square\ .$$

The next step, using ∂_2, puts one dot in a box right of the existing dot, and thus $Y_{0\,0\,r-2}$ is the sum of all strings of length r with two dots :

$$Y_{00\,r-2} = \sum \square\ldots\square\bullet\,\square\ldots\square\bullet\,\square\ldots\square\ .$$

Finally, $Y_{0^{n-1}m}$ is equal to the sum of all strings of r boxes with $n-1$ dots. In other words,

$(Isf5)$

$$Y_{0^{n-1}m} = \sum (a_1-b_1)\cdots(a_1-b_i)\,(a_2-b_{i+2})\cdots(a_2-b_j)\cdots(a_n-b_h)\cdots(a_n-b_r)\,.$$

Suppressing the void boxes, one could also write this summation as attached to a diagram of m boxes, with shifts in the indices of the b's given by the *contents* of the boxes. For more general diagrams, with $\mathbb{B} = \{0,1,2,\ldots\}$, this construction is due to Biedenharn-Louck.

Indeed, "Stirling interpolation" $\mathbb{B} = \mathbb{N} := \{0,1,2,\ldots\}$ of symmetric functions involves the specialisation of Schubert polynomials $Y_I(\mathbb{A},\mathbb{N})$, $I = i_1 \le \cdots \le i_n$, which are the *factorial Schur functions* of Biedenharn and Louck.

In that case, $R(a,\mathbb{N}_n) = a(a-1)\cdots(a-n+1)$ may be considered as a polarization of the nth-power and may be conveniently represented by the symbol $a^{[n]}$. Identity $(Isf3)$ reads now

$$(Isf6) \qquad Y_I(\mathbb{A},\mathbb{N}) = \left| a^{[k_1]},\ \ldots,\ a^{[k_n]} \right|_{a\in\mathbb{A}_n} \ / \ \Delta(\mathbb{A}_n)\ ,$$

while $(Isf5)$ becomes
$(Isf7)$

$$Y_{0^{n-1}m} = \sum (a_1)\cdots(a_1{-}i)\,(a_2{-}i{-}2)\cdots(a_2{-}j)\cdots(a_n{-}h)\cdots(a_n{-}r{+}1)\,,$$

sum on all $-1 \le i \le j \le \cdots \le h \le r$.

Expression $(Isf6)$ proves that factorial Schur functions can be characterized by their vanishing properties, as noticed by Okounkov. They are essential in the study of central elements in the universal envelopping algebra $\mathfrak{U}(\mathfrak{Gl}(\mathfrak{n}))$ (Olshanski, Okounkov,Nazarov, Molev).

One can add that factorial Schur functions can also be obtained as specializations of some Schubert polynomials in $\mathbb{A} = \{1,1,1\ldots\}$ and $\mathbb{B} = \{0,0,0\ldots\}$, the variables being now the components of the index I (and thus taking only integral values, but this sufficient to determine them). This leads to the theory of binomial determinants, as a part of bijective combinatorics so notoriously illustrated by our friend Xavier Viennot.

The relevant properties of factorial Schur functions are now related to dimensions of representations of the linear groups or unitary groups, as functions of the components of highest weight vectors.

Some operations on symmetric functions can be explicited using Newton's interpolation. Let us describe, for example, the uniform translation $\mathbb{A} \longrightarrow \mathbb{A}^+ := \{a_1{+}1,a_2{+}1,\ldots,a_n{+}1\}$. Now, divided differences are invariant under this translation, and $(Isf1)$ becomes, for any $J = j_1 \le j2 \le \cdots \le j_n\,00\ldots$, denoting $Y_{J/I}$ the image of Y_J under $\partial_{[I]}$:

$$(Isf8) \qquad Y_J(\mathbb{A}^+) = \sum_I Y_{J/I}(\{1,\ldots,1\},0,\ldots 0)\, Y_J(\mathbb{A},\{0,\ldots,0\}\ .$$

Newton interpolation of functions symmetrical in $a_1,\ldots,a_n$, uses only Schubert polynomials which are symmetrical in $a_1,\ldots,a_n$ (and have leading term a Schur function). In the case where the interpolation alphabet B is equal to $\{0,1,2,\ldots\}$, these special Schubert polynomials are called *factorial Schur functions.*

The ring of polynomials as a free module over $\mathfrak{Sym}$

We have seen that any polynomial in $\mathbb{A}$ can be uniquely expanded as a linear combination of Schubert polynomials. We skipped the proof of some simple properties of these polynomials. We are going to reinterpret Newton's formula, and prove stronger properties that those left unproved.

