Schur Functions Series

September 25, 2004

1 Symmetrizing Operators

To follow usual conventions, operators act on their right.

Def. π_ω

Factorization corresponding to a Young subgroup $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$.

Define, for two sets of positive integers,

$$\mathcal{R}(I,J) = \prod_{i\in I,\, j\in J} (x_i - x_j) \ ,$$

$$\widetilde{\mathcal{R}}(I,J) = \prod_{i\in I,\, j\in J} (1 - x_j/x_i) \ .$$

Then

$$\pi_\omega = \Theta_{k|n-k}\, \pi_{\omega'\times\omega''} \ ,$$

where $\omega' \times \omega''$ is the maximal element of $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$, and

$$\Theta_{k|n-k}(f) = \sum \sigma \left(\frac{f}{\widetilde{\mathcal{R}}([1,\ldots,k],[k+1,\ldots n])} \right) \ , \tag{1}$$

where the sum is over all the permutations which are shuffles of $[1,\ldots,k]$ and $[k+1,\ldots n]$.

Equivalently, $\Theta_{k|n-k}$ can be understood as a summation over complementary subsets $X_I : \{x_i,\, i \in I\}$, X_J, of $\{x_1,\ldots,x_\}$:

$$\Theta_{k|n-k}(f) = \sum \frac{f(X_I, X_J)}{\widetilde{\mathcal{R}}(I,J)} \ ,$$

f being a function in $\mathfrak{S}ym(k|n-k)$, i.e. a function of n variables symmetrical in the first k ones, and in the last $n-k$ ones.

Notice that $\Theta_{1|n-1}$, restricted to functions of one variable, is the Lagrange-type operator

$$f \to \sum f(a) \prod_{b\in\mathbb{A}-a} (1 - b/a)^{-1} \ .$$

2 S-series by Symmetrization

Given any series $\sum c_\lambda x^\lambda$, its image under π_ω is $\sum c_\lambda S_\lambda(X)$. This will be interesting if all the Schur functions thus obtained are different. In particular, if all the λ are dominant, then the functions obtained are indexed by partitions and there are no cancellations.

For example, one obtains

$$\prod_i (1+x_i) \prod_{i<j} (1-x_i x_j) = \pi_\omega\big((1+x_1)(1+x_2^3)(1+x_3^5)\cdots\big)$$
$$= \pi_\omega\big(1 + x^{10..} + x^{03..} + x^{13..}\big) = 1 + S_1 - S_{21} - S_{22} + \cdots ,$$

$$\prod_{i\le j} (1-x_i x_j) = \pi_\omega\big((1-x_1^2)(1-x_2^4)(1-x_3^6)\cdots\big)$$
$$= \pi_\omega\big(1 - x^{20..} - x^{04..} + x^{24..}\big) = 1 - S_2 + S_{31} - S_{33} + \cdots$$

in this way. The simplest proof of these expansions is using Vandermonde determinants, as stated in the next section.

```
Litt:=proc(n)  local i,fr; fr:=w[];
  for i from 1 to n do
    fr:=FreeConcat(fr,w[0]+w[2*i-1])
  od;
  subs(w=s,fr)
end:
Litt2:=proc(n)  local i,fr; fr:=w[];
  for i from 1 to n do
    fr:=FreeConcat(fr,w[0]-w[2*i])
  od;
  subs(w=s,fr)
end:
ACE> CLG_n(3):   # fixes the number of variables

ACE> Litt(3);
s[0,0,5]+s[1,0,5]+s[0,3,5]+s[1,3,5]+s[0,0,0]+s[1,0,0]
                                    +s[0,3,0]+s[1,3,0]
ACE> factor(Tox_n(Tos(%)));
         # One reorders the indices, then evaluate in $x$
    -(1+x3)(1+x2)(-1+x2x3)(x1+1)(-1+x1x3)(-1+x1x2)
ACE> factor(Tox_n(Tos(Litt2(3))));
```

```
(x3-1)(1+x3)(x2-1)(1+x2)(-1+x2x3)(x1-1)(x1+1)
                                   (-1+x1x3)(-1+x1x2)
```

3 S-series from Vandermonde determinants

The expansion of functions of the type

$$\prod_{i<j}(1-x_ix_j)\ ,\ \prod_{i\le j}(1-x_ix_j)\ ,\ \prod_i(1-x_i)\prod_{i<j}(1-x_ix_j)\ ,\ \ldots$$

can be obtained by taking Vandermonde determinants in the variables $x_i + 1/x_i$. In that case, the action of π_ω can be written as a determinant, and one does not need any sophisticated theory to expand these functions in terms of Schur functions (cf. notes with Fumei).

