JACOBIANS OF SYMMETRIC POLYNOMIALS

Alain Lascoux and Piotr Pragacz

Abstract

We give the Jacobian of any family of complete symmetric functions, or of power sums, in a finite number of variables.

Changes of bases of the ring of symmetric functions are usually performed using a canonical scalar product on this ring ([2], Ch.I). On the other hand, Cayley [1], Sylvester [6] and Mac Mahon [3] were rather led to characterize invariants like, e.g., the discriminant, by using differential calculus on symmetric polynomials. The aim of the present note is to write explicitly the Jacobians of different families of symmetric polynomials in n indeterminates $\mathbb{X} = \{x_1, \ldots, x_n\}$, with respect to the indeterminates, or to the elementary symmetric functions.

Define

$$\lambda_z(\mathbb{X}) := \prod_{i=1}^{n}(1 + zx_i) = \sum_{j=0}^{n} z^j\, e_j(\mathbb{X}) \quad , \quad \sigma_z(\mathbb{X}) := 1/\lambda_{-z}(\mathbb{X}) = \sum_{j=0}^{\infty} z^j h_j(\mathbb{X})\,,$$

and

$$\sum_{j=1}^{\infty} z^j\, p_j(\mathbb{X})/j := \log(\sigma_z(\mathbb{X}))\,,$$

the $e_j = e_j(\mathbb{X})$ being the *elementary symmetric functions*, the $h_j = h_j(\mathbb{X})$ being the *complete functions*, the $p_j = p_j(\mathbb{X})$ being the *power sums*.

More generally, for any $k \in \mathbb{C}$, let

$$\sigma_z(k\,\mathbb{X}) = \big(\sigma_z(\mathbb{X})\big)^k = \sum_{j=0}^{\infty} z^j h_j(k\,\mathbb{X}) = \exp\Big(\sum_{j=1}^{\infty} z^j\, p_j(k\mathbb{X})/j\Big)\,.$$

For any n-tuple $f_1, \ldots, f_n$ in the ring of symmetric polynomials in $\mathbb{X}$, one has two *Jacobians*, expressed by $(n \times n)$-determinants :

$$J(f_1, \ldots, f_n) := \left|\tfrac{\partial}{\partial_{x_i}}(f_j)\right|_{1\le i,j\le n} \qquad \text{and} \qquad J_e(f_1, \ldots, f_n) := \left|\tfrac{\partial}{\partial_{e_i}}(f_j)\right|_{1\le i,j\le n}.$$

Since

$$\tfrac{\partial}{\partial x_i}(e_j) = \sum_{r=0}^{j-1} (-1)^r x_i^r \, e_{j-1-r} \,,$$

by subtracting suitable combinations of columns in $J(e_1, \ldots, e_n)$, we get

$$J(e_1, \ldots, e_n) = \Big| \sum_{r=0}^{j-1} (-1)^r x_i^r \, e_{j-1-r} \Big|_{1 \le i,j \le n}$$

$$= \Big| (-1)^{j-1} x_i^{j-1} \Big|_{1 \le i,j \le n} = \prod_{1 \le i < j \le n} (x_i - x_j) =: \Delta(\mathbb{X}) = \Delta \,,$$

the Vandermonde determinant. Hence, for any n-tuple of symmetric polynomials, we have the factorization

$$J(f_1, \ldots, f_n) = J_e(f_1, \ldots, f_n) \cdot \Delta \,. \tag{1}$$

The following proposition gives the expressions of some J_e in terms of Schur functions $s_\nu(k\mathbb{X})$ which are defined as the determinants

$$\Big| h_{\nu_i - i + j}(k\mathbb{X}) \Big|_{1 \le i,j \le n}$$

(cf. ([2], I.3.4). This definition remains valid when $\nu = (\nu_1, \ldots, \nu_n)$ is not a partition. If $k = 1$ and $\nu_i \ge -(n-i)$ for $i = 1, \ldots, n$, then the expression

$$\Big| x_i^{\nu_j + n - j} \Big|_{1 \le i,j \le n} / \Delta$$

(cf. ([2], I.3.1)) in terms of the powers of the x_i's, gives $s_\nu(\mathbb{X})$.

Proposition 1 *For $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{N}^*)^n$, we have*

$$J_e\big(p_{\mu_1}(k\mathbb{X}), \ldots, p_{\mu_n}(k\mathbb{X})\big) = k^n \mu_1 \cdots \mu_n \, s_{(\mu_1 - n, \ldots, \mu_n - 1)}(\mathbb{X}) \tag{2}$$

and for $\mu \in \mathbb{N}^n$,

$$J_e\big(h_{\mu_1}(k\mathbb{X}), \ldots, h_{\mu_n}(k\mathbb{X})\big) = k^n s_{(\mu_1 - n, \ldots, \mu_n - 1)}\big((k+1)\,\mathbb{X}\big) \,. \tag{3}$$

(If $\mu_1 > \cdots > \mu_n \ge 1$, then $\mu_1 - n \ge \cdots \ge \mu_n - 1$ is a partition.)

Firstly, since $p_j(k\mathbb{X}) = kp_j(\mathbb{X})$, we have

$$\tfrac{\partial}{\partial x_i}\big(p_j(k\mathbb{X})\big) = k \tfrac{\partial}{\partial x_i}\big(p_j(\mathbb{X})\big) = kj \, x_i^{j-1} \,,$$

and

$$J\big(p_{\mu_1}(k\mathbb{X}),\ldots,\, p_{\mu_n}(k\mathbb{X})\big) = k^n \mu_1 \cdots \mu_n \cdot \Big| x_i^{\mu_j - 1} \Big|_{1\le i,j\le n} \,.$$

Using the expression of a Schur function in terms of the powers of the x_i's, one gets equation (2).

Secondly, using the Leibniz rule for differentiation of the product, we have

$$\tfrac{\partial}{\partial_{e_i}}\big(\sigma_z(k\,\mathbb{X})\big) = \tfrac{\partial}{\partial_{e_i}}\big(1/\lambda_{-z}(\mathbb{X})^k\big) = -k(-z)^i/\big(\lambda_{-z}(\mathbb{X})\big)^{k+1} = -k(-z)^i\sigma_z\big((k{+}1)\,\mathbb{X}\big)\,. \tag{4}$$

Thus the expression of a Schur function in terms of complete functions entails equation (3). The proposition has been proved.

For example, we have

$$(n!)^{-1}\, J(p_1,p_2,\ldots,p_n) = J(h_1,h_2,\ldots,h_n) = (-1)^{n(n-1)/2}\Delta\,,$$

and for $n=3$,

$$J_e(h_5(\mathbb{X}),h_3(\mathbb{X}),h_2(\mathbb{X})) = s_{2,1,1}(2\mathbb{X})\,.$$

Remark 1 *In analogy to (4), we record*

$$\tfrac{\partial}{\partial_{e_i}}\big(\lambda_z(k\,\mathbb{X})\big) = kz^i\lambda_z\big((k{-}1)\,\mathbb{X}\big)\,. \tag{5}$$

Remark 2 *For any three partitions μ,ν,η of the same weight d, let (μ,ν,η) be the multiplicity of the trivial representation of the symmetric group $\mathfrak{S}_d$ in the tensor product of the three irreducible representations of $\mathfrak{S}_d$ labeled by the partitions μ,ν,η ([2], I.7). Then, for any $k\in\mathbb{C}$, any partition μ of d, one has the expansion*

$$s_\mu(k\mathbb{X}) = \sum_{\nu,\eta}(\mu,\nu,\eta)s_\eta(k)s_\nu(\mathbb{X})\ ,$$

sum over all pair of partitions of weight d, $s_\eta(k)$ meaning the specialization of the Schur function s_η under $h_j \to \binom{k+j-1}{j}$, $j\ge 0$.

For example, for $n=3$, $d=4$, one has the following expression for $s_{2,1,1}(2\mathbb{X})$:

$$\big(s_4(2)+s_{3,1}(2)+s_{2,2}(2)\big)s_{2,1,1}(\mathbb{X}) + \big(s_{3,1}(2)\big)s_{2,2}(\mathbb{X}) + \big(s_{3,1}(2)+s_{2,2}(2)\big)s_{3,1}(\mathbb{X})$$

$$= 9s_{2,1,1}(\mathbb{X}) + 3s_{2,2}(\mathbb{X}) + 4s_{3,1}(\mathbb{X})$$

(in that case, all coefficients $((2,1,1),\nu,\eta)$ are 0 or 1).

Remark 3 *Scott [5], followed by Muir [4], p.148, wrongly asserted that the Jacobians of* $p_{\mu_1}(\mathbb{X}),\dots,p_{\mu_n}(\mathbb{X})$ *and* $h_{\mu_1}(\mathbb{X}),\dots,h_{\mu_n}(\mathbb{X})$ *differ by a scalar factor. The error was to deduce* $\frac{\partial}{\partial_{p_j}}(h_j)=1/j$ *from the Newton relation:*

$$jh_j=\sum_{r=0}^{j-1}p_{j-r}\,h_r\,,$$

the power sums appearing in this formula not being algebraically independent when $j>n$. *One "raison d'être" of the present note was to correct this error.*

References

[1] A. Cayley, *On symmetric functions and seminvariants* , Amer. J. of Math. **15** (1893) 1–74.

[2] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, Oxford, 2nd edition (1995).

[3] Capt. McMahon, *Operators in the theory of seminvariants*, Quart. J. Math. **20** (1885) 362–365.

[4] T. Muir, *History of the Theory of Determinants*, Vol. III, Dover reprint (1960).

[5] R. F. Scott, *On some alternating functions of n variables*, Messenger of Math. **11** (1882), 98-103.

[6] J.J. Sylvester, *Note on Capt. Mac Mahon's transformationn of the theory of invariants*, Messenger of Math. **14** (1884) 163–165.

A.L.: CNRS, Institut Gaspard Monge, Université de Marne-la-Vallée, F-77454 Marne-la-Vallée Cedex, France
Alain.Lascoux@univ-mlv.fr

P.P.: Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, PL-00950 Warsaw, Poland
pragacz@impan.gov.pl