First, we shall work with a finite alphabet $\mathbb{A}$ of cardinal n. We shall describe the structure of the ring $\mathcal{P}ol(\mathbb{A}) := \mathbb{Z}[\mathbb{B}][\mathbb{A}]$ of polynomials in $\mathbb{A}$, with coefficients involving a second alphabet $\mathbb{B}$ of the same cardinal, as a module over $\mathfrak{Sym}(\mathbb{A})$.

We define a quadratic form :

$$(Rp1) \qquad \mathcal{P}ol(\mathbb{A}) \ni f,g \Rightarrow (f,g) := fg\,\partial_\omega \ ,$$

where ∂_ω is the maximal divided difference relative to $\mathbb{A}$.

Lemma ($Rp2$). — *The divided differences ∂_i, $i = 1,\ldots,n-1$, are self-adjoint with respect to (,). Therefore, for $\mu \in \mathfrak{S}(\mathbb{A})$, ∂_μ is adjoint to $\partial_{\mu^{-1}}$.*

Moreover μ is adjoint to $(-1)^{\ell(\mu)}\mu^{-1}$.

Proof. — Given any $i < n$, factorize $\omega = \sigma_i\,\mu$. Then $(f\,\partial_i,g) = \big(f\partial_i\,g\,\partial_i\big)\partial_\mu$. Because $f\,\partial_i$ is a scalar for ∂_i, the preceeding expression is equal to

$$\big((f\partial_i)\,(g\partial_i)\big)\partial_\mu$$

and the symmetry between f and g implies that it is also equal to $(f\,,g\partial_i)$. By product, one gets the assertion for every ∂_μ.

Similarly $(f\mu\,,g) = f^\mu\,g\,\partial_\omega = f\,g^{\mu^{-1}}\,\mu\partial_\omega = (-1)^{\ell(\mu)}f\,g^{\mu^{-1}}\,\partial_\omega = (-1)^{\ell(\mu)}(f\,,g\mu^{-1})$. □

The *code* of a permutation $\mu \in \mathfrak{S}(n)$ is the vector $c[\mu] = [c_1,\ldots,c_n]$ such that $c_i := card(j > i, \mu_j < \mu_i)$. It is immediate that $\mu \longrightarrow c[\mu]$ is a bijection between $\mathfrak{S}(n)$ and the vectors $c \in \mathbb{N}^n$ such that $c \leq \rho := [n-1,\ldots,0,1]$ (i.e. $c_i \leq n-i$, $i = 1,\ldots,n$).

The *maximal* Schubert polynomial $X_\omega(\mathbb{A},\mathbb{B})$, that we choose to index by the permutation ω $(:= n\ldots1)$, is set to be

$$(Rp3) \qquad X_\omega(\mathbb{A},\mathbb{B}) := \prod_{i+j\leq n} (a_i - b_j) \ .$$

For any permutation $\mu \in \mathfrak{S}(n)$, one defines the *Schubert polynomial* X_μ by

$$(Rp4) \qquad X_\mu(\mathbb{A},\mathbb{B}) := X_\omega(\mathbb{A},\mathbb{B})\,\partial_{\omega\mu}$$

recalling that divided differences act only the a_i's.

