Pieri formulas for Demazure characters

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Abstract. We give a Pieri formula for key polynomials (also called Demazure characters). It implies an explicit decomposition of Schubert polynomials in terms of key polynomials. The constructions extend to the non-commutative cases.

1. Motivation and background

2. Notation

Given a polynomial f in the variables $x_1, x_2, \ldots, x_n$, the simple transposition s_i acts on f by exchanging the variables x_i and x_{i+1}, denoted by f^{s_i}.

The divided difference ∂_i acts on f in the following way:

$$f\partial_i = \frac{f - f^{s_i}}{x_i - x_{i+1}}. \tag{2.1}$$

The isobaric divided differences π_i and $\widehat{\pi}_i$ are defined by:

$$f\pi_i = \frac{x_i f - x_{i+1} f^{s_i}}{x_i - x_{i+1}} = f x_i \partial_i \quad \text{and} \quad f\widehat{\pi}_i = f\pi_i - f = f\partial_i x_{i+1}, \tag{2.2}$$

where x_i means "multiplication by x_i".

For any permutation σ we define the operator ∂_σ as $\partial_{r_1} \cdots \partial_{r_k}$, where $s_{r_1} \cdots s_{r_k}$ is a reduced decomposition of σ. Since the operators ∂_i satisfy the braid relations, ∂_σ is well defined. Similarly we define the operators π_σ and $\widehat{\pi}_\sigma$.

We now give the definition of Schubert polynomials and key polynomials.

For any $J \in \mathbb{N}^n$, we say that J is dominant if J is weakly decreasing. In the dominant case, we define the Schubert polynomial Y_J, the key polynomial K_J and $\widehat{K}_J$ as the same monomial $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$, namely,

$$Y_J = K_J = \widehat{K}_J = x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}. \tag{2.3}$$

If J is not dominant, then there must exist i such that $j_i < j_{i+1}$. Let $J' = [\ldots, j_{i+1}, j_i, \ldots]$. Then the Schubert polynomial Y_J, the key polynomial K_J and $\widehat{K}_J$ are recursively defined as

$$Y_J = Y_{J'}\partial_i, \quad K_J = K_{J'}\pi_i \quad \text{and} \quad \widehat{K}_J = \widehat{K}_{J'}\widehat{\pi}_i. \tag{2.4}$$

For any $J \in \mathbb{N}^n$, the polynomials Y_J, K_J, $\widehat{K}_J$ are well defined.

Suppose λ is a weakly decreasing reordering of J, then there exists a permutation σ such that $J = \lambda^\sigma$. We have

$$Y_J = Y_\lambda \partial_\sigma, \quad K_J = K_\lambda \pi_\sigma \quad \text{and} \quad \widehat{K}_J = \widehat{K}_\lambda \widehat{\pi}_\sigma. \tag{2.5}$$

Given two sets u, v of the same size, written increasingly, then $u \leq v$ if and only if $u_1 \leq v_1$, $u_2 \leq v_2$.... Let λ and μ be two partitions which are decreasing reordering of u and v respectively. Graphically, $u \leq v$ means that the diagram of λ is a subdiagram of μ. The diagram of λ is also called the diagram of u.

Given two permutations σ and ν, we say that $\sigma \leq \nu$ if and only if for any k one has $\{\sigma_1, \ldots, \sigma_k\} \leq \{\nu_1, \ldots, \nu_k\}$. This order is called the Ehresmann-Bruhat order of the symmetric group. It can be used to define an order on vectors with repeated components.

Given two vectors u, v of the same length, then $u \leq v$ if and only if for any k, $\{u_1, ..., u_k\} \leq \{v_1, ..., v_k\}$.

The Ehresmann-Bruhat order of the symmetric group also has another definition. For $\sigma \in S_n$, let $r_1 \cdots r_m$ be a reduced expression of σ. For $\nu \in S_n$, $\nu \leq \sigma$ if and only if there exists a sequence of natural numbers $1 \leq j_1 \leq \cdots \leq j_k \leq m$ such that $r_{j_1} \cdots r_{j_k}$ is one of the reduced decompositions of ν.

3. Commutation rule on operators

From the definitions of ∂_i, π_i and $\widehat{\pi}_i$, we directly get

$$\pi_i = x_i\, \partial_i, \quad \widehat{\pi}_i = \partial_i\, x_{i+1} \tag{3.6}$$

that we can rewrite as commutation rules

$$\widehat{\pi}_i\, x_i = x_{i+1}\, \pi_i, \quad x_i\, \widehat{\pi}_i = \pi_i\, x_{i+1}. \tag{3.7}$$

The following relation is immediate from the definitions of divided differences

$$x_i \partial_j = \partial_j x_i, \quad x_i \pi_j = \pi_j x_i, \quad x_i \widehat{\pi}_j = \widehat{\pi}_j x_i, \quad \text{for } |i - j| > 1. \tag{3.8}$$

Then for any $k > j$ we have

$$\pi_k\, \pi_{k-1} \cdots \pi_j = x_k \partial_k\, x_{k-1} \partial_{k-1} \cdots x_j \partial_j = x_k\, x_{k-1} \cdots x_j\ \partial_k\, \partial_{k-1} \cdots \partial_j, \tag{3.9}$$

$$\widehat{\pi}_k\, \widehat{\pi}_{k-1} \cdots \widehat{\pi}_j = \partial_k\, x_{k+1} \partial_{k-1} x_k \cdots \partial_j x_{j+1} = \partial_k\, \partial_{k-1} \cdots \partial_j\ x_{k+1}\, x_k \cdots x_{j+1}. \tag{3.10}$$

The following commutation rules are needed in the following, for $k > j$

$$\pi_k \cdots \pi_j \, x_{k+1} \cdots x_{j+1} \quad = \quad x_k \cdots x_j \, \widehat{\pi}_k \cdots \widehat{\pi}_j \tag{3.11}$$

$$x_{j+1} \cdots x_{k+1} \, \pi_j \cdots \pi_k \quad = \quad \widehat{\pi}_j \cdots \widehat{\pi}_k \, x_j \cdots x_k . \tag{3.12}$$

Let σ be a permutation with one reduced decomposition inscribed in the diagram of a partition in the South-East corner of the rectangles

$$\begin{matrix} s_{n-k} & \cdots & s_{n-1} \\ \vdots & & \vdots \\ s_1 & \cdots & s_k \end{matrix} . \tag{3.13}$$

Then applying the above commutation rules, we get

Lemma 3.1 *For any k and σ having the above reduced decomposition, the following relations hold*

$$\widehat{\pi}_\sigma x_k \cdots x_1 = (x_k \cdots x_1)^\nu \pi_\sigma, \quad \pi_\sigma x_n \cdots x_{k+1} = (x_n \cdots x_{k+1})^\nu \widehat{\pi}_\sigma, \tag{3.14}$$

where $\nu = \sigma^{-1}$.

For example we take $n = 8$, $k = 3$ and $\sigma = s_4 s_3 s_2 \, s_7 s_6 s_5 s_4 s_3$, the following planar display implies a proof of the above lemma.

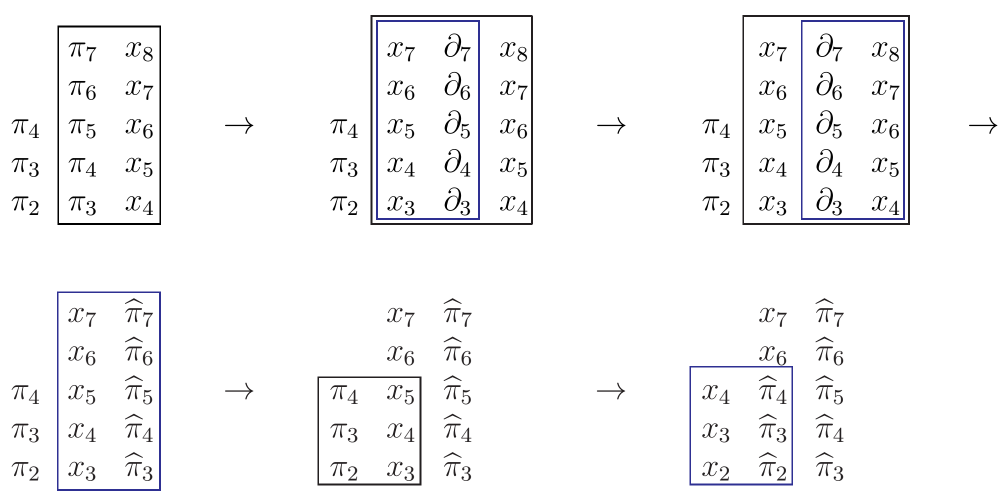

Figure 3.1 Commutations between π_σ and $x_{k+1} \cdots x_n$

We have a Leibniz formula

$$\pi_i x_i = x_{i+1} \pi_i + x_i . \tag{3.15}$$

For any $j < k$ and $m \geq 0$, iterating the above formula we get

$$\pi_j \cdots \pi_k \pi_{k+1} \cdots \pi_{k+m}\, x_j \cdots x_k = \sum_{i=j}^{k+1} x_j \cdots \bar{x}_i \cdots x_{k+1}\, \pi_j \cdots \bar{\pi}_{i-1} \cdots \pi_k \pi_{k+1} \cdots \pi_{k+m}, \tag{3.16}$$

