Bezoutians, Euclidean algorithm, and orthogonal polynomials

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Abstract

We prove a quadratic expression for the Bezoutian of two univariate polynomials in terms of the remainders for the Euclidean algorithm. In case of two polynomials of the same degree, or of consecutive degrees, this allows us to interpret their Bezoutian as the Christoffel-Darboux kernel for a finite family of orthogonal polynomials, arising from the Euclidean algorithm. We give orthogonality properties of remainders, and reproducing properties of Bezoutians.

1 Introduction

In the classical theory of elimination, or of Sturm sequences, we encounter relations which are better understood when translated in terms of orthogonal polynomials. The main goal of the present paper is to study the Bezoutians associated with a pair of univariate polynomials, and discuss their relations to the Euclidean algorithm and orthogonal polynomials. By linking Bezoutians of general pairs of polynomials, of the same degree or of consecutive degrees, to the Christoffel-Darboux kernels of the families of

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orthogonal polynomials coming from the Euclidean algorithm, we establish some reproducing properties of Bezoutians.[1]

Neither attempting to give a full survey on Bezoutians (or Bezoutian matrices) in mathematics nor quote all relevant bibliography, we note that there has recently been a revival of interest in Bezoutians because of their importance in many diverse fields of numerical and symbolical computing as well as in control theory. Bezoutians themselves are classical algebraic objects associated with a pair of univariate polynomials (nowadays, also "Bezoutian matrices" associated with a pair of polynomial matrices are studied – cf., e.g., [29] – but we shall not treat this more general notion here). This concept was invented by Bézout [1], [2], and developed by Sylvester [28], Jacobi [19], Cayley [7], and others for the purposes of elimination theory: the non-vanishing of the determinant of the Bezoutian matrix is equivalent to the fact that the polynomials are relatively prime. (As pointed out in [30], the Bezoutian matrix evolved, in fact, from Euler's work [11] in elimination theory.) Also, Bezoutian matrices turned out to be useful tools to study the real root counting problem cf., e.g., [13], and Routh-Hurwtiz (resp. Schur-Cohn) problems of counting the roots of a given polynomial in a given half-plane (resp. in a circle), cf., e.g., [14]. This is closely related to the study of stability criteria for polynomials or linear discrete-time systems (cf., e.g., [17], [20]). Bezoutian matrices show up as the inverse matrices to Hankel matrices (cf., e.g., [15], [16]) and can be used for extremal problems in set theory (cf., e.g., [31]).

As pointed out above, the present paper discusses *other* properties of Bezoutians. We describe now in more detail the content of the paper. The theorems, propositions (apart from 16 and 20), and corollaries mentioned in the following, appear to be new – we have not seen them in the literature.

In Section 2, we recall the notions associated with the Euclidean algorithm for polynomials in one variable, and the corresponding continued fraction. Besides remainders and quotients, we define successive convergents, and associated numerators and denominators and discuss relations between them.

In Section 3, we prove our basic formula expressing the Bezoutian of two polynomials as a quadratic form in their remainders (Theorem 11). This formula leads to a short proof of classical formulas of Sylvester [28] and Brioschi [5], [6] related to the theory of Sturm; this is discussed in Proposition 16. In a certain limit case, Theorem 11 gives also the Sylvester formula [28] for the Jacobian; this is the content of Proposition 20. In Section 6, for general pairs of polynomials of degrees $(n+1, n)$ and (n, n), we derive from Theorem 11 expressions presenting the Bezoutians solely in

[1]As a matter of fact, we may safely associate to the Christoffel-Darboux kernel the names of Bézout [2], Cayley [7], Sylvester [28], Hermite [18], Brioschi [5], [6], and recognize that these mathematicians treated the case of a finite discrete measure, with the help of the Euclidean algorithm.

terms of the roots of the polynomials (Propositions 26, 27, and Theorems 29 and 30).

In Section 7, we interpret, in terms of orthogonal polynomials, some identities of Brioschi [5] and show more general relations for denominators and remainders (Proposition 32 and Corollary 34). Moreover, we use the Schur function expression of orthogonal polynomials given in Proposition 31 to relate Bezoutians and Christoffel-Darboux kernels of the orthogonal polynomials coming from the Euclidean algorithm. We infer some "reproducing" congruences for Bezoutians (Theorems 38 and 41).

The appendix contains a proof of two useful properties of orthogonal polynomials using the techniques of multi-Schur functions.

2 Euclidean algorithm and continued fractions

Let $f = f(x)$ and $\varphi = \varphi(x)$ be two polynomials in $\mathbb{C}[x]$ [2], of degrees $m \geq n$ respectively. Performing the Euclidean algorithm for division (note that we use here *nonstandard* signs[3]), we get

$$f = \mathcal{Q}_0\varphi - \mathcal{R}_1, \quad \varphi = \mathcal{Q}_1\mathcal{R}_1 - \mathcal{R}_2, \quad \mathcal{R}_1 = \mathcal{Q}_2\mathcal{R}_2 - \mathcal{R}_3 \ ,...$$

$$..., \mathcal{R}_{n-2} = \mathcal{Q}_{n-1}\mathcal{R}_{n-1} - \mathcal{R}_n, \quad \mathcal{R}_{n-1} = \mathcal{Q}_n\mathcal{R}_n \ .$$

Definition 1 *The polynomial $\mathcal{R}_i$ is called the ith remainder and $\mathcal{Q}_j$ is called the jth quotient.*

Obviously, $\deg \mathcal{Q}_i = \deg \mathcal{R}_{i-1} - \deg \mathcal{R}_i$. In the generic case, $\deg \mathcal{Q}_i = 1$ and $\deg \mathcal{R}_i = n - i$ for $i = 1, ..., n$.