A permutation is sais to be *dominant* if its code I is dominant, i.e. weakly decreasing. In that case, it is easy to see that there exists at least one sequence of operations of the type

$$(\cdots j+1\ j\cdots) \longrightarrow (\cdots j\ j\cdots)$$

which leads from ρ to the code I. This means that there exists at least a chain

$$\mu^0=\omega, \mu^1, \ldots, \mu^r=\mu$$

of dominant permutations, together with a sequence of simple transpositions $\sigma^1,\ldots,\sigma^r$, such that

$$X_{\mu^{j-1}}\partial_{\sigma^j} = X_{\mu^j}\ ,\quad j=1,\ldots,r\ .$$

Supposing that $X_{\mu^{j-1}}$ is equal to the polynomial Y_K defined in $(Nis1)$, with $K=code(\mu^{j-1})=[k_1,\ldots,k_n]$, let p be the integer such that $\sigma^j=\sigma_p$. Now, because $k_p=k_{p+1}+1$, the polynomial Y_K is equal to a polynomial F symmetrical in a_p,a_{p+1}, multiplied by a single factor (a_p-b_m), for some m. Thus the image of Y_K under ∂_p is equal to F, and finally $X_{\mu^j}=F$ is the dominant polynomial indexed by the code of μ^j.

The following property shows that the maximal Schubert polynomial is a "reproducing kernel", modulo the identification (that we note by "$\equiv$") of the symmetric functions in $\mathbb{A}$ with the same symmetric functions in $\mathbb{B}$.

Proposition $(Rp5)$. — *For any element $f\in\mathcal{P}ol(\mathbb{A})$, one has*

$$\big(f(\mathbb{A}),X_\omega(\mathbb{A}^\omega,\mathbb{B})\big)\equiv f(\mathbb{B})\ ,$$

modulo the relation $\mathfrak{Sym}(\mathbb{A})=\mathfrak{Sym}(\mathbb{B})$.

Proof. — The formula is linear. Writing any monomial as a Schur function (according to (Sf10)) :

$$S_{ij\ldots hk}(\mathbb{A}, \mathbb{A}-a_1,\ldots,\mathbb{A}-a_1-\cdots-a_{n-2},\mathbb{A}-a_1-\cdots-a_{n-1})$$

we see that the expansion of such a function involves only monomials a^J with $J\le(n-1,\ldots,1,0):=\rho$, as we already have noticed in section 2. In other words, any element of $\mathcal{P}ol(\mathbb{A})$ can be written as a linear combination of monomials $a^J : J\le\rho$ with coefficients in $\mathfrak{Sym}(\mathbb{A})$. On the other hand, the expansion of $X_\omega(\mathbb{A}^\omega,\mathbb{B})$ involves only monomials a^I: $I\le(0,1,\ldots,n-1)$. Therefore, the product $f(\mathbb{A})X_\omega(\mathbb{A}^\omega,\mathbb{B})$ involves only monomials a^H: $H\le(n-1,\ldots,n-1)$. Such a monomial has a non zero image under ∂_ω iff all the exponents are different, i.e. iff the exponent is a permutation of ρ; in this case, the determinant is obtained by permutations of columns from the determinant expressing $S_{0\ldots 0}(\mathbb{A})$ and thus is equal to ± 1.

In any case, this image is independent of $\mathbb{A}$. Consequently, taking $\mathbb{A} = \mathbb{B}$ in the evaluation of the quadratic form $\big(f(\mathbb{A}), X_\omega(\mathbb{A}^\omega, \mathbb{B})\big)$ has the only effect of replacing symmetric functions in $\mathbb{A}$ by the same functions in $\mathbb{B}$. On the other hand, in the second expression (So7) of the scalar product

$$\big(f(\mathbb{A}), X_\omega(\mathbb{A}^\omega, \mathbb{B})\big) = \sum\nolimits_\mu (-1)^{\ell(\mu)} f(\mathbb{A}^\mu) X_\omega(\mathbb{A}^{\omega\mu}, \mathbb{B}) \,/\, \Delta(\mathbb{A})$$

there remains only one term under the specialization $\mathbb{A} = \mathbb{B}$, that is for $\mu = identity$, the term $f(\mathbb{B}) X_\omega(\mathbb{B}^\omega, \mathbb{B}) / \Delta(\mathbb{B}) = f(\mathbb{B})$, as required □

More generally, one can obtain divided differences by evaluating scalar products, as shown by the following proposition.