For example, we have already used in the preceding section that

$$\pi_\omega\Big((1+x_1)(1+x_2^3)(1+x_3^5)\cdots\Big) = \sum_\epsilon S_{1\epsilon_1,3\epsilon_2,5\epsilon_3,..}\ , \tag{2}$$

sum over all $\epsilon\in\{0,1\}$. In the right hand side, the Schur functions are not indexed by (decreasing) partitions and their indices must be reordered.

On the other hand, one can write $\pi_\omega\Big((1+x_1)(1+x_2^3)(1+x_3^5)\cdots\Big)$ as a determinant (this is the case for all $\pi_\omega(f_1(x_1)f_2(x_2)f_3(x_3)\cdots)$). One then recognizes that it can be transformed into a Vandermonde for the variables $x_i + 1/x_i$, and one obtains that the function computed above is

$$\prod_i((1+x_i)\prod_{i<j}(1-x_ix_j)\ . \tag{3}$$

4 Functoriality

The argument of a symmetric function does not need to be restricted to a single alphabet, one can for example use differences of alphabets. If one shows an identity on symmetric functions $f(\mathbb{A}) = g(\mathbb{A})$ valid for any $\mathbb{A}$, then it will remain valid for $A - B$.

This has the consequence that knowing the expansion, in terms of Schur functions, of such products as

$$\prod(1-a_i)^{\pm1}\prod_{i<j}(1-a_ia_j)^{\pm1} \quad \& \quad \prod(1-a_i)^{\pm1}\prod_{i\le j}(1-a_ia_j)^{\pm1}\ ,$$

implies that these expansions remain valid for products like

$$\prod(1-a_i)^{\pm 1}\prod(1-b_i)^{\mp 1}\prod_{i\le j}(1-a_ia_j)^{\pm 1}\prod_{i<j}(1-b_ib_j)^{\pm 1}\prod(1-a_ib_j)^{\mp 1}$$

involving a second alphabet $\mathbb{B}=\{b_1,b_2,\ldots\}$.

Moreover, distinguishing between letters, i.e. taking $A+X$ instead of A, where X is a finite alphabet, and expanding in terms of the letters in X, allows to put more parameters in the usual series for a single alphabet A.

I explain this point of view in the article "Littlewood's formulas for characters of orthogonal and symplectic groups".

5 Using π_ω for Induction

Let us give an example how to compute one of the Littlewood formulas by induction on the number of variables, using π_ω.

Let us take $\sigma_1(X+\Lambda^2 X)$.

Given n, let

$$F(n)=\sigma_1(X_n+\Lambda^2X_n)=\prod_{1\le i\le n}(1-x_i)^{-1}\prod_{1\le i<j\le n}(1-x_ix_j)^{-1}$$

The image of $F(n-1)$ under π_ω, with $\omega=[n,\ldots,1]$ is given by writing

$$F(n-1)=F(n)\,(1-x_n)\prod_{1\le i<n}(1-x_ix_n)=\lambda_{-1}(x_n(1+X_{n-1}))\ ,$$

and computing the image of $\lambda_{-1}(x_n(1+X_{n-1}))$ under π_ω.

One finds that this image is $1-x_1\cdots x_n$, and therefore

$$\pi_\omega\big(F(n-1)\big)=(1-x_1\cdots x_n)\,F(n)\ .$$

Supposing known that $F(n-1)=\sum_{\lambda:\ell(\lambda)\le n-1}S_\lambda(X_{n-1})$, then one has, by taking the image of this equality under π_ω :

$$F(n)=\frac{1}{1-x_1\cdots x_n}\sum_{\lambda:\ell(\lambda)\le n-1}S_\lambda(X_n)$$
$$=\sum_{\lambda:\ell(\lambda)\le n}S_\lambda(X_n)\ . \quad (4)$$

6 Filtering by the biggest part

In the case where a function $F(n)$ of the preceding type has a complicated expression, a recursion on n will not be feasible.

It will be better to start from a sum $\sum c_\lambda x^\lambda$.

Let us take the problem of finding an expression for

$$F(u, [x_1, \ldots, x_n]) := \sum u^{\lambda_1} S_\lambda(X_n) \ ,$$

i.e. of filtering the sum of Schur functions by their greatest part, the sum being over all partitions with $\ell(\lambda) \le n$.

Evidently,

$$F(u, [x_1, \ldots, x_n]) = \pi_\omega \left(\frac{1}{(1-ux_1)(1-ux_1x_2)\cdots(1-ux_1\cdots x_n)} \right) \ ,$$

the power in x_1 being the first part of the partition.

