

Gaudin functions of arbitrary level

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The theory of symmetric functions mostly relies on the Cauchy formula, that is

$$\det \left(\frac{1}{x - y} \right) = \frac{\Delta(x)\Delta(y)}{\prod (x - y)} \quad \mathbf{1}$$

“**1**” contains all the information about symmetric functions, except what concerns the plethysm.

Next: **Gaudin-Izergin-Korepin determinant**:

$$\det \left(\frac{1}{(x-y)(x-ty)} \right) = \frac{\Delta(x)\Delta(y)}{\prod (x-y)(x-ty)} \mathbf{\textcolor{red}{F}(\mathbf{x}, \mathbf{y})}$$

Useful to enumerate **ASM** , by specialization of t to a **root of unity**.

NextNext: Gaudin function of level r .

$$\det \left(\frac{1}{(x-y) \cdots (x-t^r y)} \right) = \frac{\Delta(x)\Delta(y)}{\prod (x-y) \cdots (x-t^r y)} F^r(\mathbf{x}, \mathbf{y})$$

The classical case was level $r = 1$.

How to compute ?

Answer ([Zeilberger](#)): Ask  Shalosh.

```
matrix([seq([ seq(  
    1/(x.i-y.j)/(x.i-t*y.j)/(x.i-t^2*y.j),  
                                j=1..3)] ,i=1..3)]));
```