Definition 2 *We say that (f, φ) is a general pair if $\deg \mathcal{Q}_i = 1$ for $i = 1, ..., n$.*

It is convenient to set $\mathcal{R}_{-1} := f$, $\mathcal{R}_0 := \varphi$, and $\mathcal{R}_{n+1} := 0$. Then the above sequence of equations is given compactly by the equations

$$\mathcal{R}_{i-1} = \mathcal{Q}_i\mathcal{R}_i - \mathcal{R}_{i+1} \ , \quad i = 0, \ldots, n. \tag{1}$$

We may rewrite the equation for $i = 0$ as follows:

$$\frac{\varphi}{f} = \frac{1}{\mathcal{Q}_0 - \frac{\mathcal{R}_1}{\varphi}} \ .$$

[2]The polynomials in this paper, unless otherwise explicitly stated, have complex coefficients.

[3]This variant of the Euclidean algorithm is called sometimes the *extended Euclidean algorithm.*

Iterating, we get a continued fraction:

$$\frac{\varphi}{f} = \cfrac{1}{\mathcal{Q}_0 - \cfrac{1}{\mathcal{Q}_1 - \cfrac{1}{\ddots \, \cfrac{1}{\mathcal{Q}_n}}}} \; .$$

This is a continued fraction with $n+1$ levels numbered from top to bottom with $k = 1, ..., n+1$. We recall the following definition.

Definition 3 *The intermediate fractions (stopping at the level $k = 1, ..., n$ instead of the last one) are rational functions*

$$\mathcal{N}_k/\mathcal{D}_k = \mathcal{N}_k(\mathcal{Q}_0, ..., \mathcal{Q}_{n-1})/\mathcal{D}_k(\mathcal{Q}_0, ..., \mathcal{Q}_{n-1}) \, ,$$

where we set inductively

$$\mathcal{N}_k(\mathcal{Q}_0, ..., \mathcal{Q}_{n-1}) := \mathcal{D}_{k-1}(\mathcal{Q}_1, ..., \mathcal{Q}_{n-1}) \tag{2}$$

and

$$\mathcal{D}_k(\mathcal{Q}_0, ..., \mathcal{Q}_{n-1}) := \mathcal{Q}_0\mathcal{D}_{k-1}(\mathcal{Q}_1, ..., \mathcal{Q}_{n-1}) - \mathcal{N}_{k-1}(\mathcal{Q}_1, ..., \mathcal{Q}_{n-1}) \, . \tag{3}$$

The rational function $\mathcal{N}_k/\mathcal{D}_k$ is called the kth convergent. The polynomial $\mathcal{N}_k$ is called the numerator and $\mathcal{D}_k$ is called the denominator of the kth convergent.

(Note that Equations (2) and (3) match with the equation:

$$\frac{\mathcal{N}_k(\mathcal{Q}_0, \ldots, \mathcal{Q}_{n-1})}{\mathcal{D}_k(\mathcal{Q}_0, \ldots, \mathcal{Q}_{n-1})} = \frac{1}{\mathcal{Q}_0 - \frac{\mathcal{N}_{k-1}(\mathcal{Q}_1, ..., \mathcal{Q}_{n-1})}{\mathcal{D}_{k-1}(\mathcal{Q}_1, ..., \mathcal{Q}_{n-1})}} \, . \,)$$

We have the following expressions for the first three convergents:

$$\frac{\mathcal{N}_1}{\mathcal{D}_1} = \frac{1}{\mathcal{Q}_0} \, , \quad \frac{\mathcal{N}_2}{\mathcal{D}_2} = \frac{\mathcal{Q}_1}{\mathcal{Q}_0\mathcal{Q}_1 - 1} \, , \quad \frac{\mathcal{N}_3}{\mathcal{D}_3} = \frac{\mathcal{Q}_1\mathcal{Q}_2 - 1}{\mathcal{Q}_0\mathcal{Q}_1\mathcal{Q}_2 - \mathcal{Q}_0 - \mathcal{Q}_2} \, .$$

The denominators $\mathcal{D}_k$ (resp. numerators $\mathcal{N}_k$), are in the generic case of degrees $m - n + k - 1$ and $k - 1$, respectively.

We now recall the following result.

Lemma 4 *For $i = 1, \ldots, n$,*

$$\mathcal{R}_i = \varphi\mathcal{D}_i - f\mathcal{N}_i \, . \tag{4}$$

Proof. We proceed by induction on the number of the $\mathcal{Q}_j$'s. Consider the continued fraction associated with division of φ by $\mathcal{R}_1$. This continued fraction depends on $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$, and by the induction assumption we have

$$\mathcal{R}_i = \mathcal{R}_1 \mathcal{D}_{i-1}(\mathcal{Q}_1, \ldots, \mathcal{Q}_{n-1}) - \varphi \mathcal{N}_{i-1}(\mathcal{Q}_1, \ldots, \mathcal{Q}_{n-1}) . \tag{5}$$

By Equations (2) and (3), the right-hand side of Equation (5) is equal to

$$(-f+\mathcal{Q}_0\varphi)\mathcal{N}_i(\mathcal{Q}_0, \ldots, \mathcal{Q}_{n-1}) - \varphi\big(\mathcal{D}_i(\mathcal{Q}_0, \ldots, \mathcal{Q}_{n-1}) - \mathcal{Q}_0\mathcal{D}_{k-1}(\mathcal{Q}_1, \ldots, \mathcal{Q}_{n-1})\big) . \tag{6}$$

Using Equation (2) once again, we see that the polynomial (6) equals

$$\varphi \mathcal{D}_i(\mathcal{Q}_0, \ldots, \mathcal{Q}_{n-1}) - f\mathcal{N}_i(\mathcal{Q}_0, \ldots, \mathcal{Q}_{n-1}) ,$$

which is the wanted expression for $\mathcal{R}_i$. The lemma is proved. $\square$

Corollary 5 *For $i = 1, \ldots, n$, we have the following congruence:*

$$\mathcal{R}_i \equiv \varphi \mathcal{D}_i \mod f . \tag{7}$$

Remark 6 *Alternatively, we recover this fact by observing that the denominators satisfy the recurrent equations:*

$$\mathcal{D}_{i-1} = \mathcal{Q}_i \mathcal{D}_i - \mathcal{D}_{i+1} \ , \quad i = 0, \ldots, n , \tag{8}$$

where we set $\mathcal{D}_{-1} := 0$ and $\mathcal{D}_0 := 1$. Indeed, Equations (8) and (1) give the same recurrent relations for denominators and remainders, and $\mathcal{R}_0 = \varphi \mathcal{D}_0$.

