TWISTED ACTION OF THE SYMMETRIC GROUP ON THE COHOMOLOGY OF A FLAG MANIFOLD*

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Abstract

Classes dual to Schubert cycles constitute a basis on the cohomology ring of the flag manifold $\mathcal{F}$, self-adjoint up to indexation with respect to the intersection form. Here, we study the bilinear form

$$(X\,,\,Y) := \langle X \cdot Y\,,\,c(\mathcal{F})\rangle$$

where X, Y are cocycles, $c(\mathcal{F})$ is the total Chern class of $\mathcal{F}$ and $\langle\ ,\ \rangle$ is the intersection form. This form is related to a twisted action of the symmetric group of the cohomology ring, and to the degenerate affine Hecke algebra. We give a distinguished basis for this form, which is a deformation of the usual basis of Schubert polynomials, and apply it to the computation of the Schubert cycle expansions of Chern classes of flag manifolds.

*Supported by PRC Math-Info and EEC grant n^0 ERBCHRXCT930400.

1 Introduction and preliminaries

Let V be a complex vector space of dimension n, and $\mathcal{F} = \mathcal{F}(V)$ be the variety of complete flags in V. It is well known that the cohomology ring $H^*(\mathcal{F}, \mathbb{C})$ is the quotient of the polynomial ring $\mathbb{C}[X] = \mathbb{C}[x_1, x_2, \ldots, x_n]$ by the ideal $\mathcal{I}^+$ of symmetric polynomials without constant term.

Let σ_i, $i = 1, \ldots, n-1$ be the simple transposition exchanging x_i and x_{i+1}. Denote by ∂_i the linear operator on $\mathbb{C}[x_1, \ldots, x_n]$ defined by

$$\partial_i f := \frac{f - \sigma_i f}{x_i - x_{i+1}} \tag{1}$$

(Newton's divided difference). The operators $\partial_1, \ldots, \partial_{n-1}$ induce operators on $H^*(\mathcal{F})$.

According to [1] and [4], the basis of Schubert cycles can be obtained from the class of a point $P = \frac{1}{n!}\prod_{i<j}(x_i - x_j)$ by successive applications of divided difference operators. Taking as representative of P the polynomial $X := x_1^{n-1} x_2^{n-2} \cdots x_1^0$, one obtains polynomials X_μ, $\mu \in \mathfrak{S}_n$, called Schubert polynomials, which represent the Schubert subvarieties in the cohomology ring [11]. A detailed account of the algebraic theory of Schubert polynomials can be found in Macdonald's treatise [14].

Divided differences satisfy the braid relations

$$\begin{cases} \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \\ \partial_i \partial_j = \partial_j \partial_i \qquad \text{for} \quad |i-j| > 1 \end{cases} \tag{2}$$

but the squares ∂_i^2 are null. These relations allow to define operators ∂_μ for any permutation $\mu \in \mathfrak{S}_n$: if $\mu = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m}$ is a reduced decomposition of μ, one sets $\partial_\mu = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_m}$. The result does not depend on the choice of a particular reduced decomposition of μ.

To recover an action of the symmetric group, one can take any $q \in \mathbb{C}$ and define

$$D_i := \sigma_i + q\partial_i \ , \qquad 1 \le i \le n-1 \ . \tag{3}$$

These operators still satisfy the braid relations

$$\begin{cases} D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1} \\ D_i D_j = D_j D_i \qquad (|i-j| > 1) \end{cases} \tag{4}$$

together with

$$D_i^2 = 1 \ , \tag{5}$$

so that they generate a representation of the symmetric group $\mathfrak{S}_n$ on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, as well as on the cohomology ring $H^*(\mathcal{F}, \mathbb{C})$. These operators have been considered by Cherednik and Bernstein (*cf.* [2][3]). Similar operators, acting on the equivariant K-theory of flag manifolds, have been used by Lusztig [13]. More general operators satisfying braid relations have been given in [12].

As ∂_i decreases degrees by 1, all $q \neq 0$ will give equivalent representations of $\mathfrak{S}_n$, and by homogeneity, the general case can be recovered from the case $q = 1$. For simplicity, we set $q = 1$, and write

$$s_i := \sigma_i + \partial_i \ . \tag{6}$$

We denote as above by s_μ the product of operators s_i corresponding to a permutation μ. Remark that the operator algebra generated by the s_i and the variables x_j (interpreted as operators $f \mapsto x_j f$) is isomorphic to the degenerate affine Hecke algebra considered in [2].

Schubert calculus for other classical groups can be found in the work of Fulton [7] and of Pragacz and Ratajski [15].

This paper is organized as follows. We first define certain elements (Yang-Baxter operators) of the degenerate affine Hecke algebra. Then we use them to define a bilinear form on the cohomology of a flag manifold. We exhibit a distinguished basis, called *affine Schubert polynomials*, and compute its adjoint basis. We then apply this formalism to the computation of the Schubert expansions of Chern classes.

Acknowledgements The preparation of this paper has been facilitated by the use of the program system SYMMETRICA [9] and of the Maple package SP [16].

2 Yang-Baxter operators

We shall define inductively operators $\square_\mu$ and ∇_μ associated with any permutation μ in $\mathfrak{S}_n$. Set $\square_{12\ldots n} = 1$, $\nabla_{12\ldots n} = 1$, and, if $\mu = \sigma_i\alpha$ with $\ell(\mu) = \ell(\alpha) + 1$, and $\beta = \alpha^{-1}$,

$$\begin{cases} \square_\mu = \left(s_i + \frac{1}{\beta_{i+1}-\beta_i}\right) \square_\alpha \\ \nabla_\mu = \left(s_i - \frac{1}{\beta_{i+1}-\beta_i}\right) \nabla_\alpha \end{cases} \tag{7}$$

Using the braid relations (4), one can check that this definition is consistent, *i.e.* does not depend on the chosen factorization (see [2, 3] and [6]). This follows in fact from a classical solution of the Yang-Baxter equation. In [17], C.N. Yang observed that the operators defined by $Y_i(u) = u^{-1} + \sigma_i$, where u is a scalar parameter and σ_i the transposition $(i, i+1)$ satisfy the "Quantum Yang-Baxter Equation with spectral parameter":

$$Y_i(u-v)Y_{i+1}(u-w)Y_i(v-w) = Y_{i+1}(v-w)Y_i(u-w)Y_{i+1}(u-v) \tag{8}$$

It follows that given a n-uple of parameters $\mathbf{u} = (u_1, \ldots, u_n)$, one can define for any permutation $\mu \in \mathfrak{S}_n$ an operator $R_\mu(\mathbf{u})$ by the following prescription: $R_\mu(\mathbf{u}) = Y_i(u_{\beta(i+1)} - u_{\beta(i)})R_\alpha(\mathbf{u})$, where, as above, $R_{12\ldots n} = 1$, $\mu = \sigma_i\alpha$, $\ell(\mu) = \ell(\alpha) + 1$ and $\beta = \alpha^{-1}$. Then, our operators (7) are respectively $R_\mu(\mathbf{u})$ and $R_\mu(-\mathbf{u})$, where $\mathbf{u} = (1, 2, \ldots, n)$ and σ_i is interpreted as s_i.

