Muir and the Theory of Recurrents

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Abstract

1 Introduction

The five volumes of Muir's "History ... of determinants" are without equivalent in the mathematical literature. He succeeded in giving a fairly complete and comprehensive survey of all the literature pertaining ot determinants, from the very first beginnings till his time. Only the last period (1900-1920) is not exhaustively covered, with developments in geometry and group representations hardly evoked.

Muir thought of summarizing the main properties of determinants in a single volume "A Treatise on the Theory of Determinants", which was revised and enlarged by W. Metzler, a copy of which I owe to the generosity of G.C. Rota. Thus, we can take the "Treatise" as a testimony of what Muir considered to be the main properties of determinants, as extracted from his five volumes, or his personal 300 articles on the subject.

As in many fields in mathematics, at a certain point of development, more and more specialized articles about seemingly more and special determinants, as well as generalizations without applications, were published, and this certainly discouraged mathematicians in other fields from looking at this literature. Another factor is that linear algebra, and commutative diagrams, can dispense of some of the properties of determinants in a more powerful way than manipulating algebraic identities only.

Of course the theory of determinants receives a new impetus from time to time, from combinatorics, group representations and statistical models in theoretical physics recently.

I shall examine in this text the contribution of Muir to the theory of recurrents. As a matter of fact, Muir does not define the term *recurrent*, leaving it imprecise enough to cover all type of determinants which arise in relation with linear recurrences, Taylor expansion of rational functions and so like. Thus the only working definition that I can take is that a recurrent is recurrent because it belongs to a chapter with this headline in one of the five volumes of the "History", or in the volume of the "Theory".

Recurrents can be treated using the theory of symmetric functions. Indeed many determinants occur in this last theory, and conversely, using the combinatorics of symmetric functions, and more generally, of the symmetric group, provides powerful methods to manipulate determinants. The main symmetric functions, the *Schur functions*, can be written in at least half a dozen ways as determinants, but most of their properties are proven without having recourse to the theory of determinants. In the other direction, relations as simple as Plücker relations give quadratic identities on Schur functions that one would have much difficulties to obtain from the known rules for multiplying Schur functions.

2 Symmetric functions and Determinants

Given two finite sets $\mathbb{A}, \mathbb{B}$ of variables (we say *alphabets*), one defines the complete functions $S^i(\mathbb{A}\text{–}\mathbb{B})$ of the the formal difference $\mathbb{A}\text{–}\mathbb{B}$ by the generating function (z is an extra variable)

$$\sigma_z(\mathbb{A} - \mathbb{B}) := \frac{\prod_{b\in\mathbb{B}}(1-zb)}{\prod_{a\in\mathbb{A}}(1-za)} \tag{1}$$

Similarly, the *power sums* $\Psi^i(\mathbb{A}\text{–}\mathbb{B})$ are defined by

$$\log\left(\sigma_z(\mathbb{A})\right) = \sum_{1}^{\infty} \Psi^i(\mathbb{A}\text{–}\mathbb{B})\, z^i/i\,. \tag{2}$$

The relations between the functions $\Psi^i(\mathbb{A}\text{–}\mathbb{B})$, $S^i(\mathbb{A}\text{–}\mathbb{B})$, $S^i(\mathbb{A})$, $S^i(\text{–}\mathbb{B})$ can be expressed through different determinants (not writing the argument when it is the same for all the symmetric functions appearing in the equation) :

$$(-1)^n S^n(\text{-}\mathbb{A}) = \begin{vmatrix} S^1(\mathbb{A}) & S^2(\mathbb{A}) & \cdots & S^n(\mathbb{A}) \\ S^0(\mathbb{A}) & S^1(\mathbb{A}) & \cdots & S^{n-1}(\mathbb{A}) \\ \vdots & & \vdots & \\ S^{2-n}(\mathbb{A}) & S^{1-n}(\mathbb{A}) & \cdots & S^1(\mathbb{A}) \end{vmatrix} \tag{3}$$

$$\Psi^n = \begin{vmatrix} n\,S^n & S^1 & \dots & S^n \\ (n{-}1)S^{n-1} & S^0 & \dots & S^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0\,S_0 & . & \dots & S^0 \end{vmatrix} \tag{4}$$

$$n!\,S^n = \begin{vmatrix} \Psi^1 & -1 & 0 & \dots & 0 \\ \Psi^2 & \Psi^1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Psi^{n-1} & \Psi_{n-2} & \Psi^{n-3} & \dots & 1{-}n \\ \Psi^n & \Psi^{n-1} & \Psi^{n-2} & \dots & \Psi^1 \end{vmatrix} \tag{5}$$

$$S^n(\mathbb{A}{-}\mathbb{B}) = \begin{vmatrix} S^n(\mathbb{A}) & S^1(\mathbb{B}) & \cdots & S^n(\mathbb{B}) \\ S^{n-1}(\mathbb{A}) & S^0(\mathbb{B}) & \cdots & S^{n-1}(\mathbb{B}) \\ \vdots & \vdots & & \vdots \\ S^0(\mathbb{A}) & S^{1-n}(\mathbb{B}) & \cdots & S^0(\mathbb{B}) \end{vmatrix} \tag{6}$$

One can use these determinants without reference to the theory of symmetric functions. Indeed, one can start with any formal series $f = 1 + \sum_1^\infty z^n S^n$, instead of $\sigma_z(\mathbb{A})$. The relations between the coefficients of f, f^k, $k \in \mathbb{C}$, and $\log(f)$ remain the same as when taking $\sigma_z(\mathbb{A})$ with $\mathbb{A}$ of infinite cardinality. However, factorizing formally f, i.e. deciding to write $f = \sigma_z(\mathbb{A})$ usually simplifies the algebraic manipulation of formal series, as it does in the case of polynomials, when considering their roots instead of only their coefficients.