Theorem ($Rp6$). — *For any element $f \in \mathcal{P}ol(\mathbb{A})$, any permutation μ, one has*

$$\big(f(\mathbb{A}), X_{\mu\omega}(\mathbb{A}^\omega, \mathbb{B})\big) \equiv f\, \partial_{\mu^{-1}}(\mathbb{B})$$

Proof. — One has

$$X_{\mu\omega}(\mathbb{A}^\omega, \mathbb{B}) = X_\omega(\mathbb{A}, \mathbb{B})\, \partial_{\omega\mu\omega}\, \omega = X_\omega(\mathbb{A}, \mathbb{B}) \omega \left(\omega \partial_{\omega\mu\omega}\, \omega\right) = X_\omega(\mathbb{A}, \mathbb{B})\, \omega\, \partial_\mu \ ,$$

and one can shift the divided difference ∂_μ to the left of $(\,,\,)$ thanks to ($Rp2$). Now the formula becomes the preceeding one for the function $f\, \partial_{\mu^{-1}}$ □

Corollary ($Rp7$). — *The quadratic form $(\,,\,)$ is non-degenerate. The space $\mathcal{P}ol(\mathbb{A})$ is a free module over $\mathfrak{Sym}$ with the two adjoint bases $\{X_\nu(\mathbb{A}, \mathbb{B})\}$ and $\{X_{\nu\omega}(\mathbb{A}^\omega, \mathbb{B})\}$, for all permutations μ, ν in $\mathfrak{Sym}(\mathbb{A})$.*

More precisely

$$\big(X_\nu(\mathbb{A}, \mathbb{B})\, ,\, X_{\mu\omega}(\mathbb{A}^\omega, \mathbb{B})\big) = \delta_{\mu,\nu} \ .$$

Having the basis adjoint to the Schubert polynomials, one can now expand any polynomial f in $\mathcal{P}ol(\mathbb{A})$ as a linear combination of Schubert polynomials:

$$(Rp8) \qquad f(\mathbb{A}) \equiv \sum_\mu \big(f\, ,\, X_{\mu\omega}(\mathbb{A}^\omega, \mathbb{B})\big)\, X_\mu(\mathbb{A}, \mathbb{B})$$

$$= \sum_\mu f\, \partial_{\mu^{-1}}(\mathbb{B})\, X_\mu(\mathbb{A}, \mathbb{B}) \ .$$

The preceeding formula uses the identification $\mathfrak{Sym}(\mathbb{A}) = \mathfrak{Sym}(\mathbb{B})$. When n is sufficiently big so that f belongs to the space generated by monomials a^I, $I \leq (n-1,\ldots,1,0)$, then the symmetric functions $\big(f\,, X_{\mu\omega}(\mathbb{A}^\omega,\mathbb{B})\big)$ are of degree 0 in $\mathbb{A}$, and thus the formula is exact in that case and coincide with Newton's interpolation.

One can use $(Rp8)$ to develop a Schubert polynomial itself. The following formula is a generalization of the expansion of the resultant $R(\mathbb{A},\mathbb{B})$ as a sum of products $\pm S_I(\mathbb{A})S_{I'}(\mathbb{B})$ due to Cauchy.

THEOREM $(Rp9)$. — *Let $\mathbb{A},\mathbb{B},\mathbb{C}$ be three alphabets of cardinal n, and let $\zeta \in \mathfrak{S}(n)$. Then*

$$X_\zeta(\mathbb{A},\mathbb{C}) = \sum_{\mu,\eta} X_\mu(\mathbb{A},\mathbb{B})\, X_\eta(\mathbb{B},\mathbb{C}) \ ,$$

sum on all reduced factorizations $\zeta = \mu\eta$ (i.e. factorizations such that $\ell(\mu) + \ell(\eta) = \ell(\zeta)$).