For the maximal element $\omega = (n, n-1, \ldots, 1)$ of $\mathfrak{S}_n$, one has the following factorization property (given in [6] for the case of the Hecke algebra):

Proposition 2.1 *Define polynomials* $\theta = \prod_{1\le i<j\le n} (1+x_i-x_j)$ *and* $\theta^* = \prod_{1\le i<j\le n} (1-x_i+x_j)$. *Then, for any polynomial* f,

(i) $\nabla_\omega f = \theta^* \, \partial_\omega f$

(ii) $\square_\omega f = \partial_\omega(\theta f)$.

Proof — Recall that the classes of the Schubert polynomials X_μ, $\mu \in \mathfrak{S}_n$, form a basis of $H^*(\mathcal{F}) = \mathbb{C}[X]/\mathcal{I}^+$. Given μ and i such that $\ell(\mu\sigma_i) > \ell(\mu)$, the polynomial X_μ is symmetrical in x_i and x_{i+1}. As such, it is sent to 0 by the operator $\nabla_{\sigma_i} = \sigma_i + \partial_i - 1$.

Now, for any permutation $\mu \neq \omega$, there exists an i such that $\ell(\mu\sigma_i) > \ell(\mu)$. If we choose a reduced decomposition of ω ending by σ_i, $\omega = \nu\sigma_i$, say, we see that X_μ is sent to 0 by $\partial_\omega = \partial_\nu \partial_{\sigma_i}$ and by $\nabla_\omega = \nabla_\mu \nabla_{\sigma_i}$.

Thus, ∇_ω as well as ∂_ω annihilate all Schubert polynomials X_μ for $\mu \neq \omega$. Finally, $X_\omega = x_1^{n-1} \dots x_n^0$ is sent to 1 by ∂_ω. To conclude, it remains to prove that

$$\nabla_\omega(X_\omega) = \prod_{1 \leq i < j \leq n} (1 - x_i + x_j) \ .$$

This formula can be proved by induction on n using the factorization

$$\omega_n = \sigma_1 \sigma_2 \cdots \sigma_{n-1} \omega_{n-1} \ ,$$

which gives

$$\nabla_{\omega_n} = \Big(s_1 - 1\Big) \cdots \Big(s_{n-1} - \frac{1}{n-1}\Big) \nabla_{\omega_{n-1}} \ .$$

□

3 Quadratic form

Recall that the intersection form of the cohomology ring $H^*(\mathcal{F}\,,\,\mathbb{C})$ is induced by the form on $\mathbb{C}[X]$

$$\langle f\,,\,g \rangle = \partial_\omega(fg) \mid_0 = \partial_\omega(fg \mid_{\ell(\omega)}) \tag{9}$$

where $f \mid_k$ denotes the homogeneous component of degree k of f (*cf.* [1], [5]). With respect to this form, the Schubert polynomials satisfy

$$\langle X_\mu\,,\,X_\nu \rangle = \begin{cases} 1 & \text{if} \ \ \nu = \omega\mu \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

The tangent bundle $T\mathcal{F}$ of the flag manifold has a composition sequence $\{L_i L_j^{-1}\}_{i<j}$ where $L_1, L_2, \dots, L_n$ are the tautological line bundles on $\mathcal{F}$. The total Chern class of L_i being $c(L_i) = 1 + x_i$, the total Chern class of the tangent bundle of $\mathcal{F}$ is

$$c(\mathcal{F}) = \prod_{i<j} (1 + x_i - x_j) \tag{11}$$

(see *e.g.* [8], our convention is $L_i = \xi_i^*$ in the notation of [8]). Consider now the following quadratic form on $\mathbb{C}[X]$:

Definition 3.1

$$(f\,,\,g) := \square_\omega(fg) \mid_0 \ .$$

Thus, in the cohomology ring, we see from Proposition 2.1 that

$$(f\,,\,g) = \langle f\,,\,g\,c(\mathcal{F}) \rangle = \langle f\,c(\mathcal{F})\,,\,g \rangle \ . \tag{12}$$

Lemma 3.2 *The operators* $\square_i$ *are self-adjoint with respect to the quadratic form* $(\ ,\ )$.

Proof — For any i, $\Box_\omega \Box_i = 2\Box_\omega$, since $\Box_i^2 = 2\Box_i$ and since one can find a reduced decomposition of ω ending with σ_i. Now,

$$(\Box_i f\ ,\ g) = \Box_\omega\,((\Box_i f)g)\,|_0$$

$$= \frac{1}{2}\Box_\omega\Box_i\,((\Box_i f)g)\,|_0 = \frac{1}{2}\Box_\omega\,((\Box_i f)(\Box_i g))\,|_0$$

since $\Box_i f$ is a scalar for $\Box_i$, being symmetrical in x_i, x_{i+1}. The last expression being symmetrical in f, g, this proves that $(\Box_i f\ ,\ g) = (f\ ,\ \Box_i g)$ □

4 Affine Schubert polynomials

Let $\mathcal{H}_n$ be the linear subspace of $\mathbb{C}[x_1, \ldots, x_n]$ generated by the monomials $x^I = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ such that $i_k \leq n-k$. Let Π be the projector from $\mathbb{C}[X]$ onto $\mathcal{H}_n$ associating to a polynomial P the unique representative in $\mathcal{H}_n$ of its class $\overline{P} \in \mathbb{C}[X]/\mathcal{I}^+$.

Definition 4.1 *Let $\mu \in \mathfrak{S}_n$. The* affine Schubert polynomial *of index μ is defined by*

$$Z_\mu = \Pi(\Box_{\mu^{-1}\omega} Z_\omega)$$

where $Z_\omega := X_\omega = x_1^{n-1} x_2^{n-2} \cdots x_n^0$.

Example 4.2 For $n = 3$,

$$\begin{aligned}
Z_{321} &= x_1^2 x_2 \\
Z_{312} &= x_1^2 \\
Z_{231} &= x_1 x_2 \\
Z_{213} &= x_1 - 1/2 x_1 x_2 - x_1^2 \\
Z_{132} &= x_1 + x_2 - x_1 x_2 - 1/2 x_1^2 \\
Z_{123} &= 1
\end{aligned}$$

In general, one has $Z_\omega = X_\omega$, $Z_\mu = X_\mu +$(terms of degree $> \ell(\mu)$), and $Z_{id} = 1$, the last identity being due to the fact that $\Box_\omega(Z_\omega)$ is symmetrical with term of lowest degree $X_{id} = 1$.

