TRANSITION ON GROTHENDIECK POLYNOMIALS

A. LASCOUX

CNRS, Institut Gaspard Monge, Université de Marne-la-Vallée
77454 Marne-la-Vallée Cedex, France
E-mail:Alain.Lascoux@univ-mlv.fr

Grothendieck polynomials represent Schubert varieties in the Grothendieck ring of the flag manifold (for $Gl(n, \mathbb{C})$). We describe how general Grothendieck polynomials are related to those for Grasmann manifolds, which themselves are deformations of Schur functions.

1 Introduction

Given a vector bundle V of rank n, the Grothendieck ring of classes of vector bundles of the relative flag manifold $\mathcal{F}(V)$ is generated by the classes $a_1, \ldots, a_n$ of the so-called *tautological line bundles* on $\mathcal{F}(V)$.

The structure sheaf of a Schubert variety in $\mathcal{F}(V)$, having a finite resolution by vector bundles, can be expressed as a (Laurent) polynomial in the $a_i, 1/a_i$. Explicit representatives G_σ, $\sigma \in \mathfrak{S}_n$ were defined in [16,17] under the name "Grothendieck polynomials " (we recall their definition in Eq. 16). We shall also use *codes* of permutations rather than permutations, and in that case, note $G[J]$, $J \in \mathbb{N}^n$ (codes are defined in section 2).

I gave in [12,13] two different expressions of the $G[J]$, J partition, that I recall in the last section. As usual in the theory of symmetric functions, it is appropriate to let the number of indeterminates be infinite and to obtain the finite case by specializing $x_{n+1} = 0 = x_{n+2} = \cdots$. When using these conventions, we shall write partitions decreasingly and write g_λ for the symmetric function in an infinite number of variables obtained from the symmetric function $G[J]$ (in $x_1, \ldots, x_n$), with $J = [0 \le j_1 \le \cdots \le j_n]$, $\lambda = [j_n, \ldots, j_1]$.

In general, introducing some equivalence (taking the "stable part") that can be defined in four different manners which will be described in section 5, and that we shall note $\overset{stab}{\sim}$, one can decompose any Grothendieck polynomial into a sum of Grasmannian ones

$$G_\sigma \overset{stab}{\sim} \sum c_\lambda g_\lambda \tag{1}$$

The similar decomposition for cohomology classes (represented by Schubert polynomials) embeds into the tradition of "Schubert calculus" (for example, intersection multiplicities of Schubert varieties, generalizing the so-called *Littlewood-Richardson rule*).

Buch [1] studies the algebra generated by the g_λ, and the decomposition (1) that he explicits for certain permutations. One motivation to write this text was to answer his question about positivity of the coefficients in (1) (once taking the correct normalization), and I thank him for having sent me his preprint.

Adapting what has been done for Schubert polynomials X_σ, $\sigma \in \mathfrak{S}_n$, in [15] and [18], we shall give in Eq. (32) a formula which can be translated into a tree which describes the multiplicities in (1). Moreover, this tree, as in the case of cohomology, gives a way to explicit the multiplicative structure of the Grothendieck ring of the Grassmannian, which is the problem considered by Buch[1].

To generate the tree, one needs only to describe the multiplication of Grothendieck polynomials by $1-1/a_i$, $i = 1, \ldots, n$, i.e. to describe the class of hyperplane sections of Schubert varieties. The similar rule for the cohomology ring is due to Monk (it was previously obtained by Chevalley) and leads to the explicit expression of Schubert polynomials as (positive) polynomials in the Chern classes of the tautological line bundles, as well as their stable part.

The cohomological computations can easily be extended to two sets of indeterminates, due to the extension of Monk's rule (cf.[10]), or translated into planar combinatorial constructions. The same is true for Grothendieck polynomials , but for simplicity, we shall restrict here to only one set of indeterminates.

Grothendieck polynomials are obtained from the action of the 0-Hecke algebra on the ring of polynomials. Let us emphasize that this action is not unique and depends upon several arbitrary parameters that we shall have to put into use (the action of the Iwahori-Hecke algebra of the symmetric group also depends upon several parameters, cf.[19]).

There exists quantum analogues of Grothendieck polynomials and we refer to the work of Kirillov[8] for their definition and properties.

The geometry of Grassmann varieties is well understood thanks to Schubert, Giambelli and their successors[9]. It is hoped that relating the Schubert subvarieties of a flag manifold to those of a Grassmannian will be of some help to understand those varieties, the singularities of which we still do not know how to describe [2].

In the present article, we do not consider Schubert varieties, but only the classes of their structure sheaves. Much of the combinatorics of the symmetric group that appear in my work with M.P. Schützenberger finds application to Schubert calculus, and I mentionned in this text the connections with [15–22].

2 0-Hecke algebra

The 0-Hecke algebra $\mathcal{H}$ of the symmetric group $\mathfrak{S}_n$ is the algebra generated by $T_1, \ldots, T_{n-1}$ satisfying the braid relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \& \quad T_i T_j = T_j T_i \ , \ |i-j| \neq 1$$

together with $T_i^2 = T_i$. Given arbitrary parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, such that $\alpha\delta - \beta\gamma = 1$, one gets an action of $\mathcal{H}$ on the ring $\mathbb{C}[a_1^{\pm}, \ldots, a_n^{\pm}]$ by setting

$$T_i : f \mapsto f\, T_i := \frac{(\alpha a_i + \beta)(\gamma a_{i+1} + \delta) f - (\alpha a_{i+1} + \beta)(\gamma a_i + \delta) f^{s_i}}{a_i - a_{i+1}}$$

where s_i exchanges x_i, x_{i+1} (cf.[16],[19] for a more general statement; operators on polynomials will be denoted **on the right**).

