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GREEN POLYNOMIALS AND HALL-LITTLEWOOD FUNCTIONS AT ROOTS OF UNITY

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Abstract. — We give some applications of our recent work [LLT] about Hall-Littlewood functions at roots of unity. In particular, we prove the two conjectures of N. Sultana [Su] on specializations of Green polynomials, and we generalize results of Kraskiewicz-Weyman and Molev-Tsalenko concerning characters of the symmetric group induced by maximal cyclic subgroups. We also give some new congruences for Kostka-Foulkes polynomials modulo cyclotomic polynomials.

1. Introduction

Hall-Littlewood functions are symmetric functions of a set of variables A depending on a parameter q (see [Mcd]). When $q = -1$, these functions coincide with those introduced by Schur in his theory of projective representations of the symmetric group. The investigation of Hall-Littlewood functions at other roots of unity has been initiated by A.O. Morris [Mo], in connection with problems in the modular representation theory of the symmetric group. Recently, new results on this question have been obtained [Su], [MS], [LLT]. In [Su] and [MS], one finds some conjectures about Green polynomials (which are q-analogs of the characters of symmetric groups). These conjectures are proved in section 3.

Section 4 is concerned with some combinatorial properties of certain symmetric functions discovered by Foulkes [Fo]. These functions correspond to characters of the symmetric group induced by irreducible representations of a maximal cylic subgroup. A combinatorial interpretation of their coefficients in the basis of Schur functions has been given by Kraskiewicz and Weyman [KW] in terms of congruence properties of the major index of permutations, or, which amounts to the same, charge of standard tableaux. We give here a generalization of this result, in which the

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coefficients of certain plethysms with the preceding functions are interpreted in terms of the charge of more general tableaux. This result can also be viewed as an isomorphism theorem for certain submodules of the cohomology of a generalized flag manifold, or also in terms of harmonic polynomials relative to the symmetric group.

In section 5, we derive some congruences for Kostka-Foulkes polynomials modulo cyclotomic polynomials. Such congruences were derived in [LLT] for certain columns of the q-Kostka matrix. Those presented here involve rows of the same matrix.

All the results of this paper are consequences of two theorems from [LLT], which we recall in section 2.

2. Notations and background

Our notations for symmetric functions are as in [LLT]. Partitions are for us weakly increasing vectors of nonnegative integers $I = (i_1 \le i_2 \le \ldots \le i_m)$, Schur functions of an alphabet A are denoted by $S_I(A)$, products of power-sums $\psi_{i_1}\psi_{i_2}\cdots\psi_{i_m}$ by $\psi^I(A)$, and Hall-Littlewood functions by $P_I(A;q)$, $Q_I(A;q)$, as in [Mcd]. The *modified Hall-Littlewood functions* are the $Q'_I(A;q) = Q_I(A/(1-q);q)$ (in the sense of λ-rings), that is, the Q'_I form the adjoint basis of the P_I for the standard scalar product $(\ ,\ )$ (for which the Schur functions form an orthonormal basis). The Hall-Littlewood scalar product is denoted by $\langle .,\ .\rangle_q$.

The *Green polynomials* (see [Mcd]) $X^I_J(q)$ are defined by $X^I_J(q) = \langle Q_I,\ \psi^J\rangle_q = (Q'_I,\ \psi^J)$. Given two partitions I and J, we denote by $I \vee J$ the partition obtained by reordering the concatenation of I and J, *e.g.* $(122) \vee (1125) = (1112225)$. We write I^k for $I \vee I \vee \cdots \vee I$ (k factors). If $I = (i_1, \ldots, i_r)$ we set $kI = (ki_1, \ldots, ki_r)$. Given a symmetric function F, we denote by D_F the adjoint of the linear operator $G \mapsto FG$ (multiplication by F). We shall make use of the following two results, which were established in [LLT]:

Theorem 2.1. — *Let $I = (1^{m_1}2^{m_2}\ldots n^{m_n})$ be a partition, write $m_i = kp_i + r_i$ with $0 \le r_i < k$, and $H = (1^{r_1}2^{r_2}\ldots n^{r_n})$. Then, ζ being a primitive k-th root of unity,*

$$Q'_I(A;\zeta) = Q'_H(A;\zeta)\prod_{i\ge 1}\left[Q'_{(i^k)}(A;\zeta)\right]^{p_i}\ .$$

Theorem 2.2. — *Let $\psi_k(S_n)$ denote the plethysm of the complete function S_n by the power-sum ψ_k, which is defined by the generating series*

$\sum_n z^n \psi_k(S_n(A)) = \prod_{a\in A}(1-za^k)^{-1}$. *Then, if* ζ *is as above a primitive* k*-th root of unity, one has*

$$Q'_{(n^k)}(A;\zeta) = (-1)^{(k-1)n}\psi_k(S_n(A)).$$

3. Green polynomials at roots of unity

Let $\zeta \in \mathbb{C}$ be a primitive k-th root of unity. The following two results have been conjectured in [Su] (see also [MS]):

Theorem 3.1. — *Let* $I = (1^{m_1}2^{m_2}\dots n^{m_n})$ *be a partition, with at least one part* i *with multiplicity* $m_i \geq k$*, and let* $J = (j_1,\dots,j_r)$ *be another partition,*

with at least one part $j_h \equiv 0 \pmod k$*. Then,*

$$X^I_J(\zeta) \equiv 0 \pmod k\ .$$

Theorem 3.2. — *Let* $I = (1^{m_1}\dots i^{m_i}\dots n^{m_n})$*, where for some* i*,* $m_i \geq 1$*,* $|I| = n$*,* $I' = (1^{m_1}\dots i^{m_i-1}\dots n^{m_n})$ *and suppose* $n = kr$*. Then,*

$$X_I^{(r^k)}(\zeta) = (-1)^{(k-1)j}k\ X_{I'}^{(r-j)^k}(\zeta)$$

if $i = jk$*, and* $X_I^{(r^k)}(\zeta) = 0$ *otherwise.*

Proof of Theorem 3.1: Put $I = (j^k)\vee I'$ and $J = (k\cdot H)\vee J'$, where no part of J' is multiple of k. Then, all HL-functions being understood as specialized at $q=\zeta$, $\quad X^I_J(q) = (\psi^J,\ Q'_I) = (\psi^{kH}\psi^{J'},\ Q'_{(j^k)}Q'_{I'}) \qquad$ (by Theorem 2.1)

$$= \pm(\psi^{kH}\psi^{J'},\ \psi_k(S_j)Q'_{I'}) \qquad \text{(by Theorem 2.2)}$$

$$= \pm(D_{\psi_k(S_j)}(\psi^{kH}\psi^{J'})\ ,\ Q'_{I'})$$

so that we just have to check that the coefficients of the expansion of $D_{\psi_k(S_j)}(\psi^{kH}\psi^{J'})$ in the basis (ψ^K) are in $k\mathbb{Z}$. But these coefficients can be explicitely computed: one has

$$D_{\psi_k(S_j)}(\psi^{kH}\psi^{J'}) = \psi^{J'}D_{\psi_k(S_j)}\psi^{kH}$$

and

$$D_{\psi_k(S_j)}\psi^{kH} = \sum_{|M|=j}\frac{1}{(\psi^M,\ \psi^M)}D_{\psi^{kM}}\psi^{kH}$$

with $(D_{\psi^{kM}}\psi^{kH},\ \psi^{kN}) = (\psi^{kH},\ \psi^{k\cdot(M\vee N)}) \neq 0$ only when $H = M \vee N$.

Now, let $H = (1^{a_1}2^{a_2}\ldots)$, $M = (1^{b_1}2^{b_2}\ldots)$ and $N = (1^{c_1}2^{c_2}\ldots)$. We have therefore

$$D_{\psi^{kM}}\psi^{kH} = (\psi^{kH},\ \psi^{kH})\frac{\psi^{kN}}{(\psi^{kN},\ \psi^{kN})}$$

$$= \prod_i \frac{(ki)^{a_i} a_i!}{(ki)^{c_i} c_i!}\psi^{kN}$$

whence

$$\frac{1}{(\psi^M,\ \psi^M)}D_{\psi^{kM}}\psi^{kH} = k^{\Sigma a_i - \Sigma c_i}\prod_i \frac{a_i!}{b_i! c_i!}\psi^{kN}$$

$$= k^{\ell(M)}\prod_i \binom{a_i}{b_i}\psi^{kN}$$

and this number is congruent to 0 since $\ell(M) > 0$. $\square$

Example. — With $k = 4$ ($\zeta = i = \sqrt{-1}$), we have

$$X_{224}^{11114}(i) = (Q'_{1111}Q'_4,\ \psi^4\psi^{22}) = (-1)^3(\psi_4 Q'_4,\ \psi^4\psi^{22})$$

$$(Q'_4,\ D_{\psi_4}(\psi^4\psi^{22})) = -(Q'_4,\ 4\psi^{22}) = -4(S_4,\ \psi^{22}) = -4\ .$$

To prove Theorem 3.2, we need the following lemma, which is easily established by induction [CT]:

Lemma 3.3. — *Let $F \circ G$ denote (as in* [Mcd]*) the plethysm of the two symmetric functions F and G. Then,*

$$D_{\psi_m}(F \circ G) = \sum_{d|m}\Big[\big(D_{\psi_d}F\big)\circ G\Big]\cdot\Big[\psi_d \circ \big(D_{\psi_{m/d}}G\big)\Big]\ . \qquad \square$$

Proof of Theorem 3.2: The HL-functions being as above specialized at $q = \zeta$, we have

$$X_I^{(r^k)}(\zeta) = (\psi^I,\ Q'_{(r^k)}) = (-1)^{(k-1)r}(\psi^I,\ \psi_k(S_r))$$

(by Theorem 2.2)

$$= (-1)^{(k-1)r}(\psi^{I'}\psi_i,\ \psi_k(S_r)) = (-1)^{(k-1)r}(\psi^{I'},\ D_{\psi_i}[\psi_k \circ S_r])$$

where by Lemma 3.3,

$$D_{\psi_i}[\psi_k \circ S_r] = \sum_{d|i} \Big[\big(D_{\psi_d}\psi_k\big) \circ S_r \Big] \cdot \psi_d \circ (D_{\psi_{i/d}} S_r)$$

$$= k \cdot \psi_k(D_{\psi_j} S_r) = k\psi_k(S_{r-j}) = (-1)^{(k-1)(r-j)} k Q'_{(r-j)^k}$$

if $i = kj$ and $= 0$ otherwise (by Theorem 2.2), whence $X_I^{(r^k)}(\zeta) = (-1)^{(k-1)j}(\psi^I,\ kQ'_{(r-j)^k}) = (-1)^{(k-1)j} k X_{I'}^{(r-j)^k}(\zeta)$, as required. ∎

More generally, when $I = H^k = H \vee H \vee \cdots \vee H$ (k factors), $X_J^I(\zeta)$ can be expressed by means of the value of a permutation character of $\mathfrak{S}_n$:

$$X_J^{H^k}(\zeta) = (-1)^{(k-1)|H|}(\psi^J,\ \psi_k(S^H)) = (-1)^{(k-1)|H|}(\phi_k(\psi^J),\ S^H)\ ,$$

(where ϕ_k is the adjoint of the linear operator $F \mapsto \psi_k \circ F$, and $S^H = S_{h_1} S_{h_2} \cdots S_{h_r}$) so that:

Theorem 3.4. — $X_J^{H^k}(\zeta) = 0$ *when J is not of the form kM for some partition M, and*

$$X_{kM}^{H^k}(\zeta) = (-1)^{(k-1)|H|} k^{\ell(M)} (\psi^M,\ S^H)\ .$$

In particular, $X_{kM}^{(r^k)}(\zeta) = (-1)^{(k-1)r} k^{\ell(M)}$. ∎

Examples. — *(i)* With $k = 4$ ($\zeta = i$), $X_{224}^{2222}(i) = (\psi_4(S_2),\ \psi^{224}) = 0$ and $X_{44}^{24}(i) = (S_2,\ \phi_4(\psi^{44})) = (S_2,\ 16\psi^{11}) = 16$.

(ii) With $k = 3$ ($\zeta = e^{2\pi i/3}$), $X_{339}^{222333}(\zeta) = +3^3(\psi^{113},\ S^{23}) = 3^3(\psi^{11},\ S_2 D_{\psi_3} S_3) = 27$.

4. Charge congruences, harmonic polynomials and cohomology of flag manifolds

To state our next result, we recall some definitions from number theory. For $k, n \in \mathbb{N}$, the *Ramanujan* (or *Von Sterneck*) *sum* $c(k,n)$ (also denoted $\Phi(k,n)$) is the sum of the k-th powers of the *primitive* n-th roots of unity. Its value is given by *Hölder's formula*: if $(k,n) = d$ and $n = md$, then $c(k,n) = \mu(m)\phi(n)/\phi(m)$, where μ is the Moebius function and ϕ is the Euler totient function (see *e.g.* [HW] or [NV]).

Let $P(q) = \sum_{k=0}^{n-1} a_k q^k \ \in \mathbb{Z}[q]$ be a polynomial of degree $\le n-1$. P is said to be *even modulo* n if $(i,n)=(j,n) \ \Rightarrow a_i = a_j$. The following result, due to E. Cohen ([Co], see also [De] for a simpler proof), provides a generalization of the Moebius inversion formula:

Lemma 4.1. — *The polynomial P is even modulo n iff for every divisor d of n, the residue of $P(q)$ modulo the cyclotomic polynomial $\Phi_d(q)$ is a constant $r_d \in \mathbb{Z}$. In this case, one has*

(i) $a_k = \frac{1}{n} \sum_{d|n} c(k,d) r_d$

(ii) $r_d = \sum_{t|n} c(n/d,t) a_{n/t}$. $\square$

With the aid of Ramanujan sums, we define the symmetric functions

$$\ell_n^{(k)} = \frac{1}{n} \sum_{d|n} c(k,d) \psi_d^{n/d} \ .$$

These functions were first obtained by Foulkes as Frobenius characteristics of the representations of the symmetric group induced by irreducible representations of a maximal cyclic subgroup generated by an n-cycle [Fo]. A combinatorial interpretation of the multiplicity $(S_I,\ \ell_n^{(k)})$ has been discovered by Kraskiewicz and Weyman [KW]. A proof of this result by means of Cohen's theorem (Lemma 4.1) was subsequently found by Désarménien [De]. Adapting the argument of [De] to our case, we arrive at the following generalization of the Kraskiewicz-Weyman theorem:

Theorem 4.2. — *Let Λ_i be the i-th elementary symmetric function, and for $H = (h_1, \ldots, h_m)$, $\Lambda^H = \Lambda_{h_1} \cdots \Lambda_{h_m}$. Then, the multiplicity $(S_I,\ \ell_k^{(r)} \circ \Lambda^H)$ of the Schur function S_I in the plethysm $\ell_k^{(r)} \circ \Lambda^H$ is equal to the number of Young tableaux of shape $I^{\sim}$ (conjugate partition) and evaluation H^k whose charge is congruent to r modulo k.*

Proof: Let $|I| = n$, $|H| = m$ and $d \mid k$. If $P(q) = \sum_{i=0}^{k-1} a_i q^i$ is the residue modulo $1-q^k$ of the Kostka polynomial $K_{I,H^k}(q)$, one has, for ζ a primitive d-th root of unity,

$$P(\zeta) = K_{I,H^k}(\zeta) = (-1)^{(d-1)mk/d} (\psi_d^{k/d}(S^H),\ S_I)$$

(by Theorem 2.2)

$$= (-1)^n(\psi_d^{k/d} \circ ((-1)^m S^H),\ S_I) = ((-1)^n \psi_d^{k/d} \circ \Lambda^H(-A),\ S_I(A))$$

$$= (\psi_d^{k/d}(\Lambda^H),\ S_{I^\sim}) = r_d \in \mathbb{Z}\ .$$

Hence, by Lemma 4.1 *(i)*,

$$a_i = \frac{1}{k}\sum_{d|k} c(i,d)(\psi_d^{k/d}(\Lambda^H),\ S_{I^\sim}) = (\ell_k^{(i)}(\Lambda^H),\ S_{I^\sim})\ . \qquad \square$$

Example. — With $k = 4$, $r = 2$ and $H = (2)$,

$$\ell_4^{(2)}(\Lambda^2) = S_{134} + S_{224} + S_{11114} + 2S_{1133} + 2S_{1223}$$

$$+2S_{11123} + S_{2222} + S_{11222} + 2S_{111122} + S_{1111112}\ .$$

To compute the coefficient $(S_{11123},\ \ell_4^{(2)}(\Lambda^2)) = 2$, we have to find the number of tableaux of shape $(11123)^\sim = (125)$, evaluation (2222) and charge $\equiv 2 \pmod 4$. The two tableaux satisfying these constraints are:

$$\begin{array}{|c|cccc} \hline 3 & \multicolumn{1}{|c}{} & & & \\ \hline 2 & \multicolumn{1}{|c|}{4} & & & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad\qquad \begin{array}{|c|cccc} \hline 4 & \multicolumn{1}{|c}{} & & & \\ \hline 2 & \multicolumn{1}{|c|}{3} & & & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

which both have charge equal to 6.

Similarly, the reader can check that $(S_{237},\ \ell_4^{(2)}(\Lambda^{12})) = 5$ is the number of tableaux with shape (1111233), evaluation (11112222) and charge $\equiv 2 \pmod 4$.

A related result, due to Molev and Tsalenko [MT], gives an interpretation of $\ell_n^{(k)}$ in terms of $\mathfrak{S}_n$-harmonic polynomials: let H_n^k be the Frobenius characteristic of the homogeneous component $\mathcal{H}_n^k$ of degree k of the module $\mathcal{H}_n$ of $\mathfrak{S}_n$-harmonic polynomials. Then,

$$\ell_n^{(k)} = \sum_{i \equiv k \pmod n} H_n^i\ .$$

More generally, the plethysm $\ell_k^{(j)}(\Lambda^H)$ can be expressed as a positive sum of Frobenius characteristics of modules of harmonic polynomials. To each partition J of n, it is possible to associate a certain submodule $\mathcal{H}_J$ of $\mathcal{H}_n$ which has been defined in [La] and called an *affine Specht module*. This module is isomorphic, as a graded $\mathfrak{S}_n$-module, to the rational cohomology $H^*(\mathcal{F}_J)$ of the variety of flags fixed by an unipotent of Jordan canonical form corresponding to the partition J. The action of the symmetric group on $H^*(\mathcal{F}_J)$ is defined in [HS], and the Frobenius characteristic $H_J^j = \mathrm{ch}(H^{2j}(\mathcal{F}_J)) = \mathrm{ch}(\mathcal{H}_J^j)$ is given by the equation

$$\sum_j q^j H_J^j = \sum_I \tilde{K}_{I^{\sim},J}(q) S_I$$

where the $\tilde{K}_{I,J}(q)$ are the cocharge polynomials (*i.e.* the reciprocals of the Kostka polynomials, see also [Mcd], ex. 9 p. 136).

Now, suppose $J = H^k$, and let $\tilde{K}_{I^{\sim},J}(q) \equiv \sum_{j=0}^{k-1} a_j q^j \pmod{1-q^k}$. Since $(S_I,\ H_J^i)$ is equal to the coefficient of q^j in $\tilde{K}_{I^{\sim},J}(q)$, we have

$$(S_I, \sum_{i\equiv j \pmod k} H_J^i) = a_j \ .$$

On the other hand, $\tilde{K}_{I^{\sim},J}(q) \equiv K_{I^{\sim},J}(q) \pmod{1-q^k}$ whence $a_j = (\ell_k^{(j)}(\Lambda^H),\ S_I)$. To summarize:

Theorem 4.3. — *Let $J = H^k$. Then,*

$$\ell_k^{(j)}(\Lambda^H) = \sum_{i\equiv j \pmod k} H_J^i \ . \qquad \square$$

The theorem of Molev and Tsalenko corresponds to the case $H = (1)$ *i.e.* to the case of the full flag manifold.

Example. — The character of the preceding example decomposes into three parts

$$\ell_4^{(2)}(\Lambda^2) = H_{2222}^2 + H_{2222}^6 + H_{2222}^{10},$$

the three components being

$$H_{2222}^2 = S_{111122} + S_{1111112} \ ,$$

$$H_{2222}^6 = S_{1133} + 2S_{1223} + S_{2222} + S_{11114} + 2S_{11123} + S_{11222} + S_{111122}$$

$$\text{and} \qquad H_{2222}^{10} = S_{134} + S_{224} + S_{1133}.$$

5. Further congruences for Kostka polynomials

We have given in [LLT] some congruences modulo cyclotomic polynomials for certain columns of the q-Kostka matrix. We will now present other congruences for certain rows of the same matrix.

Let ρ_n be the staircase partition $(12\dots n)$. It is well-known that for $k=2$ ($\zeta=-1$), one has $S_{\rho_n}=P_{\rho_n}$. More generally, it can be shown that if ζ is a primitive k-th root of unity, $P_{\rho_n^{k-1}}(A;\zeta)=S_{\rho_n^{k-1}}(A)$. This is a consequence of the following result:

Theorem 5.1. — *Let ζ be a primitive k-th root of 1, and I be a partition such that $\ell(I^\sim)\le n$. Then,*

$$S_{I^k\vee\rho_n^{k-1}}(A)=\sum_J K_{IJ}\ P_{J^k\vee\rho_n^{k-1}}(A;\zeta)$$

where K_{IJ} are the usual Kostka numbers. In other words, we have the following congruence modulo the cyclotomic polynomial $\Phi_k(q)$:

$$K_{I^k\vee\rho_n^{k-1},\ H}(q)\equiv\begin{cases}K_{IJ} & \text{if} \qquad H=J^k\vee\rho_n^{k-1}\\ 0 & \text{otherwise.}\end{cases}$$

This theorem is itself a direct consequence of the following three lemmas. Details about k-cores and rim hook tableaux can be found in [KSW] and [SW].

Lemma 5.2. — *Let I be a partition such that $\ell(I^\sim)\le n$.*

a) *The k-core of the partition $I^k\vee\rho_n^{k-1}$ is ρ_n^{k-1}.*

b) *The only k-rim hooks which are removable from $I^k\vee\rho_n^{k-1}$ are those of shape (1^k). After removal of such a rim hook, one gets a partition $I'^k\vee\rho_n^{k-1}$ with $\ell(I'^\sim)\le n$.* $\square$

Lemma 5.3. — *If $(Q'_H,\ S_{I^k\vee\rho_n^{k-1}})\ne0$, then $H=J^k\vee\rho_n^{k-1}$ for some partition J.*

Proof: If $(Q'_H,\ S_{I^k\vee\rho_n^{k-1}})\ne0$ then $\ell(H)\ge\ell(I^k\vee\rho_n^{k-1})$. This forces H to have a part j with multiplicity $\ge k$. Set $H=(j^k)\vee H'$. Then,

$$(Q'_H,\ S_{I^k\vee\rho_n^{k-1}})=(-1)^{(k-1)j}(Q'_{H'},\ D_{\psi_k(S_j)}S_{I^k\vee\rho_n^{k-1}})$$

(by Theorems 2.1 and 2.2)

$$=(-1)^{(k-1)j}(-1)^{(k-1)j}\sum_{I'}\alpha_{I'}(Q'_{H'},\ S_{I'\vee\rho_n^{k-1}})=\sum_{I'}\alpha_{I'}(Q'_{H'},\ S_{I'^k\vee\rho_n^{k-1}})$$

(by Lemma 5.2) where the summation is over partitions I' of weight $|I|-kj$, and the $\alpha_{I'}$ are certain nonnegative integers. By induction, we obtain after a finite number of steps

$$(Q'_H,\ S_{I^k\vee\rho_n^{k-1}}) = \sum_{H''} \alpha_{H''}(Q'_{H''},\ S_{\rho_n^{k-1}})$$

where $\alpha_{H''} \geq 0$. For this sum to be non zero, it is necessary that for some H'', $(Q'_{H''},\ S_{\rho_n^{k-1}}) \neq 0$. This implies that $\ell(H'') \geq \ell(\rho_n^{k-1})$ and that no multiplicity of H'' be $\geq k$. The only possibility is $H'' = \rho_n^{k-1}$, whence $H = H'' \vee J^k = \rho_n^{k-1} \vee J^k$. □

Lemma 5.4. — *For $|I| = |J|$ and $\ell(\tilde{I}) \leq n$,*

$$D_{\psi_k(S^J)} S_{I^k\vee\rho_n^{k-1}} = (-1)^{(k-1)|J|}\ K_{IJ} S_{\rho_n^{k-1}}\ .$$

Proof: Since $|I| = |J|$ and the k-core of $I^k \vee \rho_n^{k-1}$ is ρ_n^{k-1}, we see that

$$D_{\psi_k(S^J)} S_{I^k\vee\rho_n^{k-1}} = \beta_{IJ} \cdot S_{\rho_n^{k-1}}$$

where β_{IJ} is, up to sign, the number of k-rim hook tableaux of shape $I^k \vee \rho_n^{k-1}/\rho_n^{k-1}$ and evaluation J. From lemma 5.2, it results that the rim hooks of these tableaux are all of shape (1^k), so that $\beta_{IJ} = (-1)^{(k-1)|J|} K_{IJ}$. □

Proof of Theorem 5.1:

$$S_{I^k\vee\rho_n^{k-1}} = \sum_H (Q'_H,\ S_{I^k\vee\rho_n^{k-1}})\ P_H$$

$$= \sum_J (Q'_{J^k\vee\rho_n^{k-1}},\ S_{I^k\vee\rho_n^{k-1}}) P_{J^k\vee\rho_n^{k-1}}$$

(by Lemma 5.3)

$$= \sum_J (-1)^{(k-1)|J|} (Q'_{\rho_n^{k-1}},\ D_{\psi_k(S^J)} S_{I^k\vee\rho_n^{k-1}}) P_{J^k\vee\rho_n^{k-1}}$$

(by Theorems 2. and 2.2)

$$= \sum_J K_{IJ} P_{J^k\vee\rho_n^{k-1}}$$

(by Lemma 5.4). □

Example. — Take $k = 3$, $n = 3$, $I = (2,3)$ and $J = (1,2,2)$. Then,

$$K_{112222233333,111112222222233}(q) = q^3(1 + 2q + 6q^2 + 13q^3 + 24q^4 + 39q^5$$
$$+59q^6+81q^7+105q^8+128q^9+148q^{10}+163q^{11}+171q^{12}+172q^{13}+165q^{14}+153q^{15}+134q^{16}$$
$$+114q^{17}+92q^{18}+72q^{19}+53q^{20}+38q^{21}+25q^{22}+16q^{23}+9q^{24}+5q^{25}+2q^{26}+q^{27})$$
$$\equiv K_{23,122} = 2 \pmod{1+q+q^2}\ .$$

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