where $\bar{x}_i$ and $\bar{\pi}_{i-1}$ means the missing in the product. For this formula we have the following planar display reading the word by rows

$$\boxed{\begin{array}{ccccccc} \cdot & \pi_j & \cdots & \pi_k & \pi_{k+1} & \cdots & \pi_{k+m} \\ \cdot & x_j & \cdots & x_k & \cdot & \cdots & \cdot \end{array}} = \boxed{\begin{array}{ccccccc} \cdot & x_{j+1} & \cdots & x_{k+1} & \cdot & \cdots & \cdot \\ \cdot & \pi_j & \cdots & \pi_k & \pi_{k+1} & \cdots & \pi_{k+m} \end{array}} +$$

$$\cdots + \boxed{\begin{array}{ccccccccccc} x_j & x_{j+1} & \cdots & x_{i-1} & \cdot & x_{i+1} & \cdots & x_{k+1} & \cdot & \cdots & \cdot \\ \cdot & \pi_j & \cdots & \pi_{i-2} & \cdot & \pi_i & \cdots & \pi_k & \pi_{k+1} & \cdots & \pi_{k+m} \end{array}} + \cdots$$

$$+ \boxed{\begin{array}{ccccccccc} x_j & x_{j+1} & \cdots & x_{k-1} & x_k & \cdot & \cdot & \cdots & \cdot \\ \cdot & \pi_j & \cdots & \pi_{k-2} & \pi_{k-1} & \cdot & \pi_{k+1} & \cdots & \pi_{k+m} \end{array}}$$

Figure 3.2 Basic process I

We call the above process the basic process I. We also need basic process II and III in the following, Figure 3.3 and Figure 3.4 give their planar displays.

$$\boxed{\begin{array}{cccc} \pi_{j-1} & \pi_j & \cdots & \pi_k \\ \cdot & x_j & \cdots & x_k \end{array}} \rightarrow \boxed{\begin{array}{cccc} \cdot & x_{j+1} & \cdots & x_{k+1} \\ \pi_{j-1} & \pi_j & \cdots & \pi_k \end{array}}$$

Figure 3.3 Basic process II

$$\boxed{\begin{array}{cccccc} \cdot & \pi_j & \pi_{j+1} & \cdots & \pi_k & \cdot \\ \cdot & x_j & \pi_j & \cdots & \pi_{k-1} & \cdot \end{array}} =$$

$$\boxed{\begin{array}{cccccc} x_j & \cdot & \pi_{j+1} & \cdots & \pi_k & \cdot \\ \cdot & \cdot & \pi_j & \cdots & \pi_{k-1} & \cdot \end{array}} + \boxed{\begin{array}{ccccccc} \cdot & x_{j+1} & \pi_{j+1} & \cdots & \pi_k & \cdot \\ \cdot & \cdot & \pi_j & \cdots & \pi_{k-1} & \pi_k \end{array}}$$

Figure 3.4 Basic process III

We still take a permutation σ with one reduced decomposition inscribed in the diagram (3.13). Let $\mathcal{C}(\sigma,k)$ be the set of all the permutations which are smaller than or equal to σ for the Ehresmann-Bruhat order with maximal length in the coset class modulo $S_k \times S_{n-k}$. Then

Lemma 3.2 *For any $k>0$ and σ defined as above,*

$$\pi_\sigma\, x_1\cdots x_k = \sum_{\nu\in\mathcal{C}(\sigma,k)} (x_1\cdots x_k)^{\nu^{-1}}\, \pi_\nu. \tag{3.17}$$

Moreover we have

$$\sum_{\nu\in\mathcal{C}(\sigma,k)} (x_1\cdots x_k)^{\nu^{-1}} = K[1^k]\,\pi_{\sigma^{-1}}. \tag{3.18}$$

To illustrate this Lemma, we take $k=3$ and $\sigma = s_4s_5s_3s_4s_1s_2s_3$. Using the above three basic processes, we show the commutations between the π_i operators and monomials by induction on rows. Figure 3.5 and Figure 3.6 show how to apply the commutations inductively in a planar display.