Moreover, the elements $1 - T_i$ also satisfy the braid relations together with $(1-T_i)^2 = 1 - T_i$.

The operators

$$f \mapsto f \partial_i := \frac{f - f^{s_i}}{a_i - a_{i+1}}$$

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and the operators

$$f \mapsto f \widehat{\pi}_i := f \partial_i\, a_{i+1} = \frac{a_{i+1} f - a_{i+1} f^{s_i}}{a_i - a_{i+1}}$$

are due to Demazure [3] and called *isobaric divided differences* in [16].

Let us also use the variables $x_i := 1 - 1/a_i$, and the corresponding operators that we shall distinguish by superscripts (for example, ∂_1^x sends x_1 onto 1, and $a_1 = 1/(1-x_1)$ onto $a_1 a_2$).

Since braid relations are satisfied, to any permutation σ are associated operators $T_\sigma, \pi_\sigma^a, \pi_\sigma^x, \partial_\sigma^a, \partial_\sigma^x, \widehat{\pi}_\sigma^a, \widehat{\pi}_\sigma^x$ which can be obtained by taking any reduced decomposition of σ and evaluating the associated product of elementary operators. In particular, the operators associated to the *maximal permutation* $\omega \in \mathfrak{S}_n$, $\omega := [n, \ldots, 1]$ play a fundamental rôle.

We shall also need the isobaric divided differences corresponding to the shifted variables $a_1 - 1, a_2 - 1, \ldots, a_n - 1$ that we shall denote $D_1, \ldots, D_{n-1}$ to avoid any confusion with the preceding ones.

All these operators can be expressed in terms of Newton's divided differences only, but it greatly simplify computations, and comprehension, to introduce them instead of restricting to divided differences. Most of their properties directly arise from the following relations, which can be easily checked from their concrete action on polynomials or using the braid relations together with Leibniz formula: $f\, g\, \partial_i = f\,(g\partial_i) + (f\,\partial_i)\, g^{s_i}$.

For example, Eq. (2)–(7) are statements involving only two indices $\{i, i{+}1\}$. The operators commuting with multiplication by a function invariant under s_i, it is sufficient to check their action on the two polynomials 1 and x_i.

Lemma 1. *Divided differences satisfy the following relations :*

$$\pi_i^a = (1 - x_{i+1})\,\partial_i^x \;\; ; \;\; \widehat{\pi}_i^a = \partial_i^x\,(1 - x_i) \tag{2}$$

$$\partial_i^a = (1 - x_i)(1 - x_{i+1})\,\partial_i^x = \partial_i^x\,(1 - x_i)(1 - x_{i+1}) \tag{3}$$

$$\pi_i^x = (a_i - 1)a_{i+1}\,\partial_i^a \;\; ; \;\; \widehat{\pi}_i^x = \partial_i^a a_i(a_{i+1} - 1) \tag{4}$$

$$\partial_i^x = a_i a_{i+1}\,\partial_i^a = a_{i+1}\,\pi_i^a = \widehat{\pi}_i^a\, a_i = \partial_i^a a_i a_{i+1} \tag{5}$$

$$D_i = x_i(1 - x_{i+1})\,\partial_i^x = (a_i - 1)\,\partial_i^a \tag{6}$$

$$x_i\,\pi_i^a = \pi_i^a\, x_{i+1} + (1 - x_{i+1}) \;\; ; \;\; (x_i - 1)\,\pi_i^a = \widehat{\pi}_i^a\,(x_{i+1} - 1) \tag{7}$$

$$\pi_i^a\, D_{n-1}\cdots D_1 = D_{n-1}\cdots D_1\,\pi_{i+1}^a \;,\; i < n-1 \tag{8}$$

$$\widehat{\pi}_1\cdots\widehat{\pi}_{n-1}\,\pi_i = \pi_{i+1}\,\widehat{\pi}_1\cdots\widehat{\pi}_{n-1} \;,\; i < n-1 \tag{9}$$

$$D_\omega = (a_1 - 1)^{n-1}\cdots(a_n - 1)^0\,\partial_\omega^a = x_1^{n-1}\cdots x_n^0\,\pi_\omega^a \tag{10}$$

$$\pi_\omega^a = (1 - x_2)^1\cdots(1 - x_n)^{n-1}\,\partial_\omega^x \tag{11}$$

Only the last two relations require a comment. Indeed ∂_ω^a can be charaterized as the $\mathfrak{S}ym(a_1, \ldots, a_n)$-linear operator which annihilates all monomials $a^\lambda := a^{\lambda_1}\cdots a_n^{\lambda_n}$, $\lambda < \rho := [n{-}1, \ldots, 0]$ and sends a^ρ onto 1 (the preceding monomials are a basis of $\mathbb{C}[a_1^\pm, \ldots, a_n^\pm]$ as a free module over the ring $\mathfrak{S}ym$:= ring of symmetric functions in the a_i's or x_i's).