Proof. — To have the non nullity of $X_\zeta \partial_{\eta^{-1}}$ is the same as to require that the product $(\zeta\eta^{-1})(\eta)$ be reduced. Taking into account that $X_\zeta(\mathbb{A},\mathbb{C})$ belong to the span of monomials in $\mathbb{A}$ of degree $\leq \rho$, the theorem follows from $(Rp8)$ $\square$

All the operators that we have used are linear combinations of permutations, with coefficients which are rational functions of the a_i's.

For example,

$$\partial_1 = \frac{1}{a_1 - a_2} - \sigma_1 \frac{1}{a_1 - a_2}$$

$$\begin{aligned}\partial_1\partial_2 &= \Big(\frac{1}{a_1 - a_2} - \sigma_1 \frac{1}{a_1 - a_2}\Big)\Big(\frac{1}{a_2 - a_3} - \sigma_2 \frac{1}{a_2 - a_3}\Big) \\ &= (1-\sigma_1)\frac{1}{(a_1 - a_2)(a_2 - a_3)} + (1-\sigma_2)\frac{1}{(a_1 - a_3)(a_2 - a_3)}\end{aligned}$$

All the computations shown in these notes could be performed in the algebra of the symmetric group with rational coefficients in $\mathbb{A}$. Let us call this algebra the *algebra of divided differences*, not to confuse it with the group algebra with constant coefficients. In it we have to use the commutations relations (P denoting the operation "multiplication by P") :

$$\begin{cases} P\partial_i = (P\partial_i) + \partial_i\, P^{\sigma_i} \\ P\sigma_i = \sigma_i\, P^{\sigma_i} \end{cases}$$

The next proposition shows that we already know how to pass from permutations to divided differences, and conversely.

PROPOSITION ($Rp10$). — *Let* $\mathbb{A} = \{a_1, \ldots, a_n\}$, $\Delta := \prod_{i<j}(a_i - a_j)$. *then for any* $\mu \in \mathfrak{S}(\mathbb{A})$, *one has*

$$\begin{cases} \mu = \sum_\nu \ \partial_\nu \quad X_{\nu^{-1}}(\mathbb{A}^\mu \mathbb{A}) \\ \partial_\mu = \sum_\nu \ \nu \quad X_{\omega\mu}(\mathbb{A}, \mathbb{A}^{\omega\nu}) \end{cases}$$

Proof. — Newton's interpolation ($Rp8$) states that for any polynomial f, any μ, one has

$$f(\mathbb{A}^\mu) \equiv f\partial_\nu(\mathbb{B})\, X_{\nu^{-1}}(\mathbb{A}^\mu, \mathbb{B}) \ .$$

The left side can be interpreted as the operator $f(\mathbb{A}) \longrightarrow f(\mathbb{A}^\mu)$, that is μ, while the right side, where, of course, we can put $\mathbb{A} = \mathbb{B}$ because it does not affect the left side, is the operator $\sum \partial_\nu X_{\nu^{-1}}(\mathbb{A}^\mu \mathbb{A})$. This is exactly the first part of the claim.

Similarly, ($Rp6$) states that $f\partial_\mu$, evaluated in B, is congruent to $\sum f(\mathbb{A}^\nu) X_{\mu^{-1}\omega}(\mathbb{A}^{\omega\nu}, \mathbb{B})/\Delta(\mathbb{A}^\nu)$, and this is, for $\mathbb{A} = \mathbb{B}$, the second part of the claim, taking into account that

$$X_\nu(\mathbb{A}, ,\mathbb{B}) = (-1)^{\ell(\nu)} X_{\nu^{-1}}(\mathbb{B}, \mathbb{A}) \ ,$$

the sign being swallowed by $\Delta(\mathbb{A}^\nu) = (-1)^{\ell(\nu)}\Delta(\mathbb{A})$ $\square$

