Young's natural idempotents as polynomials

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Abstract. — Coding permutations as monomials, one obtains a compact expression of representatives of Young's natural idempotents for the symmetric group, or of q-idempotents in the Hecke algebra.

Introduction. — Young made an extensive use of the iterated factorization

$$\sum_{\mu\in\mathfrak{S}(n+1)} \mu = \left(\sum_{\mu\in\mathfrak{S}(n)} \mu\right)(1+\sigma_n+\sigma_n\sigma_{n-1}+\cdots+\sigma_n\cdots\sigma_1) \ ,$$

where σ_k is the simple transposition $(k,k+1)$ and $\mathfrak{S}(n)$ the symmetric group on n letters. For example, he writes

$$\sum_{\mu\in\mathfrak{S}(4)} \mu = (1+\sigma_1)(1+\sigma_2+\sigma_2\sigma_1)(1+\sigma_3+\sigma_3\sigma_2+\sigma_3\sigma_2\sigma_1) \ .$$

Coding permutations μ as monomials $x^{[\mu]}$, in a way we shall explain below, one has a similar factorization

$$\sum_{\mu\in\mathfrak{S}(4)} x^{[\mu]} = (1+x_1+x_1^2+x_1^3)(1+x_2+x_2^2)(1+x_3) \ , \tag{1.1}$$

but now, the product, in the ring of polynomials, has no relation whatsoever with the product in the group algebra $\mathcal{H}$ of the symmetric group, and must be considered only as a tool to give a compact form of some distinguished elements of $\mathcal{H}$.

We shall only treat the case of Young's natural idempotents [Yo], [Ru], [J-K], which are the product of two factors of the type $\sum\mu$ and $\sum(-1)^{\ell(\mu)}\mu$, summations on two Young subgroups.

We first need to introduce some combinatorial tools.

Given a variable x and an integer n, denote

$$(x,n) := 1+x+x^2+\cdots+x^{n-1} \ .$$

We embed symmetric groups $\mathfrak{S}(n)$ into each other by adding fixed points $n+1, n+2\dots$, and define $\mathcal{H}$ to be the inductive limit of $\mathcal{H}_n := \mathbb{Z}[\mathfrak{S}(n)]$.

The *code* of a permutation $\mu = (\mu_1, \ldots, \mu_n)$ is the table of inversion : $code(\mu) = [c_1, \ldots, c_n]$, where $c_i := card(j > i, \mu_j < \mu_i)$. Instead of the code we shall rather take the monomial

$$x^{[\mu]} := x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n} \ , \tag{1.2}$$

and induce by linearity a morphism P from $\mathcal{H}$ to the ring $\mathcal{P}$ of polynomials in an infinite number of variables $x_1, x_2, \ldots$. Thus every polynomial inside $\mathcal{P}$ is the image of one and only one element of $\mathcal{H}$.

It is easy to check that

$$P\Big(\sum_{\mu\in\mathfrak{S}(4)} \mu\Big) := \sum_{\mu\in\mathfrak{S}(4)} x^{[\mu]} = (1 + x_1 + x_1^2 + x_1^3)(1 + x_2 + x_2^2)(1 + x_3) \ ,$$

and more generally,

$$P\Big(\sum_{\mu\in\mathfrak{S}(n)} \mu\Big) := (x_1, n)(x_2, n-1)\cdots(x_n, 1) \ , \tag{1.3}$$

$$P\Big(\sum_{\mu\in\mathfrak{S}(n)} (-1)^{\ell(\mu)}\mu\Big) := (-x_1, n)(-x_2, n-1)\cdots(-x_n, 1) \ , \tag{1.4}$$

because a vector c in $\mathbb{N}^n$ is the code of a permutation in $\mathfrak{S}(n)$ iff $c_1 \le n-1, c_2 \le n-2, \ldots, c_n \le 0$.

We define the *(right) twisted action* of the symmetric group on the ring of polynomials $\mathcal{P}$ by

$$\forall f, g \in \mathcal{H} \ , \quad P(f) \odot g := P(fg) \ . \tag{1.5}$$

More explicitly, given any $c \in \mathbb{N}^n$ and $\sigma = (k, k+1)$ a simple transposition, the twisted action of σ on $\mathcal{P}$ satisfies

$$\begin{cases} x^c \odot \sigma = \ \cdots x_k^{c_{k+1}+1} x_{k+1}^{c_k} \cdots & \text{if } c_k \le c_{k+1} \\ x^c \odot \sigma = \ \cdots x_k^{c_{k+1}} x_{k+1}^{c_k - 1} \cdots & \text{if not.} \end{cases} \tag{1.6}$$

Equivalently, denoting $f \to f^\sigma$ the usual action of the symmetric group (interchanging x_k and x_{k+1}), one has

$$\begin{cases} x^c \odot \sigma = \ (x^c)^\sigma \, x_k & \text{if } c_k \le c_{k+1} \\ x^c \odot \sigma = \ (x^c)^\sigma / x_{k+1} & \text{if not} \end{cases} \ . \tag{1.7}$$

The symmetric group. — Given a composition $I = (i_1, \ldots, i_r) \in \mathbb{N}^r$, let us denote $\mathfrak{S}(I)$ the Young sub-group $\mathfrak{S}(i_1) \times \mathfrak{S}(i_2) \times \cdots \times \mathfrak{S}(i_r)$ of $\mathfrak{S}(n)$ permuting the letters in each successive block of respective lengths $i_1, \ldots, i_r$, where n is the *weight* of I ($:= i_1 + \cdots + i_r$).

Write $\square_I$ for the sum of elements of $\mathfrak{S}(I)$, and ∇_I for its image under the involution $\mu \longrightarrow (-1)^{\ell(\mu)}\mu$.

We say that two compositions are *conjugate* when the partitions obtained by reordering I and J are conjugate (cf. [Mcd]). For example, each of $\{331, 313, 133\}$ is conjugate to each of $\{223, 232, 322\}$, because the two partitions 331 and 322 are conjugate (i.e. their diagrams are exchanged by symmetry of the two coordinate axes).

The key point in Young's first construction of idempotents [Yo] is the following property, which was isolated by Von Neumann (cf. [Ru]):

Lemma 2.1. — *Given two conjugate compositions I, J of weight n then the space $\square_I \mathcal{H}_n \nabla_J$ is one-dimensional.*

By left and right multiplication by permutations, one obtains the elements that Young associates to the Young sub-groups determined by the rows and columns of tableaux, and which constitute his *natural* idempotents (in opposition with the *orthogonal* idempotents he obtained later [Ru], [J-K]).

We can thus restrict to considering only elements of the type $\square_I \mu \nabla_J$, where I is a decreasing partition, J its conjugate increasing composition, and μ any permutation such that $\square_I \mu \nabla_J$ is non-zero.

A *tabloid* t of shape I is a filling of each box of the diagram of I (of coordinate (i,j)) with a number $t(i,j)$. To it, we associate a polynomial

$$Pol(t) := \prod_{(i,j)} (x_{t(i,j)}\, ,\, j) \tag{2.2}$$

by taking the product on all the boxes of t.

For example,

$$Pol \begin{pmatrix} 4 & & \\ 5 & 1 & 3 \\ 2 & 6 & 7 \end{pmatrix} = \begin{matrix} (x_4, 3) & & \\ (x_5, 2) & (x_1, 2) & (x_3, 2) \\ (x_2, 1) & (x_6, 1) & (x_7, 1) \end{matrix}$$

(read the right side as a product).

The *column-tabloid* t_I^c of shape I is the tabloid obtained by numbering the boxes of I columnwise by $1, 2, 3, \ldots$, and the *row-tabloid* t_I^r, by rowwise numbering. The *weight-tabloid* w_I of shape I is the filling of the diagram of I, putting in each box the number of boxes strictly southwest of it (i.e. the box of coordinate $(i,\, j)$ is filled with the number $(i-1)(j-1)$).

Take for example $I = 3322$. Then

$$t_I^c = \begin{array}{llll} 1 & 4 & & \\ 2 & 5 & 7 & 9 \\ 3 & 6 & 8 & 10 \end{array} \ , \ t_I^r = \begin{array}{llll} 1 & 2 & & \\ 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 \end{array} \ , \ w_I = \begin{array}{llll} 0 & 2 & & \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \ .$$

Reading rowwise the tabloid t_I^c, one gets a permutation that we shall denote μ_I, and which is such that $\square_I \mu_I \nabla_J$ is different from 0.

To a partition I, we also need to associate a monomial

$$x^{<I>} := x_{\mu_1}^{c_1} \cdots x_{\mu_n}^{c_n} = x^{[\mu]} \ ,$$

where $\mu = \mu_I$, and where $(c_1, \ldots, c_n)$ is the code of μ. Equivalently, the exponent of $x^{<I>}$ is obtained by reading the tabloid w_I columnwise, from left to right.

With the preceding partition 3322, $\mu_I = 14\,2579\,36810$ has code $02\,0123\,0000$, and $x^{<I>} = x^{000\ 210\ 20\ 30} = (x_1^0 x_2^0 x_3^0)(x_4^2 x_5^1 x_6^0)(x_7^2 x_8^0)(x_9^3 x_{10}^0)$.

Lemma 2.3. — *Given a partition I, then*

$$P\left(\square_I \cdot \mu_I\right) = Pol(t_I^r) x^{<I>}$$

Proof. — The permutation μ_I is exactly the one which, when acting on the values in the tabloid t_I^c, transforms t_I^c into t_I^r. We have to check that this permutation of values inside the tabloid t_I^c corresponds exactly to the twisted action of the symmetric group on $Pol(t_I^r)$.

Factorize μ_I into cycles $\zeta^1, \ldots \zeta^n$ such that the code of ζ^k, $k = 1, \ldots, n$ is equal to $[0, \ldots, 0, c_k, 0, \ldots 0]$, where $[c_1, \ldots, c_n]$ is the code of μ_I, and suppose by induction that there exists a monomial $x^{\clubsuit}$ such that

$$P(\square_I \zeta^1 \cdots \zeta^k) = Pol(t_I^{c\,\zeta^1 \cdots \zeta^k})\, x^{\clubsuit} \ .$$

The letters $1, \ldots, k$ occupy in the tabloid $t_k := t_I^{c\,\zeta^1 \cdots \zeta^k}$ the same position as in t_I^r, and the other letters are placed columnwise in the remaining part of the diagram (the exponential notation t^ζ still means the image of the tabloid t under the permutation ζ of the letters of t).

The value $k+1$ occupies a certain box in t_I^r, of height h, which in t_k is occupied by a value p. One has to move $k+1$ to its final place by successive transpositions $(p, p-1)$, $(p-1, p-2), \ldots, (k+2, k+1)$. Suppose that an intermediate step

$$P\big(\square_I \zeta^1 \cdots \zeta^k (p, p-1) \cdots (r+1, r)\big) = Pol\big(t_k^{(p,p-1)\cdots(r+1,r)}\big) x^{\clubsuit} x_r^{p-r} \ .$$

This last polynomial is equal to

$$(x_{r-1}, h')(x_r, h)\, x_r^{p-r}\, Q\ ,$$

where the factor Q is independent of x_{r-1}, x_r. Since $h' \le p - r + 1$, the twisted action of the simple transposition $(r-1, r)$ on $(x_{r-1}, h')(x_r, h)x_r^{p-r}$ consists in exchanging x_r and x_{r-1} and multiplying by x_{r-1}.

Finally,

$$P(\square_I \zeta^1 \cdots \zeta^k \zeta^{k+1}) = Pol(t_I^{c\zeta^1 \cdots \zeta^k \zeta^{k+1}})\, x^{\clubsuit}\, x_k^{p-k}\ ,$$

and the required result follows at the last step, when obtaining the final tabloid t_I^r $\square$

For example, taking $I = 333$, an intermediate step in the computation of $P(\square_I\, \mu_I)$ will produce the polynomial

$$Pol\begin{pmatrix} 1 & 2 & 7 \\ 3 & 5 & 8 \\ 4 & 6 & 9 \end{pmatrix} x_2^2\ .$$