We refer the reader to [5] and [22] (see also [21]) for a more systematic account of the (extended) Euclidean algorithm and related objects.

3 A quadratic expression for the Bezoutian

In this section, we prove a formula expressing the Bezoutian of two univariate polynomials as a quadratic form in their remainders.

Definition 7 *If $g = g(x, y)$ is a polynomial in x,y, then we set*

$$\partial_{xy}(g) := \frac{g(x, y) - g(y, x)}{x - y} . \tag{9}$$

Given two arbitrary univariate polynomials f and φ, Bézout [1], [2] (cf. also [25], I, p. 41–52), considered quadratic expressions in the coefficients of f, φ, that he called *Équations dérivées.* Following Jacobi [19] and Cayley [7] (cf. also [25], II, p. 138), we can reinterpret Bézout's system of equations as the following polynomial.

Definition 8 *We set*

$$\operatorname{Bez}(f,\varphi) := \partial_{xy}\big(f(x)\varphi(y)\big)\,, \tag{10}$$

and call this bivariate polynomial (in x and y) the Bezoutian (of f and φ).

We refer to [21], p. 53, for some examples of Bezoutians.

We have the following two relations:

$$\operatorname{Bez}(f,\varphi) = -\operatorname{Bez}(\varphi, f)\,, \tag{11}$$

$$\operatorname{Bez}(\alpha f + \beta g, \varphi) = \alpha \operatorname{Bez}(f,\varphi) + \beta \operatorname{Bez}(g,\varphi)\,, \tag{12}$$

where α and β do not depend on x, y.

Also, the coefficients of the bivariate polynomial $\operatorname{Bez}(f,\varphi)$ are symmetric, but we shall not need this property. Basic properties of Bezoutians are collected, e.g., in [12], Chap. 8. For the purposes of the present paper, it is recommended to consult also Chapter 3 of [21].

Assume $m = \deg f \geq n = \deg \varphi$ and use the notation of Section 2. We record first the following lemma.

Lemma 9 *For $i = 0, 1, \ldots, n$,*

$$\operatorname{Bez}(\mathcal{R}_{i-1}, \mathcal{R}_i) = p_i \mathcal{R}_i(x)\mathcal{R}_i(y) + \operatorname{Bez}(\mathcal{R}_i, \mathcal{R}_{i+1})\,, \tag{13}$$

where $p_i = p_i(x,y) = \partial_{xy}(\mathcal{Q}_i(x))$.

Proof. We have

$$\frac{\mathcal{R}_{i-1}(x)\mathcal{R}_i(y) - \mathcal{R}_{i-1}(y)\mathcal{R}_i(x)}{\mathcal{R}_i(x)\mathcal{R}_i(y)} = \frac{\mathcal{R}_{i-1}(x)}{\mathcal{R}_i(x)} - \frac{\mathcal{R}_{i-1}(y)}{\mathcal{R}_i(y)}\,. \tag{14}$$

Since

$$\mathcal{R}_{i-1}(x)/\mathcal{R}_i(x) = \mathcal{Q}_i(x) - \mathcal{R}_{i+1}(x)/\mathcal{R}_i(x)\,,$$

we can rewrite the right-hand side of Equation (14) as

$$\begin{aligned} &\mathcal{Q}_i(x) - \mathcal{Q}_i(y) - \frac{\mathcal{R}_{i+1}(x)}{\mathcal{R}_i(x)} + \frac{\mathcal{R}_{i+1}(y)}{\mathcal{R}_i(y)} = \\ &\quad = (x-y)p_i + \frac{\mathcal{R}_i(x)\mathcal{R}_{i+1}(y) - \mathcal{R}_i(y)\mathcal{R}_{i+1}(x)}{\mathcal{R}_i(x)\mathcal{R}_i(y)}\,. \end{aligned} \tag{15}$$

Equating the left-hand side of Equation (14) with the right-hand side of Equation (15), multiplying by $\mathcal{R}_i(x)\mathcal{R}_i(y)$, and dividing by $x-y$, we obtain Equation (13). The lemma has been proved. □

Remark 10 *Note that the derivative of $\mathcal{Q}_i$ is the limit case:*

$$\lim_{y\to x} p_i(x,y) = \mathcal{Q}_i'(x)\,. \tag{16}$$

This will be used in Section 5.

The following theorem is basic for the present paper.

Theorem 11 *With the above notation,*

$$\mathrm{Bez}(f,\varphi) = p_0\varphi(x)\varphi(y) + \sum_{i=1}^{n} p_i\mathcal{R}_i(x)\mathcal{R}_i(y)\,. \tag{17}$$

Proof. Indeed, iterating Equation (13), we arrive at

$$\mathrm{Bez}(f,\varphi) = p_0\mathcal{R}_0(x)\mathcal{R}_0(y) + \mathrm{Bez}(\mathcal{R}_0,\mathcal{R}_1) = \cdots = \sum_{i=0}^{n} p_i\mathcal{R}_i(x)\mathcal{R}_i(y)\,.$$

This proves the theorem. □

We now discuss two special cases of this formula: the case of a general pair of polynomials (f,φ) of degrees $(n{+}1,n)$ and the case of a general pair of polynomials of the same degree n. The remainders for these two cases will be interpreted in the last two sections in terms of orthogonal polynomials, and their Bezoutian will be linked with the corresponding Christoffel-Darboux kernel.