Example 4.3 For $n = 3$,

$$\begin{aligned}
Z_{321} &= X_{321} \\
Z_{312} &= X_{312} \\
Z_{231} &= X_{231} \\
Z_{213} &= X_{213} - 1/2 X_{231} - X_{312} \\
Z_{132} &= X_{132} - X_{231} - 1/2 X_{312} \\
Z_{123} &= X_{123}
\end{aligned}$$

Theorem 4.4 *The polynomials Z_μ, $\mu \in \mathfrak{S}_n$, form a basis of $\mathcal{H}_n$. The quadratic form $(\ ,\ )$ is positive definite, and the adjoint basis of $\{Z_\mu\}$ is $\{Z_\mu^\vee\}$ where $Z_\mu^\vee = \Pi(\nabla_{\mu^{-1}\omega} X_\omega)$.*

Proof — Z_μ is a non-homogeneous polynomial with the Schubert polynomial X_μ as its term of smallest degree. Since the classes of the Schubert polynomials form a basis of $H^*(\mathcal{F})$, the same is true for the Z_μ.

The polynomials $Z_\omega^\vee Z_\mu \theta$ (for $\mu \neq id$) have no component of degree $\ell(\omega)$. Therefore, their images under ∂_ω are symmetric polynomials without constant term, which proves that for all $\mu \neq id$, $(Z_\omega^\vee\, ,\, Z_\mu) = 0$. On the other hand,

$$(Z_\omega^\vee\, ,\, Z_{12\ldots n}) = (Z_\omega^\vee\, ,\, 1)$$
$$= \partial_\omega \left(X_\omega \theta\right)|_0 = \partial_\omega \Big((X_\omega\theta)|_{\ell(\omega)}\Big)$$
$$= \partial_\omega X_\omega = X_{12\ldots n} = 1\ .$$

For the general case of a $Z_\nu^\vee$, one uses induction on the length of ν. Let ν and i be such that $\ell(\nu\sigma_i) < \ell(\nu)$. Then, for any μ,

$$(Z_\mu\, ,\, Z_\nu^\vee) = (Z_\mu\, ,\, (s_i - k) Z_\nu^\vee)$$

(where k is an appropriate constant)

$$= ((s_i - k)Z_\mu\, ,\, Z_\nu^\vee) = k'(Z_\mu\, ,\, Z_\nu^\vee) + k''(Z_{\mu\sigma_i}\, ,\, Z_\nu^\vee)$$

(for some other scalars k', k''). By induction, one can suppose $(Z_\mu\, ,\, Z_\nu^\vee) = 0$ for $\mu\nu^{-1} \neq \omega$.

One is thus reduced to study the case

$$\mu\sigma_i\nu^{-1} = \omega\ ,\quad \ell(\mu\sigma_i) > \ell(\mu)\ .$$

In that case,

$$Z_{\mu\sigma_i} = \left(s_i + \frac{1}{r}\right) Z_\mu \quad\text{and}\quad Z_{\mu\sigma_i}^\vee = \left(s_i - \frac{1}{r}\right) Z_\mu^\vee$$

for a certain integer r. Then, we check

$$(Z_{\mu\sigma_i}\, ,\, Z_{\nu\sigma_i}^\vee) = \left(\left(s_i + \frac{1}{r}\right) Z_\mu\, ,\, \left(s_i - \frac{1}{r}\right) Z_\mu^\vee\right)$$
$$= \left(\left(s_i^2 - \frac{1}{r^2}\right) Z_\mu\, ,\, Z_\nu^\vee\right) = 0\ ,$$

and

$$(Z_\mu\, ,\, Z_{\nu\sigma_i}^\vee) = \left(\left(s_i + \frac{1}{r} - \frac{2}{r}\right) Z_\mu\, ,\, Z_\nu^\vee\right)$$
$$= (Z_{\mu\sigma_i}\, ,\, Z_\nu^\vee) - \frac{2}{r}(Z_\mu\, ,\, Z_\nu^\vee) = 1 - 0\ .$$

□

Example 4.5 Again for $n = 3$,

$$\begin{aligned}
Z_{321}^\vee &= x_1^2 x_2\\
Z_{312}^\vee &= x_1^2 - 2x_1^2 x_2\\
Z_{231}^\vee &= x_1 x_2 - 2x_1^2 x_2\\
Z_{213}^\vee &= x_1 - 3/2 x_1 x_2 - 3x_1^2 + 3x_1^2 x_2\\
Z_{132}^\vee &= x_1 + x_2 - 3x_1x_2 - 3/2 x_1^2 + 3x_1^2 x_2\\
Z_{123}^\vee &= 1 - 4x_1 - 2x_2 + 6x_1x_2 + 6x_1^2 - 6x_1^2 x_2
\end{aligned}$$

5 Change of basis

The operators ∂_i are self-adjoint with respect to $\langle\ ,\ \rangle$, but σ_i is adjoint to $-\sigma_i$. This implies that $-s_i$ is adjoint to $\bar{s}_i := \sigma_i - \partial_i$.

Let us define $\overline{\square}_\mu$, $\overline{\nabla}_\mu$ to be the images of $\square_\mu$ and ∇_μ under the replacement $s_i \mapsto \bar{s}_i$. We also define

$$\overline{Z}_\mu := (-1)^{\ell(\omega\mu)} \Pi \left(\overline{\square}_{\mu^{-1}\omega} Z_\omega\right)$$

$$\overline{Z}^\vee_\mu := (-1)^{\ell(\omega\mu)} \Pi \left(\overline{\nabla}_{\mu^{-1}\omega} Z_\omega\right)$$

Then, $(-1)^{\ell(\mu)}\overline{Z}_\mu$ is obtained from Z_μ under the transformation $x_i \mapsto -x_i$, since signs in the expansion of Z_μ correspond to degree.

Lemma 5.1 *$\{\overline{Z}_{\omega\mu}\}$ is the adjoint basis of $\{Z_\mu\}$ with respect to $\langle\ ,\ \rangle$,* i.e. *one has $\langle \overline{Z}_{\omega\mu}\, , \, Z_\mu\rangle = 1$ and $\langle \overline{Z}_{\omega\mu}\, , \, Z_\nu\rangle = 0$ for $\nu \neq \mu$.*

Similarly, $\{\overline{Z}^\vee_{\omega\mu}\}$ is the adjoint basis of $\{Z^\vee_\mu\}$ for $\langle\ ,\ \rangle$.