However, $F(u, [x_1, \ldots, x_n])$ does not factorize. To write a simple expression of it, one must decompose it into simple fractions (relative to u).

This is the method followed by Desarménien (Séminaire Lotharingien) and Macdonald (see Ex.16, p.83; Example 5, p.232).

Let us add a factor $(1-u)$ and start with

$$G(u, [x_1, \ldots, x_n]) := \Big((1-u)(1-ux_1)(1-ux_1x_2)\cdots(1-ux_1\cdots x_n) \Big)^{-1}$$

For $i = 1, \ldots, n$, let ξ_i be the operator, acting only on x_i and u, such that

$$\xi_i : x_i \to 1/x_i \ , \ u \to ux_i \ .$$

Theorem 1

$$\begin{aligned}(1+\xi_1)\cdots(1+\xi_n)&\Big((1-u)^{-1} F(1, [x_1, \ldots, x_n]) \Big) \\ &= \pi_\omega \left(\frac{1}{(1-u)(1-ux_1)(1-ux_1x_2)\cdots(1-ux_1\cdots x_n)} \right) \\ &= (1-u)^{-1} F(u, [x_1, \ldots, x_n]) = \sum_k \sum_{\lambda : \lambda_1 \le k} u^k S_\lambda(X_n) \quad (5)\end{aligned}$$

Proof. We shall show the required identity by determining the residues, relative to u, of the first two functions. To avoid accumulation of indices, we shall compute only the residue for $u = 1/(x_1x_2x_3)$, which is sufficiently representative of the general case.

The coefficient of $(1-ux_1x_2x_3)^{-1}$ in the first function is

$$\xi_1\xi_2\xi_3\left(F(1,[x_1,\ldots,x_n])\right)$$
$$= F\left(1,[\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3}]\right) F(1,[x_4,\ldots,x_n])\widetilde{\mathcal{R}}(123,4\ldots n)^{-1}$$
$$= F(1,[x_1,x_2,x_3])\, F(1,[x_4,\ldots,x_n])\frac{(x_1x_2x_3)^n}{R(123,4\ldots n)} \ .$$

The second function is the image under π_ω of a $G(u,[x_1,\ldots,x_n])$. Lagrange tells us how to decompose a function of the type $\prod 1/(1-au)$. Therefore, we can decompose $G(u,[x_1,\ldots,x_n])$ as a sum of simple fractions $\sum \frac{f(a)}{R(a,A-a)}\frac{1}{1-ua}$, with the alphabet $A=\{1,x_1,x_1x_2,\ldots,x_1\cdots x_n\}$.

Explicitely, the coefficient of the term $1/(1-au)$, with $a=x_1x_2x_3$, in $G(u,[x_1,\ldots,x_n])$ is

$$a^n/R(a,\ A-a) \ .$$

The residue in $u=1/a$ is transformed by $\pi_{321\,n\ldots 4}$ as follows.

The symmetrisation in x_1,x_2,x_3 is the one of the preceding section for the variables $1/x_3,1/x_2,1/x_1$. One gets

$$\pi_{321}\Big(\big((a-1)(a-x_1)(a-x_1x_2)\big)^{-1}\Big) = F(1,[x_1,x_2,x_3]) \ .$$

The symmetrisation in $x_4,\ldots,x_n$ gives

$$\pi_{123\,n\ldots 4}\Big(a^{n-3}/R([a],[ax_4,\ldots,ax_4\cdots x_n])\Big)$$
$$= \pi_{123n\ldots 4}\Big(1/R([1],[x_4,\ldots,x_4\cdots x_n])\Big)$$
$$= F(1,[x_4,\ldots,x_n]) \ .$$

In total, the coefficient of $1/(1-ux_1x_2x_3)$ is now

$$(x_1x_2x_3)^3\, F(1,[x_1,x_2,x_3])\, F(1,[x_4,\ldots,x_n]) \ .$$

```
# Case n=5; residue of u=1/(x1x2x3)
ACE> factor(residue(1/(1-u)/(1-u*x1)/(1-u*x1*x2)
                  /(1-u*x1*x2*x3)/(1-u*x1*x2*x3*x4)
                  /(1-u*x1*x2*x3*x4*x5), u=1/x1/x2/x3));
                                     2
                               x2 x3
  - --------------------------------------------------------
```

```
     (x3 - 1)(x2 x3 - 1)(x1 x2 x3 - 1)(-1 + x4)(x4 x5 - 1)

# Symmetrisation by \pi_{321 54} : reduced decomposition [121 4]
ACE> factor(PiRdOnPol(%,[1,2,1,4]));
  2   2   2
x1  x2  x3 /((x1 x2-1)(x2 -1)(x1 x3 -1)(x4 x5 -1)(-1 + x4)(x3 -1)

    (x2 x3 - 1) (-1 + x1) (-1 + x5))
```

