# THE SYMMETRIC GROUP

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## Permutations

Given a positive integer $n$, a *permutation of order $n$* is a bijection $\sigma$ from $\{1, \ldots, n\}$ to itself.

We denote it by the list $[\sigma_1 = \sigma(1), \ldots, \sigma_n = \sigma(n)]$ of the successive images of $1, \ldots, n$.

More generally, one can use any totally ordered finite set with $n$ elements (one needs a total order, to be able to list images without specifying sources). With these conventions, erasing letters in $[\sigma_1, \ldots, \sigma_n]$ produces a permutation without having to shift the remaining entries to $1, 2, 3, \ldots$.

A *cycle* is an orbit $i$, $\sigma(i)$, $\sigma(\sigma(i))$, $\ldots$, $\sigma\big(\ldots\sigma(\sigma(i))\ldots\big)$. The *cycle decomposition* of a permutation is the collection of its different orbits.

A cycle can be thought as a collection of numbers written on a circle; considering the numbers to be beads, then a cycle is to be interpreted as a necklace. To write it one-dimensionally, one decides to begin by its smallest element $i$, and this gives the sequence $\big(i, \sigma(i), \sigma(\sigma(i)), \ldots\big)$, with $i = min(i, \sigma(i), \ldots)$. Therefore, given a bag with $k$ different beads, there are $(k-1)!$ different possible necklaces that can be made from it.

The permutation $\sigma = [7, 5, 2, 1, 10, 6, 4, 11, 8, 3, 9]$ has cycle decomposition :

$$\sigma = \begin{array}{ccc} 2 & \rightarrow & 5 \\ \uparrow & & \downarrow \\ 3 & \rightarrow & 10 \end{array} \qquad \begin{array}{ccc} & \nearrow & 7 \\ 1 & & \downarrow \\ & \nwarrow & 4 \end{array} \qquad \begin{array}{ccc} & \nearrow & 11 \\ 8 & & \downarrow \\ & \nwarrow & 9 \end{array} \qquad 6$$

It is easy to compute powers of a permutation starting its cycle decomposition. One has just to understand what is the fate of each individual cycle, independently of the others. For example, the preceding permutation has square :

$$\sigma^2 = \quad 2 \underset{\longleftarrow}{\overset{\longrightarrow}{\phantom{xx}}} 10 \qquad 3 \underset{\longleftarrow}{\overset{\longrightarrow}{\phantom{xx}}} 5 \qquad \begin{array}{ccc} & \nearrow & 4 \\ 1 & & \downarrow \\ & \nwarrow & 7 \end{array} \qquad \begin{array}{ccc} & \nearrow & 9 \\ 8 & & \downarrow \\ & \nwarrow & 11 \end{array} \qquad 6$$

The *cycle type* of a permutation is the (decreasing) list of the lengths of its cycles. A permutation is a *full cycle* if it has only one cycle.

Given a partition $\mu = [\mu_1, \ldots, \mu_r]$, then one defines $\zeta_\mu$ to be the following direct product of cycles :

$$
\begin{aligned}
\zeta_\mu &= (1, 2, \ldots, \mu_1)(\mu_1{+}1, \ldots, \mu_1{+}\mu_2) \cdots (\mu_1{+}\cdots{+}\mu_{n-1}{+}1, \ldots, \mu_1{+}\cdots{+}\mu_n) \\
&= [2, \ldots, \mu_1, 1,\ \mu_1{+}2, \ldots, \mu_1{+}\mu_2, \mu_1{+}1,\ \ldots \\
&\qquad \ldots,\ \mu_1{+}\cdots{+}\mu_{n-1}{+}2, \ldots, \mu_1{+}\mu_r, \mu_1{+}\cdots{+}\mu_{n-1}{+}1]
\end{aligned}
$$

All permutations having cycle type $\mu$ are conjugate to $\zeta_\mu$ and, conversely, a permutation conjugate to $\zeta_\mu$ has cycle type $\mu$ (the conjugates of a permutation $\zeta$ are all permutations of the type $\nu\zeta\nu^{-1}$)

Indeed, if $\mu = [3, 3, 1, 1]$ for example, given any permutation $\sigma$ of order 8, then $\sigma\zeta_{3311}\sigma^{-1}$ has cycle decomposition

$$
(\sigma_1, \sigma_2, \sigma_3)\,(\sigma_4, \sigma_5, \sigma_6)\,(\sigma_7)\,(\sigma_8)\ .
$$

Conjugating amounts to changing the values of the beads, not the cycle lengths!

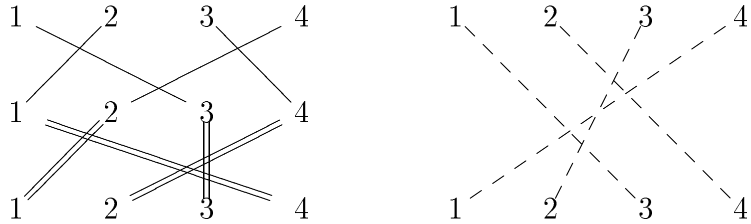
The *conjugacy class* of type $\mu$ is the subset of permutations having $\mu$ as cycle type. To count how many permutations it contains, one can reason as follows: first we have to put the beads in bags of size $\mu_1, \ldots, \mu_r$. There are $n!/(\mu_1! \cdots \mu_r!)$ possibilities. But there are bags of the same size that we must not distinguish. If $\mu = 1^{m_1}\, 2^{m_2} \ldots$, then to account for equal sizes, one has to divide by $\prod(m_i!)$. But now, with $i$ beads, one can make $(i{-}1)!$ different necklaces so that finally the order of the conjugacy class is

$$
\frac{n!}{\prod (i!)^{m_i}}\,\frac{\prod((i{-}1)!)^{m_i}}{\prod m_i!} = \frac{n!}{\prod i^{m_i}\, m_i!} \tag{1}
$$

We already have met the denominator, it is a scalar product of power sums :

$$
z_\mu = \prod i^{m_i}\, m_i! = (\Psi^\mu\,,\,\Psi^\mu)\ . \tag{2}
$$

There is another graphical representation of a permutation, by braids, which is used in knot theory and allows easy multiplication and inversion. One writes two horizontal copies of $1, 2, \ldots, n$ on top of each other, and connect each pair $i, \sigma_i$ by an edge. Multiplying permutations consists in stacking them and erasing the intermediate levels.



## Permutohedron

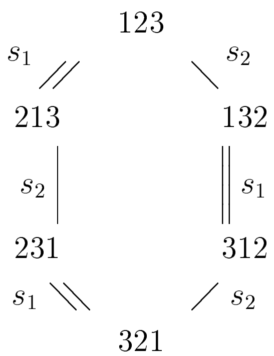
The *simple transposition* $s_i$ is the permutation with only fixed points, except for a cycle $(i, i{+}1)$. Starting with the identity permutation, and writing the multiplication by $s_i$ on the right as an edge of colour $i$ if it produces a new permutation, one gets a directed graph, the *Permutohedron* with vertices all the permutations in $\mathfrak{S}_n$. This representation of a group from a set of generators is due to Cayley. One could have chosen multiplication from the left, in which case one would have obtained the *inverse Permutohedron*, with labels exchanged by inversion from those of the Permutohedron.

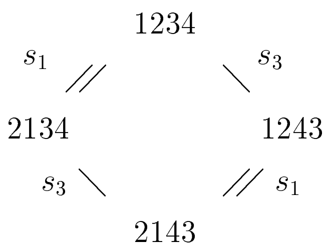
Given a permutation $\sigma$, any path from the origin to $\sigma$ is called a *reduced decomposition* of $\sigma$.

Classifying all reduced decompositions of a permutation is an interesting problem that we shall encounter in different occasions later. But already, one can notice in the Permutohedron special subgraphs : lozenges and hexagons, which account for the *braid relations*

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \tag{3}$$
$$s_i s_j = s_j s_i \qquad |i{-}j| \neq 1 \tag{4}$$

The graphical display of these relations is (taking the smallest symmetric groups in which they appear) :




Braid relations allow to deform a path in the Permutohedron without changing its end points. Let us check that two arbitrary reduced decompositions of a permutation are related by a sequence of braid relations.

First, let us observe that

$$(s_1)\,(s_2 s_1)\,(s_3 s_2 s_1)\,\cdots\,(s_{n-1}\cdots s_1) \tag{5}$$

is a reduced decomposition of the *maximal permutation* $\omega := [n, \ldots, 1]$ (we have added parentheses because we want to distinguish some factors).

Recall that a *composition* is a vector with integral non-negative components.

**Lemma 1** *Given a composition $I = [i_1, \ldots, i_r] \in \mathbb{N}^r$, $I \le [1, 2, \ldots, n{-}1]$, then the word $w_I$ which the product of the left factors of $(s_1)\,(s_2 s_1)\ldots$ of respective lengths $v_1, v_2, \ldots$ is a reduced decomposition of a permutation $\sigma$ (we shall see that it is lexicographically minimal among all reduced decompositions of $\sigma$). The composition $I$ is called the* cocode *of $\sigma$. Evaluating such words in $\mathfrak{S}_n$ gives a bijection with $\mathfrak{S}_n$.*

Now, we shall show how to transform a reduced decomposition into a lexicographically minimal one using a sequence of braid relations.

Let $w$ be a reduced decomposition of $\sigma$ in $\mathfrak{S}_n$. If $s_{n-1}$ does not occur in $w$, then $\sigma$ fixes $n$ and we are done by induction on $n$. In the contrary case, let us call *column* of length $r$ the word $s_{n-1} \cdots s_{n-r}$.

We shall iterate the operation of "canonizing" a pair (column , $s_i$), defined as follows:

$$(s_{n-1} \cdots s_{n-r}\ ,\ s_i) \to \begin{cases} (s_{i-1}\ ,\ s_{n-1} \cdots s_{n-r}) & \text{if} \quad i > n{-}r \\ (\emptyset\ ,\ s_{n-1} \cdots s_{n-r-1}) & \text{if} \quad i = n{-}r{-}1 \\ (s_{i-1}\ ,\ s_{n-1} \cdots s_{n-r}) & \text{if} \quad i < n{-}r{-}1 \end{cases}$$

The case $i = n{-}r$ cannot occur, otherwise the decomposition would not be reduced; the canonization use only braid relations.

Graphically, it says

$$\boxed{\begin{smallmatrix}7\\6\\5\\4\end{smallmatrix}}\,6 \to 5\,\boxed{\begin{smallmatrix}7\\6\\5\\4\end{smallmatrix}} \qquad ; \qquad \boxed{\begin{smallmatrix}7\\6\\5\\4\end{smallmatrix}}\,3 \to \boxed{\begin{smallmatrix}7\\6\\5\\4\\3\end{smallmatrix}} \qquad ; \qquad \boxed{\begin{smallmatrix}7\\6\\5\\4\end{smallmatrix}}\,1 \to 1\,\boxed{\begin{smallmatrix}7\\6\\5\\4\end{smallmatrix}}$$

Take now the leftmost occurrence of $s_{n-1}$ in $w$. Then $w = w'\, s_{n-1} s_i w''$, with $s_{n-1} \notin w'$. Starting with the pair $(s_{n-1}, s_i)$, we iterate canonization, and this swallows all letters on the right of the column, increasing eventually it, and concatenating letters to the left factor $w'$. The process stops when there is no more letter on the right of the column, on a word of the type $w'''\, s_{n-1} \cdots s_{n-r}$, with $s_{n-1} \notin w'''$. Canonizing $w'''$ and iterating, one gets a word which is of the type $w(v)$ for some vector $v \le [1, \ldots, n{-}1]$.

Each operation gives a word which is lexicographically smaller than the preceding one (or identical), therefore $w(v)$ is the smallest reduced decomposition among all reduced decompositions of the same permutation.

```
Canonize:=proc(left,column,i) local k;
  k:=column[nops(column];
```

```
  if i>k+1 then [[op(left),i-1],column]
  elif i=k then lprint(`NOT REDUCED`)
  elif i=k-1 then [left,[op(column),i]]
  else        [[op(left),i],column]
  fi;
end:
ItereCano:=proc(rd) local nn,i,j,left,res;
 nn:=max(op(rd));
 member(n,rd,'j');
 left:=[seq(rd[k],k=1..j-1)];
 res:=[left,[nn]];
 for k from j+1 to nops(rd)-1 do
  res:=Canonize(op(res), rd[k])
 od;
 res
end:

ACE> ItereCano([4,3,4,2,3,4,1,2,3,4]);
[]   [4]   3
[]   [4, 3]   4
[3]   [4, 3]   2
[3]   [4, 3, 2]   3
[3, 2]   [4, 3, 2]   4
[3, 2, 3]   [4, 3, 2]   1
[3, 2, 3]   [4, 3, 2, 1]   2
[3, 2, 3, 1]   [4, 3, 2, 1]   3
[3, 2, 3, 1, 2]   [4, 3, 2, 1]   4
                       [[3, 2, 3, 1, 2, 3], [4, 3, 2, 1]]
```

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**Inversions**

Because braid relations preserve the lengths of decompositions, all reduced decompositions of a permutation $\sigma$ have the same length, which is called the *length* $\ell(\sigma)$ of $\sigma$.

In fact, $\ell(\sigma)$ is the number of *inversions* of $\sigma$, i.e. the number of sub-words of type $ba$, with$b > a$, of $[\sigma_1, \ldots, \sigma_n]$. When multiplying $\sigma$ by a transposition $s_i$ such that $\ell(\sigma s_i) > \ell(\sigma)$, then one increases the set of inversions by exactly one inversion, namely $[\sigma_{i+1}, \sigma_i]$.

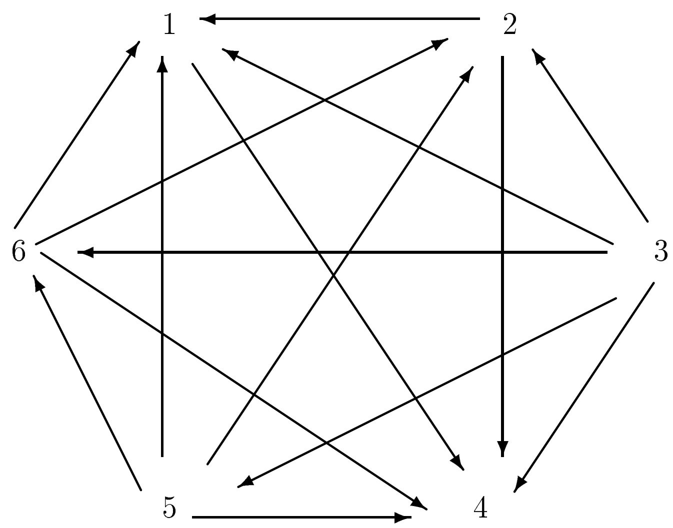
It is easy to characterize the set of pairs $[b, a]$ which are the set of inversions of a permutation. Given a permutation $\sigma \in \mathfrak{S}_n$, one associates to it a directed graph with vertices $1, \ldots, n$, such that the underlying graph is complete, with an arrow from $b$ to $a$ if $ba$ is an inversion, or from $a$ to $b$ otherwise. The graph just represents the sets of subwords of length 2 of the permutation. It is of course sufficient to know the inversions.

**Lemma 2** *A subset $\mathcal{E}$ of $\{[j, i] : n \geq j > i \geq 1\}$ is the set of inversions of a permutation iff the associated graph has no cycle.*

*Proof.* We shall show that the last (or first) component of the permutation is easy to characterize from the set of inversions. This will prove existence and unicity.

There is at least a vertex of the complete graph which is a *sink* (no arrow escapes from it) because otherwise one would have infinite paths (and this is impossible, the graph is finite and has no cycle). This sink is unique, because there is an edge between any two vertices. Erasing this sink and the arrows arriving to it, one can conclude by induction. We refer to the book of Berge [2] for more details. QED

For example, the complete graph for $\sigma = [3, 5, 6, 2, 1, 4]$ is :




The set of inversions is closed by transitivity: if $c > b > a$ and $ba$ and $cb$ are inversions, then $ca$ is also an inversion. Otherwise, one would have a cycle on $a, b, c$.

Because of this property, one does not need to write the complete graph. One just writes an arrow for primitive inversions (not resulting by transitivity from other inversions). Let $m$ be the maximal value of the end points of this graph. Then $m = \sigma_n$ and one can iterate on $n$.

```
ACE>Perm2ListInv(Perm2Inv([2,5,7,4,1,6,3]));#inversions on places for ACE!
[{1,2},{1,4},{1,5},{1,7},{3,4},{3,5},{3,6},{3,7},{4,5},{4,7},{6,7}]
```

The reduced graph corresponding to the above set of inversions, and its successive images after suppression of the maximal end point, are

$$\begin{matrix} 7 & \to & 6 & & \\ & \searrow & & \searrow & \\ 5 & \to & 4 & \to & 3 \\ & & & \searrow & \\ & & 2 & \to & 1 \end{matrix} \quad , \quad \begin{matrix} 7 & \to & 6 & & \\ & \searrow & & & \\ 5 & \to & 4 & & \\ & & & \searrow & \\ & & 2 & \to & 1 \end{matrix} \quad , \quad \begin{matrix} 7 & & & & \\ & \searrow & & & \\ 5 & \to & 4 & & \\ & & & \searrow & \\ & & 2 & \to & 1 \end{matrix} \quad , \ldots$$

from which one sees that $\sigma_7 = 3$, then $\sigma_6 = 6$, $\sigma_5 = 1, \ldots$.

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**Rothe diagram**

A permutation $\sigma$ can be represented by a matrix $M(\sigma)$, which describes its action on the vector space with basis $1, 2, \ldots, n$. Explicitly, $M(\sigma)$ has entries 1 in positions $[i, \sigma_i]$, and 0 elsewhere (taking the usual coordinates of matrices, not the Cartesian plane).

Rothe[33] found in 1800 a graphical display of the inversions of $\sigma$, starting from $M(\sigma)$ (though, of course, matrices had still to wait 50 years to appear), which leads to many combinatorial properties of permutations.

For each pair of 1's in $M(\sigma)$ in relative position $\begin{smallmatrix} 0 & \cdots & 1 \\ \vdots & & \\ 1 & & \end{smallmatrix}$ write a box $\square$ at the intersection of the up-most row and leftmost column containing these entries, thus obtaining $\begin{smallmatrix} \square & \cdots & 1 \\ \vdots & & \\ 1 & & \end{smallmatrix}$.

The planar set of such boxes is called the *Rothe diagram* of $\sigma$. The list of the number of boxes in the successive rows is called the *code* $C(\sigma)$ of $\sigma$. The morphism $\sigma \to C(\sigma)$ is a bijection from $\mathfrak{S}_n$ onto the set of integral vectors $[c_1, \ldots, c_n] \leq [0, 1, \ldots, n-1]$.

Indeed, given $c$, then $\sigma_1 = c_1 + 1$, and $[c_2, \ldots, c_n]$ is the code of a permutation $[\sigma_2, \ldots, \sigma_n]$ of $\{1, \ldots, \widehat{\sigma_1}, \ldots, n\}$.

One can read a reduced decomposition from the Rothe diagram: just number boxes in each row by consecutive numbers, starting from the number $i$ in row $i$. Now read rows from right to left, from top to bottom (interpreting $i$ as $s_i$).

The following lemma, easy to check, states that this word is a reduced decomposition. An equivalent description of it is by taking right factors, of respective lengths specified by the code, in the following reduced decomposition of $\omega$ (different from the one in eq.5):

$$\omega = (s_{n-1} \cdots s_1)\,(s_{n-1} \cdots s_2) \cdots (s_{n-1} s_{n-2})(s_{n-1}) \tag{6}$$

**Lemma 3** *Given a permutation $\sigma \in \mathfrak{S}_n$, let $v = [v_1, v_2, \ldots, v_{n-1}]$ be its code. Then the concatenation of the right factors of $(s_{n-1} \cdots s_1) \cdots (s_{n-1})(\ )$ of respective lengths $v_1, v_2, \ldots$ coincides with the word obtained from the labelling of the Rothe diagram of $\sigma$, and is a reduced decomposition of $\sigma$.*

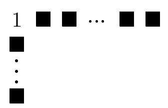
For example, the code $[3, 1, 3, 2, 3, 0, 0, 0]$ of a permutation in $\mathfrak{S}_8$ gives the reduced decomposition

$$(\bullet\bullet\bullet\bullet 321)\,(\bullet\bullet\bullet\bullet\bullet 2)\,(\bullet\bullet 543)\,(\bullet\bullet 54)\,(765)\,(\bullet\bullet)\,(\bullet)\,(\ )\ .$$

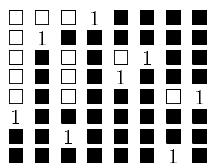
In the ACE output, boxes are numbered, each 0 is replaced by a dot, and each 1 is replaced by a cross.

```
ACE> Perm2Code([4, 2, 6, 5, 8, 1, 3, 7]);
                         [3, 1, 3, 2, 3, 0, 0, 0]

ACE> Perm2Rothe([4, 2, 6, 5, 8, 1, 3, 7]);

                 [1   2   3   x   .   .   .   .]
                 [2   x   .   .   .   .   .   .]
                 [3   .   4   .   5   x   .   .]
                 [4   .   5   .   x   .   .   .]
                 [5   .   6   .   .   .   7   x]
                 [x   .   .   .   .   .   .   .]
                 [.   .   x   .   .   .   .   .]
                 [.   .   .   .   .   .   x   .]

ACE> Perm2Rd([4, 2, 6, 5, 8, 1, 3, 7]);
                 [3, 2, 1, 2, 5, 4, 3, 5, 4, 7, 6, 5]
```

To build the Rothe diagram, instead of taking pairs of 1's, one can use the fact that there is no box right of a 1 in its row, and no box below a 1 in the same column. The Rothe diagram occupies the places which are not eliminated and which do not contain a 1.



*forbidden region*



There is another natural labelling of the boxes of the Rothe diagram, by writing in the box corresponding to an inversion $(ji)$ a variable $x_{ji}$. We shall see in the next section how to get this labelling by matrix multiplication.

$$\begin{bmatrix}
x_{41} & x_{42} & x_{43} & 1 & 0 & 0 & 0 & 0\\
x_{21} & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
x_{61} & 0 & x_{63} & 0 & x_{65} & 1 & 0 & 0\\
x_{51} & 0 & x_{53} & 0 & 1 & 0 & 0 & 0\\
x_{81} & 0 & x_{83} & 0 & 0 & 0 & x_{87} & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

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**Four diagrams**

In the matrix representing a permutation, there are in fact four types of 0's, depending on the relative positions of the 1's which are in the same row or same column: right or left, up or down. The 0's which have been replaced by the boxes of the Rothe diagram are one of the four types, let say the NW type.

Therefore, one has four Rothe diagrams, NW-Rothe, NE-Rothe,SW-Rothe, SE-Rothe, which partition the space occupied by the 0's of the matrix $M(\sigma)$.

$$\begin{array}{ll} 0 & \to 1\\ \downarrow & \\ 1 & \end{array} \quad \text{gives} \quad \begin{array}{ll} \boxed{\text{NW}} & \to 1\\ \downarrow & \\ 1 & \end{array} \quad , \quad \begin{array}{ll} 1 & \\ \uparrow & \\ 0 & \to 1 \end{array} \quad \text{gives} \quad \begin{array}{ll} 1 & \\ \uparrow & \\ \boxed{\text{SW}} & \to 1 \end{array} \quad ,$$

$$\begin{array}{rl} 1 \leftarrow & 0\\ & \downarrow\\ & 1 \end{array} \quad \text{gives} \quad \begin{array}{rl} 1 \leftarrow & \boxed{\text{NE}}\\ & \downarrow\\ & 1 \end{array} \quad , \quad \begin{array}{rl} & 1\\ & \uparrow\\ 1 \leftarrow & 0 \end{array} \quad \text{gives} \quad \begin{array}{rl} & 1\\ & \uparrow\\ 1 \leftarrow & \boxed{\text{SE}} \end{array} \quad .$$

```
Perm2fourRothe([4, 2, 6, 5, 8, 1, 3, 7]);
          3          [a    a    a    1    b    b    b    b]
          1          [a    1    b    d    b    b    b    b]
          3          [a    c    a    c    a    1    b    b]
          2          [a    c    a    c    1    d    b    b]
          3          [a    c    a    c    c    c    a    1]
          0          [1    d    b    d    d    d    b    d]
          0          [c    c    1    d    d    d    b    d]
          0          [c    c    c    c    c    c    1    d]
                      0    1    0    3    2    3    0    3
```

In ACE, boxes of are labelled $a, b, c, d$ instead of NW, NE, SW, SE.

Counting the number of $a$'s by rows, one gets $[3,1,3,2,3,0,0,0]$, that is, the code; counting the number of $d$'s by columns, one gets $[0,1,0,3,2,3,0,3]$, that is the cocode.

Rothe diagrams are related to the matrix of ranks of $M(\sigma)$; it is defined to be $\Big[r[i,j]\Big]_{1\le i,j\le n}$, $r[i,j]$ being the rank of the sub-matrix of $M(\sigma)$ taken on rows $1,\ldots,i$ and columns $1,\ldots,j$.

As shown in exercise ? they are easily obtained from the matrix giving the partial row or column sums of $M(\sigma)$.

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**Rothe diagrams by matrix multiplication**

The simplest non trivial Rothe diagram is $\left[\begin{smallmatrix}\square & 1\\ 1 & 0\end{smallmatrix}\right]$. Instead of putting a box, one can use a parameter $x$, and consider $\left[\begin{smallmatrix}x & 1\\ 1 & 0\end{smallmatrix}\right]$, or more generally, for $i$: $1\le i<n$, replace the matrix representing $M(s_i)$ by

$$T_i(x) := \begin{bmatrix} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& x & 1 &&& \\ &&& 1 & 0 &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{bmatrix}$$

Let $r$ be an integer and $I=[i_1,\ldots,i_r]\in\{1,\ldots,n{-}1\}^r$, such that $s^I := s_{i_1}\cdots s_{i_r}$ is a reduced decomposition of a permutation $\sigma$. Define $T_I(x_1,\ldots,x_r)$ to be the product

$$T_I(\mathbf{x}) = T_I(x_1,\ldots,x_r) = T_{i_1}(x_1)\cdots T_{i_r}(x_r)\ . \tag{7}$$

The matrix $T_I(\mathbf{x})$ depends on the choice of the reduced decomposition of $\sigma$. When specializing all $x_i$'s to 0, one recovers $M(\sigma^{-1})$. The combinatorial properties of the matrix $T_I(\mathbf{x})$ are studied in [20] (I have kept the conventions of this paper: the two axes of coordinates have been exchanged; equivalently, one takes $\sigma^{-1}$ instead of $\sigma$, or one reads reduced decompositions from right ot left). Let us just mention the simplest of them.