Lemma 12 *If (f,φ) is a general pair of polynomials of degrees $(n{+}1,n)$, then for $i=0,1,\ldots,n$,*

$$\mathrm{Bez}(\mathcal{R}_{i-1},\mathcal{R}_i) = \alpha_i\mathcal{R}_i(x)\mathcal{R}_i(y) + \mathrm{Bez}(\mathcal{R}_i,\mathcal{R}_{i+1})\,, \tag{18}$$

where α_i is the coefficient of x in $\mathcal{Q}_i(x)$.

Proof. This is just a special case of Lemma 9 because we have

$$\partial_{xy}(\mathcal{Q}_i(x)) = \alpha_i \quad \text{for } \; i=0,1,\ldots,n\,. \;\square$$

Iterating Equation (18), we get the following result.

Proposition 13 *If (f,φ) is a general pair of polynomials of degrees $(n{+}1,n)$, then*

$$\mathrm{Bez}(f,\varphi) = \alpha_0\varphi(x)\varphi(y) + \sum_{i=1}^{n} \alpha_i\mathcal{R}_i(x)\mathcal{R}_i(y)\,. \tag{19}$$

For example take $n=2$, $f(x)=x^3+x^2+2x+1$, and $\varphi(x)=x^2+x+1$. Then

$$\mathrm{Bez}(f,\varphi) = 1+x^2+y^2+x^2y+xy^2+x^2y^2.$$

We have $\mathcal{R}_1=-x-1$, $\mathcal{R}_2=-1$, $\mathcal{Q}_1=-x$, $\mathcal{Q}_2=x+1$, so that $\alpha_0=1$, $\alpha_1=-1$, $\alpha_2=1$, and Proposition 13 says:

$$\mathrm{Bez}(f,\varphi) = (x^2+x+1)(y^2+y+1)-(x+1)(y+1)+1\,.$$

We record the following consequence of Equations (7) and (19):

Corollary 14 *If (f, φ) is a general pair of polynomials of degrees $(n+1, n)$, then the following congruence holds modulo the ideal $(f(x), f(y))$:*

$$\mathrm{Bez}(f, \varphi) \equiv \varphi(x)\varphi(y)\big(\alpha_0 + \sum_{i=1}^{n} \alpha_i \mathcal{D}_i(x)\mathcal{D}_i(y)\big) \, . \tag{20}$$

If we specialize x and y to roots of f, then this congruence is replaced by an equality.

Similarly, we infer the following result.

Proposition 15 *If (f, φ) is a general pair of polynomials of the same degree n, then*

$$\mathrm{Bez}(f, \varphi) = \sum_{i=1}^{n} \alpha_i \mathcal{R}_i(x)\mathcal{R}_i(y) \, , \tag{21}$$

where α_i is the coefficient of x in $\mathcal{Q}_i(x)$.

4 Nullity of Sylvester's and Brioschi's determinants

In this section, we give a new simple proof of two classical identities of Sylvester and Brioschi, based on the quadratic expression for the Bezoutian from Theorem 11.

Let f be a univariate polynomial of degree $n+1$. Suppose that $(n+1)\varphi(x)$ is the derivative of $f(x)$. In his study of Sturm sequences, Sylvester [28], vol. I, p. 502 stated (without proof) the vanishing of the following determinant:

$$\left| \mathcal{D}_i(a_j)^2 \right|_{0 \le i,j \le n} \, , \tag{22}$$

where $a_0, \ldots, a_n$ are the roots[4] of f, $\mathcal{D}_i(x)$, $i = 1, \ldots, n$, are the denominators of convergents from Section 2, and $\mathcal{D}_0 = 1$.

Brioschi [5] generalized Sylvester's relation to any polynomial φ of degree n such that f and φ are relatively prime and the pair (f, φ) is general.

Proposition 16 *With these assumptions, if we set*

$$\mathcal{D}_0(x)^2 := \alpha_0 - \frac{f'(x)}{\varphi(x)} \, , \tag{23}$$

then the same determinant (22) vanishes (α_0 is the coefficient of x in $\mathcal{Q}_0(x)$).

[4] Since we are interested in the nullity of the determinant (22), we can order these roots arbitrarily.

Proof. It suffices to show that there exist complex numbers c_i, $i = 0, \ldots, n$, such that for any root a of f the following equation holds:

$$c_0 \mathcal{D}_0(a)^2 + c_1 \mathcal{D}_1(a)^2 + \cdots + c_n \mathcal{D}_n(a)^2 = 0 \; . \tag{24}$$

This follows from the expression of the specializations $x = y = a$ of the Bezoutian, divided by $\varphi(a)^2$:

$$\varphi(a)^{-2} \; \mathrm{Bez}(f, \varphi)\Big|_{x=y=a} = \frac{f'(a)\varphi(a)}{\varphi(a)^2} = \alpha_0 + \alpha_1 \mathcal{D}_1(a)^2 + \cdots + \alpha_n \mathcal{D}_n(a)^2 \, , \tag{25}$$

where Corollary 14 has been used.[5] □

5 Sylvester's formula for the Jacobian

Sylvester [28] p. 506 stated a formula relating the Jacobian of two forms in two variables (of the same degree) with the remainders and quotients. The purpose of this section is to give a simple proof of this formula, based on the quadratic expression for the Bezoutian from Theorem 11.

Let f and φ be two univariate polynomials with $\deg f \geq \deg \varphi$.

Definition 17 *We set*

$$W(f, \varphi) := f'\varphi - f\varphi' \, . \tag{26}$$

This is the simplest Wronskian. Note that

$$W(f, \varphi) = \lim_{y \to x} \mathrm{Bez}(f, \varphi) \, . \tag{27}$$

By virtue of Equation (16), we get, as the "limit $x = y$ case" of Equation (13) the following result.

Lemma 18 *With the notation of Section 2, for $i = 0, 1, \ldots, n$,*

$$W(\mathcal{R}_{i-1}, \mathcal{R}_i) = \mathcal{Q}_i' \mathcal{R}_i^2 + W(\mathcal{R}_i, \mathcal{R}_{i+1}) \, . \tag{28}$$

Iterating Equation (28), we arrive at the following formula.