For example, Muir gives several determinantal expressions of Bernoulli numbers B_n. Their exponential generating function is

$$f = \sum_0^\infty z^n B_n/n! = z(1 - \exp(-z))^{-1} \; .$$

Writing $f = \sigma_z(\mathbb{A})$ means that we introduce a "Bernoulli alphabet" $\mathbb{A}$ such that $S^n(\mathbb{A}) = B_n/n!$.

But this alphabet can be as well defined by either system of equations

$$S^n(-\mathbb{A}) \;=\; 1/(n+1)!\; n \geq 0\,, \tag{7}$$
$$\Psi^n(\mathbb{A}) \;=\; -S^n(\mathbb{A})\; n \geq 2\,, \tag{8}$$
$$S^n((n+1)\mathbb{A}) \;=\; 1\; n \geq 2,\; S^1(\mathbb{A}) = -1/2\;, \tag{9}$$

considering $1/f$ or $\log(f)$ instead of f, for the first two. The last system of equations is given by the Lagrange inversion of formal series; though a little unusual, it is the precise characterization of Bernoulli numbers which is needed by Hirzebruch [1, Lemma 1.7.1] to define Todd classes. In other terms, the last equations asssert that the coefficient of z^n in $\big(z(1-\exp(-z))^{-1}\big)^{n+1}$ is equal to 1.

Replacing in the determinants (3, 4, 5, 6) the symmetric functions of $\mathbb{A}$ by their explicit values give without computations relations involving Bernoulli numbers, inverses of factorials, or binomial coefficients from which one can deduce different special recurrents figuring in different places of Muir's History.

3 A specific example involving Catalan numbers

In [2], Theory p.731-732, Muir gives two determinants of binomial coefficients which factorize into products of consecutive odd integers.

Specifically, for $n \in \mathbb{N}$, let

$$R_n = \begin{vmatrix} \binom{1}{0} & -2 & 0 & 0 & \cdots \\ \binom{3}{1} & \binom{1}{0} & -4 & 0 & \cdots \\ \binom{5}{2} & \binom{3}{1} & \binom{1}{0} & -6 & \\ \vdots & \vdots & & & \ddots \\ \binom{2n-1}{n-1} & \binom{2n-3}{n-2} & \cdots & \cdots & \binom{1}{0} \end{vmatrix},$$

the overdiagonal being $-2, -4, \ldots, -(2n-2)$.

Muir shows by recursion on n that

$$R_n = (2n+3)(2n+5)\cdots(4n-3)(4n-1) .$$

How to interpret and generalize such a result ?

First, the determinant is of the type (5), after dividing each entry by 2. More generally, for any $k \in \mathbb{C}$, let us define M_n to be the image of the determinant (5) under the specialization $\Psi^i(\mathbb{A}_k) = k\binom{2i-1}{i-1}$ (we have indexed the alphabet by k to show its dependency upon k).

But in the case $k = 1$, $C(z) := \sigma_z(\mathbb{A}_1)$ is the generating function of Catalan numbers :

$$\Psi^i(\mathbb{A}_1) = \binom{2i-1}{i-1} \Rightarrow S^i(\mathbb{A}_1) = \binom{2i}{i-1}/(i+1) .$$

Therefore,

$$\sigma_z(\mathbb{A}_k) = (\sigma_z(\mathbb{A}_1))^k = C(z)^k = \left(\frac{1-\sqrt{1-4z}}{2z}\right)^k .$$

Using the functional equation $zC(z)^2 - C(z) + 1 = 0$, or rather,

$$zC(z)^{k+2} - C(z)^{k+1} + C(z)^k = 0 \ ,$$

one gets the relations

$$S^{n-1}(A_{k+2}) - S^n(A_{k+1}) + S^n(A_k) = 0 \ , \tag{10}$$

and this allows to determine the complete functions of A_k, when k is a positive ineger, supposing known those of $\mathbb{A}_0$ and $\mathbb{A}_1$. These expressions remain valid for any k, because they are polynomials in k, and one obtain the identity

$$S^n(\mathbb{A}_k) = \frac{k}{n+k}\binom{k+2n-1}{n} \ . \tag{11}$$

In conclusion, one has that the determinant

$$\begin{vmatrix} k\binom{1}{0} & -1 & 0 & 0 & \cdots \\ k\binom{3}{1} & k\binom{1}{0} & -2 & 0 & \cdots \\ k\binom{5}{2} & k\binom{3}{1} & k\binom{1}{0} & -3 & \\ \vdots & \vdots & & & \ddots \\ k\binom{2n-1}{n-1} & k\binom{2n-3}{n-2} & \cdots & \cdots & k\binom{1}{0} \end{vmatrix}$$

is equal to

$$\frac{k\,(k+2n-1)!}{(k+n)!},$$

the cases considered by Muir being $k = \pm 1/2$.

One can modify the last row of the determinant, by taking an extra parameter r, and replacing in this row $k\binom{2n+1-2j}{n-j}$ by

$$\binom{2n+1-2j+r}{n-j} \ , \ j = 1, \ldots, n \, .$$

An easy computation shows that the determinant still factorizes, and is equal to

$$(n+1+k+r)(n+2+k+r)\cdots(2n-1+k+r) \ . \tag{12}$$

The transformation has the same effect as a shifting k to $k+r$.

References

[1] F. Hirzebruch, *Toplogical Methods in Algebraic Geometry*, Third ed., Springer (1966).

[2] T. Muir, *Note on recurrents resolvable into a sequence of odd integers*, Trans. Royal Soc. of South Africa **8** (1919) 27–32.

[3] T. Muir, W. Metzler. *A Treatise on the Theory of Determinants*, Longmans, Green and Co (1933).