**Proposition 4** *Given $I\in\mathbb{N}^r$ such that $s^I$ is a reduced decomposition of a permutation $\sigma^{-1}$, then the matrix $T_I(\mathbf{x})$ has entries different from 0 and 1 exactly in the positions occupied by the boxes of the Rothe diagram of $\sigma$. Each polynomial entry restricts in degree 1 to a single variable.*

*Proof.* The proposition is easy to check by induction on the length of the reduced decomposition. Indeed, start from a reduced decomposition of $\sigma$. Let $M = T_I(\mathbf{x})$ be the corresponding matrix, and let $j$ be such that $\ell(\sigma s_j) > \ell(\sigma)$. Multiplying $M$ by $T_j(x)$ amounts to replace the two columns $\mathcal{C}, \mathcal{C}'$ at positions $j$, $j{+}1$ of $M$ by $x\,\mathcal{C}{+}\mathcal{C}'$, $\mathcal{C}$. All the elements in the new first column are of degree $> 1$, except the term $x\,1$. If one takes a parameter $x = x_{ji}$ recording the inversion created, then one sees that the matrix restricted to its terms of degrees 0 and 1 is the transpose of the Rothe diagram described at the end of last section. QED

Here are two succesive diagrams, for the multiplication by $s_2$, which modifies column 2 and column 3 and creates the inversion 62 :

$$\begin{bmatrix} x_{41} & x_{21} & x_{61} & x_{51} & x_{81} & 1 & \cdot & \cdot \\ x_{42} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{43} & \cdot & x_{63} & x_{53} & x_{83} & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & x_{65} & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x_{87} & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}, \quad \begin{bmatrix} x_{41} & x_{21}x_{62} + x_{61} & x_{21} & x_{51} & x_{81} & 1 & \cdot & \cdot \\ x_{42} & x_{62} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{43} & x_{63} & \cdot & x_{53} & x_{83} & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & x_{65} & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x_{87} & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

```
TT:=proc(i,n,x)
  diag(1$(i-1),matrix([[x,1],[1,0]]),1$(n-i-1))
end:
#  parameters = inversions ; input a reduced decomposition
Rd2Rothe_xx:=proc(rd) local i,j,k,n,mm,perm;
  n:=max(op(rd))+1;
  mm:=diag(1$n); perm:=[seq(i,i=1..n)];
  for i from 1 to nops(rd) do
  j:=perm[rd[i]]; k:=perm[rd[i]+1];
  mm:=multiply(mm, TT(rd[i],n, cat(x,10*k+j)));
  perm:=MultPerm(perm,SgTranspo(rd[i]));
  od;
eval(mm);
end:
```

---

### Cosets and double cosets

Given a composition $I = [i_1, \ldots, i_r]$ of $n$, let $\mathfrak{S}_I = \mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_r}$ be the Young subgroup it determines.

*Cosets* $\mathfrak{S}_n/\mathfrak{S}_I$ are equivalence classes of permutations modulo multiplication by $\mathfrak{S}_I$ on the right. It can be interpreted as cutting permutations (considered as words) into blocks of successive lengths $i_1, \ldots, i_r$, and permuting freely elements inside each block. One can also decide to write increasingly the elements inside each block, obtaining a row (=row-tableau) that one usually represents in a box.

Similarly, cosets $\mathfrak{S}_I \backslash \mathfrak{S}_n$ are obtained by cutting the set of *values* into blocks, and identifying elements inside a block, for example, giving them new names $a, b, \ldots$ (letters in a *totally ordered alphabet*).

*Double cosets* $\mathfrak{S}_I \backslash \mathfrak{S}_n / S_J$ are equivalence classes modulo the action of the two Young subgroups, and can be represented by a sequence of rows of lengths $j_1, j_2, \ldots$ using in all $i_1$ times the letter $a$, $i_2$ times the letter $b$, &c.

For example, double cosets $\mathfrak{S}_{323} \backslash \mathfrak{S}_8 / \mathfrak{S}_{44}$ are obtained by cutting permutations into two blocks of lengths 4, and identifying $1, 2, 3$ to $a$, $4, 5$ to $b$, $6, 7, 8$ to $c$.

The double coset containing $\sigma = 63715824$ can thus be coded by

$$\boxed{\text{aacc}} \otimes \boxed{\text{abbc}}$$

(we shall consider it later as a skew Young tableau, direct product of two rows, of shape $4 \otimes 4$).

One can also code such a tableau by an integral matrix of size $3 \times 2$, each row of the matrix being the degree (as a vector) of the successive rows of the tableau:

| | | $a$ | $b$ | $c$ | *row sums* |
|---|---|---|---|---|---|
| $\boxed{\text{aacc}}$ | $= a^2 b^0 c^2$ | 2 | 0 | 2 | 4 |
| $\boxed{\text{abbc}}$ | $= a^1 b^2 c^1$ | 1 | 2 | 1 | 4 |
| *column* | *sums* | 3 | 2 | 3 | |

The row sums of the matrix are the sizes of the successive lengths of the rows (i.e. are $J = 4, 4$), the column sums are the commutative evaluation of the word $a^2 b^0 c^2 a^1 b^2 c^1 \equiv a^3 b^2 c^3$.

More generally, double cosets $\mathfrak{S}_I \backslash \mathfrak{S}_n / \mathfrak{S}_J$ are in bijection with integral matrices: the $(h, k)$ entry of the matrix counts the number of occurences of letter $x_k$ in row $h$ of the corresponding skew tableau of shape $j_1 \otimes j_2 \cdots$ and commutative evaluation $x^I$.

In particular, the number of double cosets $\mathfrak{S}_I \backslash \mathfrak{S}_n / \mathfrak{S}_J$ is equal to the *number of integral matrices with row sums $I$ and column sums $J$*. This number has many interpretation, we already seen that it is equal to the scalar product of two products of complete symmetric functions, $(S^I, S^J)$.

```
ACE> GenMat([4,4],[3,2,3]);
 [3   1   0]  [3   0   1]  [2   2   0]  [2   1   1]  [2   0   2]
```

```
[0    1    3]  [0    2    2]  [1    0    3]  [1    1    2]  [1    2    1]

 [1    2    1]  [1    1    2]  [1    0    3]  [0    2    2]  [0    1    3]
 [2    0    2]  [2    1    1]  [2    2    0]  [3    0    1]  [3    1    0]
ACE> SfScalar(h4^2,h3*h2*h3);
                                              10
```

One can require each box to contain only different letters. In that case, one writes it as a *column* (i.e. a strictly decreasing sequence of letters). A direct product of columns of lengths $j_1, j_2, \ldots$ will be considered as a skew tableau of shape $1^{j_1} \otimes 1^{j_2} \otimes \cdots$. Such tableaux are in bijective correspondence with $(0,1)$-matrices with row-sums $J$ and column-sums $I$.

$$\boxed{\begin{matrix}5\\3\\2\end{matrix}} \otimes \boxed{\begin{matrix}5\\4\\3\\2\\1\end{matrix}} \otimes \boxed{\begin{matrix}5\\2\end{matrix}} \otimes \boxed{\begin{matrix}5\\3\\2\end{matrix}} \leftrightarrow \begin{bmatrix}0&1&1&0&1\\1&1&1&1&1\\0&1&0&0&1\\0&1&1&0&1\end{bmatrix}$$

(the first column is 532, which are the positions occupied by 1 in the first row of the matrix).

In our exemple, there is only one 0-1 matrix with row-sums $[3, 5, 2, 3]$ and column-sums $[1, 4, 3, 1, 4]$ (we shall see that this is because $[5332]$ and $[44311]$ are conjugate partitions) :

```
ACE> GenMat1([3,5,2,3],[1,4,3,1,4]);
                                      [0     1     1     0     1]
                                      [1     1     1     1     1]
                                      [0     1     0     0     1]
                                      [0     1     1     0     1]
ACE> GenMat([3,5,2,3],[1,4,3,1,4],'nb');
                                             2816
ACE> SfScalar(e3*e5*e2*e3, h1*h4*h3*h1*h4);
                                              1
```

The number of such matrices is equal to a scalar product between a product of elementary symmetric functions and a product of complete functions.

The case where there is only one such matrix is fundamental in the theory of representations of the symmetric group.

Let us say that two compositions are *weakly conjugate* iff the two partitions obtained by sorting them are conjugate (the word *conjugate* has a more restricted sense imposed by the theory of non-commutative symmetric functions).

**Lemma 5** *Let $I$ and $J$ be two weakly conjugate compositions. Then there is only one 0-1 matrix of row-sums $I$ and column-sums $J$.*

*Proof.* Let $r$ be the maximum of the components of $J$. Then one sees that if $j_h = r$, then all entries in the $h$-th row of the matrix must have a 1 in each position $k$ such that $i_k \neq 0$, and 0 otherwise. Suppressing these rows and substracting $r$ to each non-zero component of $I$, one gets the lemma by induction. QED

One can also interpret the matrix as coding a diagram of boxes, writing in each box its level (the diagram is obtained by transposing the 0-1 matrix, then replacing each 1 by a box, each 0 by a void) :

$$\begin{array}{cccc}
5 & 5 & 5 & 5 \\
 & 4 & & \\
3 & 3 & & 3 \\
2 & 2 & 2 & 2 \\
 & 1 & &
\end{array}$$

To the 0-1 matrix is also associated a permutation that we shall note $\zeta(I, J)$ obtained by numbering the boxes from left to right and top to bottom, and reading columnwise :

$$\begin{bmatrix} \cdot & 1 & 2 & \cdot & 3 \\ 4 & 5 & 6 & 7 & 8 \\ \cdot & 9 & \cdot & \cdot & 10 \\ \cdot & 11 & 12 & \cdot & 13 \end{bmatrix} \quad \mapsto \quad \zeta = [4, 1, 5, 9, 11, 2, 6, 12, 3, 8, 10, 13] \ .$$

---

**Subgroups associated to a tabloid**

Let $\mathcal{D}$ be a diagram of $n$ boxes in the plane. A *tabloid* of shape $\mathcal{D}$ is any numbering of the boxes of $\mathcal{D}$ with the integers $1, 2, \ldots, n$.

Let $P(t)$ be the sum of all permutations which globally preserve the rows of $t$. Then $P(t)$ is conjugate to $\square_{\omega_I}$, where $I$ is any permutation of the number of boxes in the successive rows of $\mathcal{D}$ .

For example, if

$$t = \begin{array}{cccccc}
5 & 3 & & 9 & & \\
 & & 1 & 6 & 8 & 4 \\
 & 7 & 2 & & &
\end{array}$$

and $I = [4, 2, 3]$, $\omega_I = 4321\,65\,987$ and

$$P(t) = [1684\,72\,539] \cdot \square_{432165987} \cdot [1684\,72\,539]^{-1} \ .$$

Given any composition $I$ which is a permutation of the lengths of the rows of a diagram $\mathcal{D}$ of $n$ boxes, let $\mathcal{D}_I$ be the object obtained by numbering with $1, 2, \ldots, n$ successively the boxes of $\mathcal{D}$, from left to right in each row, taking the rows in the order specified by $I$ (choosing any order between rows with the same length).

For example, for $I = [4, 2, 3]$, the numbering of the preceding diagram is

$$\mathcal{D}_{423} = \begin{array}{cccccc} 7 & 8 & & 9 & & \\ & & 1 & 2 & 3 & 4 \\ & 5 & 6 & & & \end{array} .$$

Let us note that the permutation conjugating $P(t)$ is exactly the permutation obtained by reading the boxes of $t$ in the order specified by $\mathcal{D}_{423}$.

Similarly, let $N(t)$ be the alternating sum of all permutations which preserve the columns of $t$. Then $N(t)$ is conjugate to $\nabla_{\omega(J)}$, where $J$ is any permutation of the sequence of number of boxes in the successive columns of $\mathcal{D}$.

In the case of the above tabloid, and $J = [2, 2, 1, 1, 1, 2]$ then

$$N(t') = [85\,16\,7\,3\,4\,92] \cdot \nabla_{21\,43\,5\,6\,7\,98} \cdot [85\,16\,7\,3\,4\,92]^{-1} .$$

Young took the case where $\mathcal{D}$ is the diagram of a partition $I$. From what we have just seen, we know that for any tabloid $t$ of shape $I$, the element $P(t)N(t)$ of the group algebra is equal to

$$\nu \cdot \square_{\omega_I} \cdot \nu^{-1} \cdot \eta \cdot \nabla_{\omega_J} \cdot \eta^{-1} ,$$

where $J$ is the partition conjugate to $I$ and $\nu$ and $\eta$, the two permutations obtained by reading the boxes of $t$ in a certain order. Let us note that the permutation $\nu^{-1}\eta$ does not depend upon the tableau $t$, but only of the diagram, because it is the permutation which transforms the row-reading into the column-reading.

For example, if

$$t = \begin{array}{|c|c|c} \hline 5 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & 6 & \multicolumn{1}{c}{} \\ \hline 4 & 7 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

then $\nu = [53\,26\,471]$, $\eta = [524\,367\,1]$ and $\nu\,\eta^{-1} = [135\,246\,7]$ and this permutation is the one obtained by reading columnwise

$$\begin{array}{|c|c|c} \hline 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & 4 & \multicolumn{1}{c}{} \\ \hline 5 & 6 & \multicolumn{1}{c|}{7} \\ \hline \end{array} \quad \to [135\ 246\ 7] .$$

## The group algebra of the symmetric group

Up to now, we could multiply permutations, but not add them. To recover addition, we shall work in the group algebra $\mathbb{Q}[\mathfrak{S}_n]$ of the symmetric group, with rational coefficients.

In other words, as a $\mathbb{Q}$-vector space, $\mathbb{Q}[\mathfrak{S}_n]$ is $n!$-dimensional, with a basis consisting of permutations. But moreover, two elements multiply according the multiplication of permutations :

$$\left(\sum_{\sigma\in\mathfrak{S}_n} c_\sigma\, \sigma\right)\,\left(\sum_{\nu\in\mathfrak{S}_n} d_\nu\, \nu\right) = \sum_\sigma \sum_\nu (c_\sigma\, d_\nu)\, \sigma\, \nu \ .$$

---

**Yang-Baxter elements**

Instead of handling reduced decompositions, one can now take products of factors of the type $s_i + c$, with $c \in \mathbb{Q}$. However our fundamental relation $s_1 s_2 s_1 = s_2 s_1 s_2$ is not compatible with a uniform shift :

$$\begin{aligned}(1+s_1)(1+s_2)(1+s_1) = 2 + 2s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1 \\ \neq (1+s_2)(1+s_1)(1+s_2) \ .\end{aligned}$$

Indeed $2s_1 + s_2$ is not symmetrical in $s_1, s_2$, and it implies that the elements $(1+s_1)(1+s_2)(1+s_1)$ and $(1+s_2)(1+s_1)(1+s_2)$ are different. To recover equality, one must use non constant shifts. For example,

$$(1+s_1)(\frac{1}{2}+s_2)(1+s_1) = (1+s_2)(\frac{1}{2}+s_1)(1+s_2) \ .$$

The general rule to ensure equality is due to Yang and Baxter. More precisely, we want an equality, with some constants $\alpha, \dots, \gamma'$ :

$$(s_1 + \frac{1}{\alpha})\,(s_2 + \frac{1}{\gamma})\,(s_1 + \frac{1}{\beta}) = (s_2 + \frac{1}{\alpha'})\,(s_2 + \frac{1}{\gamma'})\,(s_1 + \frac{1}{\beta'}) \ .$$

We find that we must have $\alpha = \beta'$, $\beta = \alpha'$ to ensure equality for the terms of length 2. Now, to recover symmetry in the terms of length 1 :

$$\frac{1}{\gamma}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\, s_1 + \frac{1}{\alpha\beta}\, s_2 \ ,$$

one must have

$$\gamma = \alpha + \beta \ .$$

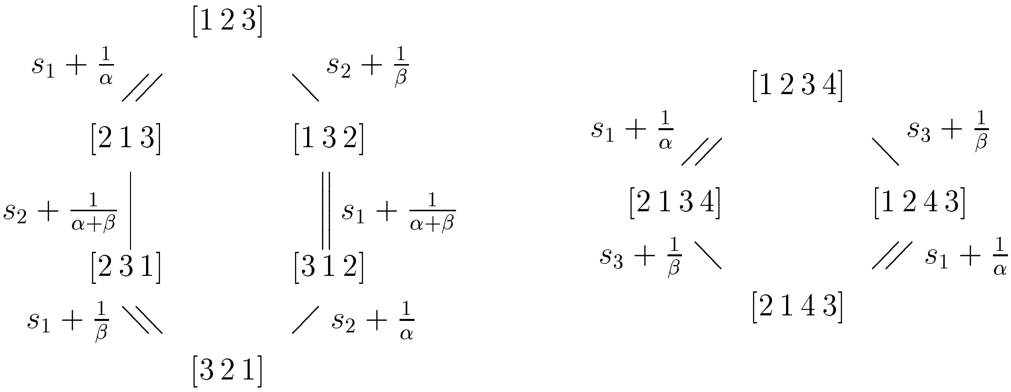
Of course, one can introduce parameters in the relation $s_i s_j = s_j s_i$, $|i-j| \neq 1$, without breaking the commutation: $(s_i + \frac{1}{\alpha})(s_j + \frac{1}{\beta}) = (s_j + \frac{1}{\beta})(s_i + \frac{1}{\alpha})$.

Finally, the braid relations have become the *Yang-Baxter relations*

$$(s_i + \frac{1}{\alpha})\,(s_{i+1} + \frac{1}{\alpha+\beta})\,(s_i + \frac{1}{\beta}) = (s_{i+1} + \frac{1}{\beta})\,(s_i + \frac{1}{\alpha+\beta})\,(s_{i+1} + \frac{1}{\alpha}) \quad (8)$$

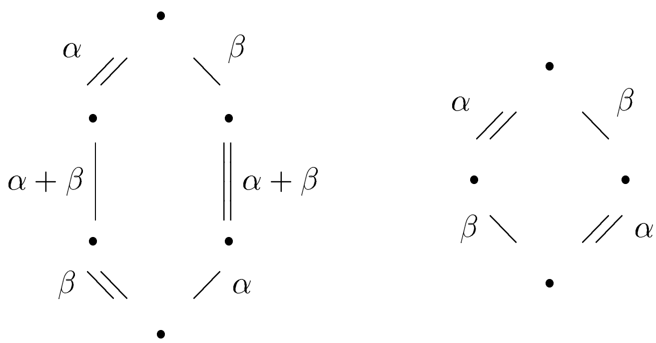
$$(s_i + \frac{1}{\alpha})\,(s_j + \frac{1}{\beta}) = (s_j + \frac{1}{\beta})\,(s_i + \frac{1}{\alpha)}\ ,\ \ |i-j| \neq 1 \quad (9)$$

Their graphical representation is easy to remember (taking $i = 1$) :



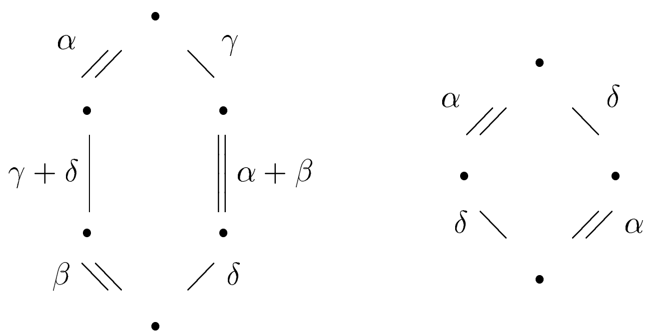

Vertical edges wear a parameter which is the sum of the two on the opposite sides, other pairs of parallel edges have the same parameter.

In short, forgetting about the labelling of vertices, the index of the simple transpositions $s_i$ being specified by the color of edges, one needs only write the parameters $\alpha$ used in the factors $s_i + 1/\alpha$ (beware that usually we are writing on edges the inverses of the parameters) :




(10)

The important constraint is that the parameter on a vertical edge is the sum of the parameters on the two opposite edges. Thus, instead of two parameters, one can take for $\mathfrak{S}_3$ four independent parameters, keeping the trivial commutation relations for lozenges :




With M.P. Schützenberger, we called this relation *Yin relation*, and we solved the problem of labelling the edges of the permutohedron in such a way that all sub-hexagons satisfy the Yin relation, and that all lozenges commute [22].

Since braid relations connect any two reduced decompositions of the same permutation, if we can label edges of a permutohedron with parameters in such a way as to satisfy relations (8,9), then all paths from the origin to a given permutation will give the same element in $\mathbb{Q}[\mathfrak{S}_n]$.

To get a coherent choice of parameters for $\mathfrak{S}_n$, Yang [42] gave the following recepee, that we can interpret as enriching the permutohedron with a second labelling of vertices, and of edges, according to the following rule :

- choose an arbitrary system of "spectral parameters" $[x_1, \ldots, x_n]$.
- label each vertex, say $\sigma$, with $[x_{\sigma_1}, \ldots, x_{\sigma_n}]$
- label the edge of color $s_i$ connecting $\sigma$ and $\sigma\, s_i$ with $x_{\sigma_{i+1}} - x_{\sigma_i}$

An edge $s_i$ with parameter $\alpha$ must be interpreted as a factor $(s_i + \frac{1}{\alpha})$ and a path must be interpreted as the product of its edges.

In summary, given parameters $[x_1, \ldots, x_n]$ all different, then all paths in the labelled permutohedron, starting from the origin to a permutation $\sigma$, give in $\mathbb{Q}[\mathfrak{S}_n]$ the same Yang-Baxter element that we shall denote $\mathcal{Y}_\sigma$ or $\mathcal{Y}_\sigma(x_1, \ldots, x_n)$. We can formulate the preceding construction as follows :

**Proposition 6** *For any choice of spectral parameters* $[x_1, \ldots, x_n]$*, all different, there exists a* Yang-Baxter basis *which is a linear basis of* $\mathbb{Q}[x_1, \ldots, x_n](\mathfrak{S}_n)$*, satisfying the following relations which caracterize it (together with normalization* $\mathcal{Y}_1 = 1$*) :*

$$\mathcal{Y}_{\sigma\, s_i} = \mathcal{Y}_\sigma \left( s_i + \frac{1}{x_{\sigma_{i+1}} - x_{\sigma_i}} \right) \quad \textit{when } \ell(\sigma s_i) > \ell(\sigma) \ . \tag{11}$$

The permutohedron for $\mathfrak{S}_3$ looks now like

$$
\begin{array}{ccccc}
 & & [x_1\, x_2\, x_3] & & \\
s_1 + \frac{1}{x_2 - x_1} & /\!/ & & \diagdown & s_2 + \frac{1}{x_3 - x_2} \\
 & [x_2\, x_1\, x_3] & & [x_1\, x_3\, x_2] & \\
s_2 + \frac{1}{x_3 - x_1} & \Big| & & \Big\| & s_1 + \frac{1}{x_3 - x_1} \\
 & [x_2\, x_3\, x_1] & & [x_3\, x_1\, x_2] & \\
s_1 + \frac{1}{x_3 - x_2} & \backslash\!\backslash & & \diagup & s_2 + \frac{1}{x_2 - x_1} \\
 & & [x_3\, x_2\, x_1] & &
\end{array}
\tag{12}
$$

---

**Quadratic form**

One usually defines quadratic forms by taking constant terms (in the case of $\mathbb{Q}[\mathfrak{S}_n]$, taking the coefficient of the identity permutation). We shall use here another convention.

Denote by $g \to \widetilde{g}$ the linear morphism on $\mathbb{Q}[\mathfrak{S}_n]$ induced by $\sigma \to \sigma^{-1}$, $\sigma \in \mathfrak{S}_n$, and by $g \cap \omega$ the coefficient of $\omega$ in $g$. For any $f, g \in Q[\mathfrak{S}_n]$, let

$$\big( f \, , \, g \big) := f \, \widetilde{g} \cap \omega \ . \tag{13}$$

The linear basis of permutations is self-adjoint with respect to this form :

$$\big( \sigma \, , \, \omega\sigma \big) = 1 \quad \& \quad \big( \sigma \, , \, \nu \big) = 0, \ \ \nu \neq \omega\sigma \ . \tag{14}$$

The next proposition, due to [24], shows that the Yang-Baxter basis is also compatible with the quadratic form. Let $\mathcal{Y}_\sigma$ be a Yang-Baxter basis for the parameters $[x_1, \ldots, x_n]$, and $\widehat{\mathcal{Y}}_\sigma$ be the Yang-Baxter basis for the reversed parameters $[x_n, \ldots, x_1]$.

**Proposition 7** *The Yang-Baxter basis* $\{\mathcal{Y}_\sigma\}$ *is adjoint to the Yang-Baxter basis* $\{\widehat{\mathcal{Y}}_{\omega\sigma}\}$*, i.e. one has*

$$\big( \mathcal{Y}_\sigma \, , \, \widehat{\mathcal{Y}}_{\omega\sigma} \big) = 1 \quad \& \quad \big( \mathcal{Y}_\sigma \, , \, \widehat{\mathcal{Y}}_\nu \big) = 0, \ \ \nu \neq \omega\sigma \ . \tag{15}$$

*Proof.* The proposition is true for $\widehat{\mathcal{Y}}_1$, because only $\mathcal{Y}_\omega$ has a term in $\omega$. Suppose it is true for $\widehat{\mathcal{Y}}_\nu$. We shall prove it for $\widehat{\mathcal{Y}}_{\nu s_i}$, $\ell(\nu s_i) > \ell(\nu)$. Indeed, $\widehat{\mathcal{Y}}_{\nu s_i} = \widehat{\mathcal{Y}}_\nu (s_i + \alpha)$ for some $\alpha$. Therefore

$$\mathcal{Y}_\sigma \, (s_i + \alpha) \widetilde{\widehat{\mathcal{Y}}_\nu} = \big( \beta \mathcal{Y}_\sigma + \gamma \mathcal{Y}_{\sigma s_i} \big) \, \widetilde{\widehat{\mathcal{Y}}_\nu} \ ,$$

for some constants $\beta, \gamma$. Therefore, one has just to consider the cases where $\sigma = \omega\nu$ or $\sigma s_i = \omega\nu$, and this is where one sees that one had to take reversed parameters for the second Yang-Baxter basis. QED

For example, let us check that $(\mathcal{Y}_{231}\,,\,\widehat{\mathcal{Y}}_{321}) = 0$, taking the parameters $[0, \alpha, \gamma = \alpha + \beta]$. Then there exists some constants $\delta, \epsilon$ such that

$$\mathcal{Y}_{231}\widehat{\mathcal{Y}}_{321} = (s_1 + \frac{1}{\alpha})\underbrace{(s_2 + \frac{1}{\gamma})\,(s_2 - \frac{1}{\beta})}_{\delta(s_2 - \frac{1}{\beta}) + \epsilon}(s_1 - \frac{1}{\gamma})(s_2 - \frac{1}{\alpha}) \ .$$

The factor of $\epsilon$ cannot contain $\omega$, and we can eliminate it. The triple $(s_1 + \frac{1}{\alpha})(s_2 - \frac{1}{\beta})(s_1 - \frac{1}{\gamma})$ can be transformed into $(s_2 - \frac{1}{\gamma})(s_1 - \frac{1}{\beta})(s_2 + \frac{1}{\alpha})$, but since $(s_2 + \frac{1}{\alpha})(s_2 - \frac{1}{\alpha})$ is a scalar, the remaining expression cannot either contain $\omega$.

---

**Special Yang-Baxter elements**

Many interesting elements of the group algebra of $\mathfrak{S}_n$ can be written in terms of Yang-Baxter elements, for different choices of parameters.