Proposition 19 *We have the following quadratic expression for the Wronskian:*

$$W(f, \varphi) = \sum_{i=0}^{n} \mathcal{Q}_i' \mathcal{R}_i^2 \, . \tag{29}$$

[5] For another proof of Sylvester's relation, see [21], Ex.2.34.

Suppose now that two (homogeneous) forms $F = F(x, y)$, $\Phi = \Phi(x, y)$ in two variables x and y are given with $N = \deg F = \deg \Phi$. Set $f = F(x, 1)$, $\varphi = \Phi(x, 1)$ and suppose $\deg f \geq \deg \varphi = n$. Following Sylvester [28], we state the following quadratic expression of the specialized Jacobian $J(F, \Phi)$ in terms of the remainders (for the division of f by φ). Its proof consists into recognizing that the Jacobian is a Wronskian.

Proposition 20 *We have the following formula for the Jacobian:*

$$\frac{1}{N} J(F, \Phi)\Big|_{y=1} = \sum_{i=0}^{n} \mathcal{Q}_i' \mathcal{R}_i^2 \, . \tag{30}$$

Proof. By the well-known identity

$$NF = x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} \, , \tag{31}$$

and Equation (29), we have

$$\sum_{i=0}^{n} \mathcal{Q}_i' \mathcal{R}_i^2 = f' \varphi - f \varphi'$$

$$= \frac{\partial F}{\partial x} \cdot \frac{1}{N} \big(x \frac{\partial \Phi}{\partial x} + y \frac{\partial \Phi}{\partial y} \big) \Big|_{y=1} - \frac{1}{N} \big(x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} \big) \frac{\partial \Phi}{\partial x} \Big|_{y=1}$$

$$= \frac{1}{N} \big(\frac{\partial F}{\partial x} \frac{\partial \Phi}{\partial y} - \frac{\partial F}{\partial y} \frac{\partial \Phi}{\partial x} \big) \Big|_{y=1} = \frac{1}{N} J(F, \Phi) \Big|_{y=1} \, .$$

The proposition has been proved. □

6 Bezoutians in terms of the roots of polynomials

In this section, we give some expressions for the Bezoutians of general pairs of monic polynomials of degrees $(n+1, n)$ and (n, n), solely in terms of the roots of the two polynomials.

First we need to introduce *multi-Schur functions.* Here are some basic definitions (for a more detailed account, we refer the reader to [21]; see also [22] and [23]).

Definition 21 *By an alphabet $\mathbb{A}$, we understand a (finite) multi-set of elements in a commutative ring.*

We shall often identify an alphabet $\mathbb{A}$ with the sum $\sum_{a \in \mathbb{A}} a$, and perform the usual algebraic operations on such elements. We shall give priority to the algebraic notation over the set-theoretic one.

Definition 22 *Given two alphabets $\mathbb{A}$, $\mathbb{B}$, the complete functions $S_i(\mathbb{A}-\mathbb{B})$ of the difference of the alphabets are defined by the generating series (with z an extra variable):*

$$\sum S_i(\mathbb{A}-\mathbb{B})z^i = \prod_{b\in\mathbb{B}}(1-bz)/\prod_{a\in\mathbb{A}}(1-az)\,. \tag{32}$$

For example, if $\mathbb{A}$ is of cardinality m, then

$$S_m(x-\mathbb{A}) = \prod_{a\in\mathbb{A}}(x-a)$$

is a monic polynomial with the multi-set of roots $\mathbb{A}$.

Definition 23 *Given $I = (i_1, i_2, \dots, i_r) \in \mathbb{N}^r$, and alphabets $\mathbb{A}_1, \dots, \mathbb{A}_r$, $\mathbb{B}_1, \dots, \mathbb{B}_r$, the multi-Schur function $S_I(\mathbb{A}_1-\mathbb{B}_1, \dots, \mathbb{A}_r-\mathbb{B}_r)$ is*

$$S_I(\mathbb{A}_1-\mathbb{B}_1, \dots, \mathbb{A}_r-\mathbb{B}_r) := \left|S_{i_k+k-h}(\mathbb{A}_k-\mathbb{B}_k)\right|_{1\le h,k\le r}\,. \tag{33}$$

For example,

$$S_{5,7,2,0,3}(\mathbb{A}-\mathbb{B}, \mathbb{C}-\mathbb{D}, \mathbb{E}-\mathbb{F}, \mathbb{G}-\mathbb{H}, \mathbb{K}-\mathbb{L}) =$$

$$= \begin{vmatrix} S_5(\mathbb{A}-\mathbb{B}) & S_8(\mathbb{C}-\mathbb{D}) & S_4(\mathbb{E}-\mathbb{F}) & S_3(\mathbb{G}-\mathbb{H}) & S_7(\mathbb{K}-\mathbb{L}) \\ S_4(\mathbb{A}-\mathbb{B}) & S_7(\mathbb{C}-\mathbb{D}) & S_3(\mathbb{E}-\mathbb{F}) & S_2(\mathbb{G}-\mathbb{H}) & S_6(\mathbb{K}-\mathbb{L}) \\ S_3(\mathbb{A}-\mathbb{B}) & S_6(\mathbb{C}-\mathbb{D}) & S_2(\mathbb{E}-\mathbb{F}) & S_1(\mathbb{G}-\mathbb{H}) & S_5(\mathbb{K}-\mathbb{L}) \\ S_2(\mathbb{A}-\mathbb{B}) & S_5(\mathbb{C}-\mathbb{D}) & S_1(\mathbb{E}-\mathbb{F}) & 1 & S_4(\mathbb{K}-\mathbb{L}) \\ S_1(\mathbb{A}-\mathbb{B}) & S_4(\mathbb{C}-\mathbb{D}) & 1 & 0 & S_3(\mathbb{K}-\mathbb{L}) \end{vmatrix}.$$