Using this fact, and the remark that all the operators that we have written are $\mathfrak{S}ym$-linear, one checks that

$$\partial_\omega^a = \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\ell(\sigma)}\,\sigma\,\frac{1}{\prod_{i<j}(a_i - a_j)} = \frac{1}{\prod_{i<j}(a_i - a_j)}\sum_{\sigma\in\mathfrak{S}_n}\sigma \tag{12}$$

$$\pi^a_\omega = a^\rho\,\partial_\omega = \frac{1}{\prod_{i<j}(1-a_j/a_i)}\sum_{\sigma\in\mathfrak{S}_n}\sigma \tag{13}$$

and, by change of variables $a_i \mapsto a_i - 1$,

$$D_\omega = \frac{1}{\prod_{i<j} 1-\frac{a_j-1}{a_i-1}}\sum_{\sigma\in\mathfrak{S}_n}\sigma = \prod_{i<j}\frac{a_i-1}{a_j-1}\sum_{\sigma\in\mathfrak{S}_n}\sigma = (a_1-1)^{n-1}\cdots(a_n-1)^0\,\partial^a_\omega \tag{14}$$

More generally, one has, for $\alpha\delta-\beta\gamma=1$,

$$T_\omega = \prod_{i<j}\frac{(\alpha a_i+\beta)(\gamma a_j+\delta)}{a_i-a_j}\sum_{\sigma\in\mathfrak{S}_n}\sigma = \prod_{i<j}(\alpha a_i+\beta)(\gamma a_j+\delta)\partial^a_\omega \tag{15}$$

In the above factorization, we consider T_ω as an operator on polynomials, not as an element of the Hecke algebra. Otherwise, we should have to work in the affine Hecke algebra.

The *Grothendieck polynomials* G_σ, $\sigma\in\mathfrak{S}_n$, are defined as follows:

$$\begin{cases} G_\omega = x_1^{n-1}\cdots x_n^0 = (1-\frac{1}{a_1})^{n-1}\cdots(1-\frac{1}{a_n})^0 \\ G_\sigma = \quad G_\omega\,\pi^a_{\omega\sigma}\end{cases} \tag{16}$$

Since Schubert polynomials are defined as the images under the ∂^x_σ of x^ρ, and since $\pi^a_i = (1-x_{i+1})\,\partial^x_i$, one sees that the term of smallest degree of G_σ is the Schubert polynomial (in the x_i's) of the same index.

Grothendieck polynomials , as well as Schubert polynomials, are a $\mathfrak{S}ym$-basis of $\mathbb{C}[a_1^\pm,\ldots,a_n^\pm] = \mathbb{C}[x_1^\pm,\ldots,x_n^\pm]$.

It is also convenient to index Grothendieck polynomials by codes of permutations, instead of permutations. The *code* of a permutation $\sigma = [\sigma^1,\ldots,\sigma^n]\in\mathfrak{S}_n$ is the vector $J=[j_1,\ldots,j_n]$ such that $j_i := \#\{k>i, \sigma^k<\sigma^i\}$. We shall write G_σ or $G[J]$ indifferently.

Grothendieck polynomials are stable with respect to the embeddings $\mathfrak{S}_n = \mathfrak{S}_n\times\mathfrak{S}_1 \hookrightarrow \mathfrak{S}_{n+1}$, and accordingly, one can freely add or suppress terminal 0's to codes without changing the polynomials : $G[J]=G[J0\ldots0]$.

Instead of starting from $G[\rho]=x^\rho$ for a given n, and letting n vary, one can as well rewrite the preceding definition as follows. Say that a vector $\lambda=[\lambda_1,\ldots,\lambda_k]$ is *dominant* if $\lambda_1\geq\cdots\geq\lambda_k\geq 0$. Then

$$\begin{cases} G[\lambda] & = & x^\lambda & \lambda \text{ dominant} \\ G[\ldots,j_k,j_{k+1},\ldots]\,\pi^a_k & = & G[\ldots,j_{k+1}-1,j_k,\ldots] & j_j > j_{k+1}\end{cases} \tag{17}$$

This convention allows to have all n's at the same time.

3 Symmetric Grothendieck Polynomials

On the Grothendieck ring of classes of vector bundles over a manifold act operators induced from the exterior powers and symmetric powers of vector bundles, that we shall denote Λ^i and S^i.

Explicitely, taking an extra variable z, the generating function of the S^i is such that

$$\sum_{i=0}^{\infty} z^i \, S^i(\mathbb{B} - \mathbb{B}') = \prod (1 - zb')/ \prod (1 - zb) \tag{18}$$

if $\mathbb{B} = \sum b$, $\mathbb{B}' = \sum b'$ are finite sums of classes of line bundles.

More generally, given any partition $K = [k_1, \dots, k_n]$, $0 \le k_1 \le \cdots \le k_n$, one has a *Schur functor* S_K, the action of which on sums $\mathbb{B} = \sum_{i=1}^n b_i$ of line bundles is

$$S_K(\mathbb{B}) = \det \left| S^{k_j+j-i}(\mathbb{B}) \right|_{1 \le i,j \le n} = \det \left| b_i^{k_j+n-j} \right|_{1 \le i,j \le n} \Big/ \det \left| b_i^{n-j} \right|_{1 \le i,j \le n} \tag{19}$$

Still a little more generally again [13], Schur functors indexed by partitions in $\mathbb{N}^n$ can be considered as morphisms from the n-th tensor power of a λ-ring into itself :

$$S_K(\mathbb{B}^1, \dots, \mathbb{B}^n) := \det \left| S^{k_j+j-i}(\mathbb{B}^j) \right|_{1 \le i,j \le n} \tag{20}$$

where $\mathbb{B}^1, \dots, \mathbb{B}^n$ are arbitrary elements of the λ-ring.