Answer of Shalosh:

```
factor(det(%));
```

```
Error, (in minor) object too large
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Answer (Newton): use my **divided differences**.

$$\frac{1}{x - z_1} \partial_1 \cdots \partial_{r-1} = \frac{1}{(x - z_1) \cdots (x - z_r)}$$

where ∂_i is the divided difference

$$f \rightarrow (f - f^{s_i})(z_i - z_{i+1})^{-1}$$

and $s_i = \text{transposition of } z_i, z_{i+1}$.

This gives us a tool to increase the number of poles of rational functions. We can [pass from Cauchy to Gaudin](#) by introducing n sets of $r+1$ variables extending the y_j variables :

$$\mathbf{y}^j := \{y_j^0 = y_j, y_j^1, \dots, y_j^r\} \ , \ j = 1, \dots, n \ ,$$

and using the products of divided differences

$$\prod_{j=1}^n \partial_0^j \cdots \partial_{r-1}^j \ ,$$

where ∂_i^j is relative to the pair y_j^i, y_j^{i+1} .

$$\begin{aligned}
& \left| \begin{array}{cc} \frac{1}{x_1 - \textcolor{red}{y}_1^0} & \frac{1}{x_2 - \textcolor{red}{y}_1^0} \\ \frac{1}{x_1 - \textcolor{blue}{y}_2^0} & \frac{1}{x_2 - \textcolor{blue}{y}_2^0} \end{array} \right| \xrightarrow{\textcolor{red}{\longrightarrow}} \left| \begin{array}{cc} \frac{1}{(x_1 - \textcolor{red}{y}_1^0)(x_1 - \textcolor{red}{y}_1^1)} & \frac{1}{(x_2 - \textcolor{red}{y}_1^0)(x_2 - \textcolor{red}{y}_1^1)} \\ \frac{1}{x_1 - \textcolor{blue}{y}_2^0} & \frac{1}{x_2 - \textcolor{blue}{y}_2^0} \end{array} \right| \\
& \xrightarrow{\textcolor{blue}{\longrightarrow}} \left| \begin{array}{cc} \frac{1}{(x_1 - \textcolor{red}{y}_1^0)(x_1 - \textcolor{red}{y}_1^1)} & \frac{1}{(x_2 - \textcolor{red}{y}_1^0)(x_2 - \textcolor{red}{y}_1^1)} \\ \frac{1}{(x_1 - \textcolor{blue}{y}_2^0)(x_1 - \textcolor{blue}{y}_2^1)} & \frac{1}{(x_2 - \textcolor{blue}{y}_2^0)(x_2 - \textcolor{blue}{y}_2^1)} \end{array} \right|
\end{aligned}$$

To get an identity, we need to compute the image of Cauchy in a second manner. The product of Cauchy by the **denominator of Gaudin** is

$$\Delta(\mathbf{x}) \left| (y_i^0)^j R(\mathbf{x}, \mathbf{y}^i - y_i^0) \right|_{i=1, \dots, n, j=0, \dots, n-1}$$

where $R(A, B)$ is the **resultant** $\prod_{a \in A} \prod_{b \in B} (a - b)$ of two sets of indeterminates.

We have now to compute the image of each entry under divided differences, having passed from rational functions to polynomials.

It is a simple exercise to arrive to a determinant of
Schur functions :

$$\det (S_{j \sqcup (\underbrace{n-1, \dots, n-1}_r)}(\mathbf{y}^i - \mathbf{x}) ,$$

with $j \sqcup \underbrace{n-1, \dots, n-1}_r = j, (n-1)^r$.

Going back to the original variables y_i , one gets a
determinant of Schur functions to express the Gaudin
function :

$$F^r(\mathbf{x}, \mathbf{y}) \Delta(\mathbf{y}) = \det (S_{j \sqcup (\underbrace{n-1, \dots, n-1}_r)}(y_i + \dots + t^r y_i - \mathbf{x}) .$$

For example, for $r = 1$, $n = 3$,

$$F^1(\mathbf{x}, \mathbf{y})\Delta(\mathbf{y}) = \begin{vmatrix} S_{022}(\textcolor{red}{y}_1 + t\textcolor{red}{y}_1 - \mathbf{x}) & S_{122}(\textcolor{red}{y}_1 + t\textcolor{red}{y}_1 - \mathbf{x}) & S_{222}(\textcolor{red}{y}_1 + t\textcolor{red}{y}_1 - \mathbf{x}) \\ S_{022}(\textcolor{blue}{y}_2 + t\textcolor{blue}{y}_2 - \mathbf{x}) & S_{122}(\textcolor{blue}{y}_2 + t\textcolor{blue}{y}_2 - \mathbf{x}) & S_{222}(\textcolor{blue}{y}_2 + t\textcolor{blue}{y}_2 - \mathbf{x}) \\ S_{022}(\textcolor{violet}{y}_3 + t\textcolor{violet}{y}_3 - \mathbf{x}) & S_{122}(\textcolor{violet}{y}_3 + t\textcolor{violet}{y}_3 - \mathbf{x}) & S_{222}(\textcolor{violet}{y}_3 + t\textcolor{violet}{y}_3 - \mathbf{x}) \end{vmatrix}$$