In case of a multi-Schur function such that

$$\mathbb{A}_1-\mathbb{B}_1 = \cdots = \mathbb{A}_r-\mathbb{B}_r = \mathbb{A}-\mathbb{B},$$

we write more compactly $S_I(\mathbb{A}-\mathbb{B})$ and call it a *Schur function.*

In case of a multi-Schur function such that for some p,

$$i_1 = \cdots = i_p = i\,,\quad i_{p+1} = \cdots = i_r = j\,,$$

$$\mathbb{A}_1-\mathbb{B}_1 = \cdots = \mathbb{A}_p-\mathbb{B}_p = \mathbb{A}-\mathbb{B}\,,\quad \mathbb{A}_{p+1}-\mathbb{B}_{p+1} = \cdots = \mathbb{A}_r-\mathbb{B}_r = \mathbb{C}-\mathbb{D}\,,$$

we write more compactly

$$S_{i^p\,;\,j^{r-p}}(\mathbb{A}-\mathbb{B};\ \mathbb{C}-\mathbb{D})\,.$$

For example, the multi-Schur function

$$S_{1,1,1,1,5,5,5}(\mathbb{A}-\mathbb{B},\mathbb{A}-\mathbb{B},\mathbb{A}-\mathbb{B},\mathbb{A}-\mathbb{B},\mathbb{C}-\mathbb{D},\mathbb{C}-\mathbb{D},\mathbb{C}-\mathbb{D})$$

is written

$$S_{1^4;5^3}(\mathbb{A}-\mathbb{B};\mathbb{C}-\mathbb{D})\,.$$

We record the following well known property (*loc.cit.*): for a partition $I \in \mathbb{N}^r$ (i.e., I is weakly increasing),

$$S_I(\mathbb{A}-\mathbb{B}) = (-1)^{|I|} S_J(\mathbb{B}-\mathbb{A})\,, \tag{34}$$

where J is the conjugate partition of I (i.e. the consecutive rows of J are equal to the corresponding columns of I).

Definition 24 *Given two alphabets $\mathbb{A}$, $\mathbb{B}$, we set*

$$R(\mathbb{A},\mathbb{B}) := \prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (a-b)\,. \tag{35}$$

We have (cf., e.g. [21] (1.4))

$$R(\mathbb{A},\mathbb{B}) = S_{n^m}(\mathbb{A}-\mathbb{B})\,, \tag{36}$$

where $m = \mathrm{card}(\mathbb{A})$ and $n = \mathrm{card}(\mathbb{B})$.

In particular, the polynomial $S_n(x-\mathbb{B})$ equals $R(x,\mathbb{B})$, and $R(\mathbb{A},\mathbb{B})$ is the resultant of the polynomials $R(x,\mathbb{A})$ and $R(x,\mathbb{B})$.

From now on, the alphabets will consist of complex numbers, and for two alphabets $\mathbb{A},\mathbb{B}$, $f(x)$ will denote the polynomial $R(x,\mathbb{A})$ and $\varphi(x)$ will denote $R(x,\mathbb{B})$.[6]

Definition 25 *We say that a pair of alphabets $(\mathbb{A},\mathbb{B})$ is general if (f,φ) is a general pair of polynomials.*

For a pair $(\mathbb{A},\mathbb{B})$ of alphabets, we set

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) := \mathrm{Bez}(f,\varphi)\,.$$

Suppose first that $\mathbb{A}$ and $\mathbb{B}$ form a general pair of alphabets of cardinalities $n{+}1$ and n.

Proposition 26 *With the above notation,*

$$\begin{aligned}\mathrm{Bez}(\mathbb{A},\mathbb{B}) &= R(x+y,\mathbb{B}) \\ &+ \sum_{i=1}^{n} (-1)^i \frac{S_{1^{n-i};(i+1)^i}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})\, S_{1^{n-i};(i+1)^i}(\mathbb{B}-y;\mathbb{B}-\mathbb{A})}{S_{i^{i-1}}(\mathbb{B}-\mathbb{A})\, S_{(i+1)^i}(\mathbb{B}-\mathbb{A})}\,.\end{aligned} \tag{37}$$

[6] We can assume as well that $\mathbb{A}$ and $\mathbb{B}$ consist of independent variables, and work over the base field of rational functions in $\mathbb{A}$ and $\mathbb{B}$. All the results that we shall obtain, remain true in this situation (and, obviously, the pair (f,φ) is general).

Proof. We use the formula for $\mathrm{Bez}(\mathbb{A},\mathbb{B})$ from Proposition 13.

Second, we invoke the following expressions for successive remainders as multi-Schur functions:

$$\mathcal{R}_i(x) = S_{1^{n-i};(i+1)^i}(\mathbb{B}-x;\mathbb{B}-\mathbb{A}) \tag{38}$$

(cf. [22], Theorem 2.6.1 and [21], Theorem 3.2.1.)

Third, the coefficients α_i admit the following expressions in terms of Schur functions in $\mathbb{A}-\mathbb{B}$ coming from [22] (2.7) p. 23 (cf. also [21] (3.8), pp. 57–58):

$$\alpha_i = \frac{(-1)^i}{S_{i^{i-1}}(\mathbb{B}-\mathbb{A})S_{(i+1)^i}(\mathbb{B}-\mathbb{A})}\,. \tag{39}$$

(In these references, this expression is given in a slightly different but equivalent form.)

Combining these three facts, Equation (37) follows. □

The polynomial $S_{1^{n-i};(i+1)^i}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$ is, up to sign, a multi-Schur presentation of the ith *subresultant* (cf. [23], Proposition 2.2). For a vast discussion of subresultants, see, e.g., [9], [24], [10], [23], and the references there.

In a similar way, for a general pair $(\mathbb{A},\mathbb{B})$ of alphabets with the same cardinality n, invoking Proposition 15, we get the following identity.