4 Stable part by restriction or symmetrization.

Let $k, n \in \mathbb{N}$, $k \ge n$, $I \in \mathbb{N}^k$. Then $G[0^{k-1}I]$ is symmetrical in $x_1, \dots, x_k$ and therefore the restriction

$$G[0^{k-1}I]\big|_{x_{k+1}=0=x_{k+2}=\cdots}$$

is a symmetric function $F[I]$ of k variables.

Similarly to the case of Schubert polynomials (cf.[15],[25]) we shall adopt the common terminology and call $F[I]$ the *stable part* of $G[I]$, writing also F_σ if σ is any permutation of code $[I0 \dots 0]$.

One needs to check some stability with respect to k, since one has only required $k \ge n$. This will be clarified by the following property.

Theorem 2. *1) Let $I \in \mathbb{N}^n$. Then*

$$G[I]D_n \cdots D_1 = G[0I] \tag{21}$$

2) Let ω be the maximal permutation of $\mathfrak{S}_n$. Then

$$G[I]D_\omega = F[I] \in \mathfrak{S}ym(x_1, \ldots, x_n) \tag{22}$$

Proof. $G[I]$ is the image of some $G[J]$, $J \in \mathbb{N}^n$ dominant, under some product of π_i^a, $i \le n-1$. However, $G[J]$ can be written as a determinant (with $\mathbb{A}_k :=: a_1 + \cdots + a_k$) :

$$G[J] = G[J0] = (a_1 \cdots a_{n+1})^{-n}\, S_{n^{n+1}}(\mathbb{A}_{n+1}{-}0,\, \mathbb{A}_n{-}j_n, \ldots\, , \mathbb{A}_1{-}j_1)\ .$$

Since $\forall i,j,k \in \mathbb{N}$, one has $S^k(\mathbb{A}_i - j)\, D_i = S^k(\mathbb{A}_{i+1} - j{-}1)$ (D_i being the isobaric divided difference with respect to the pair $(a_i{-}1, a_{i+1}{-}1)$), each D_i in the product $D_n \cdots D_1$ operates in turn on a single column of the determinant, the others being symmetrical in $(a_i{-}1, a_{i+1}{-}1)$. Therefore the image of $G[J0]$ under $D_n \cdots D_1$ is

$$(a_1 \cdots a_{n+1})^{-n}\, S_{n^{n+1}}(\mathbb{A}_{n+1} - 0,\, \mathbb{A}_{n+1} - j_n{-}1, \ldots\, , \mathbb{A}_{n+1} - j_1{-}1)$$

i.e. is equal to the image of $G[j_1+1, \ldots, j_n+1, 0]$ under $\pi_n^a \cdots \pi_1^a$ which by definition is $G[0J]$.

Now, if $G[I] = G[J]\pi_i^a \pi_j^a \cdots$, then the commutations (8) show that

$$G[0I] = G[0J]\, \pi_{i+1}^a \pi_{j+1}^a \cdots = G[J]D_n \cdots D_1 \pi_{i+1}^a \pi_{j+1}^a \cdots = G[I]D_n \cdots D_1$$

and this proves Eq. (21).

To simplify indices, let us take $k = n = 4$ to attack Eq. (22) The morphism $G[I] \mapsto G[000I]$, $I \in \mathbb{N}^4$ is $D_\nu := (D_4 \cdots D_1)\,(D_5 \cdots D_1)\,(D_6 \cdots D_1)$. The specialization $x_5 = x_6 = x_7$ of $G[000I]$ is the same as the specialization of $G[000I]D_\eta$, with $\eta = s_6 s_5 s_4 s_6 s_5 s_6$. However, $\nu\eta = \omega_7$ (= the maximal element of $\mathfrak{S}_7$), and thus is equal to $\omega_4\,(\omega_4\omega_7)$. Now, given any symmetric function of $x_1, \ldots, x_4$, applying to it any sequence of D_i and specializing $x_5 = 0 = x_6 = \cdots$ does not change the function. Therefore $G[I]\, D_{\omega_4}$, which indeed is symmetrical in $x_1, \ldots, x_4$, coincides with $G[I]D_{\omega_7}\big|_{x_5=0=x_6=\cdots}$ ☐

Notice that if I is the code of a vexillary permutation, then $G[I0^{n-1}]$ has a determinantal expression in terms of the $S^k(\mathbb{A}_i - j)$, and the morphism $G[I0^{n-1}] \mapsto G[0^{n-1}I]$ consists in replacing in the determinantal expression of $G[I0^{n-1}]$ each term $S^k(\mathbb{A}_i - j)$ by $S^k(\mathbb{A}_{i+n-1} - j{-}n{+}1)$. Specializing $x_{n+1}, x_{n+2}, \ldots$ to 0 , that is, $a_{n+1}, a_{n+2}, \ldots$ to 1, is obtained by putting $\mathbb{A}_{i+n} = \mathbb{A}_n + i$, and thus the stable part of $G[I]$ is a determinant in the $S^k(\mathbb{A}_n{-}j)$ (which is recognized to be the one expressing $G[J]$, with J= increasing reordering of I).