I remains to symmetrize between the variables in the two blocks $[x_1, x_2, x_3]$ and $[x_4, \ldots, x_n]$. This can be done with the operator $\Theta_{3|n-3}$ of the first section. This operator will transform a rational function

$$\frac{f(\{x_1, x_2, x_3\}\,,\,\{x_4, \ldots, x_n\})}{1 - ux_1x_2x_3}$$

into a rational function having simple poles $u = 1/(x_1x_2x_3)^\sigma$, for any $\sigma \in \mathfrak{S}_n/\mathfrak{S}_k \times \mathfrak{S}_{n-k}$. According to the expression (1) of $\Theta_{3|n-3}$, the operator has for effect to divide the residue in $u = 1/(x_1x_2x_3)$ by $\widetilde{\mathcal{R}}([1, \ldots, 3], [4, \ldots n])$.

In final, the coefficient of $1/(1 - ux_1x_2x_3)$ in

$$\pi_\omega\Big((x_1x_2x_3)^2\, F(1, [x_1, x_2, x_3])\, F(1, [x_4, \ldots, x_n])\Big)$$
$$= \Theta_{3|n-3}\Big((x_1x_2x_3)^2\, F(1, [x_1, x_2, x_3])\, F(1, [x_4, \ldots, x_n])\Big)$$

is

$$(x_1x_2x_3)^3\, F(1, [x_1, x_2, x_3])\, F(1, [x_4, \ldots, x_n])/\widetilde{\mathcal{R}}([1, \ldots, 3], [4, \ldots n])$$
$$= (x_1x_2x_3)^n\, F(1, [x_1, x_2, x_3])\, F(1, [x_4, \ldots, x_n])/\mathcal{R}([1, \ldots, 3], [4, \ldots n])\,.$$

One has found the same residue, for the two functions, in $u = 1/(x_1x_2x_3)$, and, by symmetry, in $u = 1/(x_ix_jx_k)$. This finishes the proof of the theorem.

Let us detail the case $n = 2$.

We start with

$$1/(1-u)(1-ux_1)(1-ux_1x_2)\ .$$

By Lagrange, it is written

$$\frac{1^2}{(1-u)(1-x_1)(1-x_1x_2)} + \frac{x_1^2}{(1-ux_1)(x_1-1)(x_1-x_1x_2)}$$
$$+ \frac{x_1^2x_2^2}{(1-ux_1x_2)(x_1x_2-1)(x_1x_2-x_1)}\ .$$

The image of the first term under $\pi_\omega = \pi_1$ is

$$\frac{1^2}{(1-u)(1-x_1)(1-x_2)(1-x_1x_2)}$$

(one factor added), and similarly, the last term becomes

$$\frac{x_1^2x_2^2}{(1-ux_1x_2)(1-x_1x_2)(1-x_1)(1-x_2)}$$

The image of the middle term does not factorize, and we must write it as a sum over $\mathfrak{S}_2$:

$$\frac{1}{(1-ux_1)(1-\frac{1}{x_1})(1-x_2)(1-\frac{x_2}{x_1})} + \frac{1}{(1-ux_2)(1-\frac{1}{x_2})(1-x_1)(1-\frac{x_1}{x_2})} .$$

The final step is just to recognize that these four terms are indeed the four terms in the expansion of

$$(1+\xi_1)(1+\xi_2)\,\left(\frac{1}{(1-u)(1-x_1)(1-x_2)(1-x_1x_2)}\right) .$$

7 Generalizing Gordon's Determinant

(See notes with Hou and Mu.)

To express the sum of all Schur functions indexed by partitions of length $\leq 2n$, Gordon wrote a determinant of order n, whose entries are linear combination of the $c_k := \sum_i S_i(\mathbb{A})S_{i+k}(\mathbb{A})$.

One can consider the same determinants, but using two alphabets $\mathbb{A}, \mathbb{B}$ and

$$c_k := \sum_i S_i(\mathbb{B})S_{i+k}(\mathbb{A}) \quad , \quad k \in \mathbb{Z} \ .$$

The new determinant is easier to evaluate than the preceding one, by induction on the cardinality of $\mathbb{A}$.

For example, for $n = 2$, one finds

$$\sum_\lambda S_\lambda B \left(1{+}S_1 B{-}S_{21}B{-}S_{22}B\right) S_\lambda\mathbb{A} = \sum_\lambda S_\lambda\mathbb{B} \left(S_\lambda\mathbb{A}{+}S_{\lambda/1}\mathbb{A}{-}S_{\lambda/21}\mathbb{A}{-}S_{\lambda/22}\mathbb{A}\right)$$

where the sum is over partitions with two parts at most.

Taking $\mathbb{B} = \mathbb{A}$, it gives an expression of the sum of all Schur functions indexed by partitions with 4 parts at most.

Let us see the construction in more details.

The determinants that one considers are determinants of order n, which can be expanded in sums of Jacobi-Trudi determinants with entries

$$c_k = \sum_i S_i(\mathbb{B})S_{i+k}(\mathbb{A})$$

instead of $h_k = S_k$. If expanded in terms of $\mathbb{A}$, such a determinant is a linear combination of Schur functions $S_\lambda(\mathbb{A})$, with $\ell(\mathbb{A}) \leq n$. Therefore, *one can restrict to an alphabet* $\mathbb{A}$ *of cardinality* n. However, one can replace Jacobi-Trudi determinants by determinants of powers of the variables $a \in \mathbb{A}$ (divided by the Vandermonde $\Delta(\mathbb{A})$). Thus, a determinant with a typical column

$$c_k, c_{k-1}, \ldots, c_{k-n+1} \ , \ k \geq 0$$

is equal, up to division by the Vandermonde, to a determinant with column

$$\left[\sum_i S_i(\mathbb{B}) a_1^{i+k}, \ldots, \sum_i S_i(\mathbb{B}) a_n^{i+k} \right]$$

i.e.

$$\left[\, a_1^k \sigma_1(a_1\mathbb{B}), \ldots, a_n^k \sigma_1(a_n\mathbb{B}) \,\right]$$

Such a determinant will reduce to a determinant of powers of the variables a_i times the function

$$\sigma_1(a_1\mathbb{B})\cdots\sigma_1(a_n\mathbb{B}) = \sigma_1(\mathbb{A}\mathbb{B}) \ .$$

Let us now take the two types of determinants used by Gordon.

The first type is

$$\Big| c_{n-i+j-1} + c_{n+i+j} \Big|_{i,j=0\ldots n-1} \ .$$

From what we said, when $|\mathbb{A}| = n$, then Gordon's determinant becomes

$$\sigma_1(\mathbb{A}\mathbb{B})\,\frac{1}{\Delta(\mathbb{A})}\,\Big| a^{n-1} + a^n,\, a^{n-2} + a^{n+1}, \ldots,\, a^0 + a^{2n-1} \Big|_{a\in\mathbb{A}} \ . \tag{6}$$

We recognize the determinant of powers, from our considerations about Littlewood formulas.

In final, Gordon's determinant is

$$\sigma_1(\mathbb{A}\mathbb{B})\,\sigma_1\big(\mathbb{A} - \Lambda^2(\mathbb{A})\big) \ .$$

For example, for $n = 2$, one obtains

$$\begin{vmatrix} c_1 + c_2 & c_0 + c_3 \\ c_0 + c_1 & c_{-1} + c_2 \end{vmatrix} \\ = \sum_{\lambda:\ell(\lambda)\le 2} S_\lambda(\mathbb{B})\,\big(S_\lambda(\mathbb{A}) + S_{\lambda/1}(\mathbb{A}) - S_{\lambda/21}(\mathbb{A}) - S_{\lambda/22}(\mathbb{A})\big) \ . \tag{7}$$

Specializing $\mathbb{B} = \mathbb{A}$, thanks to the interpretation of the determinant due to Gordon, Bender-Knuth, one gets

$$\sum_{\mu:\,\ell(\mu)\le 4} S_\mu(\mathbb{A}) = \sum_{\lambda:\ell(\lambda)\le 2} S_\lambda(\mathbb{A})\,\big(S_\lambda(\mathbb{A}) + S_{\lambda/1}(\mathbb{A}) - S_{\lambda/21}(\mathbb{A}) - S_{\lambda/22}(\mathbb{A})\big) \ . \tag{8}$$

The second type is

$$\Big| c_{n-i+j-1} - c_{n+i+j+1} \Big|_{i,j=0\ldots n-1} \ .$$

For $\mathbb{A}$ of cardinality n, we factorize it into

$$\sigma_1(\mathbb{A}\mathbb{B})\,\frac{1}{\Delta(\mathbb{A})}\,\Big| a^{n-1} - a^{n+1},\, a^{n-2} + a^{n+2}, \ldots,\, a^0 + a^{2n} \Big|_{a\in\mathbb{A}} \ . \tag{9}$$

This is also one of the determinants of Weyl and Littlewood, and Gordon's determinant is equal to

$$\sigma_1(\mathbb{A}\mathbb{B})\,\sigma_1\left(-S^2\mathbb{A}\right)\ . \tag{10}$$

This can be rewritten in terms of orthogonal Schur functions (Littlewood, p.240) :

$$\sigma_1(\mathbb{A}\mathbb{B})\,\sigma_1\left(-S^2\mathbb{A}\right) = \sum_{\lambda} S_\lambda(\mathbb{B})\,\mathcal{O}_\lambda(\mathbb{A})\ . \tag{11}$$

The expression

$$\sum_{\lambda:\,\ell(\lambda)\le n} S_\lambda(\mathbb{B})\,\mathcal{O}_\lambda(\mathbb{A}) \tag{12}$$

remains valid for any $\mathbb{A}$, $\mathbb{B}$, without restricting cardinalities.

For example,

$$c_0 - c_2 = \sum_i S_i(\mathbb{B})\mathcal{O}_i(\mathbb{A}) = \sum S_i(\mathbb{B})\left(S_i(\mathbb{A}) - S_{i-2}(\mathbb{A})\right)\ ,$$

$$\begin{aligned}\begin{vmatrix} c_1 - c_3 & c_0 - c_4 \\ c_0 - c_2 & c_{-1} - c_3 \end{vmatrix} &= \sum_{\lambda:\,\ell(\lambda)\le 2} S_\lambda(\mathbb{B})\,\mathcal{O}(\mathbb{A}) \\ &= \sum S_\lambda(\mathbb{B})\left(S_\lambda(\mathbb{A}) - S_{\lambda/2}(\mathbb{A}) + S_{\lambda/31}(\mathbb{A}) - S_{\lambda/33}(\mathbb{A})\right)\ . \end{aligned} \tag{13}$$

We have used

$$\sigma_1(-S^2) \qquad \& \qquad \sigma_1(S^1 - \Lambda^2)\ .$$

Let us see the series and the determinants corresponding to

$$\sigma_1(-S^1 - S^2)\ .$$

Since $S^2(\mathbb{A}+y) - y^2 = S^2(\mathbb{A}) + yS^1(\mathbb{A})$, one can use the series

$$\sigma_1(-S^2(\mathbb{A}+y) + y^2)\ .$$

Dividing the determinant associated to $\sigma_1(-S^2)$ by $(1-y^2)\prod_a(1-a^2)$, one obtains, after some transformation, the determinant of order $n+1$

$$\begin{vmatrix} a^0 + a^{2n} & a^1 + a^{2n-1} & \cdots & a^n + a^n \\ \vdots & \vdots & & \vdots \\ 1 & 1 & & 1 \end{vmatrix}$$

Subtracting the last column to all of them, one gets the determinant of order n

$$\left| \, a^0(1-a^n)^2,\, a^1(1-a^{n-1})^2,\, a^2(1-a^{n-2})^2,\, \ldots,\, a^{n-1}(1-a)^2 \, \right|$$

which should be equal to

$$\Delta(\mathbb{A}) \prod (1-a)^2 \prod_{i<j} (1 - a_i a_j)$$

from our computation for $\sigma_1(-S^2)$.

Indeed, subtracting each column to the one on its left, one gets in the case $n=3$,

$$\left| 1 - 2a^3 + a^6,\, a - 2a^3 + a^5,\, a^2 - 2a^3 + a^4 \right| \to (1-a) \left| 1{-}a^5,\, a(1{-}a^3),\, a^2(1{-}a) \right|$$

that we already computed.

Another determinant will be more convenient for expanding $\sigma_1(-S^1 - S^2)$ in terms of Schur functions. It is

$$\left| \, a^{n-1}(1{-}a)(1{-}a^2),\, a^{n-2}(1{-}a^2)(1{-}a^3),\, a^{n-3}(1{-}a^3)(1{-}a^4),\, \ldots,\, a^0(1{-}a^n)(1{-}a^{n+1} \, \right|$$

One evaluates this determinant by subtracting each column to the next one, and one finds that it is equal to $\Delta(\mathbb{A})\, \sigma_1(-S^1(\mathbb{A}) - S^2(\mathbb{A}))$.

We can now use this determinant to generalize Gordon, the powers appearing in the first row of the determinant being the indices in Gordon's determinant. For example, for $n=2$, one has

$$\begin{vmatrix} c_1 - c_2 - c_3 + c_4 & c_0 - c_2 - c_3 + c_5 \\ c_0 - c_1 - c_2 + c_3 & c_{-1} - c_1 - c_2 + c_4 \end{vmatrix}$$

The expansion of such determinant, will be

$$\sum_{\ell \le n} S_\lambda(\mathbb{B}) D_{\sigma_1(-S^1-S^2)} S_\lambda(\mathbb{A})$$

where

$$D_{\sum c_\mu S_\mu}(S_\lambda) = \sum c_\mu S_{\lambda/\mu} \ .$$

We are specially interested in the case $\mathbb{B} = \mathbb{A}$, which will give sums of Schur functions of length $\le 2n$.

Here are the results for $n = 2$. Only multiplicities 2 for some precise partitions. Find the law !

```
Litt4:=proc(n)  local i,fr;
  fr:=w[];
  for i from 1 to n do
    fr:=FreeConcat(fr,w[0]-w[i]-w[i+1]+w[2*i+1]);
  od;
 Tos(subs(w=s,fr))
end: #   \sigma_1 (-S^1 -S^2)
#
FiltreSchur:=proc(sf)  local tt;#   Filter according to weight
 tt:=Sf2Table(sf,s);
 convert([seq( tt[pa]*z^convert(pa,`+`)*s[op(pa)],
             pa=map(op,[indices(tt)]))], `+`)
end:
CauchyLitt4:=proc(N)   local i,sf,pa;
 sf:=subs(z=-1/z,FiltreSchur(Litt4(2)));  # Sign Changed  !!!
 taylor(
  1+convert([seq(seq(
      s[op(pa)]*SfDiff(z^(2*convert(pa,`+`))*sf,s[op(pa)]),
     pa=ListPart(i,maxlg=2)),i=1..N)],`+`),z,N+1)
end:

CauchyLitt4(9):
s[]+s[1]*z
+(2*s[1,1])*z^2
+(s[2,1]+s[1,1,1])*z^3
+(2*s[2,2]+s[2,1,1]+s[1,1,1,1])*z^4
+(s[3,2]+s[2,2,1]+s[2,1,1,1])*z^5
+(2*s[3,3]+s[3,2,1]+2*s[2,2,1,1])*z^6
+(s[3,3,1]+s[3,2,1,1]+s[2,2,2,1]+s[4,3])*z^7
```

```
+(s[4,3,1]+2*s[3,3,1,1]+s[3,2,2,1]+s[2,2,2,2]+2*s[4,4])*z^8
+(s[4,4,1]+s[5,4]+s[4,3,1,1]+s[3,2,2,2]+s[3,3,2,1])*z^9

s[4,3,2,1]+2*s[3,3,2,2]+s[5,4,1]+2*s[4,4,1,1]+2*s[5,5]

s[5,5,1]+s[5,4,1,1]+s[4,4,2,1]+s[3,3,3,2]+s[6,5]+s[4,3,2,2]

s[6,5,1]+s[5,4,2,1]+2*s[6,6]+2*s[5,5,1,1]+2*s[4,4,2,2]
  +s[4,3,3,2]+s[3,3,3,3]

s[6,5,1,1]+s[5,5,2,1]+s[4,3,3,3]+s[5,4,2,2]+s[7,6]
  +s[6,6,1]+s[4,4,3,2]

2*s[5,5,2,2]+s[5,4,3,2]+2*s[6,6,1,1]+s[6,5,2,1]
+2*s[4,4,3,3]+2*s[7,7]+s[7,6,1]

s[7,7,1]+s[7,6,1,1]+s[8,7]+s[4,4,4,3]+s[6,6,2,1]
  +s[6,5,2,2]+s[5,5,3,2]+s[5,4,3,3]              z^15

s[6,5,3,2]+2*s[5,5,3,3]+s[5,4,4,3]+s[4,4,4,4]+2*s[8,8]
  +s[8,7,1]+2*s[7,7,1,1]+s[7,6,2,1]+2*s[6,6,2,2]      z^16

s[8,8,1]+s[8,7,1,1]+s[7,7,2,1]+s[7,6,2,2]+s[6,6,3,2]
 +s[6,5,3,3]+s[5,5,4,3]+s[9,8]+s[5,4,4,4]              z^17
-------
2*s[9,9]+s[9,8,1]+s[8,7,2,1]+s[6,5,4,3]+2*s[5,5,4,4]
 +2*s[6,6,3,3]+2*s[7,7,2,2]+s[7,6,3,2]+2*s[8,8,1,1]
-------
s[10,9]+s[7,6,3,3]+s[8,7,2,2]+s[7,7,3,2]+s[8,8,2,1]
  +s[9,8,1,1]+s[9,9,1]+s[6,6,4,3]+s[6,5,4,4]+s[5,5,5,4]
-------
[9,8,2,1]+s[10,9,1]+2*s[8,8,2,2]+2*s[9,9,1,1]+2*s[10,10]
+s[8,7,3,2]+s[6,5,5,4]+s[5,5,5,5]+2*s[7,7,3,3]+s[7,6,4,3]+2*s[6,6,4,4]
-------
s[11,10]+s[8,7,3,3]+s[9,8,2,2]+s[8,8,3,2]+s[10,9,1,1]+s[9,9,2,1]
+s[10,10,1]+s[7,7,4,3]+s[7,6,4,4]+s[6,5,5,5]+s[6,6,5,4]
-------
2*s[6,6,5,5]+2*s[7,7,4,4]+2*s[10,10,1,1]+2*s[11,11]
+2*s[9,9,2,2]+2*s[8,8,3,3]+s[8,7,4,3]+s[9,8,3,2]
+s[11,10,1]+s[10,9,2,1]+s[7,6,5,4]
-------
```

```
s[6,6,6,5]+s[12,11]+s[11,11,1]+s[8,8,4,3]+s[8,7,4,4]
+s[7,7,5,4]+s[7,6,5,5]+s[11,10,1,1]+s[10,10,2,1]
+s[10,9,2,2]+s[9,9,3,2]+s[9,8,3,3]
```

8 Pfaffians and Representations of the Symmetric Group

(? note with Fumei)

One wants to compute sums of the type $\sum_\lambda S_\lambda$, sum over partition λ with even parts, such that their conjugate has also even parts. This problem has been solved by Sundquist using Pfaffians, and generalized by Ishikawa.

We find it more fruitful, and simpler, to deduce such generating series from the representations of the symmetric group indexed by partitions $[n, n]$.

The functions to compute appear now as "diagonal elements" in these representations.

9 Weyl groups of classical type

All the identities of Littlewood have an interpretation in terms of root systems of type , B, C or D, and are related to Weyl character formula.

This is how Macdonald prove them in the exercises of his book.

At our level (symmetrizing operators), it means that we have to use some extra divided differences, which for the type B_n are :

$$f \to \frac{f(x_1, x_2, \ldots) - f(1/x_1, x_2, \ldots)}{x_1 - 1/x_1}$$

or

$$f \to \frac{x_1 f(x_1, x_2, \ldots) - f(1/x_1, x_2, \ldots)/x_1}{x_1 - 1/x_1} \ .$$

These new operators act on symmetric functions in a non trivial way, and thus allow to find identities on symmetric functions, when using them, and symmetrizing again.

Alternating summations over the full group play a fundamental role, as in the case of the symmetric group. In fact these summations factor, one factor being our usual summation over the symmetric group, that is to say our operator π_ω.

10 Variations about Cauchy formula

Given two alphabets $\mathbb{A}, \mathbb{B}$, Cauchy formula asserts that

$$\sigma_1(\mathbb{A}\mathbb{B}) = \sum_\lambda S_\lambda(\mathbb{A}) S_\lambda(\mathbb{B}) \ .$$

Using other functions that Schur functions, one has more general formulas.

For example, the *orthogonal Schur function* $\mathcal{O}_\lambda(\mathbb{A})$ are defined by the generating series[1] :

$$\prod_{i\le j}(1-a_ia_j)\prod_i\prod_j(1-a_ib_j)^{-1}=\sum_\lambda \mathcal{O}_\lambda(\mathbb{A})\,S_\lambda(\mathbb{B})\ , \tag{14}$$

that is,

$$\sigma_1\big(\mathbb{A}\mathbb{B}-S^2(\mathbb{A})\big)=\sum_\lambda \mathcal{O}_\lambda(\mathbb{A})\,S_\lambda(\mathbb{B})\ . \tag{15}$$

These functions can be expressed by a determinant of complete functions, due to Weyl (th.7.9.A). For example, for a partition with three parts, this determinant will be

$$\mathcal{O}_{\lambda_1\lambda_2\lambda_3}=\begin{vmatrix} S_{\lambda_1}-S_{\lambda_1-2} & S_{\lambda_1+1}-S_{\lambda_1+1-4} & S_{\lambda_1+2}-S_{\lambda_1+2-6}\\ S_{\lambda_2-1}-S_{\lambda_2-1-2} & S_{\lambda_2}-S_{\lambda_2-4} & S_{\lambda_2+1}-S_{\lambda_2+1-6}\\ S_{\lambda_3-2}-S_{\lambda_3-2} & S_{\lambda_3-1}-S_{\lambda_3-1-4} & S_{\lambda_3}-S_{\lambda_3-6}\end{vmatrix}$$

[1] cf. Littlewood, p.240.