Proposition 27 *We have*

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) = \sum_{i=1}^{n}(-1)^i\frac{S_{1^{n-i};i^i}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})S_{1^{n-i};i^i}(\mathbb{B}-y;\mathbb{B}-\mathbb{A})}{S_{(i-1)^{i-1}}(\mathbb{B}-\mathbb{A})S_{i^i}(\mathbb{B}-\mathbb{A})}\,. \tag{40}$$

We now record the following result.

Lemma 28 *Suppose that $\mathbb{A}$ and $\mathbb{B}$ are alphabets of cardinalities m and n with $m \geq n$. For $i = 1,\ldots,n$, we have the following congruence:*

$$\mathcal{R}_i(x) \equiv S_{i^{m-n+i-1}}(\mathbb{A}-\mathbb{B}-x)\,\varphi(x) \mod f(x)\,. \tag{41}$$

Proof. It is a consequence of the explicit expression of $\mathcal{R}_i(x)$ as an element of the ideal generated by $f(x)$ and $\varphi(x)$:

$$\mathcal{R}_i(x) = (-1)^{n-i+1}S_{(m-n+i)^{i-1}}(\mathbb{B}-\mathbb{A}-x)f(x) + S_{i^{m-n+i-1}}(\mathbb{A}-\mathbb{B}-x)\varphi(x)\,. \tag{42}$$

To prove Equation (42), we proceed in the same way as in [21] (3.1.5), p. 49. We only leave to the reader to check that the signs are as stated. □

We shall now establish two expressions (44) and (45), solely in terms of roots, that are congruent to the above Bezoutians modulo the ideal $(f(x), f(y))$.

We start with a general pair of alphabets $(\mathbb{A},\mathbb{B})$ of cardinalities $n{+}1$ and n. It follows from Proposition 13 and Lemma 28 that, modulo the ideal $(f(x),f(y))$,

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) \equiv R(x{+}y,\mathbb{B})\Big(1+\sum_{i=1}^{n}\alpha_i S_{i^i}(\mathbb{A}{-}\mathbb{B}{-}x)S_{i^i}(\mathbb{A}{-}\mathbb{B}{-}y)\Big)\,. \tag{43}$$

We now record the following result.

Theorem 29 *For a general pair of alphabets $(\mathbb{A},\mathbb{B})$ of cardinalities $n+1$ and n, we have the following congruence modulo the ideal $(f(x),f(y))$:*

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) \equiv R(x{+}y,\mathbb{B})\Big(1+\sum_{i=1}^{n}(-1)^i\frac{S_{i^i}(\mathbb{A}{-}\mathbb{B}{-}x)S_{i^i}(\mathbb{A}{-}\mathbb{B}{-}y)}{S_{(i-1)^i}(\mathbb{A}{-}\mathbb{B})S_{i^{i+1}}(\mathbb{A}{-}\mathbb{B})}\Big)\,. \tag{44}$$

Proof. We combine Equations (43), (39), and (34). $\square$

We now pass to a pair of alphabets of the same cardinality.

Theorem 30 *For a general pair of alphabets $(\mathbb{A},\mathbb{B})$ of the same cardinality n, we have the following congruence modulo the ideal $(f(x),f(y))$:*

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) \equiv R(x{+}y,\mathbb{B})\Big(\frac{1}{S_1(\mathbb{A}{-}\mathbb{B})}+\sum_{i=1}^{n-1}(-1)^i\frac{S_{(i+1)^i}(\mathbb{A}{-}\mathbb{B}{-}x)S_{(i+1)^i}(\mathbb{A}{-}\mathbb{B}{-}y)}{S_{i^i}(\mathbb{A}{-}\mathbb{B})S_{(i+1)^{i+1}}(\mathbb{A}{-}\mathbb{B})}\Big)\,. \tag{45}$$

Proof. We shall deduce the present case from the one of alphabets of cardinalities $(n,n{-}1)$. For this purpose, suppose that

$$\psi(x) := S_{n-1}(x{-}\mathbb{C})$$

is the monic remainder of $\varphi(x)$ modulo $f(x)$, i.e.

$$\varphi(x) = f(x) + S_1(\mathbb{A}{-}\mathbb{B})\psi(x)\,. \tag{46}$$

Then the pair $(\mathbb{A},\mathbb{C})$ is general. We have, by combining Equations (46) and (12) (note that $S_1(\mathbb{A}{-}\mathbb{B})$ is a scalar),

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) = \mathrm{Bez}(\mathbb{A},\mathbb{C})S_1(\mathbb{A}{-}\mathbb{B})\,. \tag{47}$$

Thanks to Equation (46), the remainders of $f(x)$ by $\psi(x)$ coincide with those of $f(x)$ by $\varphi(x)$, up to powers of $u := S_1(\mathbb{A}{-}\mathbb{B})$. We get from Equation (42), equalizing degrees by an appropriate power of u,

$$S_{(i+1)^i}(\mathbb{A}{-}\mathbb{B}{-}x)\varphi(x) \equiv u^{i+1}\, S_{(i+1)^i}(\mathbb{A}{-}\mathbb{C}{-}x)\psi(x) \quad \mod \ f(x)\ .$$

Similarly, comparing the top coefficients of $S_{1^{n-i};\,i^i}(\mathbb{A}-x;\ \mathbb{A}-\mathbb{B})$ and $S_{1^{n-i};\,(i-1)^i}(\mathbb{A}-x;\ \mathbb{A}-\mathbb{C})$, we infer

$$S_{i^i}(\mathbb{A}-\mathbb{B}) = u^i S_{(i-1)^i}(\mathbb{A}-\mathbb{C}) \ .$$

Thus, using Equation (47), the congruence (44):

$$\begin{aligned}\mathrm{Bez}(\mathbb{A},\mathbb{C}) \equiv \psi(x)\psi(y)\Big(1 - \frac{S_1(\mathbb{A}-\mathbb{C}-x)S_1(\mathbb{A}-\mathbb{C}-y)}{S_{11}(\mathbb{A}-\mathbb{C})} \\ + \frac{S_{22}(\mathbb{A}-\mathbb{C}-x)S_{22}(\mathbb{A}-\mathbb{C}-y)}{S_{11}(\mathbb{A}-\mathbb{C})S_{222}(\mathbb{A}-\mathbb{C})} - \cdots\Big)\end{aligned}$$