Notice that the functions $S_{j\Box}$ are **subresultants**. Indeed $R(A, B)$ is equal to $S_{\beta\alpha}(A - B)$, with $\alpha = \text{card}(A)$, $\beta = \text{card}(B)$.

Here, we have sets of cardinality $r+1, n$, and $\Box = (n-1)^r$. However, when A, B have a letter c in common, then $S_{(\beta-1)\alpha-1}(A - B)$ **becomes the resultant of $A-c, B-c$** . This function vanishes if A, B have more than one letter in common.

Therefore, when specializing $\mathbf{x}$ into a subset of

$$\{y_1, \dots, y_n, \dots, t^r y_1, \dots, t^r y_n\},$$

then each entry of the Gaudin determinant either vanishes or becomes a resultant.

This gives enough specializations (which are either 0 or products of linear factors $y_j - t^k y_i$) to characterize the Gaudin function, as stated by the next theorem.

Theorem. $F_n^r(\mathbf{x}, \mathbf{y})$ is the *only linear combination of Schur functions* in $\mathbf{x}$ (with coefficients in $\mathbf{y}$) indexed by partitions contained in $((n-1)r)^n$, which *has the same specializations*

$$\mathbf{x} \subset \{y_1, \dots, y_n, \dots, t^r y_1, \dots, t^r y_n\}$$

than the function

$$G_n^r(\mathbf{x}, \mathbf{y}) := \frac{\Delta(\mathbf{x})}{\Delta(\mathbf{y})} \prod_i S_{\square}(y_i + \dots + t^r y_i - \mathbf{x}) .$$

Remark due to Pasquier. $F_n^1(\mathbf{x}, \mathbf{y})$ is a **factor of the (non-symmetric) Macdonald polynomial** of index $[2n-2, \dots, 2, 0, 2n-2, \dots, 2, 0]$, when $q = t^6$. We are back to the subject of the conference ! To approach its heart, let us introduce the **affine Hecke algebra**, in the pedestrian way that I favoured with **M.P. Schützenberger**, using

$$\square_i := \square_{s_i} : f \longrightarrow f \square_i = f(tx_i - x_{i+1}) \partial_i .$$

More generally, the *Euler-Poincaré characteristics* is the morphism

$$f \longrightarrow f \square_\omega := \sum_{w \in \mathfrak{S}_n} \left(f \frac{\prod_{i < j} (tx_i - x_j)}{x_i - x_j} \right)^w \in \mathfrak{Sym}(\mathbf{x}) .$$

$\square_\omega$ sends dominant monomials onto Hall-Littlewood polynomials, up to normalization. The usual generators of the Hecke algebra are $T_i := \square_i - 1$. We also need an affine operation θ , the incrementation of indices :

$$x_i \theta = x_{i+1} , \quad \text{periodicity } x_{i+n} = x_i t^{-1} .$$

We have now all the ingredients to cook-up the Gaudin-Izergin-Korepin function.

Theorem. *Let f be a function of 1 variable. Then*

$$\begin{aligned} f(x_1) R(\mathbf{x} - x_1, \mathbf{y})(1 - t\theta) \cdots (1 - t^{n-1}\theta) \square_\omega \\ = \left(\pm f(x_1) x_2 \cdots x_n \partial_1 \cdots \partial_{n-1} \right) F_n^1(\mathbf{x}, \mathbf{y}) [n]!. \end{aligned}$$

The proof consists in **testing the specializations** of both sides of the equation in the points

$$\mathbf{y} \subset \{x_1, \dots, x_n, x_2 t^{-1}, \dots, x_n t^{-1}\}.$$

Explicitely, for $n = 4$, one starts with

$$\begin{aligned}
& f(x_1) R(x_2+x_3+x_4, \mathbf{y})(1-t\theta)(1-t^2\theta)(1-t^3\theta) \\
&= f(x_1) R(x_2+x_3+x_4, \mathbf{y}) - t \begin{bmatrix} 3 \\ 1 \end{bmatrix} f(x_2) R(x_3+x_4+x_1/t, \mathbf{y}) \\
&\quad + t^3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} f(x_3) R(x_4+x_1/t+x_2/t, \mathbf{y}) \\
&\quad - t^6 f(x_4) R(x_1/t+x_2/t+x_3/t, \mathbf{y})
\end{aligned}$$

The sum under the symmetric group can be written

$$\begin{aligned}
& \sum_w \left(f(x_1) R(x_2 + x_3 + x_4, \mathbf{y}) \frac{\Delta_t(1234)}{\Delta(1234)} \right)^w \\
& - t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \sum_w \left(f(x_1) R(x_3 + x_4 + x_2 t^{-1}, \mathbf{y}) \frac{\Delta_t(2134)}{\Delta(2134)} \right)^w \\
& + t^3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \sum_w \left(f(x_1) R(x_4 + x_2 t^{-1} + x_3 t^{-1}, \mathbf{y}) \frac{\Delta_t(3214)}{\Delta(3214)} \right)^w \\
& - t^6 \sum_w \left(f(x_1) R(x_2 t^{-1} + x_3 t^{-1} + x_4 t^{-1}, \mathbf{y}) \frac{\Delta_t(4231)}{\Delta(4231)} \right)^w
\end{aligned}$$

using the Vandermonde, and the t -Vandermonde

$$\Delta_t = \prod_{i < j} (tx_i - x_j).$$

One checks the specializations of the coefficient of $f(x_1)$ in both sides. They coincide, knowing ([Lagrange interpolation](#) !) that $f(x_1)x_2x_3x_4\partial_1\partial_2\partial_3$ is equal to

$$\begin{aligned} & \frac{f(x_1)x_2x_3x_4}{R(x_1, x_2+x_3+x_4)} + \frac{f(x_2)x_1x_3x_4}{R(x_2, x_1+x_3+x_4)} \\ & + \frac{f(x_3)x_1x_2x_4}{R(x_3, x_1+x_2+x_4)} + \frac{f(x_4)x_1x_2x_3}{R(x_4, x_1+x_2+x_3)}. \end{aligned}$$

Taking $f = 1$, we shall get a statement about **Macdonald polynomials** already obtained by Warnaar.

My friend **Adriano** will allow me to use **λ -rings**. The function $\sigma_1(AB - C)$ is equal to $\prod_c (1 - c) \prod_{a,b} (1 - ab)^{-1}$, and therefore, everybody infers that

$$\sigma_1 \left(\mathbf{xy} \frac{1 - t}{1 - q} \right) = \prod_{x,y} \prod_{i \geq 0} \frac{1 - tq^i xy}{1 - q^i xy}$$

Still everybody, taken in this distinguished audience, will have recognized the **generating function of the symmetric Macdonald polynomials**. Let τ_q be the following **incrementation** of indices:

$$x_i \tau_q = x_{i+1} , \quad \text{periodicity } x_{i+n} = qx_i .$$

Inspired by the preceding computation, we now want

$$\sigma_1 \left(\mathbf{xy} \frac{1-t}{1-q} \right) (1 - t\tau_q) \cdots (1 - t^n \tau_q) \square_\omega .$$

Since

$$\mathbf{x} \frac{1-t}{1-q} \tau_q = \mathbf{x} \frac{1-t}{1-q} + x_1(t-1)$$

q disappears from the computation ! Indeed

$$\begin{aligned} & \sigma_1 \left(\mathbf{xy} \frac{1-t}{1-q} \right) (1-t\tau_q) \cdots (1-t^n\tau_q) \square_\omega \\ &= \sigma_1 (\mathbf{xy}(1-t)) (1-t\tau_0) \cdots (1-t^n\tau_0) \square_\omega \sigma_1 \left(\mathbf{xy} q \frac{1-t}{1-q} \right) \end{aligned}$$

Thus the starting function is now

$$\sigma_1(\mathbf{xy}(1-t)) = \prod (1 - txy)(1 - xy)^{-1}$$

that one rewrites, using the variables $y_i^\vee = y_i^{-1}$, as

$$R(tx_1 + \cdots + tx_n, \mathbf{y}^\vee) / R(\mathbf{x}, \mathbf{y}^\vee).$$

We are back to the functions used in the preceding theorem, which gives, as a corollary, that

$$\sigma_1(\mathbf{xy}(1-t))(1 - t\tau_0) \cdots (1 - t^n \tau_0) \square_\omega = \sigma_1(\mathbf{xy}) \tilde{F}_n^1(\mathbf{x}, \mathbf{y}) [n]!,$$

where $\tilde{F}_n^1(\mathbf{x}, \mathbf{y})$ is the Gaudin function

$$(x_1 \cdots x_n)^{n-1} F_n^1(\mathbf{x}^\vee, \mathbf{y}).$$

Notice that $\sigma_1(\mathbf{xy}(1-t))$ is the generating function of Hall-Littlewood polynomials. Reintroducing the factor in q , we get the action on the generating function of Macdonald polynomials :

$$\begin{aligned} \sigma_1 \left(\mathbf{xy} \frac{1-t}{1-q} \right) (1 - t\tau_q) \cdots (1 - t^n \tau_q) \square_\omega \\ = \sigma_1 \left(\mathbf{xy} \frac{1-t}{1-q} \right) \sigma_1(t\mathbf{xy}) \widetilde{F}_n^1(\mathbf{x}, \mathbf{y}) . \end{aligned}$$

Warnaar writes differently the LHS. Indeed, there exists commuting operators $\xi_1, \dots, \xi_n$ which act **diagonally** on the basis of non-symmetric Macdonald polynomials. The eigenvalues are $q^{\lambda_1} t^{n-1}, \dots, q^{\lambda_n} t^0$ for the polynomial M_λ indexed by $\lambda : \lambda_1 \geq \dots \geq \lambda_n \geq 0$. The symmetric functions in $\xi_1, \dots, \xi_n$ act therefore **diagonally** on the symmetric Macdonald polynomials. Knowing that $\square_\omega \xi_i \square_\omega = \square_\omega t^{n-i} \tau_q \square_\omega$, one has

$$\begin{aligned}
 P_\lambda(\mathbf{x}; q, t) \prod_{i=1}^n (1 - q^{\lambda_i} t^{n-i+1}) \\
 = P_\lambda(\mathbf{x}; q, t) (1 - t \tau_q) \cdots (1 - t^n \tau_q) \square_\omega .
 \end{aligned}$$

In final, the preceding theorem is [Warnaar's](#) theorem :

$$\sum_{\lambda} b_{\lambda} P_{\lambda}(\mathbf{x}; q, t) P_{\lambda}(\mathbf{y}; q, t) \prod_{i=1}^n (1 - q^{\lambda_i} t^{n-i+1})$$

$$= \sigma_1 \left(\mathbf{xy} \frac{1-t}{1-q} \right) \sigma_1(t\mathbf{xy}) \widetilde{F}_n^1(\mathbf{x}, \mathbf{y}) ,$$

up to notations, signs and missing factorials.

I already mentionned that the only properties of symmetric functions which were not implied by **Cauchy formula** were those related to **plethysm**. Of course **plethystic substitutions** are **change of alphabets** and have nothing to do with plethysm.

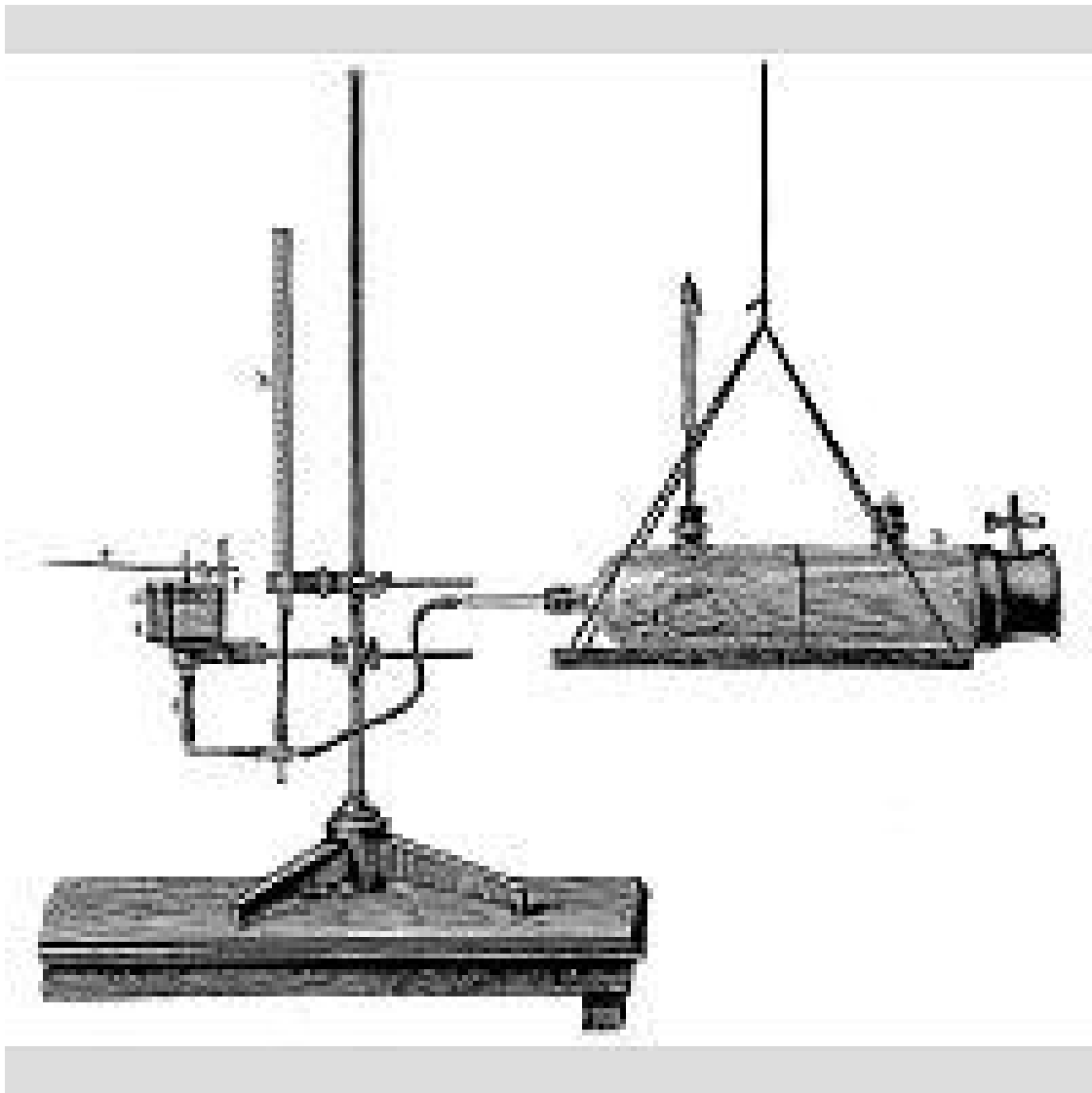
This will be evident when using the

plethysmograph of **Kronecker**

Plethysmograph according to Kronecker

(Plethysmograph nach Kronecker), 1891

vlp.mpiwg-berlin.mpg.de/technology/



Mode d'emploi (taken from [Wikipedia](#))

Plethysmograph - Wikipedia, the free encyclopedia

- [Traduire cette page]

A plethysmograph is an instrument for measuring changes in volume within an organ ...

In a traditional plethysmograph, the test subject is placed inside a ... wikipedia.org/wiki/Plethysmograph - En cache

penile plethysmograph - [Traduire cette page]

The penile plethysmograph (pluh-THIZ-muh-graf)

(PPG) is a machine for measuring changes in the

circumference of the penis. A stretchable band with

mercury ... skepdic.com/penilep.html - 30k - En cache

Plethysmograph: a disputed device -[Traduire cette page]
A genital plethysmograph (pronounced pluh-THIZ-muh-graf)
is a controversial device that measures blood flow
in the genitals. Sensing equipment is attached ...
www.tsroadmap.com/info/plethysmograph.html -

The penile plethysmograph in false allegation cases
- Cowling Investigations, Inc.
www.allencowling.com/false13.htm
- 17k - En cache - Pages similaires
plethysmograph. The American Heritage Dictionary of
the English Language: Fourth Edition. 2000.