Lattice graphs and Schubert polynomials

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Abstract. We define new planar combinatorial objects that we call lattice graphs. We generalize divided differences into operators, still satisfying braid relations, and operating on the space of lattice graphs. This allows us to interpret Schubert polynomials as sums of lattice graphs. By putting appropriate weights on lattice graphs, we recover the usual Schubert polynomials in two sets of variables.

Keywords: divided differences, Schubert polynomials, lattice graphs

1. Introduction

Schubert polynomials, defined in [12, 10, 13], were motivated by the geometry of flag manifolds, following the work of Bernstein-Gelfand-Gelfand [2] and Demazure [5]. They have been extensively studied by many authors [1, 3, 4, 6, 7, 8, 9, 17, 19, 20, 21]. More details can be found in [11, 18].

Let $v = [v_1, v_2, \ldots, v_n]$ be a permutation in the symmetric group $\mathfrak{S}_n$ on n elements $\{1, 2, \ldots, n\}$. Let s_i denote the simple transposition that interchanges i, $i+1$. It is well known that $s_1, s_2, \ldots, s_{n-1}$ generate $\mathfrak{S}_n$, and satisfy the braid relations

$$s_i s_j = s_j s_i \quad for\ |i-j| > 1, \tag{1.1}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}. \tag{1.2}$$

together with $s_i^2 = 1$. For $v \in \mathfrak{S}_n$, for $i = 1, \ldots, n$, let $c_i(v) = \#\{j : \ i < j \ and\ v_i > v_j\}$. Then $c(v) = [c_1(v), c_2(v), \ldots, c_n(v)]$ is called the *code* of v, and $\ell(v) = \sum_{i=1}^n c_i(v)$ is the *length* of v, i.e., the number of inversions of v. A decomposition $v = s_{r_1} s_{r_2} \cdots s_{r_p}$ is *reduced* iff $p = \ell(v)$. We rather use the *reduced word* $r_1 r_2 \cdots r_p$. Let $R(v)$ denote the set of all reduced words of a permutation v. The maximal permutation v_0 of $\mathfrak{S}_n$ is $[n, n-1, \ldots, 1]$. One may check that

$$(s_{n-1} \cdots s_1)(s_{n-1} \cdots s_2) \cdots (s_{n-1} s_{n-2})(s_{n-1})$$

is a reduced decomposition of v_0 (we add parentheses in order to distinguish canonical factors). The following lemma is straightforward:

Lemma 1.1 *Given a permutation $v \in \mathfrak{S}_n$, let $c = [c_1, c_2, \ldots, c_n]$ be its code, then the concatenation of right factors of the factors of*

$$(s_{n-1} \cdots s_1)(s_{n-1} \cdots s_2) \cdots (s_{n-1})(\ ),$$

of respective lengths $c_1, c_2, \ldots, c_n$, is a reduced decomposition of v.

For example, $v = [3,1,6,2,5,4]$ has code $c(v) = [2,0,3,0,1,0]$, and the above decomposition is $v = (s_2 s_1)(\ )(s_5 s_4 s_3)(\ )(s_5)(\ )$.

Let $\mathbb{Z}[a_1, a_2, \ldots, a_n]$ denote the ring of polynomials in n commutative variables with coefficients in $\mathbb{Z}$. For any $1 \leqslant i \leqslant n-1$, the *simple divided difference* ∂_i is the operator:

$$\partial_i f(a_1, a_2, \ldots, a_n) = \frac{f(\ldots, a_i, a_{i+1}, \ldots) - f(\ldots, a_{i+1}, a_i, \ldots)}{a_i - a_{i+1}} = \frac{f - f^{s_i}}{a_i - a_{i+1}}.$$

Divided differences satisfy the braid relations

$$\partial_i \partial_j = \partial_j \partial_i \quad for\ |i-j| > 1, \tag{1.3}$$
$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}. \tag{1.4}$$

together with $\partial_i^2 = 0$. Thanks to these relations, one can define ∂_v by taking any reduced word $r_1 r_2 \cdots r_p$ and putting $\partial_v = \partial_{r_1} \partial_{r_2} \cdots \partial_{r_p}$. When $r_1 r_2 \cdots r_p$ is not reduced, then $\partial_{r_1} \partial_{r_2} \cdots \partial_{r_p} = 0$. We do not follow the conventions of Lascoux-Schützenberger and write divided differences or permutations on the left of the operand.

Divided differences satisfy Leibnitz formulas, as easily seen from the definition:

$$\partial_i(fg) = (\partial_i f) g + f^{s_i}(\partial_i g) = (\partial_i g) f + g^{s_i}(\partial_i f), \tag{1.5}$$

where f and g are polynomials. In particular, symmetric functions in $x_i,\ x_{i+1}$ are scalars for ∂_i, i.e.

$$g = g^{s_i} \Longrightarrow \partial_i(fg) = g(\partial_i f).$$

Schubert polynomials in two sets of commutative variables $\mathbb{A}$ and $\mathbb{B}$ have been defined in [10], starting from a top element $X_{v_0}(\mathbb{A}, \mathbb{B})$, and using the action of the symmetry group $\mathfrak{S}_n$ on $\mathbb{A}$ only.

Definition 1.2

$$\begin{cases} X_{v_0}(\mathbb{A}, \mathbb{B}) = \prod_{1 \leqslant i,j \leqslant n,\ i+j \leqslant n} (a_i - b_j) \\ X_v(\mathbb{A}, \mathbb{B}) = \partial_{v^{-1} v_0} X_{v_0}(\mathbb{A}, \mathbb{B}) \end{cases}$$

The preceding definition implies the recursion

$$\partial_i X_v(\mathbb{A}, \mathbb{B}) = X_{v s_i}(\mathbb{A}, \mathbb{B}), \quad if\ v_i > v_{i+1}. \tag{1.6}$$

We also need the definition of compatible sequences [3]:

Definition 1.3 *For any sequence* $\mathbf{r} = (r_1 \dots r_p)$*, a* p*-tuple* $\mathbf{cp} = (cp_1, \dots, cp_p)$ *of positive integers is* $\mathbf{r}$*-compatible if*

$$cp_1 \leqslant cp_2 \leqslant \cdots \leqslant cp_p,$$
$$cp_j \leqslant r_j, \text{ and if } r_j < r_{j+1}, \text{ then } cp_j < cp_{j+1},$$
$$\text{for any } j : 1 \leqslant j \leqslant p.$$

2. Lattice Graphs

Definition 2.1 *A* lattice graph *is a collection of length* 1 *vertical edges in the* $\mathbb{N} \times \mathbb{N}$ *lattice. The edge from* (i, j) *to* $(i, j+1)$ *is denoted* $[i, j]$*(we use the South-East quadrant). Beware that, for us,* $i = 1, 2, \dots$ *is the column index and* j *is the row index; the origin is* $(1, 1)$.

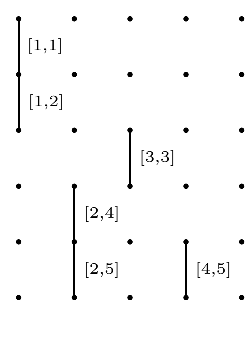


Figure 2.1 A lattice graph restricting $\mathbb{N} \times \mathbb{N}$ to a domain where all edges appear.

We shall also use three other equivalent representations of lattice graphs.

The first one is a sequence of strictly decreasing words on $1, 2, 3, \dots$ that we call *column word*, by reading upwards the columns from left to right, and replacing the edge $[i, j]$ by j. Void columns must be written as [ ]. Figure 2.1 gives $[[2, 1], [5, 4], [3], [5], [\,]]$.

The second one is a word in commutative biletters i^j (we say *biword*), replacing each edge $[i, j]$ by $(j - i + 1)^i$. The number $j - i + 1$ is called *label* of the edge $[i, j]$. Our lattice graph, with edges $[1, 1], [1, 2], [2, 4], [2, 5], [3, 3], [4, 5]$ gives the biletters $1^1, 2^1, 3^2, 4^2, 1^3, 2^4$. However, we shall write biwords using the decreasing lexicographic order (ordering first the labels, then the exponents) as a list of biletters. Figure 2.1 gives $[4^2, 3^2, 2^4, 2^1, 1^3, 1^1]$.

The third one is a 0-1 matrix $A = (a_{ij})$, where $i, j \in \mathbb{N}$. $a_{ji} = 1$ if edge $[i, j]$ belongs to lattice graph otherwise $a_{ji} = 0$. Figure 2.1 gives a lower triangular 0-1 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.7}$$

We shall label diagonals downwards, the 1-diagonal being the main diagonal.

Given s, for any k we say that two edges $[s, k]$ and $[s+1, k+1]$ are s-paired. Paired edges lie in the same diagonal.

We shall define new operators on lattice graphs, that we still call *simple transpositions* (denoted by σ) and *divided differences* (denoted by ∂) as follows (formulating them in terms of biwords):

Definition 2.2 *Given a biword $w = [w_1, w_2, \ldots, w_m]$, where $w_1, w_2, \ldots, w_m$ are biletters, then its image under the simple transposition σ_r is*

$$\sigma_r(w) = [w_1^{\sigma_r}, w_2^{\sigma_r}, \ldots, w_m^{\sigma_r}],$$

where for $w_s = (j-i+1)^i$,

$$w_s^{\sigma_r} = \begin{cases} (j-i+1)^{r+1}, & \text{if } i = r \\ (j-i+1)^r, & \text{if } i = r+1 \\ (j-i+1)^i, & \text{otherwise} \end{cases}$$

From the above definition, we see that σ_r acts only on exponents.

Equivalently, σ_r acts on a column word $\big[\ldots, y^r, y^{r+1}, \ldots\big]$, where $y^r = [a_1, \ldots, a_m]$ and $y^{r+1} = [b_1, \ldots, b_n]$, as follows:

$$\sigma_r\big[\ldots, y^r, y^{r+1}, \ldots\big] = \big[\ldots, [b_1-1, \ldots, b_n-1], [a_1+1, \ldots, a_m+1], \ldots\big].$$

Definition 2.3 *Given a biword lexicographic ordered $w = [w_1, w_2, \ldots, w_m]$, then the action of the divided difference ∂_r on it is*

$$\partial_r(w) = \sum_{i=1}^{m} \epsilon_r(w_i)\, W_i, \tag{2.8}$$

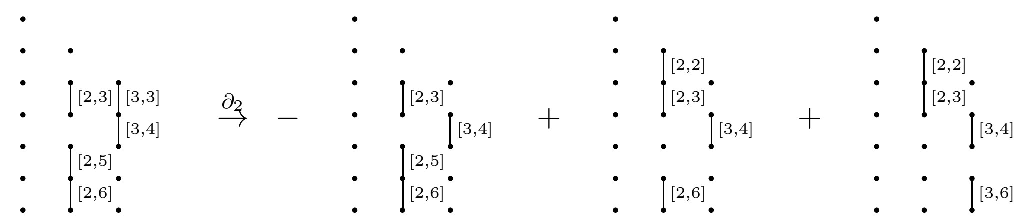
where

$$\epsilon_r(w_i) = \begin{cases} 1, & \textit{if the exponent of } w_i \textit{ is } r \\ -1, & \textit{if the exponent of } w_i \textit{ is } r+1 \\ 0, & \textit{otherwise,} \end{cases}$$

and $W_i = [w_1, \ldots, w_{i-1}, w_{i+1}^{\sigma_r}, \ldots, w_m^{\sigma_r}]$.

Paired edges will give twice the same lattice graph with different signs, so that we do not need to write them in summation (2.8).

Figure 2.2 shows an example of the action of ∂_2 in terms of lattice graphs, column words and biwords:



$$[[\ ],[6,5,3],[4,3]] \xrightarrow{\partial_2} -\ [[\ ],[6,5,3],[4]] \ +\ [[\ ],[6,3,2],[4]] \ +\ [[\ ],[3,2],[6,4]]$$

$$[5^2,4^2,2^3,2^2,1^3] \xrightarrow{\partial_2} -\ [5^2,4^2,2^3,2^2] \ +\ [5^2,2^3,2^2,1^2] \ +\ [4^3,2^3,2^2,1^2]$$

Figure 2.2 Action of a divided difference

Given a 0-1 matrix, choosing a submatrix made of 2 consecutive columns and a finite number of adjacent diagonals, we read the diagonals downwards and thus obtain a word in letters $\{00, 01, 10, 11\}$. For example the first two columns and first five diagonals of the above matrix (2.7) give the word 10 10 01 01 00.

In this representation, we define a *R-move* to be a transformation of the type

$$00\ 11\ \cdots\ 11\ 10\ \rightarrow\ 01\ 11\ \cdots\ 11\ 00,$$

with only occurrences of 11 in the middle block. This is clearly an invertible operation.

Notice that one R-move involves three main elements i, θ and m, where R-move acts on the i-th column and $(i+1)$-th column, and shifts one edge in the $(\theta+m)$-diagonal to the θ-diagonal. We shall denote such a R-move by $R_{i,\theta,m}$. Explicitly, given a lattice graph D and $i, \theta, m \geqslant 1$, if the word obtained by restricting D to $i, i+1$ columns and $\theta, \theta+1, \ldots, \theta+m$ diagonals is of type 00 11 $\cdots$ 11 10, then R-move moves the edge $[i, i+\theta+m-1]$ to $[i+1, \theta+i]$, it can be rewritten as

$$R_{i,\theta,m}(D) = D \cup \{[i+1, \theta+i]\} \setminus \{[i, i+\theta+m-1]\};$$

otherwise, we set $R_{i,\theta,m}(D) = D$.

3. Braid Relations

We will give some properties of transpositions and divided differences on lattice graphs in this section.

Proposition 3.1 *The operators σ satisfy the braid relations :*

$$\sigma_i^2 = 1, \tag{3.9}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \ \ if \ |i-j| \neq 1 \tag{3.10}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \tag{3.11}$$

Proof. It is easy to check the first two relations. As for the third relation, the two operators σ_i, σ_{i+1} act only on the columns i, $i+1$ and $i+2$ of the lattice graph. We can take $i=1$ and restrict to lattice graphs with columns 1, 2, 3. Let

$$y^1 = [a_1, \ldots, a_k], \ y^2 = [b_1, \ldots, b_m], \ y^3 = [c_1, \ldots, c_n].$$

We now compare $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$:

$$\begin{aligned}
&\sigma_1\sigma_2\sigma_1[y^1, y^2, y^3] \\
&\quad = \ \sigma_1\sigma_2\big[[b_1-1, \ldots, b_m-1], [a_1+1, \ldots, a_k+1], [c_1, \ldots, c_n]\big] \\
&\quad = \ \sigma_1\big[[b_1-1, \ldots, b_m-1], [c_1-1, \ldots, c_n-1], [a_1+2, \ldots, a_k+2]\big] \\
&\quad = \ \big[[c_1-2, \ldots, c_n-2], [b_1, \ldots, b_m], [a_1+2, \ldots, a_k+2]\big]
\end{aligned}$$

In the same way,

$$\sigma_2\sigma_1\sigma_2[y^1, y^2, y^3] = \big[[c_1-2, \ldots, c_n-2], [b_1, \ldots, b_m], [a_1+2, \ldots, a_k+2]\big].$$

Therefore we have $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. ∎

Proposition 3.2

$$\partial_i^2 = 0, \tag{3.12}$$

$$\partial_i\partial_j = \partial_j\partial_i, \quad if \ |i-j| \neq 1 \tag{3.13}$$

$$\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1} \tag{3.14}$$

Proof. We can take $i = 1$ as in Proposition 3.1 and a biword with three columns lexicographic ordered is $[w_1, w_2, \ldots, w_m]$. We now check the first relation:

$$\partial_1^2([w_1, w_2, \ldots, w_m]) = \partial_1 \left(\sum_{h=1}^{m} \epsilon_1(w_h)[w_1, \ldots, w_{h-1}, w_{h+1}^{\sigma_1}, \ldots, w_m^{\sigma_1}] \right)$$

$$\begin{aligned} = \;& \sum_{h=1}^{m} \epsilon_1(w_h) \sum_{j>h} \epsilon_1(w_j^{\sigma_1})[w_1, \ldots, w_{h-1}, w_{h+1}^{\sigma_1}, \ldots, w_{j-1}^{\sigma_1}, w_{j+1}^{\sigma_1^2}, \ldots, w_m^{\sigma_1^2}] + \\ & \sum_{h=1}^{m} \epsilon_1(w_h) \sum_{j<h} \epsilon_1(w_j)[w_1, \ldots, w_{j-1}, w_{j+1}^{\sigma_1}, \ldots, w_{h-1}^{\sigma_1}, w_{h+1}^{\sigma_1^2}, \ldots, w_m^{\sigma_1^2}] \\ = \;& 0. \end{aligned}$$

The second relation is immediate.

Let us now compute the image of the biword under $\partial_1\partial_2\partial_1$:

$$\partial_1\partial_2\partial_1([w_1, \ldots, w_m]) = \partial_1\partial_2 \left(\sum_{h=1}^{m} \epsilon_1(w_h)[w_1, \ldots, w_{h-1}, w_{h+1}^{\sigma_1}, \ldots, w_m^{\sigma_1}] \right)$$

$$\begin{aligned} = \;& \partial_1 \left(\sum_{h=1}^{m} \sum_{j<h} \epsilon_1(w_h)\epsilon_2(w_j)[w_1, \ldots, w_{j-1}, w_{j+1}^{\sigma_2}, \ldots, w_{h-1}^{\sigma_2}, w_{h+1}^{\sigma_1\sigma_2}, \ldots, w_m^{\sigma_1\sigma_2}] \right) \\ + \;& \partial_1 \left(\sum_{h=1}^{m} \sum_{j>h} \epsilon_1(w_h)\epsilon_2(w_j^{\sigma_1})[w_1, \ldots, w_{h-1}, w_{h+1}^{\sigma_1}, \ldots, w_{j-1}^{\sigma_1}, w_{j+1}^{\sigma_1\sigma_2}, \ldots, w_m^{\sigma_1\sigma_2}] \right). \end{aligned}$$

Using (3.9), the sum decomposes into six summands:

$$\begin{aligned} M_1 = & \sum_{h=1}^{m} \sum_{j<h} \sum_{j<k<h} \epsilon_1(w_h)\epsilon_2(w_j)\epsilon_1(w_k^{\sigma_2}) \\ & [w_1, \ldots, w_{j-1}, w_{j+1}^{\sigma_2}, \ldots, w_{k-1}^{\sigma_2}, w_{k+1}^{\sigma_2\sigma_1}, \ldots, w_{h-1}^{\sigma_2\sigma_1}, w_{h+1}^{\sigma_1\sigma_2\sigma_1}, \ldots, w_m^{\sigma_1\sigma_2\sigma_1}], \end{aligned}$$

$$\begin{aligned} M_2 = & \sum_{h=1}^{m} \sum_{j<h} \sum_{j<h<k} \epsilon_1(w_h)\epsilon_2(w_j)\epsilon_1(w_k^{\sigma_1\sigma_2}) \\ & [w_1, \ldots, w_{j-1}, w_{j+1}^{\sigma_2}, \ldots, w_{h-1}^{\sigma_2}, w_{h+1}^{\sigma_1\sigma_2}, \ldots, w_{k-1}^{\sigma_1\sigma_2}, w_{k+1}^{\sigma_1\sigma_2\sigma_1}, \ldots, w_m^{\sigma_1\sigma_2\sigma_1}], \end{aligned}$$

$$\begin{aligned} M_3 = & \sum_{h=1}^{m} \sum_{j<h} \sum_{k<j<h} \epsilon_1(w_h)\epsilon_2(w_j)\epsilon_1(w_k) \\ & [w_1, \ldots, w_{k-1}, w_{k+1}^{\sigma_1}, \ldots, w_{j-1}^{\sigma_1}, w_{j+1}^{\sigma_2\sigma_1}, \ldots, w_{h-1}^{\sigma_2\sigma_1}, w_{h+1}^{\sigma_1\sigma_2\sigma_1}, \ldots, w_m^{\sigma_1\sigma_2\sigma_1}], \end{aligned}$$

$$M_4 = \sum_{h=1}^{m} \sum_{j>h} \sum_{k>j} \epsilon_1(w_h)\epsilon_2(w_j^{\sigma_1})\epsilon_1(w_k^{\sigma_1\sigma_2})$$

$$[w_1, \ldots, w_{h-1}, w_{h+1}^{\sigma_1}, \ldots, w_{j-1}^{\sigma_1}, w_{j+1}^{\sigma_1\sigma_2}, \ldots, w_{k-1}^{\sigma_1\sigma_2}, w_{k+1}^{\sigma_1\sigma_2\sigma_1}, \ldots, w_m^{\sigma_1\sigma_2\sigma_1}],$$

$$M_5 = \sum_{h=1}^{m} \sum_{j>h} \sum_{h<k<j} \epsilon_1(w_h)\epsilon_2(w_j^{\sigma_1})\epsilon_1(w_k^{\sigma_1})$$

$$[w_1, \ldots, w_{h-1}, w_{h+1}^{\sigma_1}, \ldots, w_{k-1}^{\sigma_1}, w_{k+1}, \ldots, w_{j-1}, w_{j+1}^{\sigma_1\sigma_2\sigma_1}, \ldots, w_m^{\sigma_1\sigma_2\sigma_1}],$$

$$M_6 = \sum_{h=1}^{m} \sum_{j>h} \sum_{k<h} \epsilon_1(w_h)\epsilon_2(w_j^{\sigma_1})\epsilon_1(w_k)$$

$$[w_1, \ldots, w_{k-1}, w_{k+1}^{\sigma_1}, \ldots, w_{h-1}^{\sigma_1}, w_{h+1}, \ldots, w_{j-1}, w_{j+1}^{\sigma_1\sigma_2\sigma_1}, \ldots, w_m^{\sigma_1\sigma_2\sigma_1}].$$

The image under $\partial_2\partial_1\partial_2$ gives also a sum of six summands:

$$N_1 = \sum_{h=1}^{m} \sum_{j>h} \sum_{k>j} \epsilon_2(w_h)\epsilon_1(w_j^{\sigma_2})\epsilon_2(w_k^{\sigma_2\sigma_1})$$

$$[w_1, \ldots, w_{h-1}, w_{h+1}^{\sigma_2}, \ldots, w_{j-1}^{\sigma_2}, w_{j+1}^{\sigma_2\sigma_1}, \ldots, w_{k-1}^{\sigma_2\sigma_1}, w_{k+1}^{\sigma_2\sigma_1\sigma_2}, \ldots, w_m^{\sigma_2\sigma_1\sigma_2}],$$

$$N_2 = \sum_{h=1}^{m} \sum_{j<h} \sum_{k<j} \epsilon_2(w_h)\epsilon_1(w_j)\epsilon_2(w_k)$$

$$[w_1, \ldots, w_{k-1}, w_{k+1}^{\sigma_2}, \ldots, w_{j-1}^{\sigma_2}, w_{j+1}^{\sigma_1\sigma_2}, \ldots, w_{h-1}^{\sigma_1\sigma_2}, w_{h+1}^{\sigma_2\sigma_1\sigma_2}, \ldots, w_m^{\sigma_2\sigma_1\sigma_2}],$$

$$N_3 = \sum_{h=1}^{m} \sum_{j<h} \sum_{k>h} \epsilon_2(w_h)\epsilon_1(w_j)\epsilon_2(w_k^{\sigma_2\sigma_1})$$

$$[w_1, \ldots, w_{j-1}, w_{j+1}^{\sigma_1}, \ldots, w_{h-1}^{\sigma_1}, w_{h+1}^{\sigma_2\sigma_1}, \ldots, w_{k-1}^{\sigma_2\sigma_1}, w_{k+1}^{\sigma_2\sigma_1\sigma_2}, \ldots, w_m^{\sigma_2\sigma_1\sigma_2}],$$

$$N_4 = \sum_{h=1}^{m} \sum_{j<h} \sum_{j<k<h} \epsilon_2(w_h)\epsilon_1(w_j)\epsilon_2(w_k^{\sigma_1})$$

$$[w_1, \ldots, w_{j-1}, w_{j+1}^{\sigma_1}, \ldots, w_{k-1}^{\sigma_1}, w_{k+1}^{\sigma_1\sigma_2}, \ldots, w_{h-1}^{\sigma_2\sigma_1}, w_{h+1}^{\sigma_2\sigma_1\sigma_2}, \ldots, w_m^{\sigma_2\sigma_1\sigma_2}],$$

$$N_5 = \sum_{h=1}^{m} \sum_{j>h} \sum_{h<k<j} \epsilon_2(w_h)\epsilon_1(w_j^{\sigma_2})\epsilon_2(w_k^{\sigma_2})$$

$$[w_1, \ldots, w_{h-1}, w_{h+1}^{\sigma_2}, \ldots, w_{k-1}^{\sigma_2}, w_{k+1}, \ldots, w_{j-1}, w_{j+1}^{\sigma_2\sigma_1\sigma_2}, \ldots, w_m^{\sigma_2\sigma_1\sigma_2}],$$

$$N_6 = \sum_{h=1}^{m} \sum_{j>h} \sum_{k<h} \epsilon_2(w_h)\epsilon_1(w_j^{\sigma_2})\epsilon_2(w_k)$$

$$[w_1, \ldots, w_{k-1}, w_{k+1}^{\sigma_2}, \ldots, w_{h-1}^{\sigma_2}, w_{h+1}, \ldots, w_{j-1}, w_{j+1}^{\sigma_2\sigma_1\sigma_2}, \ldots, w_m^{\sigma_2\sigma_1\sigma_2}].$$

Thanks to (3.10) and

$$\epsilon_1(w^{\sigma_1\sigma_2}) = \epsilon_2(w), \quad \epsilon_2(w^{\sigma_2\sigma_1}) = \epsilon_1(w),$$
$$\epsilon_1(w^{\sigma_1}) = -\epsilon_1(w), \quad \epsilon_2(w^{\sigma_2}) = -\epsilon_2(w),$$

we get $M_k = N_k,\ for\ 1 \leqslant k \leqslant 4$ and $M_5 + M_6 = 0,\ N_5 + N_6 = 0$. Therefore $\partial_1\partial_2\partial_1 = \partial_2\partial_1\partial_2$. ∎

Similar computations give the following relations of the same type:

$$\partial_i\sigma_i + \sigma_i\partial_i = 0, \tag{3.15}$$

$$\partial_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\partial_{i+1}, \quad \partial_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\partial_i \tag{3.16}$$

$$\partial_i\sigma_{i+1}\partial_i = \partial_{i+1}\partial_i\sigma_{i+1} + \sigma_{i+1}\partial_i\partial_{i+1}, \tag{3.17}$$
$$\partial_{i+1}\sigma_i\partial_{i+1} = \partial_i\partial_{i+1}\sigma_i + \sigma_i\partial_{i+1}\partial_i. \tag{3.18}$$

These relations, together with Proposition 3.1 and 3.2, allow to define a “twisted action” of the symmetric group, lifting the action on polynomials used by Lascoux-Leclerc-Thibon [16].

Proposition 3.3 *Let $q \in \mathbb{C}$, for any i, define $T_i :\ \sigma_i + q\partial_i$. Then*

$$T_i^2 = 1, \tag{3.19}$$

$$T_iT_j = T_jT_i, \quad if\ |i-j| \neq 1 \tag{3.20}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_i. \tag{3.21}$$

Proof. (3.19) is an immediate result of (3.9), (3.12) and (3.15).

(3.20) is evident owing to (3.10), (3.13) and $\sigma_i\partial_j = \partial_j\sigma_i$.

(3.21) comes from (3.11), (3.14), (3.16), (3.17), and (3.18).

4. Schubert Polynomials and Lattice Graphs

In this section, we define Schubert polynomials in terms of lattice graphs. Our definition is different from the combinatorial description of Schubert polynomials in terms of tableaux ([14], [19]), of reduced decompositions ([3]), of rc-graphs ([1]), though connected to the preceding descriptions.

Let $\mathcal{L}$ be the $\mathbb{Z}$-module with basis the lattice graphs. Schubert polynomials $\mathbb{L}_v$, $v \in \mathfrak{S}_n$, will now be defined as elements of the space $\mathcal{L}$ as follows:

The top element $\mathbb{L}_{v_0}$ is the lattice graph corresponding to the column word $\big[[n-1,\ldots,2,1],\cdots,[n-1],[\,]\big]$, and for any $v \in \mathfrak{S}_n$,

$$\mathbb{L}_v = \partial_{v^{-1}v_0}\mathbb{L}_{v_0}. \tag{4.22}$$

Given a lattice graph, whose edges $[i,j]$ are such that $i \leqslant j$, we say that it is *reduced* if the underlying word obtained by erasing the brackets in the column word is reduced. Denote by L^v the set of all the lattice graphs which give a reduced decomposition of v.

Lemma 4.1 *Given any permutation v, any reduced lattice graph $\mathcal{D} \in L^v$, and $i, \theta, m \geqslant 1$, then $R_{i,\theta,m}(\mathcal{D}) \in L^v$.*

Proof. If $R_{i,\theta,m}(\mathcal{D}) = \mathcal{D}$, the lemma holds.

Otherwise, $R_{i,\theta,m}(\mathcal{D}) = \mathcal{D} \cup \{[i+1,\theta+i]\} \setminus \{[i,i+\theta+m-1]\}$. Let $\eta = i+\theta+m$. Restricting $\mathcal{D}$ to $i, i+1$ two columns and the diagonals between θ-diagonal and $(\theta+m)$-diagonal, the column word is

$$\big[[\eta-1,\eta-2,\ldots,\eta-m],[\eta-1,\eta-2,\ldots,\eta-m+1]\big].$$

The corresponding column word for $R_{i,\theta,m}(\mathcal{D})$ is

$$\big[[\eta-2,\eta-3,\ldots,\eta-m],[\eta-1,\eta-2,\ldots,\eta-m+1,\eta-m]\big].$$

It is straightforward, thanks to the braid relations (1.1), that the underlying permutations are the same. ∎

Lemma 4.1 shows that R-moves , or their inverses, preserve the underlying permutation.

Theorem 4.2 *For each permutation $v \in \mathfrak{S}_n$, we have*

$$\mathbb{L}_v = \sum_{\mathcal{D}\in L^v} \mathcal{D}. \tag{4.23}$$

Proof. We use decreasing induction on the length of permutations. When $v = v_0$, L^{v_0} contains only the reduced lattice graph $D_0 = \hat{L}(v_0) = \mathbb{L}_{v_0}$ and the statement is evident.

If $v \neq v_0$, let $\pi = vs_r$ and $\ell(\pi) = \ell(v) + 1$. We know that $\mathbb{L}_v = \partial_r \mathbb{L}_\pi$, and by induction , that Theorem 4.2 holds for $\mathbb{L}_\pi$.

Given any reduced lattice graph D in L^π, we denote the biword corresponding to the unpaired edges of r-th column and $(r+1)$-th column by $[w_1, w_2, \ldots, w_n]$ (ignoring the other edges). Let D_i be the lattice graph which is obtained from D by replacing $[w_1, w_2, \ldots, w_n]$ with $[w_1, \ldots, w_{i-1}, w_{i+1}^{\sigma_r}, \ldots, w_n^{\sigma_r}]$. (2.8) can be rewritten in terms of lattice graphs as

$$\partial_r D = \sum_{i=1}^{n} \epsilon_r(w_i) D_i. \tag{4.24}$$

We shall also need the underlying set $\{\partial_r D\} = \{D_i\}$.

We decompose L^π into two sets L_0^π and L_1^π as follows:

$$L_0^\pi = \{D \in L^\pi : [r, r] \notin D\}, \quad L_1^\pi = L^\pi - L_0^\pi. \tag{4.25}$$

Given $D \in L_1^\pi$, let $f(r, D)$ be the minimal integer such that $[r, f(r, D)] \in D$ and $[r, f(r, D) + 1] \notin D$. Let t^D be the maximal index such that all the labels of $w_{t^D}, \ldots, w_n$ are smaller than, or equal to, $f(r, D) - r + 1$.

Lemma 4.3 *With L_1^π, t^D and D_i defined as above, we have*

$$L^v = \bigcup_{D \in L_1^\pi} \bigcup_{i=t^D}^{n} D_i. \tag{4.26}$$

Proof. For $i : t^D \leqslant i \leqslant n$, it's clear that $\epsilon_r(w_i) = 1$, and that there exist $\alpha, \beta_1, \ldots, \beta_s$ satisfying

$$[w_i, w_{i+1}, \ldots, w_n] = \big[[\alpha, \alpha - 1, \ldots, r], [\beta_s, \beta_{s-1}, \ldots, \beta_1]\big].$$

The subword $[w_{i+1}^{\sigma_r}, w_{i+2}^{\sigma_r}, \ldots, w_n^{\sigma_r}]$ must appear in D_i, and in terms of the definition 2.3,

$$[w_{i+1}^{\sigma_r}, w_{i+2}^{\sigma_r}, \ldots, w_n^{\sigma_r}] = \big[[\beta_s - 1, \beta_{s-1} - 1, \ldots, \beta_1 - 1], [\alpha, \alpha - 1, \ldots, r + 1]\big].$$

The braid relations (1.1) entail

$$[w_i, w_{i+1}, \ldots, w_n] \ s_r = [w_{i+1}^{\sigma_r}, w_{i+2}^{\sigma_r}, \ldots, w_n^{\sigma_r}] \ .$$

Therefore, $D_i \in L^v$.

On the other hand, for every lattice graph $D' \in L^v$, let $h(r, D')$ be the minimal number such that neither $[r, h(r, D')]$ nor $[r+1, h(r, D')+1]$ belongs to D'. Among those edges with labels less than $h(r, D') - r + 1$, all the unpaired edges constitute a set $\mathcal{E}$ which lie in column $r + 1$. Otherwise it would contradict the fact that $\ell(v\, s_r) = \ell(v) + 1$. Given D', we define D by replacing all the edges in $\mathcal{E}$ by their images under σ_r, and adding a new edge $[r, h(r, D')]$. Suppose that the biword corresponding to the unpaired edges of r-th column and $r+1$-column in D is $[w_1, w_2, \ldots, w_n]$, and that the biletter corresponding to the edge $[r, h(r, D')]$ is $w_{i'}$, then $D' = D_{i'}$. ∎

Lemma 4.4 *Given π, r and the decomposition* (4.25) *of L^π, let D_i and t^D be defined as above, then we have*

$$\sum_{D \in L_1^\pi} \sum_{i=1}^{t^D - 1} \epsilon_r(w_i) D_i = 0, \tag{4.27}$$

$$\partial_r \sum_{D \in L_0^\pi} D = 0. \tag{4.28}$$

Proof. We shall show that if a lattice graph appears in (4.27) or (4.28), then it appears twice, with different signs. Then we describe an involution which allows us to eliminate these pairs.

Here we shall use 0-1 matrices instead of lattice graphs. Denote by F the flip operation on a word in the alphabet $\{00, 01, 10, 11\}$ which exchanges 01 and 10, and fixes 00 and 11. We define now a transformation Θ on words. Given any word $w = xz_1z_2 \cdots z_k y$, its image under Θ is

$$\Theta(w) = y^F z_1^F z_2^F \cdots z_k^F x^F. \tag{4.29}$$

It is clear that Θ is an involution on words in the alphabet $\{00, 01, 10, 11\}$.

We shall consider only words of the type

$$01 \quad 11 \quad \cdots \quad 11 \quad 01 \quad 11 \quad \cdots \quad 11 \quad 01 \quad 11 \quad \cdots \quad 11 \quad 00, \tag{4.30}$$

where only 11 and 01 appears in the middle block, and their images under Θ

$$00 \quad 11 \quad \cdots \quad 11 \quad 10 \quad 11 \quad \cdots \quad 11 \quad 10 \quad 11 \quad \cdots \quad 11 \quad 10\ . \tag{4.31}$$

We apply the above involution to prove (4.27). Given $D \in L_1^\pi$, we restrict D to r and $r+1$ columns. For $i, 1 \leqslant i \leqslant t^D - 1$:

If w_i corresponds to the edge $[r, \eta]$, its diagonal number is $\eta - r + 1$ and word is 10. By definition of t^D, we can find upwards the first diagonal whose word is 00. Let ξ be its label. Reading the successive columns from the ξ-diagonal to the $(\eta - r + 1)$-diagonal, we get a word u which is of type (4.31), because 01 can't appear(otherwise D is not reduced). Then we apply Θ to u, and obtain a new lattice graph D'. In fact, the action of Θ in that case is a sequence of R-moves which changes $[r, \eta]$ to $[r+1, r+\xi]$, and it doesn't change the edge $[r, r]$, so $D' \in L_1^\pi$. The action of ∂_r on the edge $[r, \eta]$ of D and on the edge $[r+1, r+\xi]$ of D' give the same lattice graph, but with different signs.

Otherwise, w_i corresponds to the edge $[r+1, \eta']$ which lies in the $(\eta' - r)-$diagonal and the diagonal word is 01. We can find downwards the first diagonal whose word is 00. Let ξ' be its label. Reading the successive columns from the ξ'-diagonal to the $(\eta' - r)$-diagonal, we get a word u' of type (4.30). By applying Θ in a similar manner, we can find a pair of lattice graphs plus the corresponding edges, and eliminate the two lattice graphs which appear in (4.27).

Relation (4.28) is also proven by using Θ, which allows us to eliminate lattice graphs by pairs. ∎

The following lemma is an immediate consequence of the above two:

Lemma 4.5 *Given two permutations π and v such that $\pi = vs_r$ and $\ell(\pi) = \ell(v) + 1$, then*

$$\partial_r L^\pi = L^v.$$

Thus we finish the proof of Theorem 4.2.

We weight each edge $[i, j]$ in the lattice by $a_i - b_{j-i+1}$. Take the weight of D as the product of the weight of all edges $[i, j] \in D$, denoted by $\omega(D)$. Then the action of divided difference on lattice graphs is compatible with the action of divided difference on monomials. The following theorem is immediate:

Theorem 4.6 *With the weight function ω, for any permutation v we have*

$$X_v(\mathbb{A}, \mathbb{B}) = \sum_{D \in L^v} \omega(D). \tag{4.32}$$

There is an obvious bijection between column words of lattice graphs L^v and RC-words of permutation v. From this bijection one can deduce that:

Theorem 4.7

$$X_v(\mathbb{A}, \mathbb{B}) = \sum_{\mathbf{r} \in R(v)} \sum_{\mathbf{cp} \in C(\mathbf{r})} \prod_{j=1}^{p} (a_{cp_j} - b_{r_j - cp_j + 1}). \tag{4.33}$$

The case of this theorem on the Schubert polynomials over one set of variables has been studied by Bergeron-Billey [1] and Fomin-Stanley [6].

For the other combinatorial construction of Schubert polynomial, we refer the reader to Lascoux-Schützenberger [15], Fomin-Kirillov [7], Kohnert [8, 9], Lenart [17].

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