Given $u\in\mathbb{N}^n$ and $i : 1\le i<n$, define $\phi_i(u)$ as follows:

$$\phi_i(u) = \begin{cases} u^{s_i}, & u_i > u_{i+1};\\ u, & u_i \leqslant u_{i+1}.\end{cases}$$

For fixed $k : 1\le k\le n$ let $\mathcal{E}_k(u)$ be the set of all the distinct permutations of u which are greater than or equal to u for the Ehresmann-Bruhat order and of minimum length in their coset class modulo $S_k\times S_{n-k}$. Then we have the following lemma

Lemma 3.3 *For any $i\ne k$ such that $u_i>u_{i+1}$, we have*

$$\mathcal{E}_k(u^{s_i}) = \{\phi_i(v)\,|\, v\in\mathcal{E}_k(u)\}. \tag{3.19}$$

Proof. Let λ be the decreasing reordering of u, and let σ be the permutation such that $\lambda^\sigma = u$. Taking another vector v, there exists a permutation ν such that $\lambda^\nu = v$. Then $v\ge u$ if and only if $\nu\le\sigma$ by checking that every simple transposition in the reduced decomposition decreases the number of inversions by 1.

Suppose $r_1\cdots r_m$ is one reduced decomposition of σ, then $r_1\cdots r_m s_i$ is one reduced decomposition of σ' where $\lambda^{\sigma'} = u^{s_i}$. For any $v=\lambda^\nu\in\mathcal{E}_k(u)$, by the subword property there exists a sequence $1\le j_1,\cdots,j_\ell\le m$ such that $r_{j_1}\cdots r_{j_\ell}=\nu$. If $\phi_i(v)=v$, we immediately get $\nu\le\sigma'$, thus $\phi_i(v)\ge u^{s_i}$. If $\phi_i(v)=v^{s_i}$, then $r_{j_1}\cdots r_{j_\ell}s_i$ is one reduced decomposition of νs_i, thus $\nu s_i\le\sigma'$, and one also gets $\phi_i(v)\ge u^{s_i}$.