Proof — As in section 4, the lemma is proved by induction on the length of μ, starting from the case

$$\langle Z_{\omega\mu}\, , \, \overline{Z}_\omega\rangle = 0 \ \text{ if } \ \mu \neq \omega\, .$$

Take i such that $\ell(\mu\sigma_i) > \ell(\mu)$. Then,

$$\langle Z_{\omega\mu\sigma_i}\, , \, \overline{Z}_\nu\rangle = \left\langle \left(s_i + \frac{1}{r}\right) Z_{\omega\mu}\, \overline{Z}_\nu \right\rangle = \left\langle Z_{\omega\mu}\, , \, \left(-\bar{s}_i + \frac{1}{r}\right) \overline{Z}_\nu \right\rangle \ .$$

Since $(-\bar{s}_i + \frac{1}{r})\overline{Z}_\nu$ is a linear combination of $\overline{Z}_\nu$ and $\overline{Z}_{\nu\sigma_i}$, the nullity of the scalar products $\overline{Z}Z_{\omega\mu\sigma_i}\, , \, \overline{Z}_\nu\rangle$ follows from those of $\langle Z_{\omega\mu}\, , \, Z_\nu\rangle$ for $\nu \neq \mu$ and $\nu \neq \mu\sigma_i$. In the special case $\nu = \mu\sigma_i$, one has

$$\langle Z_{\omega\mu\sigma_i}\, , \, \overline{Z}_\mu\rangle = \left\langle Z_{\omega\mu}\, , \, \left(-\bar{s}_i + \frac{1}{r}\right)\left(-\bar{s}_i - \frac{1}{r}\right) Z_{\mu\sigma_i} \right\rangle$$

$$= \left\langle Z_{\omega\mu}\, , \, \left(1 - \frac{1}{r^2}\right) Z_{\mu\sigma_i} \right\rangle$$

which is null. □

Example 5.2

$$\langle Z_{23514}\, , \, \overline{Z}_{41352}\rangle = \left\langle \left(s_2 + \frac{1}{2}\right) Z_{2\mathbf{53}14}\, , \, \left(-\bar{s}_2 - \frac{1}{2}\right)\overline{Z}_{4\mathbf{31}52} \right\rangle$$

$$= \left\langle \left(s_2 - \frac{1}{2}\right)\left(s_2 + \frac{1}{2}\right) Z_{25314}\, , \, \overline{Z}_{43152} \right\rangle$$

$$= \left\langle \left(1 - \frac{1}{4}\right) Z_{25314}\, , \, \overline{Z}_{43152} \right\rangle = 0\ .$$

Let $\{A_\mu\}$ and $\{B_\nu\}$ be two bases of $\mathcal{H}_n$. We denote by $M(A,B)$ the transition matrix from the basis $\{A_\mu\}$ to the basis $\{B_\nu\}$, with the convention

$$A_\mu = \sum_\nu M(A,B)_{\mu\nu} B_\nu \ . \tag{13}$$

For example, $M(Z,X)_{\mu\nu} = \langle Z_\mu \, , \, X_{\omega\nu}\rangle$ and $M(X,Z)_{\mu\nu} = (X_\mu \, , \, Z^\vee_{\omega\nu})$. These matrices have a symmetry property , thanks to the following property of the scalar product:

$$\langle \omega P \, , \, \omega Q\rangle = (-1)^{\ell(\omega)} \langle P \, , \, Q\rangle \ . \tag{14}$$

Indeed, taking into account the two identities

$$\omega(X_\mu) = (-1)^{\ell(\mu)} X_{\omega\mu\omega} \ , \quad \omega(Z_\mu) = (-1)^{\ell(\mu)} \overline{Z}_{\omega\mu\omega} \tag{15}$$

we see that the four matrices

$$M(Z,X) \ , \ M(X,Z) \ , \ M(Z^\vee, X) \text{ and } M(X, Z^\vee)$$

possess the symmetry

$$M_{\mu\nu} = M_{\omega\mu\omega,\omega\nu\omega} \ . \tag{16}$$

Furthermore, we have the following relation between these matrices and their inverses:

Theorem 5.3 *The inverse of $M(Z,X)$ is a matrix with nonnegative entries, given by*

$$M(X,Z)_{\mu\nu} = |M(Z,X)_{\nu\omega,\mu\omega}| \ .$$

Similarly,

$$M(X,Z^\vee)_{\mu\nu} = |M(Z^\vee,X)_{\nu\omega,\mu\omega}| \ ,$$

where $|\cdot|$ denotes the absolute value.

Proof — The first matrix corresponds to the expansions

$$Z_\mu = \sum_\nu \langle Z_\mu \, , \, X_{\omega\nu}\rangle X_\nu \ .$$

The inverse formulas are, according to Lemma 5.1,

$$X_\nu = \sum_\mu \langle \overline{Z}_{\omega\mu} \, , \, X_\nu\rangle Z_\mu \ .$$

But now, $\langle \overline{Z}_{\omega\mu} \, , \, X_\nu\rangle = |\langle Z_{\omega\mu} \, , \, X_\nu\rangle|$, whence the first part of the theorem. The second part is similar. □

Thus, the inverse of the matrix $M(Z,X)$ is obtained from $M(Z,X)$ by reflection through the antidiagonal $(\mu,\nu) \longrightarrow (\omega\nu, \omega\mu)$ and supression of the signs.

Corollary 5.4 *For any pair of permutations,*

$$(X_\mu \, , \, Z^\vee_\eta) = |\langle X_{\omega\mu\omega} \, , \, Z_{\omega\eta\omega}\rangle|$$

and

$$(X_\mu \, , \, Z_\eta) = |\langle X_{\omega\mu\omega} \, , \, Z^\vee_{\omega\eta\omega}\rangle| \ .$$

Indeed,

$$X_\mu = \sum_\eta (X_\mu \, , \, Z^\vee_\eta) Z_{\omega\eta}$$

but we have just seen that the coefficients of the expansion of X_μ in the basis $Z_{\omega\eta}$ are the $\langle \overline{Z}_\eta \, , \, X_\mu\rangle$. □

6 Schubert expansions of Chern classes

Let $|I|$ be a composition of n, *i.e.* $I \in \mathbb{N}^r$ with $|I| := i_1 + \ldots + i_r = n$, and let $J_1, \ldots, J_r$ be the associated decomposition of the interval $[1, n]$, that is

$$J_1 = [1, i_1] \ , \quad J_2 = [i_1 + 1, i_1 + i_2] \ , \quad \ldots \ , \quad J_r = [i_1 + \cdots + i_{r-1} + 1, n] \ .$$

Let $\mathcal{F}_I$ be the variety of flags

$$V_0 = \{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_r = V$$

such that $\dim V_k = i_1 + \cdots + i_k$. Let also $\mathfrak{S}_I$ denote the Young subgroup

$$\mathfrak{S}(J_1) \times \mathfrak{S}(J_2) \times \cdots \times \mathfrak{S}(J_r) \ \subset \mathfrak{S}_n$$

associated to the composition I. The Chern class θ^I of the tangent bundle of $\mathcal{F}_I$ is the maximal $\mathfrak{S}_I$-invariant factor of θ:

$$\theta^I = \theta / (\theta_{J_1} \theta_{J_2} \cdots \theta_{J_r})$$

where

$$\theta_{J_k} = \prod_{\substack{i<j \\ i,j \in J_k}} (1 + x_i - x_j) \ .$$

A basis of the cohomology ring $H^*(\mathcal{F}_I)$ is the set of Schubert polynomials X_μ with μ minimal in its right coset $\mu \mathfrak{S}_I$. In other words, one restricts the Schubert basis (X_μ) to those μ such that $\mu_1 < \ldots < \mu_{i_1}$, $\mu_{i_1+1} < \ldots < \mu_{i_1+i_2}$, …, $\mu_{i_1+\cdots+i_{r-1}+1} < \ldots < \mu_n$. Since $\mu_i < \mu_{i+1}$ iff X_μ is symmetrical in x_i and x_{i+1}, the Schubert basis of $H^*(\mathcal{F}_I)$ consists of those Schubert polynomials which are invariant under $\mathfrak{S}_I$.

Define the *Chern coefficient* $c_\mu[\mathcal{F}_I]$ of the variety $\mathcal{F}_I$ as the coefficient of (the class) of X_μ in the expansion of θ^I on the Schubert basis of $H^*(\mathcal{F}_I)$. In other words,

$$c_\mu[\mathcal{F}_I] = \langle \theta^I \ , \ X_{\omega\mu} \rangle \ . \tag{17}$$

As in the case of the full flag variety $\mathcal{F}$, these scalar products can be computed with the help of the scalar product $(\ ,\ )$.

Let ω_I be the maximal element of $\mathfrak{S}_I$, and $\zeta_I := \omega_I \omega$. We have seen that

$$\Box_\omega = \partial_\omega \theta = \partial_{\zeta_I} \partial_{\omega_I} \theta^I \theta_{J_1} \theta_{J_2} \cdots \theta_{J_r} \ .$$

The operator ∂_ω factorizes

$$\partial_\omega = \partial_{\zeta_i} \partial_{\omega_I}$$

so that

$$c_\mu[\mathcal{F}_I] = \partial_\omega \left(\theta^I X_{\omega\mu} \right)$$

$$= \partial_{\zeta_i} \partial_{\omega_i} \left(\theta^I X_{\omega\mu} \right) = \partial_{\zeta_i} \left(\theta^I \partial_{\omega_i} X_{\omega\mu} \right) \tag{18}$$

$$= \partial_{\zeta_I} \left(\theta^I X_{\omega\mu\omega_I} \right) = \partial_{\zeta_i} \left(\theta^I X_{\omega\mu\omega_I} \right) (\partial_{\omega_I} \theta_{J_1} \theta_{J_2} \cdots \theta_{J_r} / I!) \tag{19}$$

$$= \frac{1}{I!} \partial_\omega \left(\theta \, X_{\omega\mu\omega_I} \right) \tag{20}$$

$$= \frac{1}{I!}(1\,,\,X_{\omega\mu\omega_I})$$

Equality (18) follows from the fact that θ^I is invariant under $\mathfrak{S}_I$ and thus commutes with ∂_{ω_I}. Now, θ is of degree $\binom{n}{2}$, and ∂_ω decreases degrees by $\ell(\omega) = \binom{n}{2}$. Thus $\partial_\omega(\theta)$ is a scalar which is checked to be $n!$. More generally, by direct product, one has for the maximal element of the Young subgroup $\mathfrak{S}_I$

$$\partial_{\omega_I}\theta_{J_1}\cdots\theta_{J_r} = I! := i_1!\cdots i_r!$$

and equality (19) follows from this identity.

Since θ^I as well as $X_{\omega\mu\omega_I}$ are invariant under $\mathfrak{S}_I$, they commute with ∂_{ω_I}, which is step (20).

Summarizing, we have the following expression for the components of the Chern class of $\mathcal{F}_I$ on the Schubert basis.

Theorem 6.1 *Let $I = (i_1, \ldots, i_r)$ be a composition of n, $\mathfrak{S}_I$ and $\mathcal{F}_I$ the corresponding Young subgroup and flag variety. Let μ be a permutation which is minimum in its coset $\mu\mathfrak{S}_I$. Then, the Chern coefficient $c_\mu[\mathcal{F}_I]$ is given by*

$$c_\mu[\mathcal{F}_I] = (1\,,\,X_{\omega\mu\omega_I})/I!\ .$$

In particular, for the full flag variety (case $I = (1, 1, \ldots, 1)$), one has

$$c_\mu[\mathcal{F}] = (1\,,\,X_{\omega\mu}) = \Box_\omega(X_{\omega\mu}) \tag{21}$$

and these numbers constitute the first column of the matrix $M(X, Z^\vee)$. Equivalently, they are equal to the absolute values of the entries of the last row of $M(Z^\vee, X)$.

In the case of a Grassmann manifold $G(p, p+q) = \mathcal{F}_{(p,q)}$, the basis of $H^*(\mathcal{F}_{(p,q)})$ consists of those X_μ for which $\mu_1 < \ldots < \mu_p$ and $\mu_{p+1} < \ldots < \mu_{p+q}$ (Grassmannian permutations). In fact, for such a permutation, X_μ is equal to the Schur function indexed by the partition $(\mu_1 - 1, \mu_2 - 2, \ldots, \mu_p - p)$ on the set of variables $\{x_1, \ldots, x_p\}$. Thus, the Chern coefficient $c_\mu[\mathcal{F}_{(p,q)}]$ is

$$c_\mu[\mathcal{F}_{(p,q)}] = \Box_\omega\left(X_{(n+1-\mu_p,\ldots,n+1-\mu_1,n+1-\mu_n,\ldots,n+1-\mu_{p+1})}\right)\ . \tag{22}$$

For example, up to a factor $(2!)^2$, the Chern coefficients of $\mathcal{F}_{(2,2)}$ are $4, 16, 28, 28, 48, 24$. They are given by the absolute values of the six entries of the bottom row of the matrix $M(Z^\vee, X)$ corresponding to columns indexed by permutations $\omega\mu$ where μ is Grassmannian.

7 Tables for $n = 4$

7.1 Affine Schubert polynomials

$Z_{4321} = x_1^3x_2^2x_3$
$Z_{4312} = x_1^3x_2^2$
$Z_{4231} = x_1^3x_2x_3$
$Z_{4213} = x_1^3x_2 - 1/2\, x_1^3x_2x_3 - x_1^3x_2^2$
$Z_{4132} = x_1^3x_3 + x_1^3x_2 - x_1^3x_2x_3 - 1/2\, x_1^3x_2^2$
$Z_{4123} = x_1^3$
$Z_{3421} = x_1^2x_2^2x_3$
$Z_{3412} = x_1^2x_2^2$
$Z_{3241} = x_1^2x_2x_3 - 1/2\, x_1^2x_2^2x_3 - x_1^3x_2x_3$
$Z_{3214} = x_1^2x_2 - 2/3\, x_1^2x_2x_3 - 3/2\, x_1^2x_2^2 + 1/3\, x_1^2x_2^2x_3 - 2\, x_1^3x_2 + 2/3\, x_1^3x_2x_3 + x_1^3x_2^2$
$Z_{3142} = x_1^2x_3 + x_1^2x_2 - x_1^2x_2x_3 - 2/3\, x_1^2x_2^2 - x_1^3x_3 - x_1^3x_2$
$Z_{3124} = x_1^2 - 1/2\, x_1^2x_3 - 1/2\, x_1^2x_2 + 1/2\, x_1^2x_2x_3 - 2\, x_1^3 + 1/2\, x_1^3x_3 + 1/2\, x_1^3x_2$
$Z_{2431} = x_1x_2^2x_3 + x_1^2x_2x_3 - x_1^2x_2^2x_3 - 1/2\, x_1^3x_2x_3$
$Z_{2413} = x_1x_2^2 - 1/2\, x_1x_2^2x_3 + x_1^2x_2 - 1/2\, x_1^2x_2x_3 - 2\, x_1^2x_2^2 + 1/2\, x_1^2x_2^2x_3 - 1/2\, x_1^3x_2 + 1/4\, x_1^3x_2x_3 + 1/2\, x_1^3x_2^2$
$Z_{2341} = x_1x_2x_3$
$Z_{2314} = x_1x_2 - 2/3\, x_1x_2x_3 - x_1x_2^2 - x_1^2x_2$
$Z_{2143} = x_1x_3 + x_1x_2 - 3/2\, x_1x_2x_3 - 2/3\, x_1x_2^2 + 1/3\, x_1x_2^2x_3 + x_1^2 - 2\, x_1^2x_3 - 8/3\, x_1^2x_2 + 7/3\, x_1^2x_2x_3 + 4/3\, x_1^2x_2^2 - 1/3\, x_1^2x_2^2x_3 - 3/2\, x_1^3 + 2\, x_1^3x_3 + 7/3\, x_1^3x_2 - 2/3\, x_1^3x_2x_3 - 1/3\, x_1^3x_2^2$
$Z_{2134} = x_1 - 1/2\, x_1x_2 + 1/2\, x_1x_2^2 - x_1^2 + 1/2\, x_1^2x_2 + x_1^3$
$Z_{1432} = x_2^2x_3 + x_1x_2x_3 + x_1x_2^2 - 2\, x_1x_2^2x_3 + x_1^2x_3 + x_1^2x_2 - 2\, x_1^2x_2x_3 - 3/2\, x_1^2x_2^2 + x_1^2x_2^2x_3 - 2/3\, x_1^3x_3 - 2/3\, x_1^3x_2 + 2/3\, x_1^3x_2x_3 + 1/3\, x_1^3x_2^2$
$Z_{1423} = x_2^2 + x_1x_2 - x_1x_2^2 + x_1^2 - x_1^2x_2 - 2/3\, x_1^3$
$Z_{1342} = x_2x_3 + x_1x_3 + x_1x_2 - 2\, x_1x_2x_3 - 1/2\, x_1^2x_3 - 1/2\, x_1^2x_2 + 1/2\, x_1^2x_2x_3 + 1/2\, x_1^3x_3 + 1/2\, x_1^3x_2$
$Z_{1324} = x_2 - 1/2\, x_2x_3 - x_2^2 + x_1 - 1/2\, x_1x_3 - 5/2\, x_1x_2 + x_1x_2x_3 + 2\, x_1x_2^2 - 3/2\, x_1^2 + 1/4\, x_1^2x_3 + 9/4\, x_1^2x_2 - 1/4\, x_1^2x_2x_3 - 1/2\, x_1^2x_2^2 + x_1^3 - 1/4\, x_1^3x_3 - 1/4\, x_1^3x_2$
$Z_{1243} = x_3 + x_2 - x_2x_3 - 1/2\, x_2^2 + x_1 - x_1x_3 - 3/2\, x_1x_2 + x_1x_2x_3 + 1/2\, x_1x_2^2 - 1/2\, x_1^2 + 1/2\, x_1^2x_2$
$Z_{1234} = 1$

7.2 Adjoint polynomials

$$Z^{\vee}_{4321} = x_1^3x_2^2x_3$$
$$Z^{\vee}_{4312} = x_1^3x_2^2 - 2\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{4231} = x_1^3x_2x_3 - 2\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{4213} = x_1^3x_2 - 3/2\, x_1^3x_2x_3 - 3\, x_1^3x_2^2 + 3\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{4132} = x_1^3x_3 + x_1^3x_2 - 3\, x_1^3x_2x_3 - 3/2\, x_1^3x_2^2 + 3\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{4123} = x_1^3 - 2\, x_1^3x_3 - 4\, x_1^3x_2 + 6\, x_1^3x_2x_3 + 6\, x_1^3x_2^2 - 6\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{3421} = x_1^2x_2^2x_3 - 2\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{3412} = x_1^2x_2^2 - 2\, x_1^2x_2^2x_3 - 2\, x_1^3x_2^2 + 4\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{3241} = x_1^2x_2x_3 - 3/2\, x_1^2x_2^2x_3 - 3\, x_1^3x_2x_3 + 3\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{3214} = x_1^2x_2 - 4/3\, x_1^2x_2x_3 - 5/2\, x_1^2x_2^2 + 2\, x_1^2x_2^2x_3 - 4\, x_1^3x_2 + 4\, x_1^3x_2x_3 + 6\, x_1^3x_2^2 - 4\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{3142} = x_1^2x_3 + x_1^2x_2 - 3\, x_1^2x_2x_3 - 4/3\, x_1^2x_2^2 + 8/3\, x_1^2x_2^2x_3 - 3\, x_1^3x_3 - 3\, x_1^3x_2 + 8\, x_1^3x_2x_3 + 8/3\, x_1^3x_2^2 - 16/3\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{3124} = x_1^2 - 3/2\, x_1^2x_3 - 7/2\, x_1^2x_2 + 9/2\, x_1^2x_2x_3 + 5\, x_1^2x_2^2 - 4\, x_1^2x_2^2x_3 - 4\, x_1^3 + 9/2\, x_1^3x_3 + 25/2\, x_1^3x_2 - 12\, x_1^3x_2x_3 - 12\, x_1^3x_2^2 + 8\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{2431} = x_1x_2^2x_3 + x_1^2x_2x_3 - 3\, x_1^2x_2^2x_3 - 3/2\, x_1^3x_2x_3 + 3\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{2413} = x_1x_2^2 - 3/2\, x_1x_2^2x_3 + x_1^2x_2 - 3/2\, x_1^2x_2x_3 - 4\, x_1^2x_2^2 + 9/2\, x_1^2x_2^2x_3 - 3/2\, x_1^3x_2 + 9/4\, x_1^3x_2x_3 + 9/2\, x_1^3x_2^2 - 9/2\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{2341} = x_1x_2x_3 - 2\, x_1x_2^2x_3 - 4\, x_1^2x_2x_3 + 6\, x_1^2x_2^2x_3 + 6\, x_1^3x_2x_3 - 6\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{2314} = x_1x_2 - 4/3\, x_1x_2x_3 - 3\, x_1x_2^2 + 8/3\, x_1x_2^2x_3 - 5\, x_1^2x_2 + 16/3\, x_1^2x_2x_3 + 10\, x_1^2x_2^2 - 8\, x_1^2x_2^2x_3 + 8\, x_1^3x_2 - 8\, x_1^3x_2x_3 - 12\, x_1^3x_2^2 + 8\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{2143} = x_1x_3 + x_1x_2 - 5/2\, x_1x_2x_3 - 4/3\, x_1x_2^2 + 2\, x_1x_2^2x_3 + x_1^2 - 4\, x_1^2x_3 - 16/3\, x_1^2x_2 + 9\, x_1^2x_2x_3 + 16/3\, x_1^2x_2^2 - 6\, x_1^2x_2^2x_3 - 5/2\, x_1^3 + 7\, x_1^3x_3 + 9\, x_1^3x_2 - 12\, x_1^3x_2x_3 - 6\, x_1^3x_2^2 + 6\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{2134} = x_1 - 2\, x_1x_3 - 7/2\, x_1x_2 + 5\, x_1x_2x_3 + 9/2\, x_1x_2^2 - 4\, x_1x_2^2x_3 - 5\, x_1^2 + 8\, x_1^2x_3 + 31/2\, x_1^2x_2 - 18\, x_1^2x_2x_3 - 15\, x_1^2x_2^2 + 12\, x_1^2x_2^2x_3 + 11\, x_1^3 - 14\, x_1^3x_3 - 26\, x_1^3x_2 + 24\, x_1^3x_2x_3 + 18\, x_1^3x_2^2 - 12\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{1432} = x_2^2x_3 + x_1x_2x_3 + x_1x_2^2 - 4\, x_1x_2^2x_3 + x_1^2x_3 + x_1^2x_2 - 4\, x_1^2x_2x_3 - 5/2\, x_1^2x_2^2 + 6\, x_1^2x_2^2x_3 - 4/3\, x_1^3x_3 - 4/3\, x_1^3x_2 + 4\, x_1^3x_2x_3 + 2\, x_1^3x_2^2 - 4\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{1423} = x_2^2 - 2\, x_2^2x_3 + x_1x_2 - 2\, x_1x_2x_3 - 5\, x_1x_2^2 + 8\, x_1x_2^2x_3 + x_1^2 - 2\, x_1^2x_3 - 5\, x_1^2x_2 + 8\, x_1^2x_2x_3 + 10\, x_1^2x_2^2 - 12\, x_1^2x_2^2x_3 - 4/3\, x_1^3 + 8/3\, x_1^3x_3 + 16/3\, x_1^3x_2 - 8\, x_1^3x_2x_3 - 8\, x_1^3x_2^2 + 8\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{1342} = x_2x_3 - 2\, x_2^2x_3 + x_1x_3 + x_1x_2 - 6\, x_1x_2x_3 - 2\, x_1x_2^2 + 8\, x_1x_2^2x_3 - 7/2\, x_1^2x_3 - 7/2\, x_1^2x_2 + 25/2\, x_1^2x_2x_3 + 5\, x_1^2x_2^2 - 12\, x_1^2x_2^2x_3 + 9/2\, x_1^3x_3 + 9/2\, x_1^3x_2 - 12\, x_1^3x_2x_3 - 4\, x_1^3x_2^2 + 8\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{1324} = x_2 - 3/2\, x_2x_3 - 3\, x_2^2 + 3\, x_2^2x_3 + x_1 - 3/2\, x_1x_3 - 15/2\, x_1x_2 + 9\, x_1x_2x_3 + 13\, x_1x_2^2 - 12\, x_1x_2^2x_3 - 9/2\, x_1^2 + 21/4\, x_1^2x_3 + 73/4\, x_1^2x_2 - 75/4\, x_1^2x_2x_3 - 22\, x_1^2x_2^2 + 18\, x_1^2x_2^2x_3 + 6\, x_1^3 - 27/4\, x_1^3x_3 - 75/4\, x_1^3x_2 + 18\, x_1^3x_2x_3 + 18\, x_1^3x_2^2 - 12\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{1243} = x_3 + x_2 - 3\, x_2x_3 - 3/2\, x_2^2 + 3\, x_2^2x_3 + x_1 - 5\, x_1x_3 - 13/2\, x_1x_2 + 14\, x_1x_2x_3 + 15/2\, x_1x_2^2 - 12\, x_1x_2^2x_3 - 7/2\, x_1^2 + 11\, x_1^2x_3 + 31/2\, x_1^2x_2 - 26\, x_1^2x_2x_3 - 15\, x_1^2x_2^2 + 18\, x_1^2x_2^2x_3 + 5\, x_1^3 - 14\, x_1^3x_3 - 18\, x_1^3x_2 + 24\, x_1^3x_2x_3 + 12\, x_1^3x_2^2 - 12\, x_1^3x_2^2x_3$$
$$Z^{\vee}_{1234} = 1 - 2\, x_3 - 4\, x_2 + 6\, x_2x_3 + 6\, x_2^2 - 6\, x_2^2x_3 - 6\, x_1 + 10\, x_1x_3 + 22\, x_1x_2 - 28\, x_1x_2x_3 - 26\, x_1x_2^2 + 24\, x_1x_2^2x_3 + 16\, x_1^2 - 22\, x_1^2x_3 - 48\, x_1^2x_2 + 52\, x_1^2x_2x_3 + 44\, x_1^2x_2^2 - 36\, x_1^2x_2^2x_3 - 22\, x_1^3 + 28\, x_1^3x_3 + 52\, x_1^3x_2 - 48\, x_1^3x_2x_3 - 36\, x_1^3x_2^2 + 24\, x_1^3x_2^2x_3$$

7.3 Transition matrices with Schubert polynomials

The following matrices give the decompositions of the polynomials Z_μ and $Z_\mu^\vee$ in the basis of Schubert polynomials. Rows and columns are indexed by permutations in reverse lexicographic order:

[4321, 4312, 4231, 4213, 4132, 4123, 3421, 3412, 3241, 3214, 3142, 3124,

2431, 2413, 2341, 2314, 2143, 2134, 1432, 1423, 1342, 1324, 1243, 1234]

The bar over a number is to be interpreted as a minus sign.

7.3.1 $M(Z, X)$

The entry in row μ and column ν of the following matrix is equal to the coefficient of X_ν in Z_μ. This number is also the coefficient of $Z^\vee_{\omega\mu}$ in $X_{\omega\nu}$.

$$
\begin{array}{cccccccccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \overline{1} & \overline{1}/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \overline{1}/2 & \overline{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \overline{1} & 0 & 0 & 0 & \overline{1}/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2/3 & \overline{2} & 0 & 0 & 1/3 & \overline{3}/2 & \overline{2}/3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \overline{1} & 0 & 0 & \overline{2}/3 & \overline{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & \overline{2} & 0 & 0 & 1/2 & 0 & \overline{1}/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \overline{1}/2 & 0 & 0 & 0 & \overline{1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/2 & 1/4 & \overline{1}/2 & 0 & 0 & 1/2 & \overline{2} & 0 & 0 & 0 & 0 & \overline{1}/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{1} & \overline{2}/3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \overline{1}/3 & \overline{2}/3 & 1/3 & 2 & \overline{3}/2 & \overline{1}/3 & 4/3 & 2 & 0 & \overline{2} & 0 & 1/3 & \overline{2}/3 & \overline{3}/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \overline{1} & 0 & 1/2 & 0 & \overline{1}/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/3 & 2/3 & 0 & \overline{2}/3 & 0 & 1 & \overline{3}/2 & 0 & 0 & 0 & 0 & \overline{2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \overline{2}/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & \overline{1}/2 & 0 & 0 & 0 & \overline{2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \overline{1}/4 & 1 & 0 & \overline{1}/2 & \overline{1}/4 & 0 & 1/4 & \overline{1}/2 & 0 & 2 & 1 & \overline{1} & 0 & 0 & 0 & \overline{1} & \overline{1}/2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1 & 0 & 0 & 0 & 0 & \overline{1}/2 & \overline{1} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

7.3.2 $M(Z^\vee, X)$

The entry in row μ and column ν of the following matrix is equal to the coefficient of X_ν in $Z^\vee_\mu$. This number is also the coefficient of $Z_{\omega\mu}$ in $X_{\omega\nu}$.

$$
\begin{array}{llllllllllllllllllllllll}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & \overline{3} & \overline{3}/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & \overline{3}/2 & \overline{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{6} & 6 & 6 & \overline{2} & \overline{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & \overline{2} & 0 & 0 & 0 & 0 & \overline{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & \overline{3} & 0 & 0 & 0 & \overline{3}/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{4} & 6 & 4 & \overline{4} & 0 & 0 & 2 & \overline{5}/2 & \overline{4}/3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{16}/3 & 8/3 & 8 & 0 & \overline{3} & 0 & 8/3 & \overline{4}/3 & \overline{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & \overline{12} & \overline{12} & 8 & 9/2 & \overline{4} & \overline{4} & 5 & 9/2 & \overline{2} & \overline{3}/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & \overline{3}/2 & 0 & 0 & 0 & \overline{3} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{9}/2 & 9/2 & 9/4 & \overline{3}/2 & 0 & 0 & 9/2 & \overline{4} & 0 & 0 & 0 & 0 & \overline{3}/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{6} & 0 & 6 & 0 & 0 & 0 & 6 & 0 & \overline{2} & 0 & 0 & 0 & \overline{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & \overline{12} & \overline{8} & 8 & 0 & 0 & \overline{8} & 10 & 8/3 & \overline{2} & 0 & 0 & 8/3 & \overline{3} & \overline{4}/3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & \overline{6} & \overline{12} & 2 & 7 & \overline{5}/2 & \overline{6} & 16/3 & 7 & 0 & \overline{4} & 0 & 2 & \overline{4}/3 & \overline{5}/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{12} & 18 & 24 & \overline{12} & \overline{14} & 11 & 12 & \overline{15} & \overline{14} & 3 & 8 & \overline{3} & \overline{4} & 9/2 & 5 & \overline{3}/2 & \overline{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{4} & 2 & 4 & 0 & \overline{4}/3 & 0 & 6 & \overline{5}/2 & 0 & 0 & 0 & 0 & \overline{4} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
8 & \overline{8} & \overline{8} & 8/3 & 8/3 & \overline{4}/3 & \overline{12} & 10 & 0 & 0 & 0 & 0 & 8 & \overline{3} & 0 & 0 & 0 & 0 & \overline{2} & 1 & 0 & 0 & 0 & 0 \\
8 & \overline{4} & \overline{12} & 0 & 9/2 & 0 & \overline{12} & 5 & 9/2 & 0 & \overline{3}/2 & 0 & 8 & 0 & \overline{4} & 0 & 0 & 0 & \overline{2} & 0 & 1 & 0 & 0 & 0 \\
\overline{12} & 18 & 18 & \overline{12} & \overline{27}/4 & 6 & 18 & \overline{22} & \overline{27}/4 & 3 & 9/4 & \overline{3}/2 & \overline{12} & 10 & 6 & \overline{3} & 0 & 0 & 3 & \overline{3} & \overline{3}/2 & 1 & 0 & 0 \\
\overline{12} & 12 & 24 & \overline{4} & \overline{14} & 5 & 18 & \overline{15} & \overline{14} & 0 & 8 & 0 & \overline{12} & 9/2 & 11 & 0 & \overline{2} & 0 & 3 & \overline{3}/2 & \overline{3} & 0 & 1 & 0 \\
24 & \overline{36} & \overline{48} & 24 & 28 & \overline{22} & \overline{36} & 44 & 28 & \overline{6} & \overline{16} & 6 & 24 & \overline{20} & \overline{22} & 6 & 4 & \overline{2} & \overline{6} & 6 & 6 & \overline{2} & \overline{2} & 1
\end{array}
$$

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