We shall specially use the two cases where $[x_1, \ldots, x_n] = [1, \ldots, n]$ or $[x_1, \ldots, x_n] = [n, \ldots, 1]$, denoting

$$\Box_\sigma := \mathcal{Y}_\sigma(1, \ldots, n) \quad \& \quad \nabla_\sigma := \mathcal{Y}_\sigma(n, \ldots, 1) \ , \ \sigma \in \mathfrak{S}_n \ .$$

Let us notice that proposition 7 implies :

**Lemma 8** *$\{\Box_\sigma\}$ and $\{\nabla_{\omega\sigma}\}$ are two adjoint bases.*

The elements $\Box_\sigma$ and $\nabla_\sigma$ allow to write idempotents in the group algebra. Let us check for example that the sum of all permutations of $\mathfrak{S}_n$ is equal to $\Box_\omega = \mathcal{Y}_\omega(1, \ldots, n)$, with $\omega$ = maximal permutation = $[n, \ldots, 1]$.

Because we can start a path from the identity to $\omega$ by any simple transposition $s_i$, then $\Box_\omega$ is such that it has at least one expression with a left factor $(s_i+1)$. The two canonical reduced decompositions that we have encountered extends to two expressions of $\Box_\omega$, which are :

$$\begin{aligned}
\Box_\omega &= \Big((s_1 + \frac{1}{1})\Big)\Big((s_2 + \frac{1}{2})(s_1 + \frac{1}{1})\Big) \cdots \Big((s_{n-1} + \frac{1}{n-1}) \cdots (s_1 + 1)\Big) \\
&= \Big((s_{n-1} + \frac{1}{1}) \cdots (s_1 + \frac{1}{n-1})\Big) \cdots \Big((s_{n-1} + \frac{1}{1})(s_{n-2} + \frac{1}{2})\Big)\Big((s_{n-1} + \frac{1}{1})\Big)
\end{aligned}$$

Suppose that we have noticed that

$$\Box_{321} = \sum\nolimits_{\sigma \in \mathfrak{S}_3} \sigma$$

We want to prove that the similar property holds for

$$\square_{4321} = \square_{321}(s_3 + \frac{1}{3})(s_2 + \frac{1}{2})(s_1 + 1) \ .$$

We use the fact that $\left(\sum_{\sigma\in\mathfrak{S}_3}\sigma\right)\nu = \sum_{\sigma\in\mathfrak{S}_3}\sigma$ if $\nu \in \mathfrak{S}_3$. In other words, when multiplying $\sum_{\sigma\in\mathfrak{S}_n}\sigma$ by an expression involving only permutations in $\mathfrak{S}_n$, one can replace in this expression each permutation by 1.

Therefore

$$\square_{321}\,\frac{1}{3}\,(s_2 + \frac{1}{2})(s_1 + 1) = \square_{321}\,\frac{1}{3}\,(1 + \frac{1}{2})(1 + 1) = \square_{321} \ .$$

Similarly

$$\square_{321}\, s_3\, \frac{1}{2}\,(s_1 + 1) = \square_{321}\, s_3\, \frac{1}{2}(1 + 1) = \square_{321}\, s_3$$

Finally

$$\square_{321}(s_3 + \frac{1}{3})(s_2 + \frac{1}{2})(s_1 + 1) = \square_{321}(1 + s_3 + s_3 s_2 + s_3 s_2 s_1) \ .$$

The right hand side is indeed equal to the sum of all permutations of $\mathfrak{S}_4$, because it describes how they are obtained from permutations of $\mathfrak{S}_3$ by inserting 4 in all possible manners.

To pass from a general $\mathfrak{S}_n$ to $\mathfrak{S}_{n+1}$, one needs to write the expansion with some care. One decomposes the product $(s_n + 1/n)\cdots(s_1 + 1)$ into a sums of terms

$$s_n \cdots s_{k+1}\,\frac{1}{k}\,(s_{k-1} + \frac{1}{k-1})\cdots(s_1 + 1) \ , \ 0 \le k \le n \ ,$$

corresponding to the first time that in the succesive factors $s_i + 1/k$, one chooses to take the scalar instead of the simple transposition.

The term right of $\frac{1}{k}$ commutes with the left part, and behaves like the scalar $(1 + \frac{1}{k-1})(1 + \frac{1}{k-2})\cdots(1 + 1) = 1$ when multiplied on the left by $\sum_{\sigma\in\mathfrak{S}_n}\sigma$. Therefore one finds that

$$\square_{n+1\,n\ldots 1} = \square_{n\ldots 1}\big(s_n \cdots s_1 + s_n \cdots s_2 + \cdots + s_n + 1\big) \ ,$$

and, as for $n = 3$, this proves that $\square_{n+1\,n\ldots 1}$ is the sum of all permutations in $\mathfrak{S}_{n+1}$. Using the involution $s_i \to -s_i$, we also get that $\nabla_\omega$ is equal to the alternating sum of all permutations :

$$\square_\omega = \sum\nolimits_{\sigma\in\mathfrak{S}_n}\sigma \qquad \& \qquad \nabla_\omega = \sum\nolimits_{\sigma\in\mathfrak{S}_n}(-1)^{\ell(\sigma\omega)}\sigma \ . \tag{16}$$

We shall use later that $\square_\omega$ and $\nabla_\omega$ are the two 1-dimensional quasi-idempotents of $\mathfrak{S}_n$, and are characterized, up to a scalar, by

$$\square_\omega\, s_i = \square_\omega = s_i\, \square_\omega \quad \& \quad \nabla_\omega\, s_i = -\nabla_\omega = s_i\, \nabla_\omega \ , \ \forall i : 1 \leq i < n$$

Using this characterization would have saved us from the above computations, we performed them only to illustrate the techniques one should need to use for a general Yang-Baxter element $\square_\sigma$.

Let us see another more powerful approach, which would also apply when direct expansions in the group algebra are not feasible. It consists in identifying each element of $\mathbb{Q}[\mathfrak{S}_n]$ to an operator on polynomials; two elements coincide iff they have the same action on polynomials. If we were testing the action on the polynomials $x_1, x_2, \ldots, x_n$, then we have just said that there is no essential difference between a permutation $\sigma$ and the list $[x_{\sigma_1}, \ldots, x_{\sigma_n}]$. However, we shall use the fact that the ring of polynomials $\mathbb{Q}[x_1, \ldots, x_n]$ is a module over the ring $\mathfrak{Sym}(n)$ of symmetric polynomials. It is in fact a free module with basis the Schubert polynomials $X_\sigma$, $\sigma \in \mathfrak{S}_n$. In other words, any polynomial is a linear combination of Schubert polynomials with coefficients in $\mathfrak{Sym}(n)$ (which commute with the action of the symmetric group).

The only property that we shall need from these polynomials is that they all are symmetrical in at least one pair $x_i, x_{i+1}$, except for the maximal one $\mathbb{X}_\omega = x^\rho := x_1^{n-1} x_2^{n-2} \cdots x_n^0$.

Let us prove again that

$$\nabla_\omega = \mathcal{Y}_\omega(n, \ldots, 1) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\omega\sigma)}\, \sigma \ , \tag{17}$$

interpreting both sides as operators on polynomials.

Both sides annihilate polynomials which have at least one symmetry in a pair $x_i, x_{i+1}$ (because $(s_i - 1)$ annihilates such polynomials). Thus both sides annihilate all the Schubert polynomial, except the last one $\mathbb{X}_\omega$. Now instead of testing the action on $\mathbb{X}_\omega$, one can take the Vandermonde $\Delta := \prod_{i<j}(x_i - x_j)$. Each simple transposition acts by multiplication by –1 on $\Delta$, and therefore, $\mathcal{Y}_\omega(n, \ldots, 1)$ acts by multiplication by the scalar

$$(-1 - \frac{1}{1})(-1 - \frac{1}{2})(-1 - \frac{1}{1}) \cdots (-1 - \frac{1}{n}) \cdots (-1 - \frac{1}{1}) = \pm\, n! \ .$$

Similarly, each permutation $\sigma$ acts on the Vandermonde by $(-1)^{\ell(\sigma)}$, and their alternating sum acts by $\pm n!$. Therefore, the two operators $\nabla_\omega$ and $\sum \pm\sigma$ are equal. QED

We shall justify in more details the preceding arguments, when describing the quotient of the ring $\mathbb{Q}[x_1, \ldots, x_n]$ by the ideal generated by symmetric polynomials without constant term. This quotient is isomorphic, as a representation of $\mathfrak{S}_n$, to $\mathbb{Q}[\mathfrak{S}_n]$. Therefore identities in the group algebra can be moved to identities involving only polynomials.

We shall need the Yang-Baxter elements $\mathcal{Y}_\sigma(1, \ldots, n)$ to compute characters. In particular, we shall need the *Yang-Baxter "cycles"* of order $k$, $2 \le k \le n$, which are

$$\zeta^{[k]} := \square_{23\ldots k1} = (s_1 + \frac{1}{1})(s_2 + \frac{1}{2}) \cdots (s_{k-1} + \frac{1}{k-1})\, \frac{1}{k} \tag{18}$$

---

**Permutation modules**

From now on, let us write $\mathcal{H}$ for the group algebra $\mathbb{Q}[\mathfrak{S}_n]$ (many statements can be generalized to the *Hecke algebra*, which is a deformation of $\mathbb{Q}[\mathfrak{S}_n]$, this explains the use of $\mathcal{H}$ !).

The elements $\square_\omega$ and $\nabla_\omega$ are such that $\square_\omega \mathcal{H}$ and $\nabla_\omega \mathcal{H}$ are two 1-dimensional modules[1], called the *trivial representation* and the *alternating representation* respectively.

One cannot get much information from a 1-dimensional space, but considering the corresponding elements for Young sub-groups will be enough to generate all representations.

Let $I = [i_1, \ldots, i_r]$ be a composition of $n$, $\mathfrak{S}_I = \mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_r} \hookrightarrow \mathfrak{S}_n$ be the associated Young sub-group, and $\omega_I$ its maximal element.

From (16) one knows that

$$\square_{\omega_I} = \sum\nolimits_{\sigma \in \mathfrak{S}_I} \sigma \qquad \& \qquad \nabla_{\omega_I} = \sum\nolimits_{\sigma \in \mathfrak{S}_I} (-1)^{\ell(\sigma\omega_I)} \sigma \ . \tag{19}$$

**Proposition 9** *Given a composition $I$, the modules[2] $\square_{\omega_I} \mathcal{H}$ and $\nabla_{\omega_I} \mathcal{H}$ have basis $\{\square_\sigma,\, \sigma \in [\omega_I, \omega]\}$ and $\{\nabla_\sigma,\, \sigma \in [\omega_I, \omega]\}$ respectively, where $\sigma$ runs over all permutations in the interval $[\omega_I, \omega]$ of the permutohedron.*

*The permutations in $[\omega_I, \omega]$ are exactly those permutations which have the inversions of $\omega_I$, in other words, which are shuffles of*

$$[i_1 \ldots 1],\ [(i_2{+}i_1) \ldots (i_1{+}1)],\ \ldots,\ [(i_1{+}\cdots{+}i_r) \ldots (i_1{+}\cdots{+}i_{r-1}{+}1)] \ .$$

*The space $\square_{\omega_I} \mathcal{H}$ is isomorphic to the permutation representation of $\mathfrak{S}_n$ on words with commutative evaluation $x^I$.*

---

[1] We already used that $\square_\omega \sigma = \square_\omega$ and $\nabla_\omega \sigma = (-1)^{\ell(\sigma)} \nabla_\omega$ for any $\sigma \in \mathfrak{S}_n$.

[2] called *induced from the trivial representation or alternating representation* of $\mathfrak{S}_I$

*Proof.* The linear span of $\{\square_\sigma,\ \sigma \in [\omega_I, \omega]\}$ and $\{\square_{\omega_I}\, \omega_I\sigma,\ \sigma \in [\omega_I, \omega]\}$ is the same. Two permutations $\eta, \nu$ in the same coset $\mathfrak{S}_I\, \eta$ are such that $\square_{\omega_I}\, \eta = \square_{\omega_I}\, \nu$. Conversely, the elements $\{\square_\sigma,\ \sigma \in [\omega_I, \omega]\}$ are linearly independent, having term of highest length $\sigma$. This gives a basis of $\square_{\omega_I}\, \mathcal{H}$, and by the involution $s_i \to -s_i$, a basis of $\nabla_{\omega_I}\, \mathcal{H}$.

Checking the last statement is just a matter of rewriting the interval of the permutohedron, replacing $1, \ldots, i_1$ by $a$, $i_1{+}1, \ldots, i_1{+}i_2$ by $b$, &c. For example, one has for $I = [2, 2]$

$$
\begin{tikzcd}
 & {[2143]} \arrow[d, equal, "s_2"] & \\
 & {[2413]} \arrow[dl, "s_1"'] \arrow[dr, "s_3"] & \\
{[4213]} \arrow[dr, "s_3"'] & & {[2431]} \arrow[dl, "s_1"] \\
 & {[4231]} \arrow[d, equal, "s_2"] & \\
 & {[4321]} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & aabb \arrow[d, equal, "s_2"] & \\
 & abab \arrow[dl, "s_1"'] \arrow[dr, "s_3"] & \\
baab \arrow[dr, "s_3"'] & & abba \arrow[dl, "s_1"] \\
 & baba \arrow[d, equal, "s_2"] & \\
 & bbaa &
\end{tikzcd}
$$

The two graphs can be interpreted as furnishing a basis: $\square_{\omega_I}\eta$, and a basis of words of two modules, with the same action of the symmetric group, because the right graph is obtained from the left one by making vertices operate on the commutative monomial $x^{aabb}$ ( more simply, one can take $x^{0011}$; we need only two different exponents. It is equivalent to use monomials in commutative variables $x_1, x_2, \ldots$, or words in non-commutative letters $a, b, \ldots$, or $0, 1, \ldots$, which are interpreted as the exponents of the monomials, and of course, canot be written in any order if one does not write the variables at the same time). To be complete, one should write on the graph loops at each vertex, corresponding to the simple transpositions which preserve the vertex. QED

Notice that the stabilizer of a vertex is more evident on the right graph than the left one. For example, $x^{baab} = x_1^b x_2^a x_3^a x_4^b$, or equivalently, $x^{1001}$ are stable under $s_2$, but checking that $[4213] = \square_{2143} s_2 s_1$ is invariant requires writing

$$\square_{2143} s_2 s_1\ s_2 = \square_{2143} s_1\ s_2 s_1 = \square_{2143} s_2 s_1\ .$$

From the description in terms of words, it is clear that the spaces $\square_{\omega_I}\, \mathcal{H}$ and $\nabla_{\omega_I}\, \mathcal{H}$ have dimension the multinomial coefficient $\binom{n}{i_1, \ldots, i_r}$.

---

**Frobenius characteristic map**

There are many ways to relate the group algebra $\mathbb{Q}[\mathfrak{S}_n]$ to the ring of symmetric polynomials $\mathfrak{Sym}$. Let us follow the approach of Frobenius, which gave birth to the theory of characters.

The *Frobenius characteristic map* is the linear morphism $\mathbf{ch} : \mathbb{Q}[\mathfrak{S}_n] \to \mathfrak{Sym}$ which sends each permutation $\sigma$ of cycle type $\mu = 1^{\alpha_1} 2^{\alpha_2} \ldots$ to the product of power sums $\Psi^\mu := \Psi_1^{a_1} \Psi_2^{a_2} \cdots$.

Therefore, a conjugacy class of type $\mu$ is sent to $\frac{n!}{z_\mu} \Psi^\mu$, because $\frac{n!}{z_\mu}$ is the order of the class, with $z_\mu := 1^{\alpha_1}\alpha_1! 2^{\alpha_2}\alpha_2! \ldots$.

Frobenius' characteristic is not compatible with products of permutations, except for direct products : if $\sigma$ belongs to a Young sub-group : $\sigma = \sigma^1 \times \cdots \times \sigma^k \in \mathfrak{S}_{i_1 \times \cdots \times i_k}$, then

$$\mathbf{ch}(\sigma) = \mathbf{ch}(\sigma^1) \cdots \mathbf{ch}(\sigma^k) \ .$$

It is also compatible with cyclic permutations :

$$\mathbf{ch}(\sigma\nu \cdots \mu\, \eta) = \mathbf{ch}(\eta\, \sigma\nu \cdots \mu) \ ,$$

because the two elements that we have written are conjugate.

Since Schur functions are the fundamental basis of $\mathfrak{Sym}$, it is important to find in the group algebra elements which are sent to Schur functions under Frobenius' characteristic. We shall see later that Young idempotents possess this property. But already, we can find such elements. Let us first compute the image of a Yang-Baxter cycle.

**Lemma 10** *The image of the Yang-Baxter cycle $\zeta^{[n]}$ under* $\mathbf{ch}$ *is the complete function $S_n$.*

*Proof.* We have already seen that $\zeta^{[n]}$ decomposes into the sum of terms of the following type $(i = 0, \ldots, n{-}1)$ :

$$(s_1 + \frac{1}{1})(s_2 + \frac{1}{2}) \cdots (s_{i-1} + \frac{1}{i{-}1})\, \frac{1}{i}\, s_{i+1} \cdots s_{k-1}\, \frac{1}{k}$$

which are direct products of a Yang-Baxter cycle by a cycle. Supposing the lemma true for Yang-Baxter cycles of order less than $n$, one has

$$\mathbf{ch}(\zeta^{[n]}) = \sum_{i=0}^{n-1} S_i\, \Psi_{n-i}\, \frac{1}{n} \ .$$

But this is the Newton-Brioschi recursion between complete functions and power sums, and therefore $\mathbf{ch}(\zeta^{[n]}) = S_n$. QED

Given any composition $I = [i_1, \ldots, i_r]$, define

$$\zeta^I := \zeta^{[i_1]} \times \cdots \times \zeta^{[i_r]} \ .$$

For example,

$$\zeta^{[3,2,4]} := \zeta^{[3]} \times \zeta^{[2]} \times \zeta^{[4]} \in \mathfrak{S}(3) \times \mathfrak{S}(2) \times \mathfrak{S}(4)$$

is equal to

$$(s_1 + \frac{1}{1})(s_2 + \frac{1}{2})\frac{1}{3} \quad (s_4 + \frac{1}{1})\frac{1}{2} \quad (s_6 + \frac{1}{1})(s_7 + \frac{1}{2})(s_8 + \frac{1}{3})\frac{1}{4} \ .$$

It is convenient to extend the definition of $\zeta^{[k]}$ to negative exponents and put

$$\zeta^{[k]} = 0 \text{ for } k < 0 \quad , \quad \zeta^{[0]} = \zeta^{[1]} = 1$$

Given any increasing partition $I = [i_1, \ldots, i_r]$, let $\zeta_I$ be the determinant

$$\left|\zeta^{[i_k+k-h]}\right|_{1 \le h,k \le r} \ ,$$

where the determinant is expanded from left to right and where products are *direct products*.

**Proposition 11** *For any partition $I$, one has*

$$\mathbf{ch}(\zeta_I) = S_I \ . \tag{20}$$

*Proof.* Because of the Jacobi-Trudi formula expressing Schur functions as determinants of complete functions, the statement is equivalent to the fact that $\mathbf{ch}(\zeta^{[i,j,\ldots,k]}) = S_i S_j \cdots S_k$, but this is a direct consequence of lemma 10. QED

For example,

$$\zeta_{23} = \begin{vmatrix} (s_1+1)\frac{1}{2} & (s_1+1)(s_2+\frac{1}{2})(s_3+\frac{1}{3})\frac{1}{4} \\ 1 & (s_1+1)(s_2+\frac{1}{2})\frac{1}{3} \end{vmatrix}$$

and

$$\begin{aligned} 4!\,\mathbf{ch}(\zeta_{23}) &= \mathbf{ch}\Big(4(s_1+1)(s_3+1)(s_4+\frac{1}{2}) - 6(s_2+1)(s_3+\frac{1}{2})(s_4+\frac{1}{4})\Big) \\ &= -6\psi^{14} + 4\psi^{23} - 4\psi^{113} + 3\psi^{122} + 2\psi^{1112} + \psi^{1111} \\ &= 4!\,S_{23} \ . \end{aligned}$$

One can also find products of elementary symmetric functions. For any composition $I = [i_1, \ldots, i_k]$, let us write $\omega_I$ for the maximal element of the Young sub-group $\mathfrak{S}_{i_1 \times \cdots \times i_k}$i, and $I!$ for $i_1! \cdots i_k!$.

**Proposition 12** *The image of* $\square_{\omega_I}$ *under* **ch** *is a product of complete functions, the image of* $\nabla_{\omega_I}$*, a product of elementary symmetric functions :*

$$\mathbf{ch}\big(\square_{\omega_I}\big) = I!\, S^I \tag{21}$$
$$\mathbf{ch}\big(\nabla_{\omega_I}\big) = I!\,(-1)^{\ell(\omega_I)}\,\Lambda^I \tag{22}$$

*Proof.* Because $\square_{\omega_I}$ and $\nabla_{\omega_I}$ are direct products, using that **ch** is compatible with direct products, we have only to check the case $I=[n]$, i.e. to prove the statement for $\square_\omega$, the one for $\nabla_\omega$ being obtained from it by the involution $s_i \to -s_i$. But

$$\mathbf{ch}\left(\sum\sigma\right) = \sum_J \frac{n!}{z_J}\Psi^J = n!\,S_n\ ,$$

as is well known since Cauchy. QED

Describing the kernel of **ch** is not immediate. For example, from propositions (11,12), one gets that $6\zeta^{[3]}$ and $\square_{321}$ are both sent to $6S_3$, i.e. that

$$\mathbf{ch}\big((s_1+1)(s_2+\frac{1}{2})(s_1-1)\big) = 0\ ,$$

the next case being :

$$6\mathbf{ch}\big((s_1+1)(s_2+\frac{1}{2})(s_3+\frac{1}{3})\big) = \mathbf{ch}\big((s_1+1)(s_2+\frac{1}{2})(s_3+\frac{1}{3})(s_1+1)(s_2+\frac{1}{2})(s_1+1)\big)$$
$$\text{i.e.}\quad \mathbf{ch}\Big((s_1+1)(s_2+\frac{1}{2})(s_3+\frac{1}{3})(6-\square_{321})\Big) = 0$$

```
ACE> Perm2p:= proc(perm)
 convert( map(i->cat(p,i),Perm2CycleType(perm)),`*`)
end:
ACE> Sga2Sym:=proc(f);  # Frobenius' characteristic
   if member(whattype(f),{`+`,`*`,`^`})
        then map(Sga2Sym,f)
    elif whattype(f)=`indexed` and op(0,f)=`A`
     then    RETURN(Perm2p([op(f)]))  ;
       else f
   fi;
end:
ACE> Sga2Carre(A[3,2,1,5,4]);    # case I=[3,2]
    A[3,1,2,5,4] + A[2,1,3,5,4] + A[1,2,3,4,5] + A[1,3,2,4,5]
  + A[3,2,1,5,4] + A[2,3,1,5,4] + A[3,1,2,4,5] + A[2,1,3,4,5]
  + A[3,2,1,4,5] + A[2,3,1,4,5] + A[1,3,2,5,4] + A[1,2,3,5,4]
```

```
ACE> Toh(Sga2Sym(%));
                                    12 h3 h2
ACE> Toe(Sga2Sym(Sga2Nabla(A[3,2,1,5,4])));
                                    12 e3 e2
```

## Young normal representations and Yang-Baxter bases

We have already represesented permutations as matrices. Indeed, to handle the (non-commutative) multiplication in the group algebra of the symmetric group, the simplest tool is to realize it as a multiplication of matrices. Instead of having only one representation of dimension $n$ for $\mathfrak{S}_n$ (i.e. embedding $\mathfrak{S}_n$ into the linear group $Gl(\mathbb{C}^n)$, one can try to use other linear groups. A *representation* over $\mathbb{C}$ of dimension $N$ of $\mathfrak{S}_n$ is a morphism

$$\varphi: \ \mathfrak{S}_n \ni \sigma \to \varphi(\sigma) \in Gl(\mathbb{C}^N)$$

compatible with the mutiplication :

$$\varphi(\sigma\,\nu) = \varphi(\sigma)\,\varphi(\nu)$$

(the image of the identity being the identity matrix).

We also have implicitely used another representation, the *(left) regular representation*, which is the $n!$ dimensional representation of $\mathfrak{S}_n$ acting by left multiplication on the group algebra of itself (one has also a right regular representation).

```
ACE> Perm2RRep([2,3,1]);
                                    [ 0  0  0  0  1  0 ]
                                    [ 0  0  0  0  0  1 ]
                                    [ 0  1  0  0  0  0 ]
                                    [ 1  0  0  0  0  0 ]
                                    [ 0  0  0  1  0  0 ]
                                    [ 0  0  1  0  0  0 ]
```

Of course, with two representations, one can make a third one by taking matrices made of two diagonal blocks. One says that the resulting representation is *direct sum* of the two original ones. Thus one wants representations which are not equivalent to a direct sum.

Let us see a solution by Young to this problem for the symmetric group (his second solution, as a matter of fact). We shall rewrite it in terms of a Yang-Baxter graph, and this will make clear the connection with Yang's basis.

To any partition $\lambda$, we associate a word $y_\lambda$, by first writing the diagram of $\lambda$ as a diagram of boxes stacked in the North-East corner (the parts of $\lambda$ being the column lengths), then we fill each column with consecutive numbers, increasing upwards and starting with 0 in the bottom boxes :

$$\lambda = [3,2] \Rightarrow \begin{array}{cc} \square & \square \\ \square & \square \\ & \square \end{array} \Rightarrow \begin{array}{cc} 1 & 2 \\ 0 & 1 \\ & 0 \end{array} \Rightarrow \text{ word } \quad 10\ 210$$

Now generate a graph, by allowing all possible transpositions of adjacent letters $a, b$, with $a < b$ :

$$\begin{array}{ccc} & [10.210] & \\ & \| s_2 & \\ & [1.20.10] & \\ s_1 \diagup & & \diagdown\!\diagdown\!\diagdown s_3 \\ [210.10] & & [1.2100] \\ s_3 \diagdown\!\diagdown\!\diagdown & & \diagup s_1 \\ & [21100] & \end{array}$$

The bottom element of the graph is the weakly decreasing word commutatively equivalent to $y_\lambda$. Each edge has a color $s_i$ (transposition of components $i$, $i{+}1$).

The labels of vertices are *Yamanouchi words*[3], i.e. words $w$ such for each factorisation $w = w'w''$, then the number of occurences of $i$ is bigger or equal to the the number of occurences of $(i{+}1)$, $i = 1, 2, 3, \ldots$, in $w''$. This we write :

$$w \quad \text{Yamanouchi} \quad \Leftrightarrow \forall w = w'w'',\ |w''|_1 \geq\ |w''|_2 \geq\ |w''|_3 \geq \cdots \tag{23}$$

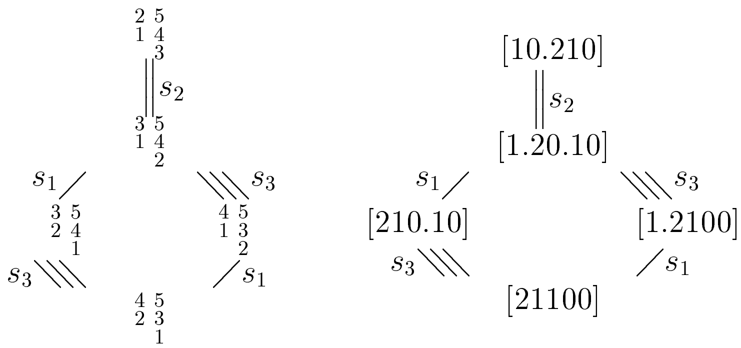
We could have also labelled vertices by permutations: take the same underlying graph, and put the maximal permutation (here [54321] at the bottom). Now the graph has become an interval in the permutohedron. However, for further applications, our present labelling is better.

Interval of the permutohedron

$$\begin{array}{ccc} & [42.531] & \\ & \| s_2 & \\ & [4.52.31] & \\ s_1 \diagup & & \diagdown\!\diagdown\!\diagdown s_3 \\ [542.31] & & [4.5321] \\ s_3 \diagdown\!\diagdown\!\diagdown & & \diagup s_1 \\ & [54321] & \end{array}$$

[3]Some people say "lattice permutations", but there is no lattice here, and permutations are hidden.

There is still another labelling, directly equivalent to the one by Yamanouchi words, which consists in interpreting each word as describing the levels occupied by the letters $1, 2, \ldots, 5$ in the diagram of $[3, 2]$ (top row is level 0) :



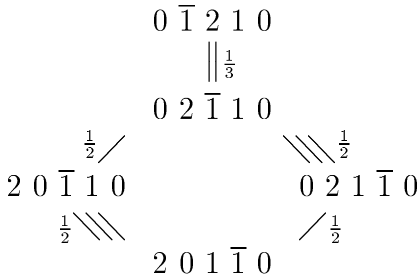

Transpositions act on the vertices of the left graph by permuting *values*.

As a Yang-Baxter graph, it is not yet totally defined, we have to choose an initial vector of spectral parameters.

We take the *content vector* $c_\lambda$, obtained by filling the diagram of $\lambda$, this time *packed in the North-West corner and the parts of* $\lambda$ *being the row lengths*, with consecutive numbers in each column, increasing upwards, in such a way to have 0 in the main diagonal. We read now the consecutive rows, *from right to left, from bottom to top* :

$$\lambda = [3,2] \Rightarrow \begin{smallmatrix} \square & \square & \square \\ \square & \square & \end{smallmatrix} \Rightarrow \begin{smallmatrix} 0 & \square & \square \\ \square & 0 & \end{smallmatrix} \Rightarrow \begin{smallmatrix} 0 & 1 & 2 \\ -1 & 0 & \end{smallmatrix} \Rightarrow \; c_\lambda = [0, -1, 2, 1, 0]$$

Now edges of the graph have not only a color $s_i$, but also a label $1/(b-a)$, for the transposition of $a$ and $b$. Labelling vertices by their corresponding content vectors (=images of the initial content vector $c_\lambda$), and writing $\overline{1}$ instead of $-1$, one gets the graph




The set of vertices of the preceding graph is the *plactic class*[4] of the word $[0\,\overline{1}\,2\,1\,0]$, we shall describe it later in the chapter about Young tableaux.

[4]The *plactic relations* are, for any triple $a < b < c$, $cab \equiv acb$, $bac \equiv bca$, and for any pair $a < b$, $baa \equiv aba$, $bab \equiv bba$. The *plactic class* of a word is its closure under plactic relations.

How to read matrices from the graph ?

The underlying vector space has a basis coded by the vertices of the graph. To represent any simple transposition $s_i$, one first erases all edges which are not labelled by $s_i$. One is left with isolated vertices (corresponding to 1-dimensional representations of $\mathfrak{S}_2$), and pairs of vertices connected by an edge, corresponding to 2-dimensional representations.

In this last case, if $\beta$ is the parameter written on the edge, then to define a two-dimensional representation of $\mathfrak{S}_n$, Young took the matrix

$$\begin{bmatrix} -\beta & 1 \\ 1-\beta^2 & \beta \end{bmatrix} .$$

In the one-dimensional case, if $i, i+1$ is a subword of the vertex, then the restriction of the representation is trivial (i.e. the matrix is 1), otherwise (if $i+1, i$ is a subword) it is the alternating representation (matrix $= -1$).

Choosing a total order on the vertices of the directed graph, compatible with its partial order, Young defined a matrix to represent the simple transposition $s_i$ by embedding these elementary matrices into a $N \times N$ matrix, putting 0 in the other places ($N$=number of vertices).

In other words, a single vertex gives a diagonal entry $\pm 1$, an edge $s_i$ connecting vertex $p$ and vertex $q$ gives a submatrix $\left[\begin{smallmatrix} -\beta & 1 \\ 1-\beta^2 & \beta \end{smallmatrix}\right]$ on rows and columns $p, q$, and all other entries are 0.

Continuing with the example, for shape $[3,2]$, here are the matrices representing $s_1, \ldots, s_4$, for the ordering (here ACE has reversed the words and the order)

$$[0,-1,1,0,2], [0,-1,1,2,0], [0,1,-1,2,0], [0,1,2,-1,0], [0,1,-1,0,2] \ :$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{3}{4} & 0 & 0 & 0 & \frac{-1}{2} \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{8}{9} & \frac{-1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{3}{4} & \frac{-1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 0 & 3/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} \end{bmatrix} \tag{24}$$

Vertices are labelled in the order $\begin{tikzcd}[sep=small] & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ 2 \arrow[dr, dash] & & 5 \arrow[dl, dash] \\ & 3 \arrow[d, dash] & \\ & 4 & \end{tikzcd}$. Edges $[1,5]$ and $[2,3]$ are labelled $s_2$, and vertex 4 is an isolated point for $s_2$. Therefore, the matrix representing $s_2$ is made of two Young-matrices of order 2, plus a matrix of order 1, placed at the positions indicated : $\begin{bmatrix} [1,1] & \cdot & \cdot & \cdot & [1,5] \\ \cdot & [2,2] & [2,3] & \cdot & \cdot \\ \cdot & [3,2] & [3,3] & \cdot & \cdot \\ \cdot & \cdot & \cdot & [4,4] & \cdot \\ [5,1] & \cdot & \cdot & \cdot & [5,5] \end{bmatrix}$. The parameters on the edges tell the precise values of the entries; the entry $[4, 4]$ is equal to 1 because component 2 of vertex 4 is smaller than component 3.

It is clear that the square of any Young matrix is the identity (because it is true for the case of order 1 and 2). It is also clear that commuting transpositions give commuting matrices. It remains essentially to check the case a six dimensional representation of $\mathfrak{S}_3$, with parameters $\alpha, \beta, \gamma = \alpha+\beta$, i.e. to check that the matrices $M_1, M_2$ candidate to represente the two simple transpositions satisfy

$$M_1 M_2 M_1 = M_2 M_1 M_2 \ .$$

Explicitely, these two matrices are

$$M_1 := \begin{bmatrix} -\alpha^{-1} & 0 & 1 & 0 & 0 & 0 \\ 0 & -(\alpha+\beta)^{-1} & 0 & 0 & 1 & 0 \\ 1-\alpha^{-2} & 0 & \alpha^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta^{-1} & 0 & 1 \\ 0 & 1-(\alpha+\beta)^{-2} & 0 & 0 & (\alpha+\beta)^{-1} & 0 \\ 0 & 0 & 0 & 1-\beta^{-2} & 0 & \beta^{-1} \end{bmatrix}$$

$$M_2 := \begin{bmatrix} -\beta^{-1} & 1 & 0 & 0 & 0 & 0 \\ 1-\beta^{-2} & \beta^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\alpha+\beta)^{-1} & 1 & 0 & 0 \\ 0 & 0 & 1-(\alpha+\beta)^{-2} & (\alpha+\beta)^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha^{-1} & 1 \\ 0 & 0 & 0 & 0 & 1-\alpha^{-2} & \alpha^{-1} \end{bmatrix} \qquad (25)$$

Arrived at this stage, instead of trying to perform the product of matrices, we remember that we have used in the preceding section another interpretation of a Yang-Baxter graph, as coding elements in the group algebra of $\mathfrak{S}_n$ (vertices can be labelled arbitrarily). We have to choose an arbitrary vector of parameters. Afterwards, we obtain $n!$ elements in the group algebra of $\mathfrak{S}_n$, by reading paths and interpreting them as products of simple factors in the group algebra.

In general, one edge is labelled $s_i + \beta$, its two vertices are $\mathcal{Y}_\sigma$ and $\mathcal{Y}_\eta = \mathcal{Y}_\sigma\,(s+\beta)$, with $\eta = \sigma s_i$. The linear span of $\mathcal{Y}_\sigma$ and $\mathcal{Y}_\eta$ is a two dimensional representation of $\mathfrak{S}_2$ acting on the right, and the matrix representing the action of $s_i$ is (reading by rows)

$$\left\{ \begin{array}{lcl} \mathcal{Y}_\sigma \to \mathcal{Y}_\sigma s_i & = & -\beta \mathcal{Y}_\sigma + \mathcal{Y}_\eta \\ \mathcal{Y}_\eta \to \mathcal{Y}_\eta s_i & = & (1-\beta^2)\mathcal{Y}_\sigma + \beta \mathcal{Y}_\eta \end{array} \right. \quad \Leftrightarrow \quad \begin{bmatrix} -\beta & 1 \\ 1-\beta^2 & \beta \end{bmatrix} . \tag{26}$$

$\boxed{\textit{This is exactly Young's matrix.}}$

It means, that if we take $\mathfrak{S}_3$ and the parameters $[0, \alpha, \alpha+\beta]$, then Yang-Baxter equation insures that $M_1 M_2 M_1 = M_2 M_1 M_2$ without any need to check it [5]. *Therefore Young's matrices represent the symmetric group.*

The explicit matrix of change of basis, with the vector of parameters $[0, a, a+b]$ is (reading the expansion of a Yang element in each row) :

```
ACE> MatYang2Perm(3,[0,a,a+b]), inverse(%);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{b} & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{a} & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{(a+b)a} & \frac{1}{a} & \frac{1}{a+b} & 1 & 0 & 0 \\ \frac{1}{b(a+b)} & \frac{1}{a+b} & \frac{1}{b} & 0 & 1 & 0 \\ \frac{ab+1}{(a+b)ba} & \frac{1}{ab} & \frac{1}{ab} & \frac{1}{b} & \frac{1}{a} & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{b} & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{a} & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{ab} & -\frac{1}{a} & -\frac{1}{a+b} & 1 & 0 & 0 \\ \frac{1}{ab} & -\frac{1}{a+b} & \frac{1}{b} & 0 & 1 & 0 \\ -\frac{ab+1}{(a+b)ba} & \frac{1}{(a+b)a} & \frac{1}{b(a+b)} & -\frac{1}{b} & -\frac{1}{a} & 1 \end{bmatrix}$$

Apart from signs, these two matrices have the same entries, but distributed differently. This fact result from the fact, seen in the preceding section, that the adjoint of a Yang-Baxter basis is a Yang-Baxter basis for the reversed parameters.

For our running example, the representation is of dimension 5, and Yang-Baxter basis is

$$\begin{tikzcd}[column sep=small] & 1 \arrow[d, "\frac{1}{3}"] & \\ & s_2+\frac{1}{3} \arrow[dl, "\frac{1}{2}"'] \arrow[dr, "\frac{1}{2}"] & \\ (s_2+\frac{1}{3})(s_1+\frac{1}{2}) \arrow[dr, "\frac{1}{2}"'] & & (s_2+\frac{1}{3})(s_3+\frac{1}{2}) \arrow[dl, "\frac{1}{2}"] \\ & (s_2+\frac{1}{3})(s_1+\frac{1}{2})(s_3+\frac{1}{2}) & \end{tikzcd}$$

[5] One nevertheless has to verify what happens in the degenerate case, when one of the differences of parameters is equal to $\pm 1$; for Young, this is the case where two consecutive integers, in a tableau, are in the same row or same column (and thus in adjacent boxes)

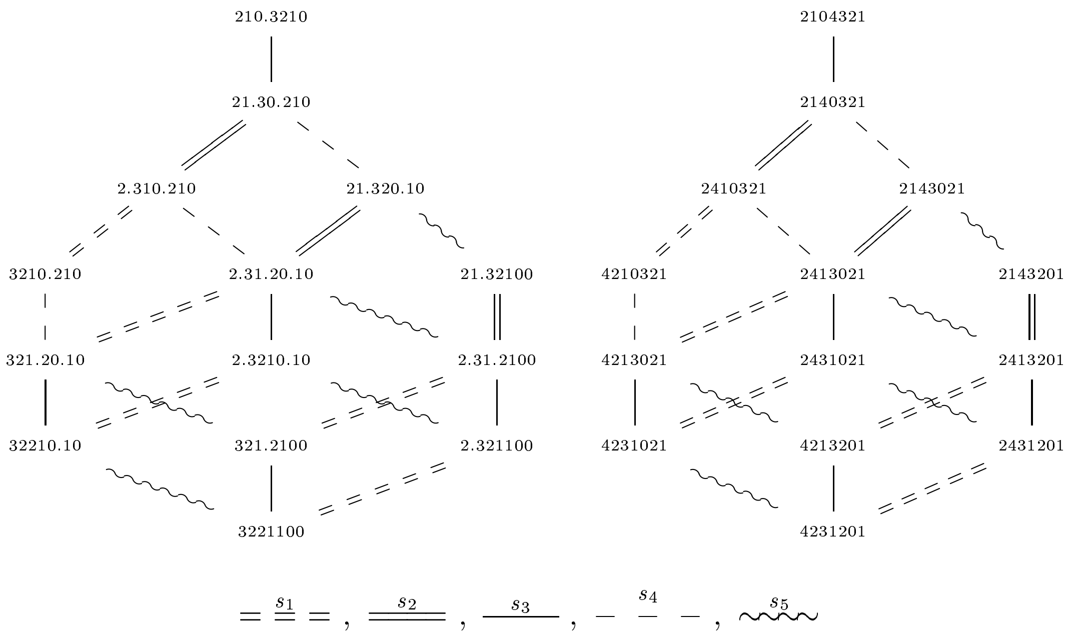
The matrices representing $s_1, \ldots, s_4$ have already been written in (24).

There are other normalizations for Young's matrices. Essentially, one can take one of the following three matrices for the case of $\mathfrak{S}_2$ (or their transposed, but of course, one must stick for $\mathfrak{S}_n$ with one type only!)

$$\begin{bmatrix} -\beta & 1 \\ 1-\beta^2 & \beta \end{bmatrix}, \quad \begin{bmatrix} -\beta & 1+\beta \\ 1-\beta & \beta \end{bmatrix}, \quad \begin{bmatrix} -\beta & \sqrt{1-\beta^2} \\ \sqrt{1-\beta^2} & \beta \end{bmatrix} \tag{27}$$

The matrix on the right is unitary, and thus is the building block of *Young orthonormal representations*. Young first obtained them by orthonormalisation of the matrices in the *natural representations*.

Here are now two copies of a bigger graph, for shape $[3,4]$; on the left, we write writing Yamanouchi words; on the right, we take the contents $+1$, to have positive numbers (this does not change differences!).




The vertices of both graphs constitute isomorphic plactic classes :

```
ACE> Free2PlaxClass(w[2,1,0,4,3,2,1]);
 w[4,2,1,0,3,2,1] +w[4,2,1,3,0,2,1] +w[4,2,1,3,2,0,1] +w[2,1,0,4,3,2,1]
+w[2,1,4,0,3,2,1] +w[2,1,4,3,0,2,1] +w[2,4,1,3,0,2,1] +w[2,4,1,0,3,2,1]
+w[4,2,3,1,2,0,1] +w[2,1,4,3,2,0,1] +w[2,4,1,3,2,0,1] +w[2,4,3,1,2,0,1]
                                     +w[4,2,3,1,0,2,1] +w[2,4,3,1,0,2,1]
ACE> nops(%) - nops(Free2PlaxClass(w[2,1,0,3,2,1,0]));
                                                  0
```

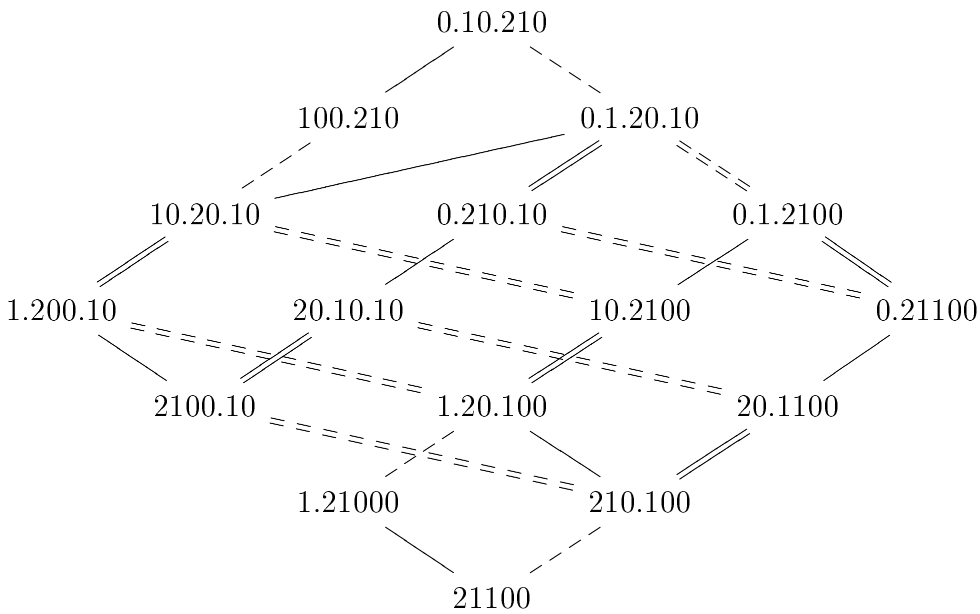
Here are the matrices representing $s_2$ and $s_3$ for this representation :

```
ACE> latex(MatRepNormal([4,3],2)),latex(MatRepNormal([4,3],3));
```

$$\left[\begin{array}{cccccccccccccc}
1/2 & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
. & 1/2 & . & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & 1/2 & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & 1/2 & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & 3/4 & -1/2 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & 3/4 & . & . & . & . & -1/2 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . \\
. & 3/4 & . & . & . & . & . & . & . & -1/2 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1/2 & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & 3/4 & -1/2 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
3/4 & . & . & . & . & . & . & . & . & . & . & . & . & -1/2
\end{array}\right]$$

$$\left[\begin{array}{cccccccccccccc}
-1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & -1 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1/3 & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & 8/9 & -1/3 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1/3 & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & 8/9 & -1/3 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1/3 & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & 8/9 & -1/3 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{array}\right]$$

We now give the 16-dimensional representation of $\mathfrak{S}_6$, for shape $[1, 2, 3]$ :



$$\overset{s_1}{\rule{2.5em}{0.4pt}}\,,\quad \overset{s_2}{=\!=\!=\!=}\,,\quad - \overset{s_3}{-} -\,,\quad = \overset{s_4}{=} =$$

## Young's natural representations

We now come to the heart of Young's work, who introduced the fundamental idea that 2-dimensional combinatorial objects were needed to work in the group algebra of the symmetric group.

Young obtained the following key property (he was using only partitions, and not compositions, but his method extends straightforwardly).

**Proposition 13** *Let $I$, $J$ be two conjugate compositions. Then $\square_{\omega_I}\mathcal{H}\,\nabla_{\omega_J}$ is a 1-dimensional module, and*

$$\square_{\omega_I}\cdot\sigma\cdot\nabla_{\omega_J}\neq 0 \Leftrightarrow \sigma=\sigma'\cdot\zeta(I,J)\cdot\sigma'',\ \sigma'\in\mathfrak{S}_I,\ \ \sigma''\in\mathfrak{S}_J\ ,$$

*where the permutation $\zeta(I,J)$ associated to a pair of conjugate compositions has been defined above.*

*Proof.* The space $\square_{\omega(I)}\mathcal{H}\,\nabla_{\omega(J)}$ is generated by the permutations $\nu$ of minimum length in their double coset $\mathfrak{S}_I\,\nu\,\mathfrak{S}_J$. These permutations are in bijective correspondence with tableaux of evaluation $1^{i_1}2^{i_2}\cdots$ and shape $j_1\otimes j_2\otimes\cdots$.

Let us check that such a tableau gives a non-zero element $\square_{\omega_I}\nu\nabla_{\omega_J}$ iff all the letters in every row separately are distinct. Indeed, suppose on the contrary that a letter occurs twice in a row. It means that there exists an integer $k$ such that $s_k\in\mathfrak{S}_I$ and such that $\nu$ and $s_k\,\nu$ give the same element modulo $\mathfrak{S}_J$. We can equivalently write

$$\nu\,\square_{\omega_J}=s_k\,\nu\,\square_{\omega_J}\ \text{or}\quad \nu\,\nabla_{\omega_J}=-s_k\,\nu\nabla_{\omega_J}\ \text{or also}\quad (1+s_k)\,\nu\,\nabla_{\omega_J}=0\ ,$$

but now, because $(1+s_k)=\square_k$ is a right factor of $\square_{\omega_I}$, this last nullity implies that of $\square_{\omega_I}\,\nu\,\nabla_{\omega_J}$.

Finally, lemma 5 tells us that there is only one tableau of evaluation $1^{i_1}2^{i_2}\cdots$ and shape $1^{j_1}\otimes 1^{j_2}\otimes\cdots$, and it corresponds to the permutation $\zeta(I,J)$. QED

For example, for $I=232$ and $J=331$, $\zeta(232,331)=136\,247\,5$, and the different combinatorial objects that we attached to the pair $I, J$ are :

$$\underset{0\text{–}1 matrix}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}} \quad \underset{diagram}{\begin{array}{|c|c|c} \hline 3 & 3 & \multicolumn{1}{|c}{} \\ \hline 2 & 2 & \multicolumn{1}{|c|}{2} \\ \hline 1 & 1 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array}} \quad \underset{numbering}{\begin{bmatrix} 1 & 2 & \\ 3 & 4 & 5 \\ 6 & 7 & \end{bmatrix}} \quad \to \ \zeta(232, 331) = 136\ 247\ 5$$

$$\boxed{\begin{matrix} 3 \\ 2 \\ 1 \end{matrix}} \otimes \boxed{\begin{matrix} 3 \\ 2 \\ 1 \end{matrix}} \otimes \boxed{2} \to \boxed{\begin{matrix} 6 \\ 3 \\ 1 \end{matrix}} \otimes \boxed{\begin{matrix} 7 \\ 4 \\ 2 \end{matrix}} \otimes \boxed{5} \ \to 631\ 742\ 5$$

the column reading of the last tableau is now the element of maximal length in its coset $\zeta(232, 331)\mathfrak{S}_{331}$ instead of being the element of minimum length.

Proposition 13 is so important in Young's theory that we now give another formulation of it in terms of Yang-Baxter elements. The situation is even simpler, we shall have a space of dimension 1 because there is only one non-vanishing element. Once more, vanishing properties come only from the fact that, for any simple transposition, $(s_i + 1)(s_i - 1) = 0$.

**Proposition 14** *Let $\lambda$ be a partition. Fill the successive columns of the diagram of $\lambda$ (in the S-W corner) (resp. of $\lambda^{\sim}$, in the N-E corner) with $1, \ldots, n$. Let $\omega'$ and $\omega''$ the column readings of these two tableaux, and $\nu$, $\eta$ their row-reading. Let $J$ be the increasing reordering of of $\lambda$, and $I = \lambda^{\sim}$ .*

*Then, for every permutation $\sigma$ greater than $\omega'$ (in the permutohedron), one has that $\Box_\sigma \nabla_\eta \neq 0$ iff $\sigma$ and $\nu$ belongs to the same double coset $\mathfrak{S}_I\, \nu\, \mathfrak{S}_J$.*

*Moreover $\Box_{\omega'}\, \mathcal{H}\, \nabla_{\omega''}$ is 1-dimensional, with basis $\Box_\nu\, \nabla_{\omega''}$, and $\Box_\nu \nabla_\eta$ is a quasi-idempotent, i.e.*

$$\Box_\nu \nabla_\eta \Box_\nu \nabla_\eta = c\ \Box_\nu \nabla_\eta\ ,\ \textit{for some } c \in \mathbb{Z},\ c \neq 0\ .$$

*Proof.* Let us take an explicit partition, say $\lambda = [3, 2, 2]$, to avoid unnecessary indices. In that case, the filled diagrams are

$$\begin{smallmatrix} 3 & 6 & \\ 2 & 5 & \\ 1 & 4 & 7 \end{smallmatrix} \to \nu = [3625147],\ \omega' = [3216547]\ \&\ \begin{smallmatrix} 2 & 4 & 7 \\ 1 & 3 & 6 \\ & & 5 \end{smallmatrix} \to \eta := [2471365],\ \omega'' = [2143765]\ .$$

We have seen in proposition 9 that $\Box_{\omega'}\, \mathcal{H}$ has a linear basis consisting of $\Box_\sigma$, $\sigma \geq \omega' = [3216547]$ in the permutohedron, i.e. $\sigma$ having subwords $321,\ 654,\ 7$, to which we shall attribute different colors. Considering $\sigma$ modulo $\mathfrak{S}_J$ amounts writing an ordered sequence of sets (call them baskets) $\{\sigma_1, \sigma_2\}$, $\{\sigma_3, \sigma_4\}$, $\{\sigma_5, \sigma_6, \sigma_7\}$ instead of a permutation. Suppose that two integers of the same color are in the basket. All the integers between them also belong to the same basket, we can suppose that the two integers, say $j, j{+}1$, are consecutive.

Therefore, $\square_\sigma$ is obtained from $\square_{\sigma'}$, where $\sigma'$ is the element of minimum length in the coset $\sigma\,\mathfrak{S}_J$, by multiplication on the right by factors of the type $s_k + \alpha$, with all $s_k \in \mathfrak{S}_J$, but one of the $\alpha$ being equal to 1 (created by the exchange of $j$ and $j{+}1$. Since $(s_k + \alpha)\nabla_{\omega''} = (-1+\alpha)\nabla_{\omega''}$, the product of these factors vanishes with $\nabla_{\omega''}$. For example, when $\sigma = [63\ 75\ 241]$, then $1, 2$ have the same color and lie in the same basket, and $\square_{6375241}\nabla_{2143765} = \square_{6375124}\,(s_5+1)(s_6+\frac{1}{3})\,\nabla_{2143765} = 0$.

To avoid nullity, one must have that all integers of the same color lie in different baskets, and once more, lemma 5 states that all permutations satisfying such property belong to the same double coset.

Moreover, $\square_{3625147}\nabla_{2143765}$ is non-zero, because it has term of highest length the product [36 25 147][21 43 765] (which is reduced).

One passes from $\square_{\omega'}$ to $\square_\nu$ by multiplication on the right by invertible factors $s_i + \frac{1}{\alpha}$, $\alpha \neq 1$; the spaces $\square_\nu\,\mathcal{H}\,\nabla_{\omega''}$ and $\square_{\omega'},\mathcal{H}\,\nabla_{\omega''}$ are the same.

Finally, $\square_\nu\nabla_\eta\square_\nu\nabla_\eta\nabla_{\omega''}$ belongs to the space $\square_{\omega'},\mathcal{H}\,\nabla_{\omega''}$. One has just to check that it is not null. We shall check later, and more easily, this type of non-vanishing properties by using the action of $\mathcal{H}$ on polynomials. QED

In the following corollary, we shall take tabloids of shape a partition, to recover statements directly adapted from those of Young.

**Corollary 15** *Given two tabloids $u$, $v$ of shape the same partition $I$, then*

*1) The space $P(u)\,\mathcal{H}\,N(v)$ is 1-dimensional. Moreover $P(u)N(u)$ and $N(u)P(u)$ are two quasi-idempotents.*

*2) If $h \in \mathcal{H}$ is such that, for all $\sigma \in \mathfrak{S}_n$ preserving the rows of $u$, one has $\sigma h = h$ and for all $\sigma \in \mathfrak{S}_n$ preserving the columns of $u$, one has $h\sigma = (-1)^{\ell(\sigma)}h$, then $h$ is proportional to $P(u)\zeta(u,v)N(v)$, where $\zeta(u,v)$ is the permutation transforming $u$ into $v$.*

*3) If $u$ and $v$ are two tabloids of shapes two different partitions $\lambda$, $\mu$, of the same number, $\mu$ being strictly higher than $\lambda$ (with respect to the natural order on partitions, for which $1^n$ is the maximum), then*

$$P(u)\,N(v) = 0 = N(v)\,P(u)\ .$$

*Proof.* The first point is a direct consequence of propositions 13,14, taking into account that all $P(u)$, for $u$ of shape $I$, are conjugate to $\square_{\omega_I}$. Similarly, all $N(v)$ are conjugate to $\nabla_{\omega_J}$, for $v$ of column-shape $J$ (= conjugate of $I$).

The invariance of $h$ with respect to multiplications on its right and its left by all the permutations belonging to some Young subgroups, implies that $h$

is equal, up to a non-zero factor, to $P(u)hN(v)$, and thus proportional to $P(u)\zeta(u,v)N(v)$.This last point is ascribed to Von Neuman by Weber [40].

There is no 0-1 matrix of row-sums the shape $I$ of $u$, and column-sums $J$, the shape (by columns) of $v$. It implies $\Box_{\omega_I}\,\mathcal{H}\,\nabla_{\omega_J}=0$, and by conjugation, $P(u)N(v)=0$. QED

We summarize now some of the different 1-dimensional spaces that we have associated to the partition $[3,3,1]$.

$$Pair[331],[223] \to \begin{bmatrix} 1&1&1\\ 1&1&1\\ 0&0&1 \end{bmatrix} \to \begin{bmatrix} 1&2&3\\ 4&5&6\\ \cdot&\cdot&7 \end{bmatrix} \to \zeta([331],[223]) = [1425367]$$

Space $\Box_{3216547}\,\mathcal{H}\,\nabla_{2143765}$, with basis $\Box_{3216547}\,[1425367]\,\nabla_{2143765}$ or $\Box_{3625147}\nabla_{2143765}$, coming from the tableau $\begin{smallmatrix} 3&6&\\ 2&5&\\ 1&4&7 \end{smallmatrix}$.

$$Pair[331],[322] \to \begin{bmatrix} 1&1&1\\ 1&1&1\\ 1&0&0 \end{bmatrix} \to \begin{bmatrix} 1&2&3\\ 4&5&6\\ 7&\cdot&\cdot \end{bmatrix} \to \zeta([331],[322]) = [1472536]$$

Space $\Box_{3216547}\,\mathcal{H}\,\nabla_{3215476}$, with basis $\Box_{3216547}\,[1472536]\,\nabla_{3215476}$ or $\Box_{3672514}\nabla_{3215476}$ coming from the contre-tableau $\begin{smallmatrix} 3&6&\\ 2&5&\\ 1&4&7 \end{smallmatrix} \to \begin{smallmatrix} 3&6&7\\ 2&5&\\ 1&4& \end{smallmatrix}$

Let us analyze more precisely the vanishing property stated in the previous proposition. Young [43], p.95, gives the following property[6], which is crucial in his characterization of idempotents, and their branching rules.

**Proposition 16** *Given two conjugate compositions $I$, $J$ and an index $k$ such that $i_k \ge i_{k+1}$, let $I^\dagger$ be the composition $[\ldots, i_k+1, i_{k+1}-1,\ldots]$ differing from $I$ only at the $k$-th and $k$+1-th components. Then*

$$\Box_{\omega_I}\cdot\Box_{\omega_{I^\dagger}}\,\mathcal{H}\,\nabla_{\omega_J} = 0 = \Box_{\omega_{I^\dagger}}\cdot\Box_{\omega_I}\,\mathcal{H}\,\nabla_{\omega_J} \tag{28}$$

*Proof.* There is no tableau of evaluation $x^{I^\dagger}$ and of shape $1^{j_1}\otimes 1^{j_2}\otimes\cdots$ and thus $\Box_{\omega_{I^\dagger}}\mathcal{H}\,\nabla_{\omega_J}=0$. QED

In the product $\Box_{\omega_I}\cdot\Box_{\omega_{I^\dagger}}$, there are many repeated factors. One can in fact replace $\Box_{\omega_{I^\dagger}}$ by a smaller factor. Indeed, suppose by conjugation that $k=1$ and write $I=[a,b,\ldots]$, $I^\dagger=[a{+}1,b{-}1,\ldots]$, with $a\ge b$, and $\tau_{ij}$ for the

[6]In terms of representations of the symmetric group, the proposition is equivalent to the fact that $Hom(S^{I^\dagger},\Lambda^J)=0$, where $S^{I^\dagger}$ denotes the representation induced by the trivial representation of the Young subgroup $\mathfrak{S}_{I^\dagger}$, and $\Lambda^J$ the representation induced by the alternating representation of $\mathfrak{S}_J$.

transposition of $i, j$. Then

$$\begin{aligned}
\omega_I &= (a, \ldots, 1,\ a+b, \ldots, a+1, a+b+1, \ldots) \\
\omega_{I^\dagger} &= (a+1, \ldots, 1,\ a+b, \ldots, a+2, a+b+1, \ldots) \text{ and} \\
\square_{a+1\cdots 1} &= \square_{a\ldots 1}(1+\tau_{1,a+1}+\cdots+\tau_{a,a+1}) \\
\square_{a+1\cdots 1} &= \square_{a\ldots 1}(s_a+\frac{1}{a})\cdots(s_1+1)
\end{aligned}$$

Thus the elements $\square_{\omega_I}\square_{\omega_I^+}$ and $\square_{\omega_I}(1+\tau_{1,a+1}+\cdots+\tau_{a,a+1})$ are equal, up to a non-zero factor, and (28) can be written

$$\square_{\omega_I}\,(1+\tau_{1,a+1}+\cdots+\tau_{a,a+1})\,\mathcal{H}\,\nabla_{\omega_J} = 0\ , \tag{29}$$

as stated by Young in QSA2, [43] p.95 (0 means here the zero-dimensional space).

Similarly, multiplying on the left by $\square_{\omega_{I^\dagger}}$, and eliminating repeated factors, one gets the nullity

$$(s_1+1)\cdots(s_a+\frac{1}{a})\,\square_{\omega_I}\,\mathcal{H}\,\nabla_{\omega_J} = 0 \tag{30}$$

used by [JMKO, prop.A2].

For example, if $I = (3,3,\ldots)$, then $I^\dagger = (4,2,\ldots)$, and one has the nullities of

$$\square_{4321\,65\ldots}\,\square_{321\,654\ldots}\,\mathcal{H}\,\nabla_{\omega(J)}$$

as well as of

$$\square_{321\,654\ldots}\,(1+\tau_{14}+\tau_{24}+\tau_{34})\,\mathcal{H}\,\nabla_{\omega(J)}$$

and of

$$(\sigma_1+1)(\sigma_2+\frac{1}{2})(\sigma_3+\frac{1}{3})\cdot\square_{321\,654\ldots}\,\mathcal{H}\,\nabla_{\omega(J)}\ .$$

Using the involution $\sigma \to (-1)^{\ell(\sigma)}\,\sigma$, and the right/left symmetry, one gets from proposition 16 the following identities, which can be identified with the lift of Plücker relations to the group algebra of $\mathfrak{S}_n$ (Young ?).

**Proposition 17** *1) Let $I$ and $J$ be two conjugate compositions, $I = (a, b, \ldots)$, with $a \geq b$. Then*

$$\begin{aligned}
\square_{\omega_J}\,\zeta(J,I)\,\nabla_{\omega_I}\,\big(1-\tau_{1,a+1}-\cdots-\tau_{a,a+1}\big) &= 0 &(31)\\
\square_{\omega_J}\,\zeta(J,I)\,\nabla_{\omega_I}\,(s_a-\frac{1}{a})\cdots(s_1-1) &= 0 &(32)
\end{aligned}$$

*2) Let $u$ be a tabloid of shape a partition. Let $A, B$ be two columns of $u$, with $B$ on the right of $A$, and $b$ an element of $B$. Then*

$$P(u)N(u) = P(u)\left(\sum_{a\in A} \tau_{a,b}\right) N(u) \ . \qquad (33)$$

For example, take $I = [3,2]$ and $J = [2,1,2]$.

```
ACE> aa:=Compo2Young([2,1,2]) &!* A[op(DeuxCompo2Perm([2,1,2],[3,2]))]
                                  &!* Compo2Young([3,2],N):
ACE> aa &!* (1-A[4,2,3,1]-A[1,4,3,2]-A[1,2,4,3]);
                                    0
ACE> aa &!* (A[1,2,4,3]-1/3) &!* (A[1,3,2]-1/2) &!* (A[2,1]-1);
                                    0
```

To illustrate the second part of the proposition, take the second and third columns of $u = \begin{smallmatrix} 6 & & \\ 3 & 5 & 7 \\ 1 & 2 & 4 \end{smallmatrix}$, and $b = 4$.

```
ACE> aa:=Word2YoungPN(w[6,3,5,7,1,2,4],P):
ACE> bb:=Word2YoungPN(w[6,3,5,7,1,2,4],N):
ACE> aa &!* (1 -A[1,4,3,2] -A[1,2,3,5,4]) &!* bb;
                                    0
```

From the first part of the proposition, one gets relations between products of minors of the Vandermonde matrix. Indeed, write $\Delta(i,j,\ldots) = \prod(x_i - x_j)$. Take now the monomial with exponent $[0^{j_1}, 1^{j_2}, 2^{j_3} \ldots]$, supposing $J$ to be a decreasing partition. It is preserved by each permutation of $\mathfrak{S}_J$, thus the action of $\Box_{\omega_J}$ is just multiplication by the order of $\mathfrak{S}_J$, one can ignore it. Now, the image of the image of the monomial under $\nabla_{omega_I}$ is a product of Vandermonde determinants, that we shall write $\Delta([I])$, because the image of $x^{012..k-1}$ under $\nabla_{k\ldots1}$ is $\Delta(1,\ldots,k)$. Therefore, equation (31) becomes

$$\Delta([I])\left(1 - \tau_{1,a+1} - \cdots - \tau_{a,a+1}\right) = 0 \ . \qquad (34)$$

For the pair $J = [2,2,1]$, $I = [3,2]$, one gets $x^{00112}\Box_{1435} = 4x^{00112}$. Under $\zeta([221],[32])$, it becomes $4x^{01201}$, which is sent by $\nabla_{32154}$ to $4\Delta(123)\,\Delta(45)$. Finally the relation is

$$\Delta(123)\,\Delta(45)(1 - \tau_{14} - \tau_{24} - \tau_{34}) = \Delta(123)\Delta(45) \\ - \Delta(423)\Delta(15) - \Delta(143)\Delta(25) - \Delta(124)\Delta(35) = 0.$$

Graphically, one represents each Vandermonde by a column, and the precedent relation is now displayed as

$$\begin{bmatrix} 3 & \\ 2 & 5 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & \\ 2 & 5 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 3 & \\ 4 & 5 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 4 & \\ 2 & 5 \\ 1 & 3 \end{bmatrix} = 0 \ .$$

In fact, this relation, called *Plücker relation*[7], is valid for minors of any matrix, and not only minors of the Vandermonde matrix.

We shall see with more details in the next section that Young's identities imply minor identities.

[7]Plücker relations are quadratic relations between minors of order $n$ of an $n\times\infty$ matrix, obtained (before Plücker) in the $18^{th}$ century through eliminations in systems of linear equations. Writing $[A]$ for the minor taken on columns specified by an ordered set $A$, and $[A]\,[B]$ for the product of two minors, choosing some arbitrary $b$ in $B$, Plücker states :

$$[A]\,[B] = \sum_{a\in A} \tau_{a,b}\big([A]\,[B]\big)\ .$$

Notice that we have also written

$$[13]\,[24]\left(s_1 - \frac{1}{2}\right)(s_2 - 1) = 0$$

which is not a standard form of a Plücker relation.

We are now ready to study the fundamental modules $\Box_{\omega_I}\,\zeta(I,J)\,\nabla_{\omega_J}\,\mathcal{H}$ or, equivalently (by inversion of permutations, and the involution $s_i \to -s_i$), $\mathcal{H}\,\Box_{\omega_I}\,\zeta(I,J)\,\nabla_{\omega_J}$.

Let us describe, as a first example, $\Box_{2143}s_2\nabla_{2143}\,\mathcal{H}$. Let $v_1 = \Box_{2143}s_2\nabla_{2143}$, $v_2 = \Box_{2143}s_2\nabla_{2143}s_2$. They have different leading terms, and thus are linearly independent. We know the images of $v_1$ under multiplication by a simple transposition: $v_1s_1 = v_1s_3 = v_1$, $v_1s_2 = s_2$. However, computing $v_2s_1 = v_1s_2s_1 = v_1s_1s_2s_1$ is less evident. It is necessary to use Young's relation (31) :

$$\Box_{2143}\,s_2\nabla_{2143}\big(1-\tau_{13}-\tau_{23}\big)=0,$$

which shows that $v_1\tau_{13} = v_2s_1$ belongs to the linear span of $v_1, v_2$ a similar computation giving also $v_2s_3$. Thus the module is a 2-dimensional representation of $\mathfrak{S}_4$.

To describe the general case, recall that a *standard Young tableau* of shape $\lambda$ is a filling of the boxes of the diagram of $\lambda$ with unrepetited consecutive numbers $1, 2, \ldots$, in such a way that columns strictly decrease, and rows increase. Let us write $\mathfrak{Tab}(\lambda)$ for the set of standard tableaux of column shape $\lambda$.

We shall identify for the moment a tableau with the *permutation obtained by reading its successive columns, from left to right.*

The first description that Young [Y1] gave of a representation of the symmetric group, is the following (slightly adapted to our conventions) :

**Theorem 18** *Let $I$ be an increasing partition, $J$ be the conjugate decreasing partition.*

*Then $\Box_{\omega_I}\,\zeta(I,J)\,\nabla_{\omega_J}\,\mathcal{H}$ is a representation of the symmetric group with basis*

$$\{\Box_{\omega_I}\,\zeta(I,J)\,\nabla_t \;:\; t\in\mathfrak{Tab}(J)\} \quad or \quad \{\Box_{\omega_I}\,\zeta(I,J)\,\nabla_{\omega_J}\,\omega_J\,t \;:\; t\in\mathfrak{Tab}(J)\}\ .$$

*Proof.* The elements $\Box_{\omega_I}\,\zeta(I,J)\,\nabla_t$ are linearly independent, because their leading terms $\omega_I\zeta(I,J)\,t$ are different. One passes from $\{\Box_{\omega_I}\,\zeta(I,J)\,\nabla_t\}$ to $\{e_t := \Box_{\omega_I}\,\zeta(I,J)\,\nabla_{\omega_J}\,\omega_J\,t\}$ by a triangular matrix, we shall rather show that $\{e_t\}$ span the module, i.e. that any $e_\sigma := \Box_{\omega_I}\,\zeta(I,J)\,\nabla_{\omega_J}\,\omega_J\sigma$, $\sigma\in\mathfrak{S}_n$, is a linear combination of $e_t$.

If one codes $e_\sigma$ by the tabloid obtained from $e_{\omega_J}$ by replacing in the first tableau $\omega_J$, $1, 2, \ldots$ by $\sigma_1, \sigma_2, \ldots$, then one has to show that every tabloid $u$ is a linear combination of standard tableaux.

Of course, one can commute the entries in each column separately, it just introduces a sign. Thus one can suppose that tabloids have decreasing columns. If $u$ is not a tableau, then there is at least one violation, i.e. two adjacent entries $a, b$ in the same row, with $a > b$. Starting with the violation $a, b$ which the furthest in the North-East, and interchanging $b$ with all the elements of the preceding columns thanks to relation (31), and repeating this process, Young shows that one can *straighten* the tabloid into a sum of tableaux. The delicate point is that one creates in general new violations by correcting one, and one has to make sure that the algorithm converges instead of looping. We shall avoid totally this analysis by producing, as Young did 30 years later, orthonormal bases. There is no more straightening now, but only evaluation of scalar products. We shall also, in the next section, gives another description of natural representations, where straightening is replaced by evaluation of scalar products. QED

Let us check the module $\square_{13254}\,\zeta(122,32)\,\nabla_{32154}\,\mathcal{H}$. There are 5 tableaux, and 5 tabloids, with decreasing columns, which are not tableaux. One has, modulo the space generated by the five tableaux,

$$\begin{smallmatrix}4&\\3&5\\2&1\end{smallmatrix} = \begin{smallmatrix}4&\\3&5\\2&\textcircled{1}\end{smallmatrix} := \begin{smallmatrix}4&\\3&5\\2&1\end{smallmatrix}\Big(\tau_{12}+\tau_{13}+\tau_{14}\Big) = \begin{smallmatrix}4&\\3&5\\1&2\end{smallmatrix} + \begin{smallmatrix}4&\\1&5\\2&3\end{smallmatrix} + \begin{smallmatrix}1&\\3&5\\2&4\end{smallmatrix} \equiv 0\ ,$$

writing the letter which is to be exchanged inside a disk. Similarly, exchanging 4 and 5, $\begin{smallmatrix}5&\\3&4\\2&\textcircled{1}\end{smallmatrix} \equiv 0$. Now,

$$\begin{smallmatrix}5&\\4&2\\3&\textcircled{1}\end{smallmatrix} \equiv \begin{smallmatrix}5&\\4&2\\1&3\end{smallmatrix}\ ,\quad \begin{smallmatrix}5&\\4&\textcircled{2}\\1&3\end{smallmatrix} \equiv \begin{smallmatrix}5&\\4&1\\2&3\end{smallmatrix}\ ,\quad \begin{smallmatrix}5&\\4&1\\2&\textcircled{3}\end{smallmatrix} \equiv \begin{smallmatrix}5&\\4&1\\3&2\end{smallmatrix}\ .$$

Combining these relations with the preceding ones, one gets that

$$\begin{smallmatrix}5&\\4&2\\3&1\end{smallmatrix} \equiv 0 \equiv \begin{smallmatrix}5&\\4&3\\2&1\end{smallmatrix} \equiv \begin{smallmatrix}5&\\4&3\\1&2\end{smallmatrix}\ .$$

Already on that small example, one sees that one has to combine several Plücker relations to remove one violation. In order to do that Young wrote relations more efficient than (31), which involve summing on several letters at a time[8]. We shall detail them later. In the present case, one has

$$\begin{smallmatrix}3&\\2&5\\1&4\end{smallmatrix} - \begin{smallmatrix}4&\\2&5\\1&3\end{smallmatrix} + \begin{smallmatrix}4&\\3&5\\1&2\end{smallmatrix} + \begin{smallmatrix}5&\\2&4\\1&3\end{smallmatrix} - \begin{smallmatrix}5&\\3&4\\1&2\end{smallmatrix} + \begin{smallmatrix}5&\\4&3\\1&2\end{smallmatrix} = 0\ ,$$

and this gives in one step that the tabloid $\begin{smallmatrix}5&\\4&3\\1&2\end{smallmatrix}$ belongs to the span of tableaux.

[8] they are called *Garnir relations*[14].

## Polynomial representations of the symmetric group

As a linear space on which acts $\mathfrak{S}_n$, the group algebra $\mathcal{H} = \mathbb{Q}[\mathfrak{S}_n]$ is isomorphic to the linear span of $x^\rho := x_1^{n-1}x_2^{n-2}\cdots x_n^0$, the correspondence exchanging $\sigma$ and $(x^\rho)^\sigma$.

If instead of $\rho$, one takes another weight $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$, then the linear span $\mathbb{V}^\lambda$ of the orbit of $x^\lambda$ is of dimension $n!/dim(\mathfrak{Stab}(\lambda)) = n!/\alpha_1!\,\alpha_2!\cdots$, instead of $n!$, denoting by $\mathfrak{Stab}(\lambda)$ the subgroup of $\mathfrak{S}_n$ leaving $\lambda$ invariant, and writing $\lambda = 1^{\alpha_1}2^{\alpha_2}\cdots$.

The $\mathcal{H}$-spaces $\mathbb{V}^\lambda$ are called *permutation representations*. We already met these representations, as spaces $\square_{\omega_I}\mathcal{H}$. One can start from these representations to build the irreducible ones, but this approach does not furnish explicit irreducible representation matrices.

We shall see that the images of some permutation modules modulo symmetric functions are irreducible modules, and that it is easy to write the matrices representing the action of the symmetric group on these quotient modules.

---

Let $\mathbb{H}$ be the quotient space $\mathbb{Q}[x_1, \ldots, x_n]/\mathfrak{Sym}_+$, of polynomials modulo the ideal $\mathfrak{Sym}_+$ generated by polynomials without constant term. Here, the use of the letter $\mathbb{H}$ comes from the fact that this space, also called *coinvariant space of the symmetric group* is isomorphic, as an $\mathfrak{S}_n$-space, to the space of *harmonic polynomials*.

As a vector space, $\mathbb{H}$ is of dimension $n!$. It is easy to see by induction on $n$ that it has a basis consisting of monomials $x^\lambda$, $\lambda \leq \rho = [n{-}1, \ldots, 0]$ (i.e. $\lambda_1 \leq n{-}1$, $\lambda_2 \leq n{-}2, \ldots$), and that it is isomorphic to the regular representation of $\mathfrak{S}_n$[9].

A better linear basis of $\mathbb{H}$ consists of the *Schubert polynomials*

$$\{X_\sigma = Y_J,\, \sigma \in \mathfrak{S}_n,\ \ J = code(\sigma)\}\ ,$$

that one index indifferently with permutations, or with codes (some properties are better seen, or some computations are easier, on one indexing than on the other, cf. []).

[9]But the ring of polynomials has a grading (=total degree in the $x_i$'s) and a product that has, a-priori, no natural counterpart in $\mathbb{Q}[\mathfrak{S}_n]$. Similarly, one does not see what corresponds to the multiplication of permutations at the level of polynomials. We shall need idempotents to see the correspondence.

Recall that every polynomial in $x_1, \ldots, x_n$ is a linear combination of Schubert polynomial indexed by permutations belonging to symmetric groups of arbitrary order, $\mathfrak{S}_m$ being identified with its embedding $\mathfrak{S}_m \times \mathfrak{S}_1$ into $\mathfrak{S}_{m+1}$. The ideal $\mathfrak{Sym}_+$ is the linear span of all Schubert polynomials indexed by permutations which cannot be restricted to $\mathfrak{S}_n$. Therefore, once a polynomial is expressed into the Schubert basis, one gets a representative of it in $\mathbb{H}$ by just annhilating all Schubert polynomials indexed by permutations which move at least one value $> n$.

We shall need congruences [10] which come from decomposing $\mathbb{X} = \{x_1, \ldots, x_n\}$ into any pair of disjoint subsets $\mathbb{X}'$, $\mathbb{X}''$.

For any positive integer $k$ and any subset $\mathbb{X}' \subseteq \mathbb{X}$, let us note $\mathbb{X}'^k := \{x^k,\, x \in \mathbb{X}'\}$.

**Lemma 19** *Let $\mathbb{X} = \mathbb{X}' \cup \mathbb{X}''$, and $f(\mathbb{X}')$ be any symmetric functions in $\mathbb{X}'$. Then*

$$f(\mathbb{X}') \equiv f(-\mathbb{X}'') \qquad \text{mod } \mathfrak{Sym}_+ \tag{35}$$

*In particular, for any positive integers $k, r$, writing $\Psi_J$ for the monomial symmetric function of index the partition $J$, one has*

$$(-1)^r \Psi_{k^r}(\mathbb{X}') \equiv \sum\nolimits_J \Psi_J(\mathbb{X}'') \ , \tag{36}$$

*sum over all partitions of $kr$, with all parts multiple of $k$.*

*Proof.* Since for any $k$, $\Psi^k(\mathbb{X}') = \Psi^k(\mathbb{X}) - \Psi^k(\mathbb{X}'') \equiv \Psi^k(-\mathbb{X}'')$, the first statement is true for power sums, hence for any symmetric function. In particular,

$$(-1)^r \Psi_{1^r}(\mathbb{X}') \equiv S_r(\mathbb{X}'') = \sum\nolimits_J \Psi_J(\mathbb{X}'') \ ,$$

sum over all partitions of $r$. Since a function remains symmetrical after substitution of $x_i$ by $x_i^k$, $i = 1, \ldots, n$, raising variables to power $k$ produces the second statement from the preceding equation. QED

Let $u = [u_1, \ldots, u_{\in} \mathbb{N}^n$, $u \leq \rho$ be weakly decreasing. Then the monomial (called *dominant*) $x^J$ is equal to the Schubert polynomial $Y_u$. Suppose that moreover $u$ is such that there exists $i, r : i + r \leq n - 1$ such that

$$u_i = n\text{–}i,\ u_{i+1} = u_{i+2} = \cdots = u_{i+r} = n\text{–}i\text{–}r \ .$$

Then, from Monk's rule [], one easily obtains the following property :

[10] recall that for any power sum $\Psi^k$ and any alphabet $\mathbb{X}$, $\Psi^k(-\mathbb{X}) := -\Psi^k(\mathbb{X})$. This defines the fundamental involution $\mathbb{X} \to -\mathbb{X}$ of the ring $\mathfrak{Sym}(\mathbb{X})$. In particular, for an elementary symmetric function $\Lambda^k$, one has $\Lambda^k(-\mathbb{X}) = (-1)^k S^k(\mathbb{X})$.

**Lemma 20** *For any polynomial $f$ in $r$ variables,*

$$f(x_{i+1},\dots,x_{i+r})\,Y_u \equiv 0 \mod \mathfrak{Sym}_+(\mathbb{X})$$
$$\Leftrightarrow\; f(x_{i+1},\dots,x_{i+r}) \equiv 0 \mod \mathfrak{Sym}_+(x_{i+1},\dots,x_{i+r})\;. \qquad (37)$$

---

In particular, $x_i^n \equiv 0$, because it is equal to $S_n(x_i) \equiv \pm S_{1^n}(X - x_i) = 0$, the nullity coming from the fact that $X'' = X - x_i$ has only $n$–1 letters. Similarly, $S_{n-1,n-1}(X_2) \equiv 0$, $S_{n-2,n-2,n-2}(X_3) \equiv 0$ &c., where $X_2, X_3, \dots$ are subsets of $X$ of order $2, 3, \dots$ respectively.

---

Quadratic form Computations in the quotient ring are made easy by defining a quadratic form [11]

$$(f,g) = (fg,1) = (1,fg) := \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\ell(\sigma)}\,(fg)^\sigma\,\frac{1}{\Delta}\Big|_{x_1=0=x_2=\cdots}\,, \qquad (38)$$

where $\Delta := \prod_{i<j}(x_i - x_j)$ is the Vandermonde. In other words, given two polynomials $f, g$, one builds an alternating function from their product. Its quotient by $\Delta$ is a symmetric function, the constant term of which one finds by specializing all $x_i$'s to 0 (this amounts evaluating modulo the ideal $\mathfrak{Sym}_+$).

Explicitely, the scalar product of two monomials $x^u, x^v$ is

$$(x^u, x^v) = (x^{u+v}, 1) = \begin{cases} (-1)^{\ell(\sigma)} & if\ u+v\ = [\dots 210]^\sigma \\ 0 & \text{otherwise} \end{cases} \qquad (39)$$

In this set-up, to evaluate the quadratic form, one has to test whether the vector $u+v$ is a permutation of $[n\text{–}1, \dots, 1, 0]$ or not, and if so, keep the sign of the permutation.

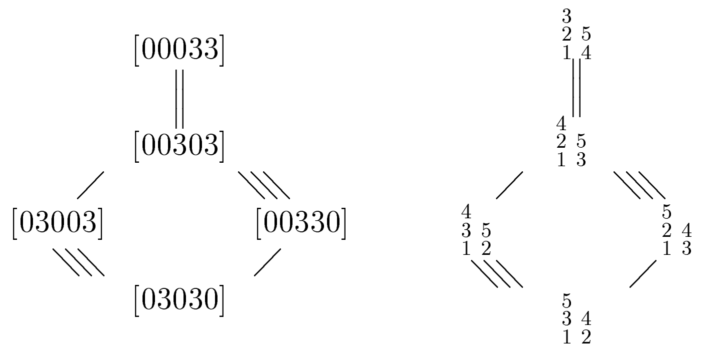
Let us now compare the modules $\nabla_{2143}s_2\square_{2143}\,\mathcal{H}$, $x^{1100}\,\mathcal{H}$, $x^{2200}\,\mathcal{H}$. Their generator is invariant under $s_1, s_3$, and the action of $s_2$ produces an element which is not proportional to it. But now $x^{1100}s_2s_3 = x^{1001}$ is not a combination of $x^{1100}, x^{1010}$, though $\nabla_{2143}s_2\square_{2143}s_2s_1$ is a combination of $\nabla_{2143}s_2\square_{2143}$, $\nabla_{2143}s_2\square_{2143}s_2$. However,

$$x^{2002}+x^{2020}+x^{2200} = x_1^2(x_2^2+x_3^2+x_4^2) = x_1^2\big((x_1^2+x_2^2+x_3^2+x_4^2)-x_1^2\big) \equiv -x_1^4 \equiv 0\,,$$

[11] We could have twisted this form, as we did for $\mathcal{H}$, by taking the product of $f(x_1,\dots,x_n$ with $g(x_n,\dots,x_1)$. This is the convention that we first chosed in the theory of Schubert polynomials, to have them constitute a self-adjoint basis.

and similarly $x^{0220}$ also belongs to the span of $x^{2200}, x^{2020}$. Therefore, the two modules $\nabla_{2143} s_2 \square_{2143}\, \mathcal{H}$ and $x^{2200}\, \mathcal{H}$ are isomorphic. On the contrary, $x^{1100}\, \mathcal{H}$ is 5-dimensional, and not irreducible.

Let us take a bigger example, with the 5-vertices graph corresponding to shape $[3,2]$ that we already wrote many times. We write a copy of the graph starting with the vertex $[00033]$ :




Both graphs have 5 vertices, related by the same permutations to the top one. Notice however, that $s_1, s_2, s_4$ act by –1 on the top tableau, and by 1 on the top monomial. We have described the space $\begin{smallmatrix}3&\\2&5\\1&4\end{smallmatrix}\, \mathcal{H}$ by writing how tabloids decompose in the basis of tableaux. To the tabloid $\begin{smallmatrix}3&\\2&5\\1&4\end{smallmatrix}\, \sigma$ we associate the monomial $x^{00033}\, \sigma$. Taking into account signs, we have corresponding equations

$$0 = \begin{smallmatrix}4&\\3&5\\2&1\end{smallmatrix}\Big(1 - \tau_{12} - \tau_{13} - \tau_{14}\Big) \;\longleftrightarrow\; x^{30003}\Big(1 + \tau_{12} + \tau_{13} + \tau_{14}\Big) = 0\ ,$$

but now, the equation on the right is easier to check, because it reads

$$x^{30003} + x^{03003} + x^{00303} + x^{00033} \equiv -x^{00003}x^{00003} = -x^{00006} \equiv 0\ .$$

More interesting, the more sophisticated relation expressing the tabloid $\begin{smallmatrix}5&\\4&3\\1&2\end{smallmatrix}$ corresponds to

$$x^{03300} + x^{03030} + x^{03003} + x^{00330} + x^{00303} + x^{00033} \equiv 0\ ,$$

but this is true because the left-hand side is a symmetric function of $x_2, \ldots, x_5$. Therefore, according to (**??**), it is congruent to a symmetric function of $x_1$ of degree 6, and it must be null (already $x_1^5 \equiv 0$).

Let us now evaluate the scalars products of the vertices of the left graph with vertices $[01201], [01021], [00121], [01012], [00112]$, obtained by reading from right to left the vertices of the original graph for shape $[3,2]$.

The quadratic form is expressed by the following matrix (0 are replaced by dots) :

$$Q = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 \end{bmatrix} \quad .$$

We know from (39) that we must obtain a matrix of $0, \pm 1$. That the diagonal has no zero entry is clear, because we started with a decomposition of $\rho =$ [01234] into [00033] + [01201], and acted with the same transpositions on both vectors. But what is remarkable is that the matrix is lower triangular.

Now, we can obtain relations by just evaluating scalar products. For example, the expansion of $x^{30003}$ is obtained from the matrix, and the evaluation of $(x^{30003}, x^{01201}) = (x^{31204}, 1)$, $(x^{30003}, x^{01021}) = (x^{31024}, 1)$, $(x^{30003}, x^{00121}) = (x^{30124}, 1)$, $(x^{30003}, x^{01012}) = (x^{31015}, 1) = 0$, $(x^{30003}, x^{00112}) = (x^{30115}, 1) = 0$.

In the general case, we proceed as follows.

Given any two vectors $u, v \in \mathbb{N}^n$, let $\binom{u}{v}$ be the word in the biletters $\binom{u_i}{v_i}$.

Given a partition $\lambda$ of $n$, recall that we have defined a word $y_\lambda$ by filling its diagram (packed in the North-East corner, parts of $\lambda$ being column lengths) with successive integers, starting from 0 in the bottom boxes, and then, reading successive columns from left to right. Let $u_\lambda = \rho - v_\lambda = [n{-}1 - v_1, n{-}2 - v_2, \ldots, 0 - v_n]$.

$$\lambda = [422] \to \underset{y_\lambda = [10\,10\,3210]}{\begin{array}{ccc} 1 & 1 & 3 \\ 0 & 0 & 2 \\ & & 1 \\ & & 0 \end{array}} \to u_\lambda = [76543210]{-}[10103210] \quad \underset{u_\lambda = [66\,44\,000]}{\begin{array}{cccc} 6 & 6 & & \\ 4 & 4 & & \\ 0 & 0 & 0 & 0 \end{array}}$$

Let $V_\lambda$ be the linear span of the orbit of $x^{y_\lambda}$ under $\mathfrak{S}_n$, and $U_\lambda$ be the linear span of the orbit of $x^{u_\lambda}$.

**Lemma 21** *Let $\lambda$ be a partition, $u$ be any permutation of $u_\lambda$, $v$ be any permutation of $y_\lambda$. Then*

$$(x^u\,,\,x^v) \neq 0 \quad \textit{iff} \quad \begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix} \ \textit{commutatively}\ .$$

*Proof.* $(x^u\,,\,x^v) \neq 0$ (and $= \pm 1$) is equivalent to the fact that $u + v$ is a permutation of $\rho$. But it is possible to get the components $0, 1, \ldots, \lambda_1{-}1$ only as $0+0, 0+1, \ldots, 0+(\lambda_1{-}1)$. This determines $\lambda_1$ letters of the biword $\binom{u}{v}$, and one proceeds by induction on the length of $\lambda$. QED

Take now the graph $\Gamma_\lambda$ with its first labelling (the plactic class of $y_\lambda$), and a copy of it, that we denote $\Gamma_\lambda^\clubsuit$, taking now $u_\lambda$ as top vertex. Denote by $\clubsuit$ the involution exchanging the labellings of vertices (so that $u_\lambda = y_\lambda^\clubsuit$).

**Lemma 22** *Let $u$ be a vertex of $\Gamma_\lambda^\clubsuit$, $v$ be a vertex of $\Gamma_\lambda$. Then $(x^u, x^{u^\clubsuit}) = \pm 1$. If $(x^u, x^v) \neq 0$, then $v$ is smaller than $u^\clubsuit$ for the lexicographic order, and $u$ is smaller than $v^\clubsuit$ for the right-lexicographic order.*

*Proof.* First, $u + u^\clubsuit$ is a permutation of $\rho$. This proves the first assertion. Secondly, from lemma 21, we have that in the case of non nullity of $\binom{u}{v}$, for any $b$, then the set of biletters $\binom{b}{\bullet}$ in $\binom{u}{v}$ is the same as in $\binom{u_\lambda}{y_\lambda}$. But in this last biword, the bottom letters are $\ldots, 2, 1, 0$. Therefore, any permutation of the biletters $\cdots \binom{b}{2}\binom{b}{1}\binom{b}{0}$ will give a word in the bottom letters lexicographically smaller than ..................... QED

For example, for .......................... but only $v =$ ......... are vertices of $\Gamma_\lambda$.

A more detailed analysis of the restriction of the quadratic form to $U_\lambda \times V_\lambda$ is made in [3].

Taking the lexicographic order on the vertices of $\Gamma_\lambda$, and its image under $\clubsuit$ on the vertices of $\Gamma_\lambda^\clubsuit$, one deduces from lemma 22 the following proposition.

**Proposition 23** *Given a partition $\lambda$ of $n$, then the restriction of the canonical quadratic form of $\mathbb{Q}[x_1, \ldots, x_n]/\mathfrak{Sym}_+$ to $U_\lambda \times V_\lambda$ is a triangular matrix $Q_\lambda$, with a diagonal entries in $\{1, -1\}$.*

## Jucys elements and the center of the group algebra of $\mathfrak{S}_n$

Given a non-commutative algebra, there are many ways to relate it to the commutative world. In this section, we shall treat the related questions of describing the *center of the group algebra of* $\mathfrak{S}_n$ (i.e. the set of elements which commutes with every permutation) and finding a *maximal commutative subalgebra.*

If $g \in \mathbb{Q}[\mathfrak{S}_n]$ commutes with every permutation $\sigma$, then

$$g = \sigma\, g\, \sigma^{-1} = \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} g = \sigma\, g\, \sigma^{-1} \; .$$

Therefore, $g$ is a linear combination of conjugacy classes, and conversely, any conjugacy class belongs to the center $\mathcal{Z}_n$ of $\mathbb{Q}[\mathfrak{S}_n]$. As a result, one has :

**Lemma 24** *Conjugacy classes are a linear basis of the center* $\mathcal{Z}_n$ *of* $\mathbb{Q}[\mathfrak{S}_n]$. *The product of two conjugacy classes is a sum of conjugacy classes.*

Describing the multiplication constants of $\mathcal{Z}_n$ is not a straightforward matter, we shall see later how to do it with differential operators. Multiplication of classes can be translated into an operation on symmetric function, which is implemented into ACE :