In other words, if $I \in \mathbb{N}^n$ is the code of a vexillary permutation, then $G[I]\, D_\omega = G[J]$. This leads to define, for a general $I \in \mathbb{N}^n$, a *G-key polynomial* $KG[I]$ by

$$\begin{cases} KG[I] & = & G[I] & I \text{ dominant} \\ KG[\dots, j_k, j_{k+1}, \dots]\, D_k & = & KG[\dots, j_{k+1}, j_k, \dots] & j_k > j_{k+1} \end{cases} \tag{23}$$

Using that $D_i D_\omega = D_\omega$, $i < n$, one gets that $KG[I]\, D_\omega = G[J]$, with $J=$ increasing reordering of I. Thus a decomposition of the type

$$G[I] = \sum c_{I,J} KG[J] \tag{24}$$

induces a decomposition

$$G[I]\, D_\omega = \sum c_{I,J} KG[J]\, D_\omega \tag{25}$$

of the stable part of $G[I]$ into Grassmannian Grothendieck polynomials.

In the case of Schubert polynomials, I gave with Schützenberger such a decomposition in terms of "Demazure characters", but it involves a combinatorics of tableaux which will be avoided here (cf.[21],[26]).

A Grothendieck polynomial indexed by a permutation $\sigma = \nu \times \eta$ belonging to a Young subgroup $\mathfrak{S}_m \times \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{m+n}$ factorizes into two Grothendieck polynomials (proposition 6.7 of [14]). It is clear, on the definition by restriction, that the stable part of G_σ factorizes into the product of the stable parts of G_ν and G_η. In particular, if ν and η are vexillary, of respective codes J, K, which reorders into the partitions λ, μ, then

$$G_\sigma \overset{stab}{\sim} G[J]\, G[K] \overset{stab}{\sim} g_\lambda g_\mu \ ,$$

and therefore, as in the case of cohomology, any algorithm producing the stable part of a Grothendieck polynomial allows to describe products of Grassmannian Grothendieck polynomials. Instead of symmetrizing operators, or restrictions, we shall prefer, in the next section, using *transitions*.

Grothendieck polynomials satisfy a recursion with respect to $\mathbb{C}[x_1, \dots, x_n] = \mathbb{C}[x_1] \otimes \mathbb{C}[x_2, \dots, x_n]$ (cf.[17]) that Fomin and Kirillov[6] have written as a generating function in the 0-Hecke algebra. This allows them to define in the same fashion stable Grothendieck polynomials :

Let T_i be the generators of the 0-Hecke algebra (with $T_i^2 = -T_i$), and for any n, r: $n > r$, let

$$\mathcal{A}_{n,r}(x) := (1 + xT_{n-1}) \cdots (1 + xT_r)$$

Definition[6]. For any n, and any $\sigma \in \mathfrak{S}_n$, the *stable Grothendieck polynomials* $\widetilde{F}_\sigma$ are the coefficients of the generating function

$$\mathcal{A}_{n,1}(x_1)\cdots\mathcal{A}_{n,1}(x_n) = \sum\nolimits_{\sigma\in\mathfrak{S}_n} \widetilde{F}_\sigma\, T_\sigma \ . \tag{26}$$

On the other hand, the generating function[6] of Grothendieck polynomials G_σ, $\sigma \in \mathfrak{S}_{2n-1}$, is

$$\mathcal{G}_{2n-1} := \mathcal{A}_{2n-1,1}(x_1)\,\mathcal{A}_{2n-1,2}(x_2)\cdots\mathcal{A}_{2n-1,2n-1}(x_{2n-1}) \ . \tag{27}$$

Now, if $T_iT_j\cdots T_k$ is equal to T_σ, $\sigma \in \mathfrak{S}_n$, σ having code I, then $T_{i+n-1}T_{j+n-1}\cdots T_{k+n-1}$ is equal to $T_{\sigma'}$, σ' having code $[0^{n-1}I]$. Therefore, the coefficient of $T_{\sigma'}$ in the specialization $T_1 = 0 = \cdots = T_{n-1}$; $x_{n+1} = 0 = \cdots = x_{2n-1}$ of $\mathcal{G}_{2n-1}$ is the same as the coefficient of T_σ in $\mathcal{A}_{n,1}(x_1)\cdots\mathcal{A}_{n,1}(x_n)$, that is, one has the identity

$$\widetilde{F}_\sigma = G[0^{n-1}I]\,\big|_{x_{n+1}=0=\cdots=x_{2n-1}} = F_\sigma \ . \tag{28}$$

In fact, Grothendieck polynomials already appear at the level of the "twisted" group algebra of the symmetric group (i.e. the algebra generated by the s_i's, x_j's), as coefficients in the expansion of isobaric divided differences $\pi_\sigma\ \pi_\sigma$ in the basis of permutations (cf.[22]). Yang-Baxter equation in the 0-Hecke algebra also lead to similar expansions ([6], [11]).

5 Transition on Grothendieck polynomials

Let σ be a permutation J its code, and r be the length of J (i.e. $j_r > 0$, $j_i = 0\ \forall i > r$). Let ν be the permutation of code $[j_1, \ldots, j_{r-1}, j_r - 1]$. Recall that for what concerns Schubert polynomials in $x_1, x_2, \ldots$, one has the equation

$$X_\sigma = x_r X_\nu + \sum\nolimits_{i\in\mathcal{I}(\sigma)} X_{\tau_{ji}\nu} \ , \tag{29}$$

sum over all transpositions $\tau_{ji} : j := \nu_r, i \in \mathcal{I}(\sigma)$ (i.e. i such that such that $\ell(\tau_{ji}\nu) = \ell(\nu) + 1$).

These equations, called *transition* in [15], refine the Ehresmann-Bruhat order on the symmetric group and give the expression of Schubert polynomials as a positive sum of monomials; they also furnish the stable part of them as a sum of Schur functions, if one cancels the successive terms $x_r X_\nu$, when iterating Eq. (29).

One has similar transitions for Grothendieck polynomials . But first, one needs to explicit products of G_σ by x_r's. In [14], I described the $G_\sigma/(a_1\cdots a_r)$. It requires a small adaptation to get products by $1/a_r = 1 - x_r$ instead. On the other hand, I cannot relate products $G_\sigma(1-x_r)$ to the products $a^\lambda G_\sigma$, λ dominant, considered in [7] under the name *Pieri formula.* Geometrically these last products are classes of restriction of ample line bundles over Schubert subvarieties, and not intersections as in our case.

Let us keep the same hyptheses as in Eq. (29), ordering the values $i_1 < i_2 < \cdots < i_k$, $i \in \mathcal{I}(\sigma)$, appearing in the summation. Given any $\sigma \in \mathfrak{S}_n$, and any pair of integers $j, i \le n$, we shall write $\tau_{ji} \star G_\sigma$ for $G_{\tau_{ji}\sigma}$, considering that a transposition τ_{ji} acts on indices of Grothendieck polynomials present on its right.

Proposition 3. *Let σ be a permutation. With the hypothesis of Eq. (29), one has*

$$G_\sigma = G_\nu + (x_r - 1)\,(1-\tau_{ji_k}) \star \cdots \star (1-\tau_{ji_1}) \star G_\nu\ . \tag{30}$$

Proof. Suppose first that $\sigma_1 > \cdots \sigma_{r-1}$. If $j_{r-1} \neq 0$, then $G[j_1, \ldots, j_r] = x_1 \cdots x_r G[j_1 - 1, \ldots, j_r - 1]$ and one is reduced to study a polynomial of smaller degree. The fundamental case to study is $J = [j_1, \ldots, j_{r-2}, 0, j_r]$, $j_{r-2} > 0$.

By induction on r, we know that

$$\begin{gathered} G[j_1, \ldots, j_{r-2}, j_r + 1] = G[j_1, \ldots, j_{r-2}, j_r] + \\ (x_{r-1} - 1)(1-\tau_{ji_k}) \star \cdots \star (1-\tau_{ji_2}) \star G[j_1, \ldots, j_{r-2}, j_r] \end{gathered} \tag{31}$$

the transpositions appearing in this formula being the same as in Eq. (30), but for τ_{j1} which is missing (the hypothesis $j_{r-2} > 0$ implies $i_1 = 1$).

Taking the image of (31) under π^a_{r-1} with the help of (7), one gets $G[j_1, \ldots, j_{r-2}, 0, j_r] =$

$$\begin{aligned}
&G[j_1, \ldots, j_{r-2}, 0, j_r{-}1] + (1{-}\tau_{ji_k}) \star \cdots \star (1{-}\tau_{ji_2}) \star G[j_1, \ldots, j_{r-2}, j_r, 0]\widehat{\pi}^a_{r-1}(x_r{-}1) \\
&= G[j_1, \ldots, j_{r-2}, 0, j_r - 1] + \cdots \star (1-\tau_{ji_2}) \\
&\qquad \star\big(G[j_1, \ldots, j_{r-2}, 0, j_r - 1] - G[j_1, \ldots, j_{r-2}, j_r, 0]\big) \\
&= G[j_1, \ldots, j_{r-2}, 0, j_r - 1] + (1-\tau_{ji_k}) \cdots \star (1-\tau_{ji_2}) \star (1-\tau_{j1}) \\
&\qquad \star G[j_1, \ldots, j_{r-2}, 0, j_r - 1]\,(x_r - 1)
\end{aligned}$$

which is the required expression for those special J.

A general $G[J']$, $J' \in \mathbb{N}^r$, will be obtained from a $G[J]$, $j_1 \ge \cdots \ge j_{r-1}$ by repeated use of the π^a_k, $i \le r-2$.

Suppose that Eq. (30) is true for a permutation σ. Let $p \le r-2$ be an integer such that $\nu_p > \nu_{p+1}$. Then, except when both ν_p and ν_{p+1} belong to $\mathcal{I}(\sigma)$, the image of (30) under π^a_p is a decomposition

$$G_{\sigma s_p} = G_{\nu s_p} + (x_r - 1)\,(1-\tau_{ji_k}) \star \cdots \star (1-\tau_{ji_1}) \star G_{\nu s_p}$$

of the type required by the theorem. On the other hand, if $\nu_p = \beta, \nu_{p+1} = \alpha \in \mathcal{I}(\sigma)$, one writes the RHS of (30) as a summation

$$(x_r - 1) \sum \eta' \star (1 - \tau_{j\beta})(1 - \tau_{j\alpha}) \star \eta'' \star G_\nu \ ,$$

sum over certain pairs of permutations η', η''. Thus terms occur in four-tuples

$$(x_r - 1)\big(G_{\cdots\beta\alpha\cdots j\cdots} - G_{\cdots j\alpha\cdots\beta\cdots} - G_{\cdots\beta j\cdots\alpha\cdots} + G_{\cdots j\beta\cdots\alpha\cdots}\big) \ ,$$

writing only the indices at places $p, p+1, r$. Now, the image of such a term under π_p^a is

$$(x_r - 1)\big(G_{\cdots\alpha\beta\cdots j\cdots} - G_{\cdots\alpha j\cdots\beta\cdots}\big)$$

and therefore the decomposition of $G_{\sigma s_p}$ is obtained from the one of G_σ by replacing the double factor $(1-\tau_{j\beta})(1-\tau_{j\alpha})$ by the single one $(1-\tau_{j\beta})$, and exchanging ν for νs_p. □

Specializing x_r to 0, one finally gets

Theorem 4. *1) Let σ be a permutation, $\sigma_1 = 1$. Using conventions (29), the stable part of G_σ is equal to that of*

$$\Big(1 - (1 - \tau_{ji_k}) \star \cdots \star (1 - \tau_{ji_1})\Big) \star G_\nu \ . \tag{32}$$

2) There exists positive integers m_J^σ such that

$$G_\sigma \overset{stab}{\sim} \sum\nolimits_J (-1)^{\ell(\sigma)-|J|} \ m_J^\sigma \, G[J] \ , \tag{33}$$

sum over all partitions J.

Proof. One gets 1) from a transition by specializing x_r to 0. Instead of having to embed $\mathfrak{S}_n$ into $\mathfrak{S}_1 \times \mathfrak{S}_n$, we have choosen the hypothesis $\sigma_1 = 1$.

From the theory of Schubert polynomials [18], one knows that all the permutations appearing when iterating sufficiently many times a Schubert-transition are vexillary. Now, in the case where σ is vexillary, the set $\mathcal{I}(\sigma)$ is of cardinality 1, and every long sequence of transitions starting from σ encounter a grassmannian permutation (with code obtained by reordering increasingly the code of σ).

There are more permutations appearing in Eq. (30) than Eq. (29). To adapt the argument of [18], one has to modify the weight function $(m-r)\ell(\sigma)$, where m is the first index such $\sigma_m > \sigma_{m+1}$. Buch proposes the function $\sum_{i\geq 1} 2^i j_{i+m}$, where $[j_1, j_2, ..]$ is the code of σ. This function vanishes for grasmannian permutations, and strictly decreases in a transition $\sigma \mapsto \eta$, η being any permutation appearing on the RHS of (30).

In summary, the stable part of G_σ belongs to the space generated by the stable parts of the G_γ, γ grassmannian. However, each product by a transposition appearing in a transition increases length (of permutations) by 1, and signs are as stated, since $\ell(\sigma) = |code(\sigma)|$ for any permutation σ (this positivity was conjectured by Buch [1], as we already mentionned in the introduction). $\square$

As an example of decomposition given by theorem 4, let us take $\sigma = [1,5,2,4,3,9,6,7,8]$; this implies $J = [0,3,0,1,0,3]$, $r = 6$. One has

$$X_{152439678} = x_6 X_{152438679} + X_{15248367} + X_{15283467} + X_{18243567} \ ,$$

and therefore $\mathcal{I} = \{3,4,5\}$,

$$G_{152439678} = G_{152438679} + (x_6 - 1)(1-\tau_{85}) \star (1-\tau_{84}) \star (1-\tau_{83}) \star G_{152438679} \ .$$

Expression (32) develops in this case into
$(1-\tau_{85})\star(1-\tau_{84})\star(1-\tau_{83})\star G_{152438679} = (1-\tau_{85})\star(1-\tau_{84})\big(G_{152438679} - G_{152483679}\big) = (1-\tau_{85}) \star \big(G_{152438679} - G_{152483679} - G_{152834679} + G_{152843679}\big) = G_{152438679} - G_{152483679} - G_{152834679} + G_{152843679} - G_{182435679} + G_{182453679} + G_{182835679} - G_{182543679}$.

Iterating transitions, and using codes now, one finally gets a sum
$G[030103] \overset{stab}{\sim} G[0601] + G[0502] + G[034] - G[053] - G[0602] + G[05011] + G[0412] + G[0331] - G[0431] - G[05021] - G[06011] - G[0512] - G[0341] + G[0531] + G[06021]$
of vexillary Grothendieck polynomials , which implies, by sorting indices,
$G[030103] \overset{stab}{\sim} g_{61} + g_{52} + g_{43} - g_{53} - g_{62} + g_{511} + g_{421} + g_{331} - 2g_{431} - 2g_{521} - g_{611} + g_{531} + g_{621}$.

One has a similar decomposition of the stable part of Schubert polynomials in terms of Schur functions (cf.[17],[4]). This decomposition is a direct by-product (just sorting increasingly all indices) of the decomposition of Schubert polynomials in terms of key polynomials (cf.[20],[26]), which involves finding all Young tableaux (in the alphabet $s_1, s_2, \ldots$) which are reduced deompositions of a given permutation.

Fomin and Greene[5] obtain a decomposition of the stable part of G_σ in terms of Schur functions, by enumerating tableaux in the T_i's which are decompositions, in the 0-Hecke algebra, of a given T_σ. Reconstructing Eq. (33) from their result, however, is not possible since many cancelations occur : Schur functions are not the appropriate symmetric functions for what concerns the Grothendieck ring.

We shall give elsewhere a decomposition of Grothendieck polynomials in terms of G-key polynomials. Let us mention that Grothendieck polynomials are given by a simple statistics on alternating sign matrices. This also will be written elsewhere.

6 Variations among determinantal expressions

There are many determinantal expressions of classes of Schubert subvarieties of Grassmannians, that is, of g_λ, λ partition, or of $G[J]$, J increasing.

Let us illustrate them on a concrete example, the case of a determinantal variety defined by the vanishing of minors of order 2 of a matrix of order 4. With the notations used in this text, the Grothendieck polynomial which is to be explicited is $G_{3412} = G[0022]$.

By definition, it is

$$(1-1/a_1)^5(1-1/a_2)^4(1-1/a_3)^1\,\pi^a_{4321} \tag{34}$$

$$= (1-1/a_1)^4(1-1/a_2)^4\,\pi^a_{4321} \tag{35}$$

$$= (a_1-1)^4(a_2-1)^4 a_3^4 a_4^4\,\pi^a_{4321}\,\frac{1}{(a_1a_2a_3a_4)^4} \tag{36}$$

$$= (a_1-1)^4(a_2-1)^4 a_3^4 a_4^4\, a_1^3 a_2^2 a_3^1\,\partial^a_{4321}\,\frac{1}{(a_1a_2a_3a_4)^4} \tag{37}$$

$$det\left|\, a_i^3(a_1-1)^4\,,\ a_i^2(a_1-1)^4\,,\ a_i^5\,,\ a_i^4 \,\right|\ \frac{(a_1a_2a_3a_4)^{-4}}{\Delta(\mathbb{A})} \tag{38}$$

with $\Delta(\mathbb{A}) := \prod_{i<j}(a_i - a_j)$.

The equality between Eq. (34) and Eq. (35) comes from the fact that $(1-1/a_1)\pi^a_1 = 1$, and that $\pi^a_{4321}\,\pi_1 = \pi^a_{4321}$.

Since ∂^a_{4321} is given by an alternating summation on the symmetric group, (37) is the expansion of (38). The quotient is a multi-Schur function

$$\begin{vmatrix} S^4(\mathbb{A}-4) & S^5(\mathbb{A}-4) & S^6(\mathbb{A}-4) & S^7(\mathbb{A}-4) \\ S^3(\mathbb{A}-4) & S^4(\mathbb{A}-4) & S^5(\mathbb{A}-4) & S^6(\mathbb{A}-4) \\ S^2(\mathbb{A}) & S^3(\mathbb{A}) & S^4(\mathbb{A}) & S^5(\mathbb{A}) \\ S^1(\mathbb{A}) & S^2(\mathbb{A}) & S^3(\mathbb{A}) & S^4(\mathbb{A}) \end{vmatrix} \frac{1}{(a_1a_2a_3a_4)^4} \tag{39}$$

and this the expression that is given in [12] (specialized to the case one of the vector bundles is trivial of rank 4).

In chapter 2 of my thesis [13], I used determinants with entries more general than symmetric powers. In the case considered, it gives:

$$G[0022] = \begin{vmatrix} 1-\frac{1}{a_1a_2a_3a_4} & a_1+a_2+a_3+a_4-4 \\ \frac{1}{a_1a_2a_3a_4}\left(4-\frac{1}{a_1}-\frac{1}{a_2}-\frac{1}{a_3}-\frac{1}{a_4}\right) & 1-\frac{1}{a_1a_2a_3a_4} \end{vmatrix}\,. \tag{40}$$

One can see, more easily than using Grothendieck polynomials , that this last determinant is equal to the class of the complex resolving submaximal minors (*Eagon-Northcott complex*).

Using now variables x_i's, and relation (11), one has:

$$G[0022] = x_1^5 x_2^4 (1-x_2) x_3 (1-x_3)^2 (1-x_4)^3 \, \partial_{4321}^x \tag{41}$$

$$= \left| x_i^5 \; x_i^4 (1-x_i) \; x_i (1-x_i)^2 \; (1-x_i)^3 \right| \frac{1}{\Delta(X)} \tag{42}$$

$$= \pm \begin{vmatrix} S^2(X) & S^3(X) & S^4(X) & S^5(X) \\ S^2(X-1) & S^3(X-1) & S^4(X-1) & S^5(X-1) \\ S^0(X-2) & S^1(X-2) & S^2(X-2) & S^3(X-2) \\ S^0(X-3) & S^1(X-3) & S^2(X-3) & S^3(X-3) \end{vmatrix} . \tag{43}$$

The equivalence of (38) and (42), or (39) and (43) is in fact a mere change of variables $x_i \mapsto 1 - \frac{1}{a_i}$, using $\Delta(X) = \prod((1-\frac{1}{a_i}) - (1-\frac{1}{a_j})) = (-1)^{\binom{n}{2}} \Delta(\mathbb{A})(a_1 \cdots a_n)^{-n+1}$

The expression (43) is given by Lenart [24](th. 2.4).

Proposition 3.11 of [14] states that, more generally, Grothendieck polynomials are still given by determinants in the case of *vexillary permutations*.

Note, however, that when the partition J has repeated parts, then one can slightly modify the determinant expressing $G[J]$ without changing its value. This is already seen at the level of the equality of (34) and (35).

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