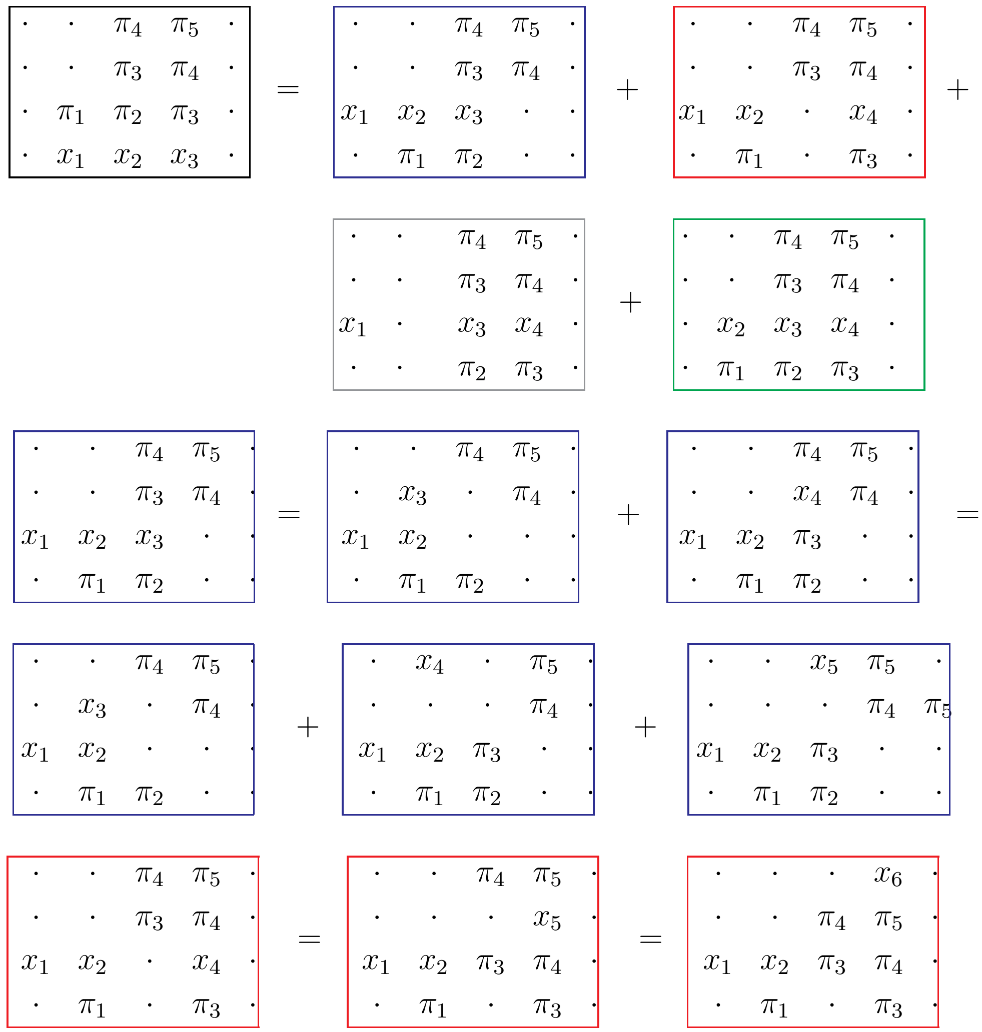

Figure 3.5 (I) Commutations between π_σ and $x_1 \cdots x_k$

Since v is of minimum length in the coset class modulo $S_k \times S_{n-k}$, then ν must have the maximum length among all the vectors smaller than σ in the coset class, that is to say, for any $t \neq k$ the permutation νs_t with length increased by 1 is not smaller than σ. Let $\phi_i(v) = \lambda^{\nu'}$. For the case of $\phi_i(v) = v$ if there exists an integer $t \neq k$ such that $r_{j_1} \cdots r_{j_\ell} s_t$ as a subword sequence of $r_1 \cdots r_m s_i$ is reduced. Then $t \neq i$, otherwise it is contrary to $v_i \leq v_{i+1}$. Thus $r_{j_1} \cdots r_{j_\ell} s_t$ is a subword sequence of $r_1 \cdots r_m$. This is a contrary to the minimum length of v. For the case of $\phi_i(v) = v^{s_i}$ if there exists an integer $t \neq k$ such that $r_{j_1} \cdots r_{j_\ell} s_i s_t$ as a subword sequence of $r_1 \cdots r_m s_i$ is reduced. Then we must have a subword sequence of $r_1 \cdots r_m$ with length greater than ℓ such that it is also reduced. This is also a contrary to the minimum length of v. Thus we obtain $\phi_i(v)$ must have the minimum length among all the vectors greater than u^{s_i} in the coset class.

Conversely, following the above arguments, one can check that for any $v' \in \mathcal{E}_k(u^{s_i})$ there exists a vector $v \in \mathcal{E}_k(u)$ such that $v' = \phi_i(v)$. ∎

4. Pieri formula for key polynomials

In this section, we will prove the following two theorems:

Theorem 4.1 *Given $J \in \mathbb{N}^n$, let k be such that $j_1 \geq \cdots \geq j_k$, $j_{k+1} \geq \cdots \geq j_n$. Then*

$$\widehat{K}_J\, x_k \cdots x_1 = K_{J+[1^k\, 0^{n-k}]} \tag{4.20}$$

and

$$K_J\, x_n \cdots x_{k+1} = \widehat{K}_{J+[0^k\, 1^{n-k}]}. \tag{4.21}$$

Proof. $\widehat{K}_J$ and $\widehat{K}_J$ are the images of $x^\lambda = K_\lambda = \widehat{K}_\lambda$ (with $\lambda =$ decreasing reordering of J), under some $\widehat{\pi}_\sigma$ or π_σ respectively. These two operators can be realized as a sequence of isobaric divided differences inscribed in the diagram of a partition in the South-East corner of the rectangles

$$\begin{matrix} \widehat{\pi}_{n-k} & \cdots & \widehat{\pi}_{n-1} \\ \vdots & & \vdots \\ \widehat{\pi}_1 & \cdots & \widehat{\pi}_k \end{matrix} \qquad \text{or} \qquad \begin{matrix} \pi_{n-k} & \cdots & \pi_{n-1} \\ \vdots & & \vdots \\ \pi_1 & \cdots & \pi_k \end{matrix}\ .$$

By Lemma 3.1, we have

$$\widehat{\pi}_\sigma x_k \cdots x_1 = (x_k \cdots x_1)^\nu \pi_\sigma, \quad \pi_\sigma x_n \cdots x_{k+1} = (x_n \cdots x_{k+1})^\nu \widehat{\pi}_\sigma, \tag{4.22}$$

where $\nu = \sigma^{-1}$. One concludes by checking that multiplying x^λ by one of the two monomials corresponds to adding $[1^k\, 0^{n-k}]$ or $[0^k\, 1^{n-k}]$ to J. ∎

Theorem 4.2 *Given $J \in \mathbb{N}^n$, then for any k we have*

$$K_J\, x_1..x_k = \sum_{I \in \mathcal{E}_k(J)} K[I + [1^k, 0^{n-k}]], \tag{4.23}$$

sum over all I which are $\geq J$ (for the Ehresmann-Bruhat order) and of minimum length in their coset class modulo $S_k \times S_{n-k}$.

Proof. Let $u = (j_1, \ldots, j_k)$ and $v = (j_{k+1}, \ldots, j_n)$. First we prove the special case where both u and v are decreasing. Let λ be the decreasing reordering of J, then there exists a permutation σ with one reduced decomposition inscribed in the diagram (3.13) such that $J = \lambda^\sigma$. Now from the definition of key polynomials we deduce that

$$K_J\, x_1 \cdots x_k = (x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}\, \pi_\sigma)\, x_1 \cdots x_k = (x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n})\, \pi_\sigma\, x_1 \cdots x_k. \qquad (4.24)$$

By Lemma 3.2, we have

$$K_J\, x_1 \cdots x_k = (x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}) \sum_{\nu \in \mathcal{C}(\sigma,k)} (x_1 \cdots x_k)^{\nu^{-1}}\, \pi_\nu = \sum_{I \in \mathcal{E}_k(J)} K[I + [1^k, 0^{n-k}]]. \qquad (4.25)$$

For general J we don't have u and v are decreasing. Let u' and v' be decreasing reordering of u and v respectively. Then K_J is obtained from $K[u', v']$ by applying a series of isobaric divided differences (where π_k is not a factor). Notice that $K[J]$ is symmetric in x_i and x_{i+1} if $j_i < j_{i+1}$, which means that in this case we have $K_J \pi = K_J$. By successively using Lemma 3.3, we get the desired theorem. ∎

5. Decomposition of Schubert polynomials

The preceding Pieri formula allows us to give a generation of Schubert polynomials, providing their decomposition in terms of key polynomials.

Theorem 5.1 *Given $J \in \mathbb{N}^n$, let k be minimum such that $j_k = 0$ (if it does not exist, replace J by $J0$). Let $I = [j_1 - 1, \ldots, j_{k-1} - 1, j_{k+1}, \ldots, j_n]$. Then*

$$Y_J = Y_I\ \pi_{n-1} \cdots \pi_k\ x_{k-1} \cdots x_1. \qquad (5.26)$$

Proof. Notice that $x_{k-1} \cdots x_1$ commutes with $\pi_{n-1} \cdots \pi_k$, and therefore

$$\begin{aligned} Y_I\ \pi_{n-1} \cdots \pi_k\ x_{k-1} \cdots x_1 &= Y_I (x_{n-1} \cdots x_k)(x_{k-1} \cdots x_1) \partial_{n-1} \cdots \partial_k \\ &= Y_{I+1^{n-1}}\, \partial_{n-1} \cdots \partial_k = Y_J. \end{aligned}$$

∎

Consequently the decomposition of Y_I in terms of key polynomials gives the decomposition of Y_J. Let

$$Y_I = \sum_u c_u\, K_u, \qquad (5.27)$$

Then

$$\begin{aligned} Y_J &= \left(\textstyle\sum_u c_u\, K_u\right) \pi_{n-1} \cdots \pi_k\ x_{k-1} \cdots x_1 \\ &= \textstyle\sum_u (c_u\, K_u)\ \pi_{n-1} \cdots \pi_k\ x_{k-1} \cdots x_1 \\ &= \textstyle\sum_u c_u\, (K_{u'}\ x_{k-1} \cdots x_1) \\ &= \textstyle\sum_u c_u \sum_v K_v = \sum_{v'} c_{v'} K_{v'} \end{aligned}$$

An immediate consequence of this theorem is that any Schubert polynomials can be expressed as the image of 1 under the action of a product of isobaric divided differences and monomials $x_1 x_2 \cdots x_k$.

6. Noncommutative Pieri formula

Given an infinite totally ordered alphabet A with noncommutative letters, let $A_n = \{a_1, \ldots, a_n\}$, and let $\mathfrak{Plar}$ be the plactic algebra of A.

Let $\mathfrak{Sym}(A_n)$ be the linear span of the Schur functions in A_n. We denote the elementary symmetric function of order k in A_n by $\Lambda^k(A_n)$.

Define the *Schubert Module* to be the subspace of $\mathfrak{Plar}$:

$$\mathfrak{Schub} = \cdots \otimes \mathfrak{Sym}(A_3) \otimes \mathfrak{Sym}(A_2) \otimes \mathfrak{Sym}(A_1)$$

linearly generated by finite products of symmetric polynomials (respecting the order $\ldots, A_3, A_2, A_1$).

It is known that $\mathfrak{Sym}(A_n)$ has a basis $\{P_J \mid J \leq [\ldots, 3, 2, 1, 0]\}$ [?] where for $J \in \mathbb{N}^n$

$$P_J = \Lambda^{j_1}(A_{n-1}) \cdots \Lambda^{j_n}(A_0). \tag{6.28}$$

Let ev be a map which sends a_i to x_i for each i, then $\{ev(P_J)\}$ is a basis of $\mathfrak{Pol}$.

Let λ be a partition, then $\cdots a_3^{\lambda_3} a_2^{\lambda_2} a_1^{\lambda_1}$ is defined as the free key polynomials corresponding to λ, which belongs to $\mathfrak{Schub}$. The operators π_i^a and $\widehat{\pi}_i^a$ introduced in [?] are the operators which act on $\mathfrak{Plar}$, and they preserve the module $\mathfrak{Schub}$. Notice that they don't satisfy the braid relations, i.e., there is no operator π_σ^a or $\widehat{\pi}_\sigma^a$ on the free algebra. However we can still define the free key polynomials recursively as shown in [?]

$$K_J = K_{J'} \pi_i^a \quad \text{and} \quad \widehat{K}_J = \widehat{K}_{J'} \widehat{\pi}_i^a, \tag{6.29}$$

where $J_i < J_{i+1}$ and $J' = J^{s_i}$. Thus the free key polynomials constitute another basis of $\mathfrak{Schub}$.

The following figure shows the relations between these several algebras.

Notice that Pieri formulas (4.20) and (4.23) are also true in the Schubert module, because both the divided differences $\pi_i^a, \widehat{\pi}_i^a$ and the standard monomial $a_k \cdots a_1$ preserve the Schubert module. These corresponding formulas are called noncommutative Pieri formulas. By Theorem 5.1, we also have the decomposition of free Schubert polynomial in terms of free key polynomials.

$$\boxed{\begin{array}{ccccc} \cdot & \cdot & \pi_4 & \pi_5 & \cdot \\ \cdot & \cdot & \pi_3 & \pi_4 & \cdot \\ x_1 & \cdot & x_3 & x_4 & \cdot \\ \cdot & \cdot & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & \cdot & \pi_4 & \pi_5 & \cdot \\ \cdot & \cdot & \pi_3 & \pi_4 & \cdot \\ \cdot & x_2 & x_3 & x_4 & \cdot \\ \cdot & \pi_1 & \pi_2 & \pi_3 & \cdot \end{array}} =$$

$$\boxed{\begin{array}{ccccc} \cdot & \cdot & \pi_4 & \pi_5 & \cdot \\ \cdot & x_3 & x_4 & \cdot & \cdot \\ x_1 & \cdot & \pi_3 & \cdot & \cdot \\ \cdot & \cdot & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & \cdot & \pi_4 & \pi_5 & \cdot \\ \cdot & x_3 & \cdot & x_5 & \cdot \\ x_1 & \cdot & \cdot & \pi_4 & \cdot \\ \cdot & \cdot & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & \cdot & \pi_4 & \pi_5 & \cdot \\ \cdot & \cdot & x_4 & x_5 & \cdot \\ x_1 & \cdot & \pi_3 & \pi_4 & \cdot \\ \cdot & \cdot & \pi_2 & \pi_3 & \cdot \end{array}} +$$

$$\boxed{\begin{array}{ccccc} \cdot & \cdot & \pi_4 & \pi_5 & \cdot \\ \cdot & x_3 & x_4 & \cdot & \cdot \\ \cdot & x_2 & \pi_3 & \cdot & \cdot \\ \cdot & \pi_1 & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & \cdot & \pi_4 & \pi_5 & \cdot \\ \cdot & x_3 & \cdot & x_5 & \cdot \\ \cdot & x_2 & \cdot & \pi_4 & \cdot \\ \cdot & \pi_1 & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & \cdot & \pi_4 & \pi_5 & \cdot \\ \cdot & \cdot & x_4 & x_5 & \cdot \\ \cdot & x_2 & \pi_3 & \pi_4 & \cdot \\ \cdot & \pi_1 & \pi_2 & \pi_3 & \cdot \end{array}} =$$

$$\boxed{\begin{array}{ccccc} \cdot & x_4 & \cdot & \cdot & \cdot \\ \cdot & x_3 & \cdot & \pi_5 & \cdot \\ x_1 & \cdot & \pi_3 & \cdot & \cdot \\ \cdot & \cdot & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & \cdot & x_5 & \cdot & \cdot \\ \cdot & x_3 & \pi_4 & \pi_5 & \cdot \\ x_1 & \cdot & \pi_3 & \cdot & \cdot \\ \cdot & \cdot & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & \cdot & \cdot & x_6 & \cdot \\ \cdot & x_3 & \pi_4 & \pi_5 & \cdot \\ x_1 & \cdot & \cdot & \pi_4 & \cdot \\ \cdot & \cdot & \pi_2 & \pi_3 & \cdot \end{array}} +$$

$$\boxed{\begin{array}{ccccc} \cdot & x_4 & x_5 & \cdot & \cdot \\ \cdot & \cdot & \pi_4 & \cdot & \cdot \\ x_1 & \cdot & \pi_3 & \pi_4 & \cdot \\ \cdot & \cdot & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & x_4 & \cdot & x_6 & \cdot \\ \cdot & \cdot & \cdot & \pi_5 & \cdot \\ x_1 & \cdot & \pi_3 & \pi_4 & \cdot \\ \cdot & \cdot & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & \cdot & x_5 & x_6 & \cdot \\ \cdot & \cdot & \pi_4 & \pi_5 & \cdot \\ x_1 & \cdot & \pi_3 & \pi_4 & \cdot \\ \cdot & \cdot & \pi_2 & \pi_3 & \cdot \end{array}} +$$

$$\boxed{\begin{array}{ccccc} \cdot & x_4 & \cdot & \cdot & \cdot \\ \cdot & x_3 & \cdot & \pi_5 & \cdot \\ \cdot & x_2 & \pi_3 & \cdot & \cdot \\ \cdot & \pi_1 & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & \cdot & x_5 & \cdot & \cdot \\ \cdot & x_3 & \pi_4 & \pi_5 & \cdot \\ \cdot & x_2 & \pi_3 & \cdot & \cdot \\ \cdot & \pi_1 & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & \cdot & \cdot & x_6 & \cdot \\ \cdot & x_3 & \pi_4 & \pi_5 & \cdot \\ \cdot & x_2 & \cdot & \pi_4 & \cdot \\ \cdot & \pi_1 & \pi_2 & \pi_3 & \cdot \end{array}} +$$

$$\boxed{\begin{array}{ccccc} \cdot & x_4 & x_5 & \cdot & \cdot \\ \cdot & \cdot & \pi_4 & \cdot & \cdot \\ \cdot & x_2 & \pi_3 & \pi_4 & \cdot \\ \cdot & \pi_1 & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & x_4 & \cdot & x_6 & \cdot \\ \cdot & \cdot & \cdot & \pi_5 & \cdot \\ \cdot & x_2 & \pi_3 & \pi_4 & \cdot \\ \cdot & \pi_1 & \pi_2 & \pi_3 & \cdot \end{array}} + \boxed{\begin{array}{ccccc} \cdot & \cdot & x_5 & x_6 & \cdot \\ \cdot & \cdot & \pi_4 & \pi_5 & \cdot \\ \cdot & x_2 & \pi_3 & \pi_4 & \cdot \\ \cdot & \pi_1 & \pi_2 & \pi_3 & \cdot \end{array}}$$

Figure 3.6 (II) Commutations between π_σ and $x_1 \cdots x_k$

$$
\begin{tikzcd}
\mathfrak{Schub} \arrow[r, hook] \arrow[d] & \mathfrak{Plax} \arrow[dl] \\
\mathfrak{Pol} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathfrak{Sym}(A_n) \arrow[r, hook] \arrow[d, "\wr"'] & \mathfrak{Plax} \arrow[d] \\
\mathfrak{Sym}(X_n) \arrow[r, hook] & \mathfrak{Pol}
\end{tikzcd}
$$