```
ACE> ProdCC:= proc(cc1,cc2)  ## Enter 2 linear combination of classes
  Toc( SfCCProd(cc1, cc2))
 end:
ACE> ProdCC( c[3,2,1],c[4,2]);
   36 c[4,1,1]+ 48 c[2,2,2]+ 24 c[2,1,1,1,1]+ 24 c[3,2,1]+ 30 c[6]
```

Let us now look at commutative sub-algebras. We shall follow Jucys [19], who described a *Gelfand-Zetlin* basis of the group algebra of successive symmetric groups. Recall that we embedded symmetric groups of successive orders

$$\mathfrak{S}_1 \hookrightarrow \mathfrak{S}_2 \simeq \mathfrak{S}_2 \times \mathfrak{S}_1 \hookrightarrow \mathfrak{S}_3 \simeq \mathfrak{S}_3 \times \mathfrak{S}_1 \ldots \mathfrak{S}_{n-1} \simeq \mathfrak{S}_{n-1} \times \mathfrak{S}_1 \hookrightarrow \mathfrak{S}_n \; ,$$

by identifiying a permutation of $\mathfrak{S}_{k-1}$ to the permutation of $\mathfrak{S}_k$ obtained by adding to it a fixed point $k$.

The Gelfand-Zetlin sub-algebra $\mathcal{J}_n \subset \mathcal{H}_n$, $n = 1, 2, \ldots$ is the algebra generated by the successive centers $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ of $\mathcal{H}_1, \ldots, \mathcal{H}_n$. It is clear that is commutative. We shall see later that it is a maximal commutative sub-algebra of $\mathcal{H}_n$.

Since the sum of all transpositions belong to the center, then $\mathcal{J}_n$ contains all elements $\sum_{i,j:i<j\leq k} \tau_{ij}$, or equivalently the *Jucys elements*[12]

$$\xi_n := \sum_{1\leq i<j<n} \tau_{in}$$

(it is convenient to put $\xi_1 = 0$).

One can now use the Jucys elements to describe the center. We have already said that it is linearly spanned by conjugacy classes. However, as an algebra, Farahat and Highman [8] have shown that the sums $\sum_{I:\ell(I)=n-k} \mathcal{C}_I$, $1 \leq k \leq n{-}1$ generate it. Jucys proved the more precise following property :

**Theorem 25** *For each $k$ : $1 \leq k \leq n{-}1$, the elementary symmetric function $\Lambda^k(\xi_1,\ldots,\xi_n)$ of degree $k$ in the Jucys elements is equal to the sum $\sum_{I:\ell(I)=n-k} \mathcal{C}_I$ of all conjugacy classes of length $n{-}k$ (i.e. indexed by partitions of length $n{-}k$).*

For example,

$$\begin{array}{rcl}
\Lambda^1 & = & \mathcal{C}_{21\ldots1} \\
\Lambda^2 & = & \mathcal{C}_{221\ldots1} + \mathcal{C}_{31\ldots1} \\
\Lambda^3 & = & \mathcal{C}_{321\ldots1} + \mathcal{C}_{41\ldots1} \\
\cdots & & \cdots \\
\Lambda^{n-1} & = & \mathcal{C}_n
\end{array}$$

*Proof.* . By induction on $n$, one has essentially to describe the effect of multiplication of a conjugacy class of $\mathfrak{S}_{n-1}$ by a transposition $\tau_{in}$. If $\sigma \in \mathfrak{S}_{n-1}$ has a cycle $(a,\ldots,b,i,c\ldots)$, then $\tau_{in}\,\sigma$ will differ from it just by the cycle $(a,\ldots,b,\ i,n,\ c\ldots)$.

On the other hand,

$$\Lambda^k(\xi_1+\cdots+\xi_n) = \Lambda^k(\xi_1+\cdots+\xi_{n-1}) + \xi_n\,\Lambda^{k-1}(\xi_1+\cdots+\xi_{n-1})\ .$$

The first term corresponds to adding a cycle constituted by $n$ only, the second, to the multiplication by the sum of all transpositions involving $n$. QED

Given Jucys' result , a natural question is : how to express the conjugacy classes, which constitute the natural linear basis of the center, as symmetric polynomials in the $\xi_j$'s ?

[12] the terminology Jucys-Murphy, or Murphy element is also used, the last one being predominent, but contrary to the historical order; our choice is meant to reestablish a more balanced average of citations.

We shall give, in the next proposition, a linear basis of products of elementary symmetric functions. For $I$ a partition of $n$, and $J := [j_1, j_2, \dots, j_k]$ the conjugate partition, let

$$\Lambda\langle I\rangle := \Lambda^{j_2} \cdots \Lambda^{j_k} = \Lambda^{j_2,\dots,j_n}$$

("product" of the columns of the diagram of $I$ minus its first column).

**Proposition 26** *The $\Lambda\langle I\rangle$, $|I| = n$, constitute a linear basis of $Sym(\xi_1, \dots, \xi_n)$, and the matrix of change of basis is triangular by blocks (taking the natural order of partitions on the conjugacy classes $\mathcal{C}_J$, and taking an order on the $\Lambda\langle I\rangle$'s, such that the weight $|I|$ is increasing).*

For example, for $n = 4, 5, 6$, ACE computes

$$\begin{array}{cc} & \begin{matrix} c_{1111} & c_{211} & c_{22} & c_{31} & c_{4} \end{matrix} \\ \begin{matrix} \Lambda\langle 0\rangle \\ \Lambda\langle 1\rangle \\ \Lambda\langle 11\rangle \\ \Lambda\langle 2\rangle \\ \Lambda\langle 3\rangle \end{matrix} & \begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ 6 & . & 2 & 3 & . \\ . & . & 1 & 1 & . \\ . & . & . & . & 1 \end{bmatrix} \end{array} \quad \begin{array}{cc} & \begin{matrix} c_{11111} & c_{2111} & c_{221} & c_{311} & c_{32} & c_{41} & c_{5} \end{matrix} \\ \begin{matrix} \Lambda\langle 0\rangle \\ \Lambda\langle 1\rangle \\ \Lambda\langle 11\rangle \\ \Lambda\langle 2\rangle \\ \Lambda\langle 21\rangle \\ \Lambda\langle 3\rangle \\ \Lambda\langle 4\rangle \end{matrix} & \begin{bmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ 1 & . & 2 & 3 & . & . & . \\ . & . & 1 & 1 & . & . & . \\ . & 9 & . & . & 4 & 6 & . \\ . & . & . & . & 1 & 1 & . \\ . & . & . & . & . & . & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{matrix} c_{111111} & c_{21111} & c_{2211} & c_{3111} & c_{222} & c_{321} & c_{411} & c_{33} & c_{42} & c_{51} & c_{6} \end{matrix} \\ \begin{matrix} \Lambda\langle 0\rangle \\ \Lambda\langle 1\rangle \\ \Lambda\langle 11\rangle \\ \Lambda\langle 2\rangle \\ \Lambda\langle 111\rangle \\ \Lambda\langle 21\rangle \\ \Lambda\langle 3\rangle \\ \Lambda\langle 22 \\ \Lambda\langle 31\rangle \\ \Lambda\langle 4\rangle \\ \Lambda\langle 5\rangle \end{matrix} & \begin{bmatrix} 1 & . & . & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . & . \\ 15 & . & 2 & 3 & . & . & . & . & . & . & . \\ . & . & 1 & 1 & . & . & . & . & . & . & . \\ . & 51 & . & . & 6 & 9 & 16 & . & . & . & . \\ . & 14 & . & . & 3 & 4 & 6 & . & . & . & . \\ . & . & . & . & 1 & 1 & 1 & . & . & . & . \\ 85 & . & 28 & 37 & . & . & . & 11 & 12 & 20 & . \\ . & . & 13 & 12 & . & . & . & 6 & 7 & 10 & . \\ . & . & . & . & . & . & . & 1 & 1 & 1 & . \\ . & . & . & . & . & . & . & . & . & . & 1 \end{bmatrix} \end{array}$$

The inverse matrices are

$$\begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ 6 & . & -1 & 3 & . \\ -6 & . & 1 & -2 & . \\ . & . & . & . & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ 10 & . & -1 & 3 & . & . & . \\ -10 & . & 1 & -2 & . & . & . \\ . & 9/2 & . & . & -1/2 & 3 & . \\ . & -9/2 & . & . & 1/2 & -2 & . \\ . & . & . & . & . & . & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . & . \\ 15 & . & -1 & 3 & . & . & . & . & . & . & . \\ -15 & . & 1 & -2 & . & . & . & . & . & . & . \\ . & -4 & . & . & 2 & -7 & 10 & . & . & . & . \\ . & 13 & . & . & -3 & 10 & -12 & . & . & . & . \\ . & -9 & . & . & 1 & -3 & 3 & . & . & . & . \\ 54 & . & -7 & 18 & . & . & . & 3/5 & -8/5 & 4 & . \\ -67 & . & 9 & -19 & . & . & . & -4/5 & 9/5 & -2 & . \\ 13 & . & -2 & 1 & . & . & . & 1/5 & -1/5 & -1 & . \\ . & . & . & . & . & . & . & . & . & . & 1 \end{bmatrix}$$

---

**Commutation relations**

It is immediate to check that the Jucys elements satisfy the following relations

$$\xi_{k+1}\, s_k - s_k\, \xi_k = 1 \quad \& \quad \xi_k\, s_i = s_i\, \xi_k \ , \ |k-i| > 1 \ . \tag{40}$$

Suppose now that $h \in \mathcal{H}_n$ is an eigenvector for $\xi_1, \dots, \xi_n$, with eigenvalues $c_1, \dots, c_n$ :

$$h\, \xi_i = h\, c_i \ , \ 1 \le i \le n \ . \tag{41}$$

Then one can easily generate other eigenvectors :

**Lemma 27** *Let $h \in \mathcal{H}_n$ be a simultaneous eigenvector for the Jucys elements, with different eigenvalues $c_1, \dots, c_n$. Then, for any $i : 1 \le i < n$, $g := h\,(s_i + \frac{1}{c_{i+1}-c_i})$ is also eigenvector, with the same eigenvalues as $h$, except*

$$g\, \xi_i = g\, c_{i+1} \quad \& \quad g\, \xi_{i+1} = g\, c_i \ .$$

*Proof.* Because of the commutations (40), $g$ is an eigenvector for all $\xi_j$, $j \neq i, i{+}1$, with same eigenvalues as $f$. On the other hand

$$h\left(s_i + \frac{1}{c_{i+1}-c_i}\right)\xi_i = h\left(xi_{i+1}\, s_i + 1 + \xi_i \frac{1}{c_{i+1}-c_i}\right.$$
$$= h\left(c_{i+1}(s_i + \frac{1}{c_{i+1}-c_i}) + \frac{-c_{i+1}+c_{i+1}-c_i+c_i}{c_{i+1}-c_i}\right) = g\, c_{i+1}\ ,$$

and the product $g\xi_{i+1}$ is determined by the fact that $\xi_i + \xi_{i+1}$ commutes with $s_i$. QED

We shall need more commutation relations.

**Lemma 28** *Let $i,j,k$ three integers, $1 \leq i \leq k \leq j$; let $g_i, g_{i+1}, \ldots, g_j$ belong to the algebra generated by the Jucys elements. Then*

$$s_i g_i s_{i+1} s_{i+1} \cdots s_j g_j\, \xi_k = \xi_{k+1}\ s_i g_i s_{i+1} g_{i+1} \cdots s_j g_j - s_i g_i s_{i+1} \cdots g_{k-1}\, \emptyset\, g_k s_{k+1} \cdots s_j g_j\ . \quad (42)$$

*Proof.* The element $\xi_k$ commutes with all the factors of the right of $s_i$; now, $s_i\xi_i = \xi_{i+1}s_i - 1$ and $\xi_{i+1}$ is free to reach the extreme left. QED

---

**PNP and Jucys elements**

To find orthogonal idempotents starting from natural representations, Young used the elements[13] $P(t)N(t)P(t)$, for tableaux with rows filled of consecutive integers, instead of taking $P(t)N(t)$ or $N(t)P(t)$.

We are going to show that these elements have many remarkable properties; they are, in particular, two-sided eigenvectors with respect to the Jucys elements.

Recall Young's relations[14] :

$$N(t)P(t)\left(1 + \sum_{a\in A} \tau_{a,b}\right) = 0\ , \quad (43)$$

for any choice of two rows of $t$, $a$ varying over all the entries of a row, and $b$ belonging to a row of not bigger length.

Given an integer $k$, then the Jucys element $\xi_k$ is the sum of all transpositions which exchange $k$ with elements of lower rows, plus transpositions

[13] $P(t)N(t)P(t) \neq 0$, because $P(t)N(t)P(t)N(t) = n!/dim(\lambda)P(t)N(t)$, when $t$ is of shape $\lambda$.

[14] no signs, because we took $N(t)P(t)$ instead of $P(t)N(t)$; Young's relations involve two consecutive parts of a composition, or equivalently by conjugation, any pair of rows of a tableau.

exchanging $k$ with elements in the same row. For each of the lower rows, one has $N(t)P(t)\sum_a \tau_{ak} = -N(t)P(t)$, and each transposition preserving the row where $k$ lies is such that $N(t)P(t)\,\tau_{ak} = N(t)P(t)$. Summing up, and using left and right multiplication, one obtains the following proposition :

**Proposition 29** *Let $t$ be a tableau with rows filled of consecutive integers (call it a* bottom tableau*). Then $P(t)N(t)P(t)$ is a two-sided eigenvector of all the Jucys elements, with*

$$\xi_i\, P(t)N(t)P(t) = c(i,t)P(t)N(t)P(t) = P(t)N(t)P(t)\,\xi_i \ , \tag{44}$$

*where $c(i,t)$ is the content of the box of $t$ containing $i$, i.e. the distance of $i$ to the main diagonal.*

*In particular, all the Jucys elements commute with $P(t)N(t)P(t)$.*

---

**Yang-Baxter graphs and eigenvectors**

Given a partition $\lambda$, we shall take again the Yang-Baxter graph $\Gamma_\lambda$, vertices being labelled by the content vector. We shall still label each edge $v \to v\, s_i$ by the factor

$$\left(s_i + \frac{1}{\beta}\right) \frac{1}{\sqrt{1-\beta^{-2}}} \ ,$$

where $\beta = v_i - v_{i+1}$ (notice the change of sign !).

Let $t_\lambda$ be the bottom tableau of shape $\lambda$. For any other tableau $t$ of the same shape, define $\theta(t_\lambda, t)$ to be the product of the edges of any path $t_\lambda \to t$, and $\widetilde{\theta}(t_\lambda, t)$ to be its image under the antiautomorphism of $\mathcal{H}$ induced by $\sigma \to \sigma^{-1}$. If $\theta(t_\lambda, t) = \left(s_i + \frac{1}{\alpha}\right)\frac{1}{\sqrt{1-\alpha^{-2}}} \cdots \left(s_k + \frac{1}{\gamma}\right)\frac{1}{\sqrt{1-\gamma^{-2}}}$, then $\widetilde{\theta}(t_\lambda, t) = \left(s_k + \frac{1}{\gamma}\right)\frac{1}{\sqrt{1-\gamma^{-2}}} \cdots \left(s_i + \frac{1}{\alpha}\right)\frac{1}{\sqrt{1-\alpha^{-2}}}$.

Let $e_{t_\lambda t_\lambda}$ be the idempotent proportional to $P(t_\lambda)N(t_\lambda)P(t_\lambda)$. For any pair $t, t'$ of tableaux of shape $\lambda$, define

$$e_{tt'} = \widetilde{\theta}(t_\lambda, t)\, e_{t_\lambda t_\lambda}\, \theta(t_\lambda, t') \ . \tag{45}$$

Then

**Theorem 30** *The $n!$ elements $e_{tt'}$, $t, t'$ standard tableaux of the same shape, are (non zero) two-sided eigenvectors of Jucys elements, with eigenvalues*

$$\xi_i\, e_{t,t'} = c(i,t)\, e_{t,t'} \qquad \& \qquad e_{t,t'}\, \xi_i = e_{t,t'}\, c(i,t') \ . \tag{46}$$

*They are a linear basis of $\mathcal{H}$ and multiply as matrix units, i.e.*

$$e_{t,t'}\, e_{t',t''} = e_{t,t''} \qquad \& \qquad e_{t,t'}\, e_{u,t''} = 0 \ \textit{if}\ u \neq t' \tag{47}$$

*Proof.* For two tableaux of shape $\lambda$, the $e_{t,t'}$ are obtained by multiplying the $e_{t_\lambda t_\lambda}$ with invertible factors, and therefore are non-zero. The commutation relations given by Lemma 27 show that

$$\xi_i \, e_{t,t'} = c(i,t) \, e_{t,t'} \qquad \& \qquad e_{t,t'} \, \xi_i = e_{t,t'} \, c(i,t') \ ,$$

because it is true for $e_{t_\lambda t_\lambda}$. Notice that it implies that the Jucys elements commute with all the diagonal elements $e_{tt}$. Moreover, the $e_{tt}$ are also idempotents.

Since the "content vectors" are all different, the $e_{t,t'}$ are linearly independent, and therefore, constitute a linear basis of $\mathcal{H}$.

Since $P(t)N(t)\mathcal{H}P(t')N(t') = \{0\}$ if $t$ and $t'$ are not of the same shape, then $e_{t,t'} \, e_{u,u'} = 0$ if $t$ and $u$ have not the same shape.

Given any tableau of shape $\lambda$, a product $e_{t_\lambda t} \, e_{tt_\lambda} = \left(\widetilde{\theta}(t_\lambda t)\right)^{-1} e_{tt}e_{tt} \left(\theta(t_\lambda t)\right)^{-1}$ is different from 0. Multiplying it on the right and on the left by invertible factors, it implies that any product $e_{t't}e_{tt''}$ is non zero. But it is a left and right eigenvector of the Jucys elements, and therefore, proportional to $e_{t't''}$. Since $e_{tt}$ is an idempotent, $\widetilde{\theta}(t_\lambda t) \, e_{t_\lambda t_\lambda}\theta(t_\lambda t)\widetilde{\theta}(t_\lambda t)e_{t_\lambda t_\lambda} = \widetilde{\theta}(t_\lambda t)e_{t_\lambda t_\lambda}\theta(t_\lambda t)$, and $e_{t_\lambda t_\lambda}\theta(t_\lambda t)\widetilde{\theta}(t_\lambda t)e_{t_\lambda t_\lambda} = \theta(t_\lambda t)e_{t_\lambda t_\lambda}$. One concludes that the factor of proportionality is 1.

In the case of four tableaux $t', t, u, t''$ of shape $\lambda$, one writes the product $e_{t't}e_{uu'}$ as $e_{t't}e_{tt}e_{uu}e_{ut''}$. If $t \neq u$, at least one $\xi_i$ has a different eigenvalue on $e_{tt}$ and $e_{uu}$. But $\xi_i \, e_{tt}e_{uu} = e_{tt} \, \xi_i \, e_{uu}$, and therefore the products $e_{tt}e_{uu}$ and $e_{t't}e_{uu'}$ are null.

**Yang Polynomials**

Given a partition $\lambda$, let

$$u := [(\lambda_2 + \cdots + \lambda_n)^{\lambda_1},\ (\lambda_3 + \cdots + \lambda_n)^{\lambda_2},\ (\lambda_4 + \cdots + \lambda_n)^{\lambda_3},\ \ldots, 0^{\lambda_n}]\ ,$$

and let $t_\lambda$ be the bottom tableau of shape $\lambda$. The monomial $x^u$ is equal to the Schubert polynomial $Y_u$, and is invariant under the Young subgroup $\mathfrak{S}_\lambda$.

In fact, from the theory of Schubert polynomials, one easily obtain the following characterization of $Y_u$.

**Lemma 31** *Let $f$ be the class of an homogenous polynomial of degree $|u|$ in the quotient ring $\mathfrak{H}\mathfrak{H} := \mathbb{C}[x_1, \ldots, x_n]/\mathfrak{Sym}_+$. If $f$ is invariant under $\mathfrak{S}_\lambda$ , then it is proportional to the Schubert polynomial $Y_u$.*

Denote by $\lambda! = \lambda_1!\,\lambda_2! \cdots$ the order of $\mathfrak{S}_\lambda$. The element $P(t_\lambda)N(t_\lambda)P(t_\lambda)$ is a quasi-idempotent :

$$(P(t_\lambda)N(t_\lambda)P(t_\lambda))^2 = \frac{n!\,\lambda!}{dim(\lambda)}\, P(t_\lambda)N(t_\lambda)P(t_\lambda)\ .$$

Let $e_\lambda$ be the idempotent proportional to $P(t_\lambda)N(t_\lambda)P(t_\lambda)$:

$$e_\lambda = \frac{dim(\lambda)}{n!\,\lambda!} P(t_\lambda)N(t_\lambda)P(t_\lambda)\ .$$

**Lemma 32** *The class of the monomial $x^u$ in $\mathfrak{H}\mathfrak{H}$ is invariant under $e_\lambda$.*

*Proof.* Clearly $PNPx^u$ is invariant under the left multiplication by elements of $\mathfrak{S}_\lambda$. Therefore $e_\lambda\, x^u$ is proportional to $x^u$. But the two modules $\mathcal{H}\, x^u$ and $\mathcal{H}\, P(t_\lambda)N(t_\lambda)P(t_\lambda)$ are isomorphic, and therefore the equality

$$P(t_\lambda)N(t_\lambda)P(t_\lambda)\ P(t_\lambda)N(t_\lambda)P(t_\lambda) = c\, P(t_\lambda)N(t_\lambda)P(t_\lambda)$$

implies that $P(t_\lambda)N(t_\lambda)P(t_\lambda)\, x^u = c\, x^u$ for the same constant $c$. QED

Let $\Gamma_\lambda$ be the normalized Yang-Baxter graph for the partition $\lambda$. Any edge $v \to v\, s_i$ is now labelled

$$\Big(s_i + \frac{1}{\alpha}\Big)\, \frac{\alpha}{\sqrt{\alpha^2 - 1}} \qquad \text{with} \quad \alpha = |v_i - v_{i+1}|\ .$$

Given any tableau $t$ of shape $\lambda$, let $\widetilde{\theta}_t$ and $\theta_t$ be respectively a path from $t$ to $t_\lambda$, and a path from $t_\lambda$ to $t$ (evaluated, as usual, as the product of the labels of its edges).

Define, for any tableau $t$ the *Yang polynomial* $b_t$ to be

$$b_t := \widetilde{\theta}_t\, x^u\ . \tag{48}$$

**Theorem 33** *Let $\lambda$ be a partition, and $\lambda! = \lambda_1!\,\lambda_2!\cdots$ be the order of $\mathfrak{S}_\lambda$. Then for any pair of tableaux of shape $\lambda$, one has*

$$P(t_\lambda)\theta_{t'}\,b_t \equiv 0\ ,\ t \neq t' \qquad \& \qquad \frac{1}{\lambda!}P(t_\lambda)\theta_{t'}\,b_t \equiv x^u\ . \tag{49}$$

*Proof.* $P(t_\lambda)\theta_{t'}\,b_t$ is proportional to $x^u$, therefore, proportional to $P(t_\lambda)N(t_\lambda)P(t_\lambda)\theta_{t'}\widetilde{\theta}_t P(t_\lambda)N(t_\lambda)P(t_\lambda)x^u$. The fact that the product of the two idempotents $\widetilde{\theta}_{t'}\,e_\lambda\,\theta_{t'}$ and $\widetilde{\theta}_t\,e_\lambda\,\theta_t$ is 0 or $\widetilde{\theta}_t\,e_\lambda\,\theta_t$ whether $t \neq t'$, or not implies the theorem. QED

**Corollary 34** *Every polynomial in the space $\mathcal{H}\,x^u$ decomposes uniquely in $\mathfrak{H}\mathfrak{H}$ as*

$$f \equiv \sum_{t\in\mathfrak{Tab}(\lambda)} \left(\frac{1}{\lambda!\,x^u}P(t_\lambda)(\theta_t\,f)\right)\,b_t\ . \tag{50}$$

## Young idempotents as limits of Yang-Baxter elements

We have already seen that Young matrices are a solution of Yang-Baxter equation, and that Young orthogonal idempotents involve products of factors of the type

$$s_k + \frac{1}{c(t,k) - c(t,k+1)}$$

in which the contents $c(t,k)$ of standard tableaux appear (we have implicitely used the fact that that no two consecutive contents are equal).

Another way to relate Young idempotents and Yang-Baxter elements is due to Jucys and has been developped by Cherednik.

It consists mostly in taking only one Yang-Baxter element, the element indexed by $\omega$, and in specializing in it the vector of spectral parameters to all the content vectors of standard tableaux. But, because contents are not all different, the specialization is not straightforward and must be obtained as a limit.

Thus, let us first "polarize" the contents by using extra variables $\epsilon_1, \ldots, \epsilon_n$ and puting, for a letter $k$ in row $i$ of a tableau $t$

$$c^+(k,t) := c(k,t) + \epsilon_i \ . \tag{51}$$

We can now use these modified contents as parameters in Yang-Baxter elements, since they are all different. Jucys [19] gave an inprecise version of the following theorem[15], but this was corrected by Cherednik [**?**].

**Theorem 35 (Jucys-Cherednik)** *Let $\lambda$ be a partition of $n$, $t$ be the bottom tableau of shape $\lambda$. Then the limit of*

$$Y_\omega(c^+(1,t), \ldots, c^+(n,t))$$

*when all the $\epsilon_i$ tend towards 0, exists and is equal, up to a non zero factor, to the Young element*

$$P(t)\, N(t)\, P(t)\, \omega \ .$$

Nazarov has made a detailed analysis of why the above Yang-Baxter element has no pole (and extended his analysis to cover the spin representations of the symmetric group). Independently, [JMKO] characterized the left and

[15] he took different $\epsilon$'s for each box of a diagram, and let all $\epsilon$'s tend independently towards 0. The limit does not exist in general for unrelated $\epsilon$'s.

right ideals generated by the limit of $Y_\omega$ and thus provided another proof of Theorem **??**.

We prefer to show that one can avoid getting involved into sophisticated studies of limits, by just extending Yang-Baxter relations in such a way as to cover the limit case.

We first notice that the limit of

$$(s_1 \pm 1)\,(s_2 + \frac{1}{\epsilon})\,\left(s_1 + \frac{1}{\epsilon \mp 1}\right)$$

for $\epsilon \to 0$ exists and is equal to $(s_1 \pm 1)\,(s_2 s_1 \mp s_2 - 1)$, and, by symmetry, is also equal to $(s_2 s_1 \mp s_1 - 1)\,(s_2 \pm 1)$. Therefore, one has

$$(s_1 \pm 1)\,(s_2 s_1 \mp s_2 - 1) = (s_2 s_1 \mp s_1 - 1)\,(s_2 \pm 1)\ . \tag{52}$$

Thus, factors of the type $(s_1+1)(s_2 s_1 \mp s_2 - 1)$ can be used instead of the troublesome $(s_1+1)(s_2+\frac{1}{\epsilon})\ \left(s_1 + \frac{1}{\epsilon\mp 1}\right)$.

To be able to handle these special factors in Yang-Baxter products, we need to replace commutations of the following type (the numbers 1,2,3 stand for any triple of consecutive numbers, and $\alpha$ is a non-zero constant) :

$$\begin{aligned}\Big((s_3+\frac{1}{\alpha})(s_2+\frac{1}{\epsilon-1})(s_1+\frac{1}{\epsilon})\Big)\ \Big((s_3+\frac{1}{\epsilon-\alpha-1})(s_2+\frac{1}{\epsilon-\alpha})\Big) = \\ \Big((s_2+\frac{1}{\epsilon-\alpha-1})(s_1+\frac{1}{\epsilon-\alpha})\Big)\ \Big((s_3+\frac{1}{\epsilon-1})(s_2+\frac{1}{\epsilon})(s_1+\frac{1}{\alpha})\Big)\end{aligned} \tag{53}$$

by

$$\begin{aligned}\Big((s_3+\frac{1}{\alpha})(s_2 s_1 - s_1 - 1)\Big)\ \Big((s_3+\frac{1}{-\alpha-1})(s_2+\frac{1}{-\alpha})\Big) = \\ \Big((s_2+\frac{1}{-\alpha-1})(s_1+\frac{1}{-\alpha})\Big)\ \Big((s_3 s_2 - s_2 - 1)(s_1+\frac{1}{\alpha})\Big)\end{aligned} \tag{54}$$

Yang-Baxter relations will be replaced by (52), (54) whenever special factors are involved. We still can write enough Yang-Baxter type products to produce elements which can be characterized by some of their right and left possible factors (in the group algebra)[16].

We are now ready to define an element depending on parameters $x_1,\ldots,x_n$ which can be specialized to the contents of any standard tableau with $n$ boxes.

[16]Recall for example that the Yang-Baxter element $Y_\omega(x_1,\ldots,x_n)$ is, up to a scalar, the only element in $\mathcal{H}_n$ which admits any $s_i + 1/(x_{i+1}-x_i)$ as a right or left factor.

**Definition 36** *Given a partition $J$ of $n$, let $w = (\sigma_{n-1}\cdots\sigma_1)(\sigma_{n-1}\cdots\sigma_2)\cdots(\sigma_{n-1})$ be the maximal*[17] *reduced decomposition of $\omega_n$. Then the* Young-Yang element of index *$J$, $\mathbb{Y}_J$, is obtained as follows.*

*Write the Yang-Baxter product corresponding to $w$, specialize the parameters to the contents of the diagram of $J$, but replace each double factor of the type $(s-1)(s'-1/0)$, with $s$ and $s'$ consecutive simple transpositions, by a factor $(ss'-s-1)$.*

This definition makes sense because for the choosen reduced decomposition, a factor $ss'-1/0$ has always as a left neighbour a factor $s-1$. One can in fact characterize all the reduced decompositions of $\omega$ having this property, and consequently one is able to use all of them instead of taking only a canonical reduced decomposition.

However, we need only have enough decompositions to be able to characterize the Young-Yang element by its possible left and right factors $s_i + 1/\alpha$ of degree 1, and we shall not consider all possible factorizations of the Young-Yang element.

**Lemma 37** *Let $\mathbb{Y}_J$ be a Young-Yang element. Let $[a+1,\ldots,a+k]$ be any row of the bottom tableau $t_J$ of shape $J$. Then $(s_{a+1}+1),\ldots,(s_{a+k}+1)$ are left factors of $\mathbb{Y}_J$ in the group algebra. Moreover, if the row is not the top one, then*

$$(s_{a+k}-\frac{1}{k})(s_{a+k-1}-\frac{1}{k-1})\cdots(s_{a+1}-1)$$

*is a left factor of $\mathbb{Y}_J$.*

*Proof.* Suppose by induction that the lemma is true for the partition $I$ obtained from $J$ by decreasing by 1 its smallest part. Let $\mathbb{Y}_{I^\dagger}$ the image of $\mathbb{Y}_I$ under the embedding $\mathfrak{S}_{n-1}\mapsto\mathfrak{S}_1\times\mathfrak{S}_{n-1}$, $n=|J|$. Then

$$\mathbb{Y}_J = (s_{n-1}+\star)\cdots(s_1+\star)\,\mathbb{Y}_{I^\dagger}\ ,$$

where there are in the extra left factor, as many factors of the type $(ss'-s-1)$ as there are entries other than $n$ in the diagonal occupied by $n$ in $t_J$. More precisely, apart from the case of the top row, the subfactor involving the indices $a+1,\ldots,a+k$ is of the type

$$(s_{a+k}+\frac{1}{b-k})\cdots(s_{a+b+2}+\frac{1}{-2})\,(s_{a+b+1}s_{a+b}-s_{a+b}-1)\,(s_{a+b-1}+\frac{1}{1})\cdots(s_{a+1}+\frac{1}{b-1})\,.$$

We know by induction on $n$ that $(s_{a+2}+1),\ \ldots\ ,(s_{a+k}+1)$ are left factors of $\mathbb{Y}_{I^\dagger}$ , and so is $(s_{a+2}-\frac{1}{k})(s_{a+3}-\frac{1}{k-1})\cdots(s_{a+k+1}-\frac{1}{1})$.

[17] with respect to the lexicographic order

But the extended Yang-Baxter equations allow us to commute these factors with $(s_{a+k}+1/(b\text{–}k))\cdots(s_{a+1}+1/(b\text{–}1))$, producing just a decrease by one of the indices of the simple transpositions $s$. QED

For example, let $J=[5,6,10]$. The tableau $t_J$ and its contents are

$$\begin{array}{|c|c|c|c|c|c|}\hline 17 & \cdots & 21 \\ \cline{1-3} 11 & \cdots & 15 & 16 \\ \cline{1-4} 1 & \cdots & 5 & 6 & \cdots & 10 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline -2 & \cdots & 2 \\ \cline{1-3} -1 & \cdots & 3 & 4 \\ \cline{1-4} 0 & \cdots & 4 & 5 & \cdots & 9 \\ \hline \end{array}$$

The Young-Yang element factorizes into

$$(s_{20}+\frac{1}{2-1})(s_{19}+\frac{1}{2-0})(s_{18}+\frac{1}{2-(\text{–}1)})(s_{17}+\frac{1}{2-(\text{–}2)})$$
$$(s_{16}+\frac{1}{2-4})(s_{15}s_{14}-s_{14}-1)(s_{13}+\frac{1}{2-1})(s_{12}+\frac{1}{2-0})(s_{11}+\frac{1}{2-(\text{–}1)})$$
$$(s_{10}+\frac{1}{2-9})\cdots(s_5+\frac{1}{2-4})(s_4s_3-s_3-1)(s_2+\frac{1}{2-1})(s_1+\frac{1}{2-0})$$

times $Y_{4,6,10}$.

We can now present an alternative to the result of Jucys and Cherednik.

**Theorem 38** *Let $J$ be a partition, $t_J$ be the bottom tableau of shape $J$. Then the Young-Yang element $Y_J$ is equal to $P(t_J)\,N(t_J)\,P(t_J)\,\omega$, up to a non-zero scalar.*

*Proof.* We shall check that $Y_J$ satisfies the left and right vanishing properties which characterize $P(t_J)\,N(t_J)\,P(t_J)$.

Let $i$ be an integer such that $\sigma_i\in\mathfrak{S}_J$. Then $s_i+1$ is a left factor of $\mathbb{Y}_J$ and therefore $(s_i-1)\,\mathbb{Y}_J$ is null for all those $i$.

Take now a row of $t_J$: according to Lemma 37, one can factorize on the left $y:=(s_{a+k}-\frac{1}{k})(s_{a+k-1}-\frac{1}{k-1})\cdots(s_{a+1}-1)$. But the product $(s_{a+1}+1)(s_{a+2}+\frac{1}{2}\cdots(s_{a+k}+\frac{1}{k})\,y$ reduces to $(s_{a+1}+1)(1-\frac{1}{k^2})\cdots(1-\frac{1}{2^2})(s_{a+1}-1)$, and thus is is null : the two modules $\mathcal{H}\,\mathbb{Y}_J$ and $\mathcal{H}\,P(t)\,N(t)$ are isomorphic.

The same reasoning applies to the factors that one can extract from the right of $\mathbb{Y}_J$, up to the reversal of the parameters $c(1,t_J),\ldots c(n,t_J)$. These right and left vanishing properties show that $\mathbb{Y}_J$ is equal, up to a factor, to $P(t_J)\,N(t_J)\,P(t_J)\,\omega$. The leading term of $\mathbb{Y}_J$ is $\omega$, and thus the coefficient of $\omega\omega=1$ in $P(t)N(t)P(t)$ determines the factor of proportionality. QED

We have already seen that $P(t_J)N(t_J)P(t_J)$ is equal to the product of $P(t_J)N(t_J)$ by invertible factors $s+1/c$, $c\notin\{0,\pm1\}$. Thus from $\mathbb{Y}_J$, one can get $P(t_J)N(t_J)$. This corresponds to starting with a reduced decomposition of another permutation than the maximal one, or extracting factors from the right of $\mathbb{Y}_J$. One could as easily produce the idempotents corresponding to other tableaux than the bottom one $t_J$. All these expressions are directly obtained from $\mathbb{Y}_J$ and we shall not bother the reader with unnecessary details.

For example, when $J = [2, 2]$, the content vector is $[0, 1, -1, 0]$, specialization of $[0, 1, -1, \epsilon]$. The Young-Yang element is

$$(s_3+1)\Big(s_2s_1 - s_1 - s_1\Big)(s_3 - \frac{1}{2})(s_2-1)(s_3+1)$$

which is indeed the expression obtained from the maximal Yang-Baxter element

$$(s_3 + \frac{1}{1+\epsilon})\Big((s_2 + \frac{1}{\epsilon-1})(s_1 + \frac{1}{\epsilon})\Big)(s_3 - 1/2)(s_2-1)(s_3+1)\ .$$

Using the comutations ? , it can also be written

$$(s_3+1)(s_1+1)\Big(s_2s_1 - s_2 - s_1\Big)(s_3 - \frac{1}{2})(s_2-1)(s_3+1)\ ,$$

expression which shows that $P(t_{22})$ can be factorized on its left (we have to multiply it by the permutation $\omega = [4, 3, 2, 1]$ to be able to factorize $P(t_{22})$ on the right).

On the other hand, the element $PNP$, for the tableau $\begin{smallmatrix}3&4\\1&2\end{smallmatrix}$, is equal to

$$\square_{2143}\, s_2 \nabla_{2143}\, s_2 \square_{2143} = 4 + \cdots$$

and therefore

$$4\, \mathbb{Y}_{22}\, \omega = P(t_{22}) N(t_{22}) P(t_{22})\ .$$

Gelfand-Zetlin bases, and Jucys-Murphy construction

The requirement to build recursively objects for successive symmetric groups $\mathfrak{S}_n$ compatible with the embeddings $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$ is a very strong condition, and determines these objects as soon as one fixes them for the case of $\mathfrak{S}_2$.

Let us first look at orthogonal idempotents.

Taking all standard tableaux $t$, Young associated to them quasi-idempotents $P(t)N(t)$. These quasi-idempotents are not mutually orthogonal; for a pair of tableaux $t, t'$ of the same shape, one can have non-zero products $P(t)N(t)P(t')N(t')$.

Young described complicated branching rules for these quasi-idempotents [18] We describe this construction of Young in the appendix.

Moreover, Young gave an orthogonalisation process, but unfortunately his description is complicated, not really fully explicited and not canonical because he leaves the choice of different orderings of tableaux.

Thrall [38] gave a simpler construction which is clearly compatible with the branching process.

For a standard tableau $t$ of $n$ letters, let $t\backslash n$ denote the tableau obtained by erasing the letter $n$.

Following Thrall, define elements $f_t$, where $t$ runs over all standard tableaux, recursively by :

$$c(t)\, f_{t\backslash n}(t)\, P(t)N(t)\, f_{t\backslash n}\ , \tag{55}$$

putting $e_1 = 1$ for the tableau with 1 box, $c(t)$ being a certain specific non-zero constant that we shall make precise later.

For any pair of tableaux $t, t'$ of the same shape, let now

$$f_{tt'} = c(t,t')\, f_t\, \zeta(t,t')\, f_{t'} \quad \& \quad f_{tt} = f_t\,, \tag{56}$$

where $\zeta(t,t')$ is the permutation $(t)^{-1}t'$ (considering standard tableaux as permutations), and $c(t,t')$ some specific constant.

**Theorem 39 (Thrall)** *The products $f(t)f(t')$ are null if $t \neq t'$.*

*For tableaux of the same shape $u, t, t', u'$, then the products $e_{ut}\, e_{t'u'}$ are null if $t \neq t'$.*

*There exist of choice of the constants $\alpha(t)$, $\alpha(t,t')$ such that $f(t)^2 = f(t)$, and $f_{ut}\, f_{tu'} = f_{uu'}$, i.e. such that the $f_{tt'}$ are matrix units.*

[18] Of course, at the level of characters only, the rule is very simple, reducing to adding a new box to a diagram, in all possible positions. But this does not give the relations between the elements $P(t)N(t)$ and $P(t')N(t')$ when $t'$ is obtained by adding a letter to $t$.

*Proof.* If $t$ and $t'$ have not the same shape, then we have seen that $P(t)N(t)\,\mathcal{H}\,P(t')N(t') = 0$, and therefore $f_t f_{t'} = 0$. Supposing now $t$ and $t'$ to have the same shape, but be different, then $t\backslash n$ is not equal to $t'\backslash n$, and by induction on $n$ one has $f_{t\backslash n} f_{t'\backslash n} = 0$. This implies in turn that $f_t f_{t'} = 0$ and $f_{ut}\, f_{t'u'} = 0$.

To check that the elements $f(t)$ are non zero, and that there exist constants such that they are idempotents, we shall show that they are non-zero multiples of the elements $e_{tt}$ seen in the preceding section.

Let $\theta_n = \sum_{1\le i<j\le n}$. We have seen Young quasi-idempotents $P(t)N(t)$ and $N(t)P(t)$, for any tableau $t$, are two-sided eigenvectors for this central element :

$$\theta_n\, P(t)N(t) = P(t)N(t)\,\theta_n = (\sum_i c(i,t))\, P(t)N(t)\ . \tag{57}$$

Notice that the sum of all contents $\sum_i c(i,t)$ does not depend on $t$, but only on the shape $\lambda$ of $t$. Let us denote it $c(\lambda)$.

We shall now exploit one of the main feature of Gelfand-Zetlin bases. In the present case, it will be that the property satisfied by $(\theta_n,\, t)$, for $t$ with $n$ boxes, will be valid for any pair $(\theta_n,\, t)$.

**Proposition 40** *The $f_{tt'}$ are two-sided eigenvectors for the Jucys elements. One has, for all $i = 1, \ldots, n$,*

$$\xi_i\, f_{tt'} = c(i,t)\, f_{tt'}\ \ ,\ \ f_{tt'}\,\xi_i = f_{tt'}\, c(i,t')\ . \tag{58}$$

*Proof.* Supposing the theorem to be true for $n-1$, then we know that $f_{tt'}$ is a two sided eigenvector for $\xi_1, \ldots, \xi_{n-1}$, with eigenvalues the contents of $1, \ldots, n$-1. But we know from (57) that $f_{tt'}$ is also a two sided eigenvector for $\theta_n$ with eigenvalue $c(\lambda)$, and it implies that it is a two-sided eigenvector for $\xi_n$ with the contents of $n$ in $t$ and $t'$ as eigenvalues on the left and on the right respectively. QED

**Corollary 41** *Thrall elements $f_{tt'}$ are non zero-multiple of the units $e_{tt'}$ defined in ?*

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### Jucys-Murphy construction of idempotents

The characterization of the elements $e_{tt}$ as two-sided eigenvectors for all the $\xi_i$'s naturally lead to another recursive definition of them, due to Jucys, then Murphy.

Given a standard tableau $u$ with $n-1$ boxes, let $\mathfrak{Sons}(u) := \{v : v\backslash n = u\}$ be the set of all standard tableaux obtained by adding a letter $n$ to $u$. Given some $t \in \mathfrak{Sons}(u)$, we call the other $v \in \mathfrak{Sons}(u)$ the *brothers* of $t$ and denote their set $\mathfrak{Broth}(t)$.

Define recursively elements $g_t$ indexed by standard tableaux by :

$$g_t := g_{t\backslash n} \prod_{v \in \mathfrak{Broth}(t)} \frac{\xi_n - c(n,v)}{c(n,t) - c(n,v)} \ . \tag{59}$$

**Theorem 42** *The $g_t$ are two-sided eigenvectors for the Jucys-Murphy elements and coincide with Young's orthogonal idempotents $e_{tt}$.*

*Proof.* Let $\phi(x_1,\ldots,x_n)$ be any polynomial in $n$ variables. Then, because the $e_{uv}$ are eigenvectors of the Jucys elements, one has for every pair of tableaux $(u,v)$ with at least $n$ boxes :

$$\begin{aligned} g_t(\xi_1,\ldots,\xi_n)\, e_{uv} &= g_t(c(1,u),\ldots,c(n,u))\, e_{uv} \ , \\ e_{uv}\, g_t(\xi_1,\ldots,\xi_n) &= e_{uv}\, g_t(c(1,v),\ldots,c(n,v)) \ , \end{aligned}$$

i.e. the product of $e_{uv}$ by $g_t$ is obtained by replacing each $\xi_i$ by the content of $i$ in $u$ or $v$.

But, by construction $g_t(c(1,u),\ldots,c(n,u))$ vanish if the restriction of $u$ to $n$ boxes does not coincide with $t$. Since $\{e_{u,v}\}$, where $(u,v)$ runs over all pairs of standard tableaux of the same shape with $n$ boxes is a linear basis of $\mathcal{H}_n$, the $g_t$ coincide with the $e_{tt}$ up to a scalar factor. However, $e_{tt}$ is invariant by multiplication by any factor $(\xi_i - y)/(c(i,t) - y)$, for any $y \neq c(i,t)$. Therefore

$$e_{tt} g_t = g_t e_{tt} = e_{tt}$$

and $g_t$ coincide with $e_{tt}$. QED

Both Jucys and Murphy kept in the expression of the $g_t$ in terms of Jucys elements unnecessary factors, defining elements $f'_t$ by the following recursion

$$g'_t := f'_{t\backslash n} \frac{\xi_n - n}{c(n,t) - n} \cdots \frac{\widehat{\xi_n - c(n,t}}{0} \cdots \frac{\xi_n + n}{c(n,t) + n} \ , \tag{60}$$

with the factor having 0 in denominator omitted.

Of course, because $g_t$ is an eigenvector, multiplying it by extra factors $(\xi_i - y)/(c(i,t) - y)$ does not change its value, as long as $y \neq c(i,t)$ and therefore $g'_t = g_t$.

Jucys and Murphy's proof that the $g'_t$ are simultaneous eigenvectors, without assuming previous knowledge of the $e_{tt}$, mostly reduces to checking the following lemma :

**Lemma 43** *Let $P_n(x) := (x-n)\cdots x \cdots (x+n)$. Then for any $i \le n$,*

$$P_n(\xi_i) = 0 \ .$$

*Proof.* By induction on $n$, one has $P_n(\xi_i) = 0$, $i < n$. The central element $P_n(\xi_1) + \cdots + P_n(\xi_n)$ is thus equal to $P_n(\xi_n)$, and it is characterized by the value of its restriction to a copy of each irreducible representation. But we have already seen that Young's relations give in each representation elements which are eigenvectors of all $\xi_i$'s, with which one can check the extra nullity of $P_n(\xi_n)$. QED

Jucys and Murphy expression of idempotents clearly encodes the fact that they are eigenvectors, with eigenvalues the contents. It also shows that the $e_{tt'}$ are a Gelfand-Zetlin basis. However, it has disadvantages. Developing expressions (59) or (60) in $\mathcal{H}_n$ is costly, because of the number of factors, and because each Jucys element is a sum of transposition.

On the other hand, with Yang-Baxter elements, we need only compute, for each shape $\lambda$, the element $P(t_\lambda)N(t_\lambda)P(t_\lambda)$ (which is obtained by enumeration of double cosets, and not by products in $\mathcal{H}_n$), and then we obtain the other idempotents by multiplication by factors of the type $(s_i + 1/x)$ according to the Yang-Baxter graph $\Gamma_\lambda$.

# Notes

## Spaces of Tableaux and Garnir relations

Many spaces that we obtained can be identified with linear spans of tabloids of a given shape, the symmetric group acting by permutation of the entries of the tabloids.

Tabloids are not linearly independent, and we want the relations that they satisfy to be such that tableaux be a linear basis. Young gave such relations, using different approaches depending on the type of representations he was describing.

The simplest relations have been described by Garnir [14], but in fact, are special cases of classical relations between minors, and also of generalizations of the Lagrange interpolation formula.

Suppose for the moment that tabloids satisfy the following relations :

1. They are invariant under permutations preserving rows.

2. Given two consecutive rows of lengths $p, q$ $(p \leq q)$ of a tabloid $t$, let $A$ be the entries of the first row, $B \cup C$ the entries of the second row, with $card(A) + card(C) = q + 1$. Then

$$\sum_{\sigma \in \mathfrak{S}(A \cup C)} \sigma\,(t) = 0 \ . \tag{61}$$

Because tabloids are invariant under permutations in rows, instead of summing on the group $\mathfrak{S}(A \cup C)$, one needs only enumerate complementary subsets (written increasingly) $A', C'$, replacing $A$ by $A'$ and $C$ by $C'$ in $t$. For example, for $p = 2, q = 3$, $A = \{a_1, a_2\}$, $A = \{c_1, c_2\}$, the sum on the symmetric group $\mathfrak{S}(a_1, a_2, c_1, c_2)$ is equal to $2! \times 2!$ the sum on pairs of complementary subsets of cardinality 2.

$$\begin{smallmatrix} a_1 & a_2 & \\ b & c_1 & c_2 \end{smallmatrix} + \begin{smallmatrix} a_1 & c_1 & \\ b & a_2 & c_2 \end{smallmatrix} + \begin{smallmatrix} a_1 & c_2 & \\ b & a_2 1 & c_1 \end{smallmatrix} + \begin{smallmatrix} a_2 & c_1 & \\ b & a_1 & c_2 \end{smallmatrix} + \begin{smallmatrix} a_2 & c_2 & \\ b & a_1 & c_1 \end{smallmatrix} + \begin{smallmatrix} c_1 & c_2 & \\ b & a_1 & a_2 \end{smallmatrix} = 0 \ .$$

Let $\lambda$ be a partition, $\mathfrak{Tbl}(\lambda)$ the set of tabloids of shape $\lambda$. Then the following proposition shows that Garnir relations are sufficient to express any tabloid in terms of tableaux.

**Proposition 44** *Let $n$ be an integer. Let $V$ be an $\mathfrak{S}_n$-module, linearly spanned by tabloids invariant under permutations in rows, and satisfying relations (61). Then any tabloid is a linear combination of tableaux of the same shape.*

*Proof.* Define a violation ............ Taking an extremal violation (with respect to the order on the plane, that is there is no other violation in its North-East corner), then using Garnir relation ... QED

Notice that we have not forbidden tableaux to be linearly dependent. Usually, one first checks directly their independence. The usual strategy for the second step is to prove that there is no non-zero morphism between spaces of tabloids of different shapes, and to use some general results on representations of finite groups on $\mathbb{C}$, to conclude that the above spaces are irreducible representations, and that every irreducible representation of the symmetric group is isomorphic to one of such spaces. However, we prefered to explain how Young constructed explicit representations without any knowledge of group theory.

Let us review the different interpretations of Garnir's relations, or of similar relations, that we have encountered.

Let us begin with the group algebra. We take an increasing partition $J$, and $I$ a composition (weakly) conjugate to $J$. Filling the diagram of $J$ with consecutive numbers, from top to bottom, we isolate a pair of consecutive rows of lengths $p, q$, $p \le q$,

$$[\ldots, \alpha, \ldots, \beta]\ ,\ [\beta{+}1, \ldots, \gamma, \ldots]$$

taking $\alpha, \gamma$ in such a way that $\gamma - \alpha \ge q$. Then, one has (**??**)

$$\nabla_{\omega_I}\, \mathcal{H}\, \square_{\omega'}\, \square_{\omega_J} = 0\ , \tag{62}$$

where $\omega' = [1, \ldots, \alpha{-}1,\ \gamma, \ldots, \alpha,\ \gamma{+}1, \ldots]$ (and therefore, $\square_{\omega'} = \sum_{\sigma \in \mathfrak{S}(\alpha,\ldots,\gamma)}$). Because one can factor $\sum_{\eta \in \mathfrak{S}(\alpha,\ldots,\beta)} \eta \times \sum_{\nu \in \mathfrak{S}(\beta+1,\ldots,\gamma)} \nu$ on the left of $\square_{\omega_J}$, the preceding relation can be reduced to

$$\nabla_{\omega_I}\, \mathcal{H} \sum_{\sigma} \sigma\, \square_{\omega_J} = 0\ , \tag{63}$$

where $\sigma$ runs over cosets representatives of $\mathfrak{S}(\alpha, \ldots, \gamma)/\mathfrak{S}(\alpha, \ldots, \beta) \times \mathfrak{S}(\beta{+}1, \ldots, \gamma)$.

For example, writing the letters which commute freely between themselves in boldface, one has the following relation ($\alpha = 5, \beta = 7, \gamma = 12$) :

$$0 = \sum \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} \\ \hline \mathbf{8} & \mathbf{9} & \mathbf{10} & \mathbf{11} & \mathbf{12} & 13 & 14 \\ \hline 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ \hline \end{array} \quad \leftrightarrow \quad \nabla_{\omega_{44432221}}\, \mathcal{H}\, \square_{1234\ 12\ldots5\ 15\ldots22}\, \square_{\omega_{3478}} = 0\ .$$

Multiplying from the right the left hand side by any permutation $\nu$, one gets a relation involving permutations of letters in consecutive rows of an arbitrary tabloid of shape $J$.

We also described representations in the quotient ring $\mathbb{C}[x_1, \ldots, x_n]/\mathfrak{Sym}_+$, and needed Garnir relations in this set-up. To a partition $J$ we associated a monomial $x^u$, with $u = [\cdots (\lambda_1 + \lambda_2)^{\lambda_3},\ \lambda_1^{\lambda_2},\ 0^{\lambda_1}]$, $\lambda =$ decreasing reordering of $J$, and we obtained

$$x^u \,\square_{\omega'} \equiv 0 \ , \tag{64}$$

taking the same definition of $\omega'$ as in (62). Again, one can reduce the summation to cosets representatives of $\mathfrak{S}(\alpha, \ldots, \beta) \times \mathfrak{S}(\beta{+}1, \ldots, \gamma) \backslash \mathfrak{S}(\beta{+}1, \ldots, \gamma)$, in which case the action of $\square_{\omega'}$ can be interpreted as a summation on complementary subsets of fixed cardinality.

There are other possible equivalent families of relations than (64). One has for example the following lemma.

**Lemma 45** *Let $\lambda$ be a decreasing partition of $n$, $\mu$ be the vector $\mu = [, 0, \lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3, \ldots]$, $u = [\mu_1^{\lambda_1}, \mu_2^{\lambda_2}, \mu_3^{\lambda_3}, \ldots]$. For any integer $j < \ell(\lambda)$, any $k : 1 \leq k \leq \lambda_{j+1}$, let $v$ be the vector obtained from $u$ by changing the block $\mu_{j+1}^{\lambda_{j+1}}$ into $\mu_j^k \, \mu_{j+1}^{\lambda_{j+1}-k}$. Then*

$$x^u \equiv x^v \, \psi_{\lambda_j^k}(x_{\mu_j+1}, \ldots, x_{\mu_j+\lambda_j}) \qquad \mathrm{mod}\ \mathfrak{Sym}(x_1, \ldots, x_n) \ . \tag{65}$$

For example, for $\lambda = [5, 4, 3, 3]$, and $j = 2$, $k = 2$, one has $\mu = [0, 5, 9, 12]$,
$u = [0, 0, 0, 0, 0,\ 5, 5, 5, 5,\ 9, 9, 9,\ 12, 12, 12]$,
$v = [0, 0, 0, 0, 0,\ 5, 5, 5, 5,\ \mathbf{5}, \mathbf{5}, 9,\ 12, 12, 12]$. The relation is

$$x^u \equiv x^v \, \Psi_{44}(x_6, x_7, x_8, x_9) \qquad \mathrm{mod}\ \mathfrak{Sym}(x_1, \ldots, x_{15}) \ .$$

In other words, one has increased by 4 in all possible manners the exponents in $x^v$ of two of the indeterminates $x_6, x_7, x_8, x_9$.

Identify now a column $A = [a_1, \ldots, a_k]$ with the Vandermonde $\Delta(A) = \prod_{i<j}(x_{a_i} - x_{a_j})$, and a tabloid with the product of its columns. For the two-columns tableau $t = \begin{array}{cc} \beta & \\ \vdots & \delta \\ \vdots & \vdots \\ \alpha & \gamma \\ \vdots & \vdots \\ 1 & \beta{+}1 \end{array}$ the original Garnir relations, relative to Specht representations, are

$$\sum\nolimits_{\sigma} (-1)^{\ell(\sigma)} \sigma\Big(\Delta(\beta, \ldots, 1)\, \Delta(\delta, \ldots, \beta{+}1)\Big) = 0 \ . \tag{66}$$

As before, the summation can be reduced to cosets representatives, and one still has nullity when taking two arbitrary columns (consecutive or not, but ordered by length) in a tabloid. The other columns introduce a constant factor, and by conjugation, one passes from $1, \dots, \delta$ to any set of $\delta$ integers.

For example, taking columns 1 and 3, and summing on subsets of $\{2, 3, 7, 8\}$, one has the following nullity :

$$\begin{smallmatrix} \mathbf{3} & 6 & 9 & \\ \mathbf{2} & 5 & \mathbf{8} & 11 \\ 1 & 4 & \mathbf{7} & 10 \end{smallmatrix} - \begin{smallmatrix} \mathbf{7} & 6 & 9 & \\ \mathbf{2} & 5 & \mathbf{8} & 11 \\ 1 & 4 & \mathbf{3} & 10 \end{smallmatrix} + \begin{smallmatrix} \mathbf{8} & 6 & 9 & \\ \mathbf{2} & 5 & \mathbf{7} & 11 \\ 1 & 4 & \mathbf{3} & 10 \end{smallmatrix} + \begin{smallmatrix} \mathbf{7} & 6 & 9 & \\ \mathbf{3} & 5 & \mathbf{8} & 11 \\ 1 & 4 & \mathbf{2} & 10 \end{smallmatrix} - \begin{smallmatrix} \mathbf{8} & 6 & 9 & \\ \mathbf{3} & 5 & \mathbf{7} & 11 \\ 1 & 4 & \mathbf{2} & 10 \end{smallmatrix} + \begin{smallmatrix} \mathbf{8} & 6 & 9 & \\ \mathbf{7} & 5 & \mathbf{3} & 11 \\ 1 & 4 & \mathbf{2} & 10 \end{smallmatrix} = 0 \, .$$

The above relations are implied by the following generalization of Lagrange interpolation, given by Sylvester (?). For two alphabets $\mathbb{A}$, $\mathbb{B}$, write $\mathcal{R}(\mathbb{A}, \mathbb{B})$ for the resultant $\mathcal{R}(\mathbb{A}, \mathbb{B}) := \prod_{a \in \mathbb{A}, b \in \mathbb{B}} (a - b)$.

**Lemma 46** *Let $n, n', n''$ be three integers, $n = n' + n''$, and $\mathbb{X}$ be an alphabet of cardinality $n$. Let $\mathfrak{Sym}(n', n'')$ be the space of polynomials symmetrical in the first $n'$ indeterminates, and also symmetrical in the last $n''$ ones. Then the following morphism[19] from $\mathfrak{Sym}(n', n'')$ to $\mathfrak{Sym}(n)$*

$$f \to \sum\nolimits_{\mathbb{X}' \cup \mathbb{X}'' = \mathbb{X}} f(\mathbb{X}', \mathbb{X}'') / \mathcal{R}(\mathbb{X}', \mathbb{X}'') \tag{67}$$

*sends polynomials of degree $< n'n''$ to $0$, and sends $(x_1 \cdots x_{n'})^{n''}$ to $1$.*

A more general property is the following "exchange lemma" for resultants, that is useful in rational interpolation [**?**].

**Lemma 47** *Let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be three alphabets of respective cardinals $n, n', n''$, $n = n' + n''$. Then*

$$\sum_{\mathbb{A} = \mathbb{A}' \cup \mathbb{A}''} \frac{\mathcal{R}(\mathbb{A}', \mathbb{B})}{\mathcal{R}(\mathbb{A}', \mathbb{C}) \mathcal{R}(\mathbb{A}', \mathbb{A}'')} = \frac{R(\mathbb{B}, \mathbb{C})}{R(\mathbb{A}, \mathbb{C})} \tag{68}$$

$$\sum_{\mathbb{A} = \mathbb{A}' \cup \mathbb{A}''} \frac{\mathcal{R}(\mathbb{A}', \mathbb{B}) \mathcal{R}(\mathbb{A}'')}{\mathcal{R}(\mathbb{A}', \mathbb{A}'')} = \mathcal{R}(\mathbb{B}, \mathbb{C}) \, , \tag{69}$$

*sum over all disjoint decompositions $\mathbb{A} = \mathbb{A}' \cup \mathbb{A}''$ of $\mathbb{A}$, with card$(\mathbb{A}') = n'$.*

The Vandermonde determinants in (66) are minors of the same Vandermonde matrix $\left[ x_i^j \right]_{j \geq 0, 1 \leq i \leq n}$, and the relation they satisfy results in fact from

[19] This is the Gysin morphism, for the cohomology of a relative Grassmannian. It can in fact be rewritten as a summation on the full symmetric group $\mathfrak{S}_n$, in which case it becomes evident [**?**]

quadratic relations on minors of order $n$ of any $n\times\infty$ matrix[20] $M$, due, once more, to Sylvester.

Given a set of integers $A$ of cardinality $n$, write $[A]$ for the minor of $M$ taken on columns specified by $A$. Then one has the following relation, generalizing Plücker relations.

**Lemma 48** *Let $M$ be an $n\times\infty$ matrix, $\alpha,\beta$ be integers such that $\alpha+\beta=n$. Then*

$$[1,\ldots,\alpha\text{-}1,\alpha,\ldots,n]\,[n\text{+}1,\ldots,\beta,\ \beta\text{+}1,\ldots,2n]\sum\nolimits_{\sigma\in\mathfrak{S}(\alpha,\ldots,\beta)}(-1)^{\ell(\sigma)}\sigma=0\ , \tag{70}$$

*where the permutations act on the indices of the pair of minors.*

As always, one can restrict to a summation on cosets representatives, and take sets of columns which are not $1,2,3,\ldots$. In other words, let $A,B,C$ be three sets of integers, with $card(B)\geq n+1$. Then, one has the following quadratic relation :

$$\sum_{B'\cup B''=\mathbb{B}}\pm[A,B']\,[B'',C]=0\ , \tag{71}$$

sum over all direct decompositions $B=B'\cup B''$, with $card(B')=n-card(A)$, $card(B'')=n-card(C)$, the sign being the sign of the permutation $(B',B'')\to B$.

There is in fact a direct connection between spaces of minors and Young decompositions of spaces of polynomials according to their types of symmetry. It involves using *Schur functors*[21] and representations of the linear group instead of representations of the symmetric group. As a matter of fact, such a connection is provided by the *Schur-Weyl duality*, but Young was already using the action of his idempotents on the tensor space $\oplus_n V^{\otimes n}$.

Thus, let us take a vector space $V$ , interpreting integers as vectors in $V$, and the columns of the matrix $M$ as elements of the exterior power $\wedge^n(\mathbb{V})$.

Now, a product of two minors is an element of $\wedge^n(\mathbb{V})\otimes\wedge^n(\mathbb{V})$, but Garnir relations show that it belongs to the component $S_{2^n}(V)$, where $S_{2^n}$ is the Schur functor of index the partition $2^n$. We shall refer to the book of Fulton [13] for an elementary introduction to representations of the linear group.

[20] a matrix of order $n\times m$ can be considered as an infinite matrix by adjoining to it a $n\times\infty$ null matrix.

[21] I was using in my thesis Schur functions with modules as arguments, instead of sets of variables, but Verdier made me use the term *Schur functor*, which remained.

# References


[1] ACE, S. Veigneau. *an Algebraic Environment for the Computer algebra system MAPLE*, `http://phalanstere.univ-mlv.fr/~ace` (1998).

[2] C. Berge. *Principes de Combinatoire*, Dunod (); *Principles of Combinatorics*

[3] C. Carré, A. Lascoux, B. Leclerc. *Turbo straightening*, Int J. of Algebra and Comp. (1992) 275-290.

[4] J. Desarmenien. *An algorithm for the Rota straightening formula*, Discrete M. **30** (1980) 51–68.

[5] J. Desarmenien, J. King, G.C. Rota. *Invariant Theory, Young bitableaux and combinatorics*, Adv. in M. **27** (1978) 63–92.

[6] M. Demazure. *Une formule des caractères* , Bull. Sc. Math., **98** (1974) 163–172.

[7] G. Duchamp, D. Krob, A. Lascoux, B. Leclerc, T. Scharf, J.Y. Thibon. *Euler-Poincaré characteristics and polynomial representations of Iwahori-Hecke algebras*, Pub. RIMS (Kyoto), **31** (1995) 179–201.

[8]

[9] S. Fomin and C. Greene, *Noncommutative Schur functions and their applications*, Discrete Math.

[10] G.S. Frame, G. de B. Robinson, and R.M. Thrall, *The hook graphs of* $S_n$, Canadian J. Math. **6** (1954), 316-324.

[11] F.G. Frobenius, *Über die Charaktere der symmetrischen Gruppe*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin (1900), 516-534 (Reprinted in Gesammelte Abhandlungen, **3**, 148-166).

[12] W. Fulton and J. Harris, *Representation theory, A first course*, Graduate Texts in Mathematics **129**, Springer-Verlag, New York, 1991.

[13] W. Fulton, *Young Tableaux*, London M.S. Student Text 35, Cambridge University Text (1997).

[14] H. Garnir, *Théorie de la représentation des groupes symétriques*, Mémoires de la Soc. Royale des Sc. de Liège **10** (1950).

[15] R. Geck, S. Kim, *Bases for the Bruhat-Chevalley order on all finite Coxeter groups*, J. of Algebra **197** (1997) 278–310.

[16] G. James. *The representation theory of the symmetric group*, Springer L.N. **682**.

[17] G. James, A. Kerber. *The representation theory of the symmetric group*, Encyclopedia of Maths, vol **16** Addison-Wesley (1981).

[18] Jimbo, Miwa...

[19] A. Jucys, *Symmetric polynomials and the center of the symmetric group rings*, Rep. Math. Phys. **5** (1974) 107–112.

[20] C. Kassel, A. Lascoux, C. Reutenauer, *Factorizations in Schubert cells*, Adv. in Maths **150** (2000) 1–35.

[21] A. Lascoux, *Cyclic permutations on words, tableaux and harmonic polynomials*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), Manoj Prakashan, Madras, 1991, 323-347.

[22] A. Lascoux. *Potentiel Yin sur le groupe symétrique,* Séminaire Lotharingien de Combinatoire **38** (1996)

[23] *Young's natural idempotents as polynomials*, Annals of Combinatorics **1** (1997) 91–98.

[24] A. Lascoux, B. Leclerc and J.Y. Thibon. *Flag Varieties and the Yang-Baxter Equation* , Letters in Math. Phys., **40** (1997) 75–90.

[25] D.E. Littlewood. *The construction of invariant matrices*, Proc. London M.S. **43** (1937) 226–240.

[26] D.E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, Oxford University Press, New York 1940.

[27] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford University Press, Oxford, 2nd edition (1995).

[28] F.D. Murnaghan, *The characters of the symmetric group*, Amer. J. Math. **59** (1937), 739-753.

[29] G.E. Murphy, *On the representation theory of the symmetric groups and associated Hecke algebras*, J. Algebra **152** (1992), 492-513.

[30] T. Nakayama, *On some modular properties of irreducible representations of a symmetric group I and II*, Jap. J. Math. **17** (1940), 165-184, 411-423.

[31] A. Okounkov, A. Vershik, *A new approach to representation theory of symmetric groups*, Selecta Math., **2** (1996) 581–605.

[32] D.P. Robbins, H. Rumsey, *Determinants and Alternating sign matrices*, Adv. in Maths (1986) 169–184.

[33] H.A.Rothe, *Ueber permutationen, in Beziehung auf die Stellen ihrer Elemente*, Sammlung combinatorisch -analytischer Abhandlungen, Leipzig, (cited by Muir, History of determinants, vol1) (1800).

[34] G-C. Rota. *Combinatorial Theory and Invariant Theory*, Notes by L.Guibas, Bowdoin College (1971).

[35] D.E. Rutherford, *Substitutional Analysis*, Edinburgh, at the University Press, 1948.

[36] I. Schur, *Über eine Klasse von Matrizen, die sich einer gegeben Matrix zuordnen lassen*, Dissertation, 1901; (Reprinted in Gesammelte Abhandlungen **1**, 1-72).

[37] W. Specht. *Die irreduziblen darstellungen der symmetrischen gruppe*, Math. Z. **39** (1935) 606–711.

[38] R.M. Thrall. *Young's semi normal representation of the symmetric group*, Duke M. **8** (1941) 611–624.

[39] A. Vershik. *Local algebras and a new version of Young's orthonormal form*, Topics in Algebra, Banach Center Pub., **26** (1990) 467–473.

[40] Weber, *Algebra*

[41] H. Weyl, *The classical groups, their invariants and their representations*, Princeton University Press, Princeton, 1946.

[42] C.N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. **19** (1967), 1312– 1315.

[43] A. Young. *The Collected Papers of Alfred Young*, University of Toronto Press (1977).

[Y1] A. Young, *On Quantitative Substitutional Analysis* (First paper) Proc. London Math. Soc. **33** (1900), 97-146.

[Y2] A. Young, *On Quantitative Substitutional Analysis* (Second paper) Proc. London Math. Soc. **34** (1902), 361-397.

[Y3] A. Young, *On Quantitative Substitutional Analysis* (Fifth paper) Proc. London Math. Soc. (2) **31** (1930), 273-288.

[Y4] A. Young, *On Quantitative Substitutional Analysis* (Sixth paper) Proc. London Math. Soc. (2) **34** (1931), 196-230.

[Y5] A. Young, *On Quantitative Substitutional Analysis* (Eighth paper) Proc. London Math. Soc. (2) **37** (1934), 441-495.