becomes

$$\begin{aligned}\mathrm{Bez}(\mathbb{A},\mathbb{B}) \equiv R(x+y,\mathbb{B})\Big(\frac{1}{S_1(\mathbb{A}-\mathbb{B})} - \frac{S_2(\mathbb{A}-\mathbb{B}-x)S_2(\mathbb{A}-\mathbb{B}-y)}{S_1(\mathbb{A}-\mathbb{B})S_{22}(\mathbb{A}-\mathbb{B})} \\ + \frac{S_{33}(\mathbb{A}-\mathbb{B}-x)S_{33}(\mathbb{A}-\mathbb{B}-y)}{S_{22}(\mathbb{A}-\mathbb{B})S_{333}(\mathbb{A}-\mathbb{B})} - \cdots\Big).\end{aligned}$$

This proves the theorem. □

Note that the congruences (44) and (45) become equalities, when specializing x and y to roots of f.

7 Euclidean remainders versus orthogonal polynomials

In this section, we investigate two families of orthogonal polynomials (we refer to [3], [4], and [26] for basic information on orthogonal polynomials, needed in the present section). The first family comes from the Euclidean algorithm for division of a pair of polynomials of degrees $(n+1, n)$, and the second one comes from division of a pair of polynomials of degrees (n, n).

Assume first that $(\mathbb{A},\mathbb{B})$ is a pair of alphabets of cardinalities $n+1$ and n (and follow the convention of the previous section). One finds in the book of Brioschi [5], p. 167, the following identities (we invoke the notation of Section 2 and write $\mathbb{A}-a$ for $\mathbb{A}\setminus\{a\}$). For a fixed i,

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i(a)\mathcal{N}_i(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,, \tag{48}$$

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i(a)\mathcal{D}_{i-1}(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,, \tag{49}$$

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i^2(a)\,\mathcal{Q}_i(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,. \tag{50}$$

These three identities make use of the following functional on $\mathbb{C}[x]$:

$$\mu : g(x) \mapsto \mu\big(g(x)\big) := \sum_{a\in\mathbb{A}} g(a) \frac{\varphi(a)}{R(a, \mathbb{A}-a)} \ . \tag{51}$$

The functional μ is an incarnation of the Lagrange interpolation in the points $a \in \mathbb{A}$ (cf., e.g., [3] and [21]). It is characterized by the fact that it sends each x^i, $i \in \mathbb{N}$, onto the complete function $S_i(\mathbb{A}-\mathbb{B})$.

Given a linear functional, it is known how to write an orthogonal basis in terms of *moments* (cf. [26] and [3]). These expressions are still valid in the case of our functional μ, restricted to the space of polynomials of degree $\le n$, and the following proposition is a rewriting, in terms of Schur functions, of the classical expressions of orthogonal polynomials in terms of Hankel determinants involving moments.

Proposition 31 *The Schur polynomials*

$$P_k = S_{k^k}(\mathbb{A}-\mathbb{B}-x)\,, \tag{52}$$

$k = 0, 1, \ldots, n$, form a unique (up to normalization) orthogonal basis with respect to the functional μ, of polynomials of respective degrees $0, 1, \ldots, n$.

(For the reader's convenience, we recall a proof in Appendix 8.)

By combining the congruence (7) with Lemma 28, we thus get the following basic result.

Proposition 32 *For any fixed $0 \le i < j \le n$, the following relations hold:*

$$\sum_{a\in\mathbb{A}} a^i \mathcal{D}_j(a) \frac{\varphi(a)}{R(a, \mathbb{A}-a)} = 0\,, \tag{53}$$

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i(a)\mathcal{D}_j(a) \frac{\varphi(a)}{R(a, \mathbb{A}-a)} = 0\,. \tag{54}$$

Brioschi's relation (49) is just a particular case of Equation (54) with consecutive i and j.

We can replace in Equation (48) the polynomial $\mathcal{N}_i$, which is of degree $i-1$, by any polynomial of degree strictly less than i, because the set $\{\mathcal{D}_i\}$ forms an orthogonal basis. Since by Equation (8) we have

$$\mathcal{Q}_i \mathcal{D}_i = \mathcal{D}_{i-1} + \mathcal{D}_{i+1}\,,$$

Equation (50) results from the orthogonality of $\mathcal{D}_i$ with $\mathcal{D}_{i+1}$, and with $\mathcal{D}_{i-1}$.

Remark 33 *Let f be a univariate polynomial of degree $n+1$. Suppose that $(n+1)\varphi(x)$ is the derivative of $f(x)$ In [6], Brioschi gives the following relations: for fixed $0 \le i < j \le n$,*

$$\sum_{a\in\mathbb{A}} a^i \mathcal{D}_j(a) = 0\,. \tag{55}$$

The sum looks different from Equation (53) because the term

$$\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = \frac{\varphi(a)}{f'(a)} = \frac{1}{n+1} \tag{56}$$

can be erased.

We infer from Proposition 32 and Equation (7):

Corollary 34 *For any fixed $0 \le i < j \le n$, the following relations hold:*

$$\sum_{a\in\mathbb{A}} a^i \mathcal{R}_j(a) \frac{1}{\varphi(a)R(a,\mathbb{A}-a)} = 0\,, \tag{57}$$

$$\sum_{a\in\mathbb{A}} \mathcal{R}_i(a)\mathcal{R}_j(a) \frac{1}{\varphi(a)R(a,\mathbb{A}-a)} = 0\,. \tag{58}$$

Remark 35 *Note that*

$$\frac{f'(x)}{f(x)} = \sum_{a\in\mathbb{A}} \frac{1}{x-a}\ , \tag{59}$$

so that the case treated by Sylvester and Brioschi is the case of orthogonal polynomials for a discrete uniