CRYSTAL GRAPHS AND q-ANALOGUES OF WEIGHT MULTIPLICITIES FOR THE ROOT SYSTEM A_n*

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Abstract

We give an expression of the q-analogues of the multiplicities of weights in irreducible $\mathfrak{sl}_{n+1}$-modules in terms of the geometry of the crystal graph attached to the corresponding $U_q(\mathfrak{sl}_{n+1})$-modules. As an application, we describe multivariate polynomial analogues of the multiplicities of the zero weight, refining Kostant's generalized exponents.

1 Introduction

There exist interesting q-analogues of the multiplicities of weights in the irreducible representations of the classical Lie algebras. In the general case, these polynomials have been defined by Lusztig [22]. For the root system A_n, they coincide with the Kostka-Foulkes polynomials $K_{\lambda\mu}(q)$ (*cf.* [23]), which are the coefficients of the Schur symmetric functions $s_\lambda(X)$ on the basis of Hall-Littlewood functions $P_\mu(X;q)$. As recently shown by Kirillov and Reshetikhin [12], they are also the generating functions for the sum of quantum numbers of the Bethe vectors of certain integrable models in Statistical Mechanics. Also, the specialization at roots of unity of particular Kostka-Foulkes polynomials gives the decomposition coefficients of certain plethysms [17].

A combinatorial interpretation of the Kostka-Foulkes polynomials has been given in [19] (see also [24],[23]), where they were identified as the Poincaré polynomials of natural subsets of the *plactic monoid* regarded as a ranked poset.

The plactic monoid is the multiplicative structure on the set of Young tableaux which reflects the Robinson-Schensted correspondence.

As observed by Date-Jimbo-Miwa [3], the tensor products of the Gelfand-Tsetlin bases of the quantum enveloping algebra $U_q(\mathfrak{sl}_{n+1})$ at $q = 0$ are also described by the Robinson-Schensted correspondence.

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This was the starting point of Kashiwara's theory of *crystal bases*, which are canonical bases of the integrable modules of the algebras $U_q(\mathfrak{g})$, $\mathfrak{g}$ being any symmetrizable Kac-Moody algebra.

The action of the generators of $U_q(\mathfrak{g})$ on the crystal basis at $q = 0$ is described by a combinatorial object, called the *crystal graph* [9]. This reduces a large part of the representation theory of $U_q(\mathfrak{g})$ to combinatorial questions. For example, one can read the resolution of a tensor product into its irreducible components by looking at the connected components of its crystal graph.

The main theorem of the present paper gives a description of the q-multiplicities $K_{\lambda\mu}(q)$, for the root system A_n, in terms of the geometry of the crystal graph associated to the irreducible $U_q(\mathfrak{sl}_{n+1})$-module V_λ. The essential tool is an operation of the Weyl group on the crystal graph, which for A_n coincides with the canonical action of the symmetric group on the plactic monoid defined in [18]. The fixed points of this action play a particular rôle, and lead to the definition of multivariate polynomials, which refine the generating functions of Kostant's generalized exponents for $SL(n+1, \mathbf{C})$.

The main results have been announced in [17].

We would like to thank Andrei Zelevinsky for many stimulating discussions.

2 Crystal graphs

Recall that the quantum enveloping algebra $U_q(\mathfrak{sl}_{n+1})$ is the $\mathbf{Q}(q)$-algebra generated by elements e_i, f_i, t_i, t_i^{-1}, $1 \le i \le n$, subject to the following relations [8][4]

$$[t_i, t_j] = 0\ , \qquad t_i t_i^{-1} = t_i^{-1} t_i = 1 \tag{1}$$

$$[t_i, e_j] = [t_i, f_j] = 0 \ \text{ for } |i-j| > 1 \tag{2}$$

$$t_j e_i = q^{-1} e_i t_j\ , \quad t_j f_i = q f_i t_j \ \text{ for } |i-j| = 1 \tag{3}$$

$$t_i e_i = q^2 e_i t_i\ , \quad t_i f_i = q^{-2} f_i t_i \tag{4}$$

$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}} \tag{5}$$

$$[e_i, e_j] = [f_i, f_j] = 0 \ \text{ for } |i-j| > 1 \tag{6}$$

$$e_j e_i^2 - (q+q^{-1}) e_i e_j e_i + e_i^2 e_j = f_j f_i^2 - (q+q^{-1}) f_i f_j f_i + f_i^2 f_j = 0 \text{ for } |i-j| = 1\ . \tag{7}$$

For a $U_q(\mathfrak{sl}_{n+1})$-module M and $\lambda \in \mathbf{Z}^{n+1}$, set $M_\lambda = \{u \in M \mid t_i u = q^{\lambda_i - \lambda_{i+1}} u\}$. Elements of M_λ are called weight vectors (of weight λ), and a weight vector is said to be primitive if it is annihilated by the e_i's. A highest weight $U_q(\mathfrak{sl}_{n+1})$-module is a module M containing a primitive vector v such that $M = U_q(\mathfrak{sl}_{n+1})\, v$. We denote by V_λ the irreducible highest weight module with highest weight λ.

The module M is said to be integrable if each M_λ is finite-dimensional, $M = \bigoplus_\lambda M_\lambda$, and for any i, M is a direct sum of finite dimensional representations of the subalgebra isomorphic to $U_q(\mathfrak{sl}_2)$ generated by e_i, f_i, t_i and t_i^{-1}. By the representation theory of $U_q(\mathfrak{sl}_2)$, for any integrable $U_q(\mathfrak{sl}_{n+1})$-module M, one has the decomposition

$$M = \bigoplus_{\lambda} \bigoplus_{0 \le k \le \lambda_i - \lambda_{i+1}} f_i^{(k)} (\operatorname{Ker} e_i \cap M_\lambda) \tag{8}$$

where $f_i^{(k)} = \frac{q-q^{-1}}{q^k-q^{-k}} f_i^k$. Then, Kashiwara [9] defines endomorphisms $\tilde{e}_i$, $\tilde{f}_i \in \mathrm{End}\, M$ by

$$\tilde{f}_i(f_i^{(k)}u) = f_i^{(k+1)}u \quad \text{and} \quad \tilde{e}_i(f_i^{(k)}u) = f_i^{(k-1)}u \tag{9}$$

for $u \in \mathrm{Ker}\,(e_i) \cap M_\lambda$ and $0 \le k \le \lambda_i - \lambda_{i+1}$ (by convention $f_i^{(k)} = 0$ for $k < 0$).

Let $\mathcal{A}$ be the subalgebra of $\mathbf{Q}(q)$ formed by the rational functions without pole at $q = 0$. Kashiwara introduces the $\mathcal{A}$-lattice in V_λ

$$L(\lambda) = \sum_{i_1,i_2,\ldots,i_r} \mathcal{A}\, \tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_r}\, u_\lambda \tag{10}$$

where u_λ is a highest weight vector of V_λ, and shows that the set

$$B(\lambda) = \{\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_r}\, u_\lambda \bmod qL(\lambda) \,|\, 1 \le i_1,\ldots,i_r \le n\}\backslash\{0\} \tag{11}$$

is a basis of the $\mathbf{Q}$-vector space $L(\lambda)/qL(\lambda)$. The pair $(L(\lambda), B(\lambda))$ is called a (lower) *crystal basis* of V_λ. One has

$$\tilde{e}_i L(\lambda) \subset L(\lambda)\,, \quad \tilde{f}_i L(\lambda) \subset L(\lambda)\,, \tag{12}$$

so that $\tilde{e}_i$ and $\tilde{f}_i$ induce operators in $L(\lambda)/qL(\lambda)$ still denoted by $\tilde{e}_i$, $\tilde{f}_i$. Their action on $B(\lambda)$ is strikingly simple, namely,

$$\tilde{e}_i B(\lambda) \subset B(\lambda) \cup \{0\}\,, \quad \tilde{f}_i B(\lambda) \subset B(\lambda) \cup \{0\}\,, \tag{13}$$

and for $u, v \in B(\lambda)$,

$$\tilde{f}_i u = v \iff \tilde{e}_i v = u\,. \tag{14}$$

The *crystal graph* Γ_λ associated to V_λ is the coloured oriented graph whose vertices are the elements of $B(\lambda)$, and whose arrows of colour i describe the action of $\tilde{f}_i$:

$$u \stackrel{i}{\longrightarrow} v \iff \tilde{f}_i u = v\,.$$

More generally, there exist crystal bases and crystal graphs for any integrable module M with highest weights. These objects are compatible with tensor products, in the sense that if (L_1, B_1) and (L_2, B_2) are crystal bases for the modules M_1 and M_2 then $(L_1 \otimes L_2, B_1 \otimes B_2)$ is a crystal basis of $M_1 \otimes M_2$. Moreover, the decomposition of $M_1 \otimes M_2$ into irreducible representations is given by the decomposition of its crystal graph into connected components, two submodules being isomorphic if and only if the corresponding graphs are isomorphic (as coloured graphs). In particular, applying this process to the r-th tensor power of the fundamental representation V whose crystal graph is

$$1 \stackrel{1}{\longrightarrow} 2 \stackrel{2}{\longrightarrow} \cdots n-1 \stackrel{n-1}{\longrightarrow} n \stackrel{n}{\longrightarrow} n+1$$

(where for brevity the basis vector v_i is denoted by its index i), one obtains by identifying the isomorphic connected components an equivalence relation on the set of words labeling the vertices of the crystal graph of $V^{\otimes r}$. This relation is the plactic equivalence, described in the next section.

3 The plactic monoid

Let $A = \{a_1 < a_2 < \ldots < a_{n+1}\}$ denote a totally ordered alphabet of noncommutative indeterminates, and consider the free monoid A^* generated by A. The Robinson-Schensted correspondence associates to a word $w \in A^*$ a pair $(P(w), Q(w))$ of Young tableaux of the same shape. It was shown by Knuth [13] that the equivalence on A^* defined by

$$w \equiv w' \iff P(w) = P(w')$$

is generated by the relations

$$zxy \equiv xzy \ , \quad yxz \equiv yzx \ , \quad yxx \equiv xyx \ , \quad yxy \equiv yyx \ ,$$

for any $x<y<z$ in A. The quotient set $A^*/\equiv$, which by definition is in one-to-one correspondence with the set of Young tableaux on A, is therefore endowed with a multiplicative structure reflecting the Robinson-Schensted construction. The monoid $A^*/\equiv$ is called the *plactic monoid*. We denote by $\mathrm{Tab}\,(\lambda, \mu)$ the set of tableaux with shape λ and weight μ, *i.e.* such that the number of occurences of the letter a_i is μ_i.

As can be deduced from the detailed description of crystal graphs given in [10], two vertices $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}$ and $v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_r}$ of the crystal graph of $V^{\otimes r}$ are identified (for the equivalence mentioned at the end of the preceding section) if and only if the words $a_{i_1} a_{i_2} \cdots a_{i_r}$ and $a_{j_1} a_{j_2} \cdots a_{j_r}$ are plactically congruent. In fact, the Knuth relations amount to the identification of the two copies of $V_{(2,1)}$ lying in $V^{\otimes 3}$.

The algebraic properties of the plactic monoid have been investigated in [18].

Although this will not be used in the sequel, it is worthwhile to mention at this point that the above approach to the plactic monoid allows to define a similar object for any finite dimensional complex simple Lie algebra. For example, the plactic monoid of type C_n is the quotient of the free monoid on the ordered alphabet

$$\bar{A} + A = \{\bar{a}_n < \bar{a}_{n-1} < \ldots < \bar{a}_1 < a_1 < \ldots < a_n\}$$

by the relations (15) to (18)

$$\text{for } \ x < y < z \ \text{ and } \ x \neq \bar{z} \ , \qquad \begin{cases} yxz & \equiv & yzx \\ xzy & \equiv & zxy \end{cases} \tag{15}$$

$$\text{for } \ x < y \ \text{ and } \ x \neq \bar{y} \ , \qquad \begin{cases} yxx & \equiv & xyx \\ yyx & \equiv & yxy \end{cases} \tag{16}$$

$$\text{for } \ i \leq n-1 \ \text{ and } \ \bar{a}_i \leq x \leq a_i \ , \qquad \begin{cases} a_i\bar{a}_i x & \equiv & \bar{a}_{i+1}a_{i+1}x \\ x a_i \bar{a}_i & \equiv & x\bar{a}_{i+1}a_{i+1} \end{cases} \tag{17}$$

For $w = x_1 x_2 \cdots x_k$ with $x_1 > x_2 > \ldots > x_k$, set

$$E_w = \{(x_p, x_q) \mid x_p = a_i \ , \ x_q = \bar{a}_i \ , \ q - p < k + i - n\} \ .$$

Then,

$$w \equiv \hat{w} \tag{18}$$

where $\hat{w}$ is the word obtained from w by erasing all pairs $(x_p, x_q) \in E_w$. These relations are obtained by identifying the isomorphic connected components of the crystal graph of

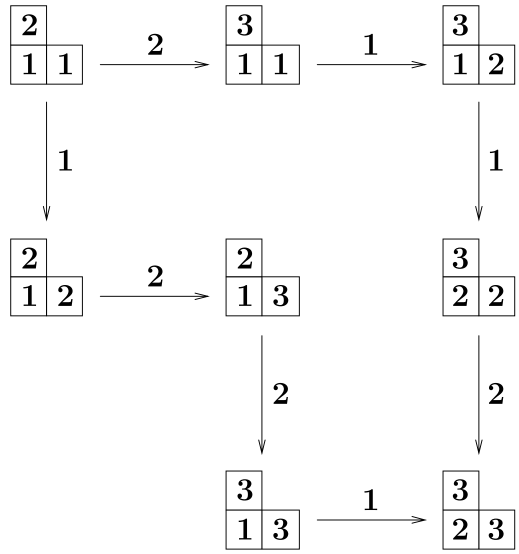

Figure 1: Crystal graph of the $U_q(\mathfrak{sl}_3)$-module $V_{(2,1)}$

$V^{\otimes 3}$ where V is the vector representation of $U_q(\mathfrak{sp}_{2n})$. Distinguished representatives of the equivalence classes are provided by the tableaux of [10], up to a simple reindexation due to the use of a different ordering. One can deduce from this a Robinson-Schensted type algorithm, sending a word w on $\bar{A}+A$ onto a pair $(P(w), Q(w))$, where $P(w)$ is the above mentioned representative of the class of w, and $Q(w)$ is an oscillating tableau, describing the intermediate shapes arising at the succesive stages of the algorithm. This bijection is different from the one given by Berele [1].

Now returning to the A_n case, we introduce following [18, 20, 21] linear operators ϵ_i, ϕ_i, σ_i, i=1, …, n acting on the free associative algebra $\mathbf{Q}\langle A\rangle$ in the following way. Consider first the case of a two-letter alphabet $A = \{a_i, a_{i+1}\}$. Let $w = x_1 \cdots x_m$ be a word on A. Bracket every factor $a_{i+1}\, a_i$ of w. The letters which are not bracketed constitute a subword w_1 of w. Then, bracket every factor $a_{i+1}\, a_i$ of w_1. There remains a subword w_2. Continue this procedure until it stops, giving a word $w_k = x_{j_1} \cdots x_{j_{r+s}}$ of the form $w_k = a_i^r\, a_{i+1}^s$ for some integers r, s. The image of w_k under ϵ_i, ϕ_i or σ_i is given by

$$\epsilon_i(a_i^r a_{i+1}^s) = \begin{cases} a_i^{r+1} a_{i+1}^{s-1} & (s \geq 1) \\ 0 & (s = 0) \end{cases}, \quad \phi_i(a_i^r a_{i+1}^s) = \begin{cases} a_i^{r-1} a_{i+1}^{s+1} & (r \geq 1) \\ 0 & (r = 0) \end{cases},$$

$$\sigma_i(a_i^r a_{i+1}^s) = a_i^s a_{i+1}^r \,.$$

The image of the initial word w is then $w' = y_1 \cdots y_m$ where $y_k = x_k$ for $k \notin \{j_1, \ldots, j_{r+s}\}$. For instance, if

$$w = (a_2 a_1) a_1 a_1 a_2 (a_2 a_1) a_1 a_1 a_1 a_2 \,,$$

we shall have

$$w_1 = \,.\,.\, a_1 a_1 (a_2 \,.\,.\, a_1) a_1 a_1 a_2 \,,$$

$$w_2 = \,.\,.\, a_1 a_1 \,.\,.\,.\,.\, a_1 a_1 a_2 \,.$$

Thus,

$$\epsilon_1(w) = a_2a_1\mathbf{a_1a_1}a_2a_2a_1a_1\mathbf{a_1a_1a_1}\,,$$

$$\phi_1(w) = a_2a_1\mathbf{a_1a_1}a_2a_2a_1a_1\mathbf{a_1a_2a_2}\,,$$

$$\sigma_1(w) = a_2a_1\mathbf{a_1a_2}a_2a_2a_1a_1\mathbf{a_2a_2a_2}\,,$$

where the letters printed in bold type are those of the image of the subword w_2. Finally, in the general case, the action of the operators ϵ_i, ϕ_i, σ_i on w is defined by the previous rules applied to the subword consisting of the letters a_i, a_{i+1}, the remaining letters being unchanged. We also define an involution denoted by Ω_2. It is the *anti*-automorphism of the algebra $\mathbf{Q}\langle A\rangle$ such that $\Omega_2(a_i) = a_{n-i+2}$.

All these operators are compatible with the plactic congruence $\equiv$ and therefore induce operations on tableaux that we shall also denote by ϵ_i, ϕ_i, σ_i and Ω_2.

Let λ be a partition of length $\leq n+1$, and let y_λ be the corresponding Yamanouchi tableau, *i.e.* the only tableau of shape and weight λ. It is known [21] that the products $\phi_{i_1}\cdots\phi_{i_r}$ applied to y_λ generate the set $\mathrm{Tab}\,(\lambda\,,\,\cdot)$ of all tableaux of shape λ over A. Define a coloured graph G_λ on $\mathrm{Tab}\,(\lambda\,,\,\cdot)$ whose edges of colour i describe the action of ϕ_i:

$$t \xrightarrow{i} t' \iff \phi_i(t) = t'\,.$$

Comparing this definition with [10], one sees that G_λ is none other than the crystal graph Γ_λ of V_λ. In other words, the operators ϵ_i, ϕ_i coincide with the endomorphisms $\tilde{e}_i$, $\tilde{f}_i$ of $V^{\otimes r}$ "at $q=0$", in the identification $a_{i_1}\cdots a_{i_r} \longleftrightarrow v_{i_1}\otimes\cdots\otimes v_{i_r}$.

It follows from the definition of Ω_2 that

$$t \xrightarrow{i} t' \iff \Omega_2(t') \xrightarrow{n+1-i} \Omega_2(t)\,.$$

Hence, the crystal graph has a symmetry of order 2 obtained by reversing its arrows and changing their colour from i to $n+1-i$.

We end this section by defining another coloured graph issued from the plactic monoid. It would be interesting to have also an interpretation of this graph in the framework of crystal bases.

The operation of conjugation $h \to g^{-1}hg$ in a group is replaced in a monoid by the circular permutation $w = uv \to vu$. In the plactic monoid, a special kind of conjugation called *cyclage* has been introduced in [18] (see also [15]). It is defined as follows. Let t, t' be two tableaux of weight μ, where μ is a partition, *i. e.* $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n+1} \geq 0$. For $i \geq 2$, write $t \xrightarrow{i} t'$ if and only if there exists u in $A^*/\!\equiv$ such that $t = a_i\,u$ and $t' = u\,a_i$. In this case, t' is said to be a *cyclage* of t. In this way, the set $\mathrm{Tab}\,(\cdot\,,\mu)$ of all tableaux of weight μ is given the structure of a connected oriented (coloured) graph H_μ, whose transitive closure is a partial order with minimal element the row tableau $t_\mu := a_1^{\mu_1}a_2^{\mu_2}\cdots a_{n+1}^{\mu_{n+1}}$. The cocharge $co(t)$ of a tableau t of weight μ is defined as its rank in the poset $\mathrm{Tab}\,(\cdot\,,\,\mu)$, that is, the number of cyclages needed to transform t into the row tableau t_μ. The maximal value of the cocharge on $\mathrm{Tab}\,(\cdot\,,\,\mu)$ is $\|\mu\| = \sum_i (i-1)\mu_i$. The *charge* of t is $c(t) := \|\mu\| - co(t)$.

A tableau t which is not a row tableau admits in general several cyclages. We call *initial cyclage*, and we denote by $\mathcal{C}(t)$, the cyclage obtained by cycling the first letter of

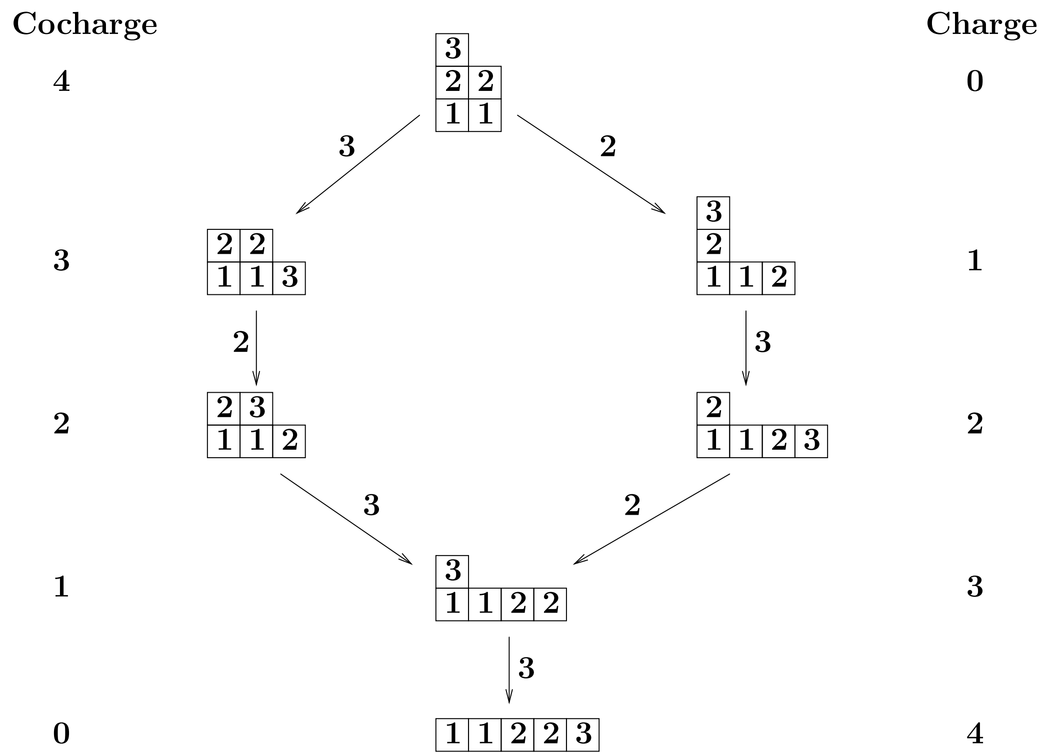

Figure 2: The cyclage graph $H_{(2,2,1)}$ for A_2

the row reading of t. In Figure 2, all arrows except the one labelled 2 in the top right corner correspond to initial cyclages. Deleting from H_μ all the arrows corresponding to non initial cyclages, one is left with a tree T_μ whose root is the row tableau t_μ.

More generally, the initial cyclage of a tableau t whose weight is not a partition is defined in exactly the same way. The initial cyclage commutes with the operators σ_i [15]:

$$\sigma_i(\mathcal{C}(t)) = \mathcal{C}(\sigma_i(t))$$

for any tableau t whose shape is not a row. Thus, the image by σ_i of the tree T_μ is an isomorphic tree $T_{\sigma_i(\mu)}$, and it is natural to extend the definition of the charge to tableaux of arbitrary weight by requiring that $c(\sigma_i(t)) = c(t)$.

4 Kostka-Foulkes polynomials

Let R be the root system of a finite dimensional complex simple Lie algebra $\mathfrak{g}$, W its Weyl group, R^+ the set of positive roots and ρ the half-sum of positive roots. One can define a q-analogue $\mathcal{P}_q$ of Kostant's partition function by

$$\prod_{\alpha\in R^+} (1 - qe^\alpha)^{-1} = \sum_{\mu\in P} \mathcal{P}_q(\mu) e^\mu$$

where P is the weight lattice of $\mathfrak{g}$. Substituting this q-analogue to the ordinary partition function in Kostant's weight multiplicity formula, one defines

$$K_{\lambda\mu}(q) = \sum_{w\in W} (-1)^{\ell(w)} \mathcal{P}_q((\lambda+\rho)w - \mu - \rho)$$

which, as shown by Lusztig [22], turns out to be a polynomial in q with nonnegative integer coefficients. In the general case, the only known proof of this property comes from the interpretation of the $K_{\lambda\mu}(q)$ as (renormalized) Kazhdan-Lusztig polynomials for the affine Weyl group. However, for $R = A_n$, they coincide with the Kostka-Foulkes polynomials, which are the coefficients of the expansion

$$s_\lambda(X) = \sum_{\mu} K_{\lambda\mu}(q) P_\mu(X;q) \tag{19}$$

of the Schur functions on the basis of Hall-Littlewood functions [23]. In this case, there exists a combinatorial description of the $K_{\lambda\mu}(q)$, which implies the positivity of their coefficients.

Theorem 4.1 [19, 24] *The Kostka-Foulkes polynomial $K_{\lambda\mu}(q)$ is the generating function of the charge on* $\mathrm{Tab}\,(\lambda,\mu)$*:*

$$K_{\lambda\mu}(q) = \sum_{t\in \mathrm{Tab}\,(\lambda,\mu)} q^{c(t)} \ .$$

5 Crystal graphs and q-multiplicities

If one restricts the crystal graph Γ_λ to its edges of colour i, one obtains a decomposition of this graph into *strings of colour* i. Each tableau t belongs to a unique string of colour i, possibly reduced to t.

The graph Γ_λ can also be decomposed in a different way. Indeed, the operators σ_i satisfy Moore-Coxeter relations [18], which allows to define an action of the Weyl group $W = \mathbf{S}_{n+1}$ on the crystal graph Γ_λ and to decompose it into *orbits*. We shall denote by $\mathcal{O}_t$ the orbit of the tableau t under $\mathbf{S}_{n+1}$.

Note that a similar action of the Weyl group on the crystal graph can be defined for the other root systems. In geometric terms, the i-th generator of the Weyl group sends a vertex to its mirror image through the middle of its string of colour i. For example, on the graph of Figure 3 one can see four orbits. Reading tableaux columnwise, these are the two fixed points 2143 and 3142, a 6-element orbit {2121, 3131, 4141, 3232, 4242, 4343}, and a 12-element orbit constituted of the remaining tableaux.

As in the classical case, the multiplicity of the weight μ in V_λ is equal to the number $K_{\lambda\,\mu}$ of tableaux of shape λ and weight μ, that is, to the number of orbits $\mathcal{O}_t$ parametrized by these tableaux.

By definition, the charge is constant along the orbits $\mathcal{O}_t$, and can therefore be regarded as a statistic on the set of these orbits.

We shall now introduce a different statistic $d(t)$ on tableaux, which reflects the geometry of the crystal graph around t. Denote by $d_i(t)$ the *exponent of t in direction i*, defined by

$$d_i(t) = \min\{\max\{k \mid \epsilon_i^k(t) \neq 0\},\, \max\{k \mid \phi_i^k(t) \neq 0\}\} \, .$$

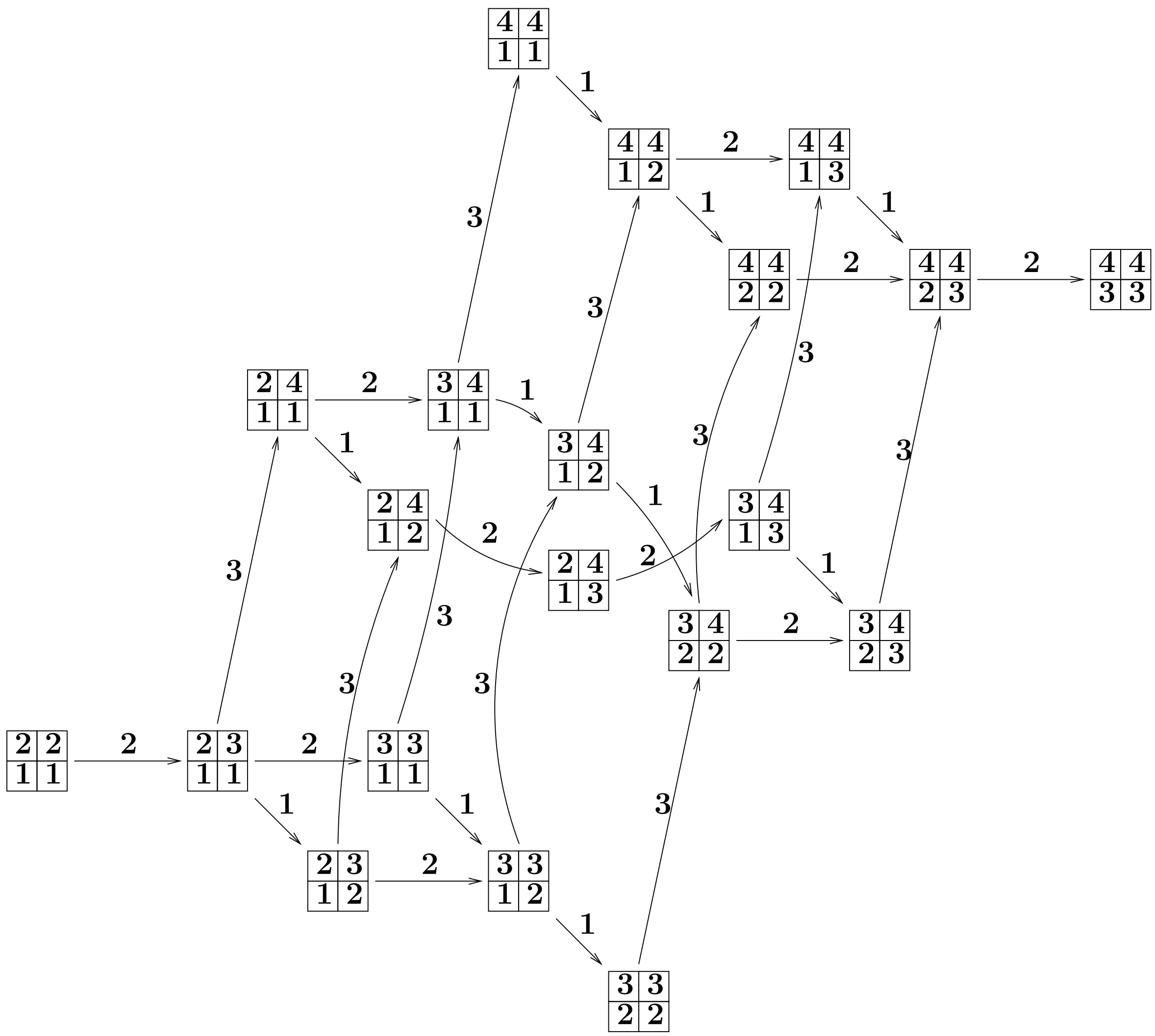

Figure 3: Crystal graph of the $U_q(\mathfrak{sl}_4)$-module $V_{(2,\,2)}$

In other words, $d_i(t)$ is the distance from t to the nearest end of its string of colour i. Define then

$$d(t) = \sum_{i=1}^{n} i\, d_i(t)\,.$$

The statistic $d(t')$ is not constant for t' in the orbit $\mathcal{O}_t$. But one has

Theorem 5.1 (i) *For any tableau t, the arithmetic mean of the integers $d(t')$, $t' \in \mathcal{O}_t$, is an integer denoted by $b(t)$.*

(ii) *$b(t)$ is equal to the charge $c(u)$ of the image u of t under the involution Ω_2.*

(iii) *The Kostka-Foulkes polynomial $K_{\lambda\,\mu}(q)$ is equal to*

$$K_{\lambda\,\mu}(q) = \sum_{t \in \mathrm{Tab}\,(\lambda,\,\mu)} q^{b(t)}\,.$$

Example 5.2 We take $n = 3$, $\lambda = (3, 2)$, $\mu = (2, 1, 1, 1)$. There are three tableaux in $\mathrm{Tab}\,(\lambda,\,\mu)$

$$t = \begin{array}{|c|c|c|}\cline{1-2} 3 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 2 \\ \hline \end{array} \qquad u = \begin{array}{|c|c|c|}\cline{1-2} 2 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 3 \\ \hline \end{array} \qquad v = \begin{array}{|c|c|c|}\cline{1-2} 2 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 4 \\ \hline \end{array}$$

The orbit of t is made of the following tableaux t'

$$\begin{array}{|c|c|c|}\cline{1-2} 3 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\cline{1-2} 3 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\cline{1-2} 3 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\cline{1-2} 4 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

whose numbers $d(t')$ are respectively equal to 4, 4, 1, 3. Similarly, the orbit $\mathcal{O}_u$ is made of the following tableaux u'

$$\begin{array}{|c|c|c|}\cline{1-2} 2 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\cline{1-2} 2 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\cline{1-2} 2 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\cline{1-2} 2 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 3 & 4 \\ \hline \end{array}$$

whose numbers $d(u')$ are all equal to 2. Finally, the orbit $\mathcal{O}_v$ is made of the tableaux v'

$$\begin{array}{|c|c|c|}\cline{1-2} 2 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\cline{1-2} 2 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\cline{1-2} 3 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\cline{1-2} 3 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline \end{array}$$

whose numbers $d(v')$ are respectively 5, 3, 4, 4. Thus, $b(t) = 3 = c(v)$, $b(u) = 2 = c(u)$, $b(v) = 4 = c(t)$, and one has $\Omega_2(t) \in \mathcal{O}_v$, $\Omega_2(u) \in O_u$, $\Omega_2(v) \in \mathcal{O}_t$. As a result, we get $K_{\lambda\mu}(q) = q^2 + q^3 + q^4$.

Proof — We establish (ii) which clearly implies (i) and (iii). Let $d'(t) = d(\Omega_2(t)) = \sum_{1\le i\le n}(n-i+1)d_i(t)$ by the symmetry property of Γ_λ mentionned in section 3. Denote by $b'(t)$ the arithmetic mean of the integers $d'(t')$ for $t' \in \mathcal{O}_t$. More generally, given a subset S of $\mathcal{O}_t$, we denote by $B'(S)$ the arithmetic mean of the $d'(t')$ for $t' \in S$.

The proof proceeds by induction. One first checks by direct computation that for the row tableau t_μ of weight the partition μ, there holds

$$b'(t_\mu) = \|\mu\|\,. \tag{20}$$

Therefore, from the definition of $c(t)$, it is sufficient to prove that

$$\mathcal{C}(t) = s \quad \Longrightarrow \quad b'(s) = b'(t) + 1\,, \tag{21}$$

where $\mathcal{C}(t)$ is the initial cyclage of t (see section 3). To this end, we decompose the orbit $\mathcal{O}_t$ into chains in the following way. For two tableaux $u,\,v \in \mathcal{O}_t$, write $u \rightsquigarrow v$ if there exists an $i \in \{2,\,\ldots,\,n+1\}$ such that $\sigma_i(u) = v$ and a_{i+1} (resp. a_i) is the first letter of the row reading of u (resp. of v). The connected components of the resulting graph are linear graphs called *chains*. Now, if $\delta = \mathcal{C}(\gamma)$ is the set of tableaux obtained by applying the initial cyclage operator to a chain γ of $\mathcal{O}_t$, then

$$B'(\delta) = B'(\gamma) + 1\,. \tag{22}$$

Indeed, it is possible to describe explicitely the difference between the n-uple of exponents $\mathbf{d}(t) = (d_1(t), \ldots, d_n(t))$ of a tableau t of the chain, and that of its image $\mathcal{C}(t)$. It turns out that this difference vector is zero, except at the ends of the chain.

More precisely, let $\gamma = \{u_i \rightsquigarrow u_{i-1} \rightsquigarrow \ldots \rightsquigarrow u_h\}$ be a chain, where the indices have been chosen such that a_j is the first letter of u_j, and let $\mathcal{C}(u_j) = v_j$. Note that since u_i is not a row tableau, one has $h \geq 2$. Let $(\mathbf{e}_j)$ be the canonical basis of $\mathbf{Z}^n$. We also set for convenience $\mathbf{e}_{n+1} = 0$. Then, the difference vectors are as follows if $i > h$

$$\begin{aligned} \mathbf{d}(v_i) - \mathbf{d}(u_i) &= -\mathbf{e}_i && (23)\\ \mathbf{d}(v_k) - \mathbf{d}(u_k) &= 0, \quad i > k > h && (24)\\ \mathbf{d}(v_h) - \mathbf{d}(u_h) &= \mathbf{e}_{h-1} && (25) \end{aligned}$$

and if $i = h$

$$\mathbf{d}(v_i) - \mathbf{d}(u_i) = \mathbf{e}_{i-1} - \mathbf{e}_i\,. \qquad (26)$$

This clearly implies (22). □

Remark

As observed by Terada [27], the standard polynomials $K_{\lambda,(1^n)}(q)$ can also be interpreted in terms of the Kazhdan-Lusztig basis of the irreducible S_n-module W_λ.

6 Refinement of the generalized exponents

The polynomials $K_{\lambda,(k^{n+1})}(q)$ are of particular importance for representation theory. The coefficient of q^d in $K_{\lambda,(k^{n+1})}(q)$ is equal to the multiplicity of the irreducible SL_{n+1}-module V_λ in the homogeneous component of degree d of the affine coordinate ring of the variety of nilpotent matrices in $\mathfrak{sl}_{n+1}$ [7], or equivalently of the space of SL_{n+1}-harmonic polynomials [6]. Similar polynomials can be defined for any reductive complex algebraic groups [14]. The exponents of the nonzero terms of $K_{\lambda,(k^{n+1})}(q)$ have been called by Kostant the *generalized exponents* of the module V_λ.

In the case where μ is a partition of rectangular shape $\mu = (k^{n+1})$, the previous construction becomes simpler because all the elements of $\mathrm{Tab}\,(\lambda, (k^{n+1}))$ remain fixed under $\mathbf{S}_{n+1}$. One can then define polynomials in several variables

$$\mathbf{K}_{\lambda,(k^{n+1})}(x_1, \ldots, x_n) = \sum_{t \in \mathrm{Tab}\,(\lambda,(k^{n+1}))} \prod_{i=1}^{n} x_i^{d_i(t)}$$

such that $K_{\lambda,(k^{n+1})}(q) = \mathbf{K}_{\lambda,(k^{n+1})}(q, q^2, \ldots, q^n)$.

Example 6.1 (i) $\mathbf{K}_{(3,3,2),(2,2,2,2)}(x_1,\, x_2,\, x_3) = x_1x_2 + x_1x_3 + x_2x_3$.

(ii) $\mathbf{K}_{(5,2,1),(2,2,2,2)}(x_1,\, x_2,\, x_3) = x_2x_3^2 + x_2^2x_3 + x_1x_3^2 + 2\,x_1x_2x_3 + x_1x_2^2 + x_1^2x_3 + x_1^2x_2$.

(iii) $\mathbf{K}_{(4,3,1),(2,2,2,2)}(x_1,\, x_2,\, x_3) = x_1x_2 + x_2x_3 + x_2^2 + x_1x_3^2 + x_1^2x_3 + 2\,x_1x_2x_3$.

These polynomials admit a rather simple description in terms of plactic operations.

Theorem 6.2 *Let $t \in \mathrm{Tab}\,(\lambda, (k^{n+1}))$. Then,*

(i) *There exists a unique tableau u of minimal weight such that in the plactic monoid, $t\,.\,u$ is a Yamanouchi tableau. Moreover, u is itself a Yamanouchi tableau and if y is the Yamanouchi tableau of weight (k^{n+1}), one has*

$$t\,u \equiv u\,y \equiv y\,u\ .$$

(ii) *Denote by* $\nu(t)$ *the weight of the tableau* u *defined in* (i), *and for a partition* μ, *set* $x_\mu = x_{\mu_1} x_{\mu_2} \cdots x_{\mu_r}$. *Then,*

$$\mathbf{K}_{\lambda,(k^{n+1})}(x_1,\ldots,x_n) = \sum_{t \in \mathrm{Tab}\,(\lambda,(k^{n+1}))} x_{\nu'(t)} \ ,$$

where $\nu'(t)$ *denotes the conjugate of the partition* $\nu(t)$.

Example 6.3 With $\lambda = (4,2)$ and $\mu = (2,2,2)$, one has the following equalities in the plactic monoid

$$\begin{array}{l}\boxed{3}\boxed{3}\\ \boxed{1}\boxed{1}\boxed{2}\boxed{2}\end{array} \,.\, \begin{array}{l}\boxed{1}\boxed{1}\end{array} = \begin{array}{l}\boxed{3}\boxed{3}\\ \boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\end{array} \ , \qquad \begin{array}{l}\boxed{2}\boxed{3}\\ \boxed{1}\boxed{1}\boxed{2}\boxed{3}\end{array} \,.\, \begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{1}\end{array} = \begin{array}{l}\boxed{3}\boxed{3}\\ \boxed{2}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\end{array} \ ,$$

$$\begin{array}{l}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{3}\boxed{3}\end{array} \,.\, \begin{array}{l}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\end{array} = \begin{array}{l}\boxed{3}\boxed{3}\\ \boxed{2}\boxed{2}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\end{array} \ .$$

so that

$$\mathbf{K}_{(4,2),(2,2,2)}(x_1,x_2) = x_1x_1 + x_2x_1 + x_2x_2 = s_{(2)}(x_1,x_2) \ .$$

Proof — The proof relies on the fact that for a tableau t of rectangular weight (k^{n+1}), $d_i(t)$ is equal to the greatest integer s such that $\epsilon_i^s(t) \neq 0$. Suppose that the rightmost occurence of a letter a_{i+1} in t is *free*, *i.e.* can be transformed into a_i by an application of ϵ_i. A tableau u such that $t\,u$ is Yamanouchi will then necessarily contain the column $c_i = a_i \cdots a_2 a_1$. Now the tableau $t\,c_i$ has one violation of the Yamanouchi condition less than t, and $d_i(t\,c_i) = d_i(t) - 1$. Iterating the process, one arrives at a Yamanouchi tableau $t\,u$, and the minimal tableau u constructed is this way is clearly unique. The class of u in the plactic monoid is a product of mutually commuting columns c_i, and each c_i appears with multiplicity $d_i(t)$. □

Example 6.4 With

$$\mathrm{t} = \begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{3}\end{array}$$

one has

$$\begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{3}\end{array} \,.\, \begin{array}{l}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\end{array} = \begin{array}{l}\boxed{3}\boxed{3}\\ \boxed{2}\boxed{2}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\end{array}$$

which corresponds to the following pairings

$$2\ 1\ 1\ 2\ 3\ 3\ \ .\ 2\ 1\ 2\ 1\ 1$$

The monomial associated to t is $x_2x_2x_1$, obtained by reading the lengths of the columns of u.

If $t' = \Omega_2(t)$, one has $d_i(t') = d_{n-i+1}(t)$, so that the polynomials $\mathbf{K}_{\lambda\mu}$ are symmetrical in any pair of variables with complementary indices (x_i, x_{n+1-i}). For certain families of partitions, they are even symmetric in the whole set of variables $x_1, x_2, \ldots, x_n$. It will be convenient to adopt the following notations:

Let k be an integer ≥ 1, β a partition of length $r \leq n$ whose parts are $\leq k$, and α a partition of the same weight than β, and length $\leq n+1$. Define a partition $\lambda = [\alpha, \beta]^k_{n+1}$ of length $n+1$ by

$$\lambda = (\alpha_1 + k, \ldots, \alpha_r + k, k, k, \ldots, k, k - \beta_s, \ldots, k - \beta_1)\ .$$

When $k = \beta_1$ we simply write $\lambda = [\alpha, \beta]_{n+1}$.

Theorem 6.5 (i) *Using these notations, when the partition α is a row $\alpha = (m)$, we have*

$$\mathbf{K}_{[(m),\beta]^k_{n+1},(k^{n+1})}(x_1, \ldots, x_n) = s_\beta(x_1, \ldots, x_n)\,.$$

(ii) *Similarly, when $\beta = (m)$,* $\mathbf{K}_{[\alpha,(m)]^k_{n+1},(k^{n+1})}(x_1, \ldots, x_n) = s_\alpha(x_1, \ldots, x_n)\,.$

(iii) *In any case, for $k \geq \max(\alpha_1, \beta_1)$,*

$$\mathbf{K}_{[\alpha,\beta]^k_{n+1},(k^{n+1})}(x_1, \ldots, x_n) = \mathbf{K}_{[\beta,\alpha]^k_{n+1},(k^{n+1})}(x_1, \ldots, x_n)\,.$$

Proof — Set $\lambda = [\alpha, \beta]^k_{n+1}$. Since one clearly has

$$\mathbf{K}_{\lambda+(r^{n+1}),((k+r)^{n+1})} = \mathbf{K}_{\lambda,(k^{n+1})} \tag{27}$$

one may always assume that $k \geq \max(\alpha_1, \beta_1)$, and (i) will be a consequence of (ii) and (iii).

To establish (ii), one remarks that when $\beta = (m)$ is a row, the free letters of a tableau $t \in \mathrm{Tab}\,(\lambda, (k^{n+1}))$ are exactly those filling the piece of shape α lying outside the rectangle, and this piece can be any tableau of weight m over $2, 3, \ldots, n+1$. Hence $\mathbf{K}_{[\alpha,(m)]^k_{n+1},(k^{n+1})}(x_1, \ldots, x_n) = s_\alpha(x_1, \ldots, x_n)\,.$

Finally, point (iii) follows from the existence of an exponent-preserving map on tableaux exchanging $\mathrm{Tab}\,([\alpha, \beta]^k_{n+1}, (k^{n+1}))$ and $\mathrm{Tab}\,([\beta, \alpha]^k_{n+1}, (k^{n+1}))$.

Let $t \in \mathrm{Tab}\,([\alpha, \beta]^k_{n+1}, (k^{n+1}))$. This map is defined for k sufficiently large ($k \geq \max(\alpha_1, \beta_1)$), but one can also consider it as a well-defined involution on $\mathfrak{sl}_{n+1}$-tableaux (which are ordinary tableaux modulo the relation $a_{n+1}a_n \cdots a_1 \equiv 1$). To construct the image t' of t by this involution, complete the part of t lying in the rectangle (k^{n+1}) by writing on the top of each column its complementary subset, in decreasing order from bottom to top. One obtains in this way the mirror image of a tableau of shape β. Put this tableau at the right of an initially empty rectangle (k^{n+1}), and fill its top right corner by the mirror image of the part of t lying inside α. Now fill the columns by the complementary subsets of the corresponding columns, and finally erase the mirror α-tableau of the top right corner. This correspondence is obviously self-inverse (for k large enough), and it is not difficult to check that it preserves the exponents. For example, with $k = 5$, $n = 4$, $\alpha = (4, 2, 2)$, $\beta = (5, 3)$ and

$$
\mathrm{t} = \begin{array}{ccccccccc}
5 & 5 & & & & & & & \\
3 & 3 & 4 & 4 & 4 & \mathbf{5} & \mathbf{5} & & \\
2 & 2 & 2 & 3 & 3 & \mathbf{3} & \mathbf{4} & & \\
1 & 1 & 1 & 1 & 1 & \mathbf{2} & \mathbf{2} & \mathbf{4} & \mathbf{5}
\end{array}
$$

one constructs sucessively

$$
\begin{array}{ccccccccc}
\mathbf{4} & \mathbf{4} & \mathbf{3} & \mathbf{2} & \mathbf{2} & & & & \\
5 & 5 & \mathbf{5} & \mathbf{5} & \mathbf{5} & & & & \\
3 & 3 & 4 & 4 & 4 & 5 & 5 & & \\
2 & 2 & 2 & 3 & 3 & 3 & 4 & & \\
1 & 1 & 1 & 1 & 1 & 2 & 2 & 4 & 5
\end{array}
\qquad \text{and} \qquad
\begin{array}{cccccccccc}
5 & \mathbf{5} & \mathbf{4} & \mathbf{2} & \mathbf{2} & & & & & \\
4 & 4 & 5 & \mathbf{4} & \mathbf{3} & & & & & \\
3 & 3 & 3 & \mathbf{5} & \mathbf{5} & & & & & \\
2 & 2 & 2 & 3 & 4 & 5 & 5 & 5 & & \\
1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 4
\end{array}
$$

so that

$$
\mathrm{t'} = \begin{array}{cccccccccc}
5 & & & & & & & & & \\
4 & 4 & 5 & & & & & & & \\
3 & 3 & 3 & & & & & & & \\
2 & 2 & 2 & 3 & 4 & \mathbf{5} & \mathbf{5} & \mathbf{5} & & \\
1 & 1 & 1 & 1 & 1 & \mathbf{2} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{4}
\end{array}
$$

One can check that both t and t' have exponents $\mathbf{d}(t) = \mathbf{d}(t') = (2,\ 1,\ 2,\ 1)$. $\square$

One can also give an explicit formula in the case of the standard weight $\mu = (1^{n+1})$. Set $t_i = x_i\,(1 - x_i)^{-1}$, i=1, $\ldots$, n. Then:

$$
\frac{\mathbf{K}_{\lambda,(1^{n+1})}(x_1, \ldots, x_n)}{\prod_{1\le i\le n}(1 - x_i)} = \sum_{\mu} K_{\lambda'\mu} \sum_{I \in \mathbf{S}(\mu)} t_{i_1} t_{i_1+i_2} \cdots t_{i_1+\cdots+i_{r-1}}\,, \tag{28}
$$

where $\mathbf{S}(\mu)$ denotes the set of all compositions $I = (i_1, \ldots, i_r)$ obtained by permutation of the parts of $\mu = (\mu_1, \ldots, \mu_r)$.

This formula is in fact the commutative image of an identity in the algebra of noncommutative symmetric functions, defined in [5]. Using the notations of [5], one can check that the generating function

$$
\sum_{\lambda} \mathbf{K}_{\lambda,(1^{n+1})}(x_1, \ldots, x_n) s_\lambda
$$

is the commutative image of

$$
\mathcal{K}_{n+1}(x_1, \ldots, x_n) = \sum_{|I|=n+1} \left(\prod_{d \in D(I)} x_d \right) R_I
$$

where for a composition $I=(i_1,\ldots,i_r)$, $D(I)=\{i_1,i_1+i_2,\ldots,i_1+\cdots+i_{r-1}\}$ and R_I is the associated noncommutative ribbon Schur function. Expanding on the basis of products of complete symmetric functions, one finds

$$\mathcal{K}_{n+1}=\sum_{|J|=n+1}\left(\sum_{I\geq J}(-1)^{\ell(J)-\ell(I)}\prod_{d\in D(I)}x_d\right)S^J$$

and it remains to show that the coefficient of S^J is this expression is equal to

$$\prod_{d\in D(J)}x_d\prod_{e\notin D(J)}(1-x_e)\ ,$$

which follows from a straightforward argument of inclusion-exclusion.

Remarks

1. The stability property (27) simply means that the **K**-polynomials are associated to $\mathfrak{sl}_{n+1}$ rather than to $\mathfrak{gl}_{n+1}$. In particular, for $\lambda=((k-1)^{n+1})+\nu$ where $|\nu|=n+1$, one has $\mathbf{K}_{\lambda,(k^{n+1})}=\mathbf{K}_{\nu,(1^{n+1})}$, a 'first layer formula' in the sense of [26]. It would be interesting to know whether there exists multivariate analogs of the other polynomials considered in [26], associated to the exterior algebra of $\mathfrak{sl}_{n+1}$ and to the Macdonald complex.

2. Putting $x_k=q^k$ in Theorem 6.5, one recovers a known formula ([6], Prop. 4.2). For the general case, there is an asymptotic formula of Stanley [25]

$$f_{\alpha\beta}(q):=\lim_{n\to\infty}K_{[\alpha,\beta]_{n+1},(\beta_1^{n+1})}(q)=s_\alpha * s_\beta(q,q^2,q^3,\ldots) \tag{29}$$

where $*$ is the internal product of symmetric functions. When neither α nor β is a row, $\mathbf{K}_{[\alpha,\beta]_{n+1},(\beta_1^{n+1})}$ is not symmetric, and (29) cannot be interpreted as a specialization of a symmetric identity. Multivariate analogues of the asymptotic multiplicities $f_{\alpha\beta}(q)$ have been considered in [2]. Other properties of the generalized exponents can be found in [11].

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