For example, for $\mu = \sigma_1\sigma_2$, one has $\mathbb{A}^{\sigma_1\sigma_2} = \{a_3, a_2, a_1\} := \mathbb{A}'$, and

$$\sigma_1\sigma_2 = 1 + \partial_1 X_{213}(\mathbb{A}', \mathbb{A}) + \partial_2 X_{132}(\mathbb{A}', \mathbb{A}) + \partial_1\partial_2 X_{312}(\mathbb{A}', \mathbb{A})$$

$$= 1 + \partial_1(a_3 - a_1) + \partial_2(a_3 + a_2 - a_1 - a_2) + \partial_1\partial_2(a_3 - a_1)(a_3 - a_2) \ .$$

Symmetrically,

$$\partial_1\partial_2\Delta = 1X_{213}(\mathbb{A}, \mathbb{A}^\omega) + \sigma_1 X_{213}(\mathbb{A}, \mathbb{A}^{\omega\sigma_1}) + \sigma_2 X_{213}(\mathbb{A}, \mathbb{A}^{\omega\sigma_2}) + \sigma_1\sigma_2 X_{213}(\mathbb{A}, \mathbb{A}^{\omega\sigma_1\sigma_2})$$

$$= 1(a_1 - a_3) + \sigma_1(a_1 - a_3) + \sigma_2(a_1 - a_2) + \sigma_1\sigma_2(a_1 - a_2) \ .$$

Without Schubert polynomials, it would have been more difficult to obtain ($Rp10$) in the algebra of divided differences (cf. [KK ?]).

For example, to compute ∂_μ, one has to start from a reduced decomposition $\mu = \sigma_i\sigma_j \cdots \sigma_k$, and write accordingly

$$\partial_\mu = (1 - \sigma_i)\frac{1}{a_i - a_{i+1}} + (1 - \sigma_j)\frac{1}{a_j - a_{j+1}} + \cdots + (1 - \sigma_k)\frac{1}{a_k - a_{k+1}} \ .$$

Now, one has to push all the coefficients either on the right, or on the left according to one's preferences, and sum all terms $\sigma_i^{\varepsilon_1}\sigma_j^{\varepsilon_2}\cdots$, $\varepsilon \in \{0,1\}$, which are decompositions of the same permutation.

With this method, it is not appearant that the entries of the inverse matrix which expresses the μ in terms of the ∂_ν are the same, up to multiplication by Δ and twist of indices. As a by-product however, one gets a relation between Schubert polynomials and the Ehresmann-Bruhat order on the symmetric group.

This order is defined by :
$\nu \leq \mu$ iff, given an arbitrary reduced decomposition of μ, $\mu = \sigma_i \sigma_j \cdots$, then there exists at least one choice of ε such that $\sigma_i^{\varepsilon_1} \sigma_j^{\varepsilon_2} \cdots = \nu$.

In other words, we have just observed that in the expansion of $\partial_\mu = (1-\sigma_i)\frac{1}{a_i - a_{i+1}} + (1-\sigma_j)\frac{1}{a_j - a_{j+1}} + \cdots$, and of $\mu = (1+\partial_i(x_{i+1}-x_i))(1+\partial_j(x_{j+1}-x_j)\cdots$, only permutations $\nu \leq \mu$ appear.

Comparing with $(Rp10)$, one obtains the implication $\Leftarrow$ of the following lemma :

Lemma $(Rp11)$. — *For any pair of permutations ν, μ, one has the equivalence*

$$\nu \leq \mu \;\Leftrightarrow\; X_{\nu^{-1}}(\mathbb{A}^\mu, \mathbb{A}) \neq 0 \;.$$

Given two alphabets $\mathbb{A}$, $\mathbb{B}$ of cardinal n, the ring $K[\mathbb{A}]$, with $K = \mathbb{Z}[\mathbb{A}]$, is a free-module over $\mathfrak{Sym}(\mathbb{A})$. Its Schubert basis $X_\mu(\mathbb{A},\mathbb{B})$ is in bijection with permutations μ in $\mathfrak{S}(n)$. The adjoint basis, with respect to the scalar product $(f,g) := fg\partial_\omega$, is $\{X_{\mu\omega}(\mathbb{A}^\omega,\mathbb{B})\}$.

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