To put 3 into its final position, we have to take the image of the preceding polynomial under the sequence of transpositions $(7, 6)$, $(6, 5)$, $(5, 4)$, $(4, 3)$, i.e. taking only the relevant factors, compute the image of

$(x_3, 2)(x_5, 2)(x_7, 3)$:

$\xrightarrow{\odot(7,6)} (x_3, 2)(x_5, 2)\, x_6\, (x_6, 3)$

$\xrightarrow{\odot(6,5)} (x_3, 2)\, x_5^2\, (x_5, 3)(x_6, 2)$

$\xrightarrow{\odot(5,4)} (x_3, 2)\, x_4^3\, (x_4, 3)(x_6, 2)$

$\xrightarrow{\odot(4,3)} x_3^4\, (x_3, 3)(x_4, 2)(x_6, 2)$

which is the factor in $Pol\begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \\ 5 & 7 & 9 \end{pmatrix}$ involving the variables $x_3, \ldots, x_7$.

Given two strictly increasing sequences of integers $B = (\beta_1, \ldots, \beta_r)$, $\Gamma = (\gamma_1, \ldots, \gamma_r)$ of the same length, denote $x_B^\Gamma := x_{\beta_1}^{\gamma_1} \cdots x_{\beta_r}^{\gamma_r}$ and

$$\nabla(B, \Gamma) := \sum_{\nu \in \mathfrak{S}(r)} (-1)^{\ell(\nu)} (x_B^\Gamma)^\nu\, x_B^{[\nu]}\ , \tag{2.4}$$

where $x_B^{[\nu]} = x_{\beta_1}^{c_1} \cdots x_{\beta_r}^{c_r}$, with $[c_1, \ldots, c_r] = code(\nu)$.

Theorem (2.5). — *Given a (decreasing) partition I, let J be its conjugate increasing composition. Then the image of the idempotent $\Box_I\,\mu_I\,\nabla_J$ as a polynomial is*

$$1\odot\Box_I\odot\mu_I\odot\nabla_J = P\big(\Box_I\,\mu_I\,\nabla_J\big) = Pol(t_I^r)\prod_i \nabla(B^i,\Gamma^i)\ ,$$

where $B^1, B^2, \ldots$ are the successive rows of t_I^r (reading downwards), and $\Gamma^1, \Gamma^2, \ldots$, the rows of the weight tabloid w_I.

Proof. — The polynomial $Pol(t_I^r)x^{<I>}$ factorizes into factors corresponding to each row of t_I^r, and ∇_J also factorizes into operators acting on the elements of one row only. Thus take any integer h and consider the row of height h in t_I^r. Up to a shift of indices, let $1, 2, \ldots r$ be the entries in this row. The h-th row of w_I is 0, $(h-1)$, $2(h-1), \ldots, (r-1)(h-1)$. The factor of $Pol(t_I^r)x^{<I>}$ corresponding to row h is equal to

$$f = (x_1,h)\cdots(x_r,h)x_1^0x_2^{h-1}\cdots x_r^{(r-1)(h-1)}\ .$$

Since each (x_i,h) is of degree $\leq h-1$, the image of f under the twisted action of the permutation μ is equal to $f^\mu\, x^{[\mu]}$, and thus is equal to

$$(x_1,h)\cdots(x_r,h)\left(x_1^0x_2^{h-1}\cdots x_r^{(r-1)(h-1)}\right)^\mu x^{[\mu]}\ .$$

By product, we get the statement of the theorem $\Box$

For example,

$$P\Big(\Box_{3321}\,\mu_{3321}\,\nabla_{234}\Big) = Pol\left(\begin{array}{l}1\,2\\ 3\,4\,5\\ 6\,7\,8\,9\end{array}\right)\nabla(12,02)\,\nabla(345,012) =$$

$$\begin{aligned}&= (x_1,3)(x_2,3)(x_2^2-x_1^3)(x_3,2)(x_4,2)(x_5,2)\\ &(x_3^0x_4^1x_5^2 - x_3^0x_4^3x_5^1 - x_3^2x_4^0x_5^2 + x_3^4x_4^0x_5^1 + x_3^2x_4^3x_5^0 - x_3^4x_4^2x_5^0)\\ &(1-x_6+x_6^2-x_6^3)(1-x_7+x_7^2)(1-x_8)\end{aligned}$$

Hecke algebra. — The preceding construction easily extends to the q-deformation of the algebra of the symmetric group. Recall that the Hecke algebra $\mathcal{H}_n^q$ of the symmetric group $\mathfrak{S}(n)$ has linear basis T_μ, $\mu\in\mathfrak{S}(n)$, and its multiplicative structure is determined by

$$\begin{aligned}T_\mu T_\sigma &= T_{\mu\sigma} && \text{if } \ell(\mu\sigma)>\ell(\mu)\\ &= (q-1)T_\mu + qT_{\mu\sigma} && \text{otherwise,}\end{aligned}$$

for any permutation μ and any simple transposition σ.

Under the morphism (still denoted P) $T_\mu \longrightarrow x^{[\mu]} := P(T_\mu)$, the space of polynomials $\mathcal{P}$ in an infinite number of variables $x_1, x_2, \ldots$ can be identified with the limit of $P(\mathcal{H}_m^q)$, $m \to \infty$ and inherits, from the right multiplication of $\mathcal{H}_m^q$ by $\mathcal{H}_n^q$, $m \geq n$, of a structure of right $\mathcal{H}_n$-module. In particular, the submodule $1\,\mathcal{H}_n^q$ has linear basis x^I, $i_1 \leq n-1, i_2 \leq n-2, \ldots, i_n \leq 0$.

More explicitly, given $I \in \mathbb{N}^n$ and $\sigma = (k,\, k+1)$ a simple transposition, the action of $\mathcal{H}_n^q$ on $\mathcal{P}$ satisfies

$$\text{(3.2)} \qquad \begin{cases} x^I T_\sigma = \ (x^I)^\sigma\, x_k & \text{if } i_k \leq i_{k+1} \\ x^I T_\sigma = \ (q-1)x^I + q(x^I)^\sigma / x_{k+1} & \text{if not.} \end{cases}$$

We have characterized in [LS1], [LS2] the rational local operators acting on the ring of polynomials and satisfying the braid relations (they also satisfy an Hecke relation $(T_k - q_1)(T_k - q_2) = 0$). The above action (3.2) does not belong to this family, because it uses the non rational projections π_k^+ and π_k^- :

$$\pi_k^+(x^I) \quad = \quad \begin{cases} x^I & \text{if } i_k \leq i_{k+1} \\ 0 & \text{if not} \end{cases}$$

$$\pi_k^-(x^I) \quad = \quad \begin{cases} 0 & \text{if } i_k \leq i_{k+1} \\ x^I & \text{if not} \end{cases}$$

Let $\square_I^q$ denote the sum $\sum_{\mu \in \mathfrak{S}(I)} T_\mu$ and ∇_I^q be its image under the morphism $T_\mu \longrightarrow (-1/q)^{\ell(\mu)} T_\mu$.

For example,

$$\square_3^q = T_{123} + T_{213} + T_{132} + T_{231} + T_{312} + T_{321} \ ,$$

$$\nabla_3^q = T_{123} - q^{-1} T_{213} - q^{-1} T_{132} + q^{-2} T_{231} + q^{-2} T_{312} - q^{-3} T_{321} \ .$$

We have

$$\text{(3.3)} \qquad P(\square_r^q) = P(\square_r) = (x_1, r) \cdots (x_r, 1)$$

and

$$\text{(3.4)} \qquad P(\nabla_r^q) = (-x_1/q, r) \cdots (-x_r/q, 1) \ .$$

We still have the property that, given two conjugate compositions I, J of weight n, $\square_I^q \mathcal{H}_n^q \nabla_J^q$ is a 1-dimensional module (cf. [DKLLST], more general intertwining spaces are described by [D-K]);

From this property, one can write q-idempotents and describe the irreducible representations of the Hecke algebra in characteristic 0, for generic q (see [D-J]).

Taking into account that ∇_J^q differs from ∇_J by the appearance of powers of q, we define, for two sets B, Γ of integers

$$\nabla^q(B,\Gamma) := \sum_{\mu\in\mathfrak{S}(r)} (-q)^{-\ell(\mu)} (x_B^\Gamma)^\mu \, x_B^{[\mu]} \tag{3.5}$$

(for $q = 1$, one recovers 2.4).

The computations that we made in the case of the symmetric group remain valid, because at each step we had to compute the image under a simple transposition $(k, k+1) = \sigma$ of a polynomial f such that $\pi_k^-(f) = 0$, in which case the action of T_σ coincides with the twisted action of σ. Therefore we have :

Theorem (3.6). — *Given a (decreasing) partition I, let J be its conjugate increasing composition. Then the image of the q-idempotent $\square_I^q \, \mu_I \, \nabla_J^q$ as a polynomial is*

$$P(\square_I^q \, \mu_I \, \nabla_J^q) = Pol(t_I^r) \prod_i \nabla^q(B^i, \Gamma^i) \ ,$$

where $B^1, B^2, \ldots$ are the successive rows of t_I^r (reading downwards), and $\Gamma^1, \Gamma^2, \ldots$, the rows of the weight tabloid w_I.

For example, in the case of partition $I = (3, 3, 2)$, the corresponding idempotent for the symmetric group $\mathfrak{S}(8)$ can be written as the polynomial

$(x_6^2 - x_6 + 1)(1 + x_2 + x_2^2)(1 + x_5)(1 + x_4)(1 + x_3)(x_3^4 x_4^2 - x_3^4 x_5 - x_3^2 x_4^3 + x_3^2 x_5^2 + x_4^3 x_5 - x_4 x_5^2)(1 + x_1 + x_1^2)(x_1^3 - x_2^2)(-1 + x_7)$,

and the corresponding q-idempotent for the Hecke algebra is (up to normalization by a power of q)

$(1 + x_2 + x_2^2)(x_6^2 + q^2 - q x_6)(1 + x_5)(1 + x_4)(1 + x_3)(q x_3^4 x_5 - x_3^4 x_4^2 - q^2 x_3^2 x_5^2 + q x_3^2 x_4^3 + x_4 x_5^2 q^3 - x_4^3 x_5 q^2)(1 + x_1 + x_1^2)(x_1^3 - x_2^2 q)(q - x_7)$.

Both polynomials code a summation on the 40320 permutations of $\mathfrak{S}(8)$ that the sytem ACE [Ve] readily accepted to perform, and factorize, producing the above TEX output that MAPLE transformed by a random ordering of exponents.

References

[D-J] R.Dipper, G.James, *Representations of Hecke algebras of general linear groups*, Proc. London M.S. **52** (1986) 20–52.
[D-K] G.Duchamp, S.Kim, *Intertwining spaces associated with q-analogues of the Young symmetrizers in the Hecke algebra* , Banach Center Publications **36**, Warszawa (1996) 61–70.
[DKLLST] G.Duchamp, D.Krob, A.Lascoux, B.Leclerc, T.Scharf, J-Y.Thibon, *Euler-Poincaré characteristic and polynomial representation of Iwahori-Hecke algebra*, Pub. RIMS **31** (1995) 179–201.
[J-K] G.D.James, A.Kerber, The representation theory of the symmetric group, Addison-Wesley (1981).
[LS1] A.Lascoux & M.P.Schützenberger, *Symmetry and Flag manifold*, Invariant theory , Springer L.N. **996** (1983) 118–144.
[LS2] A.Lascoux & M.P.Schützenberger, *Symmetrisation operators on polynomial rings*, Funct. Anal i prigogi **21** (1987) 77-78.
[Mcd] I.G.Macdonald, Symmetric functions and Hall polynomials, Oxford Math. Mono (1979).
[Ru] D.E.Rutherford, Substitutional analysis, Edinburgh, the University Press (1948).
[Yo] A.Young, Collected work, University of Toronto Press (1977).
[Ve] S.Veigneau, ACE, *an algebraic combinatorics environment for the computer algebra system MAPLE*, Thesis (first chapter), Université de Marne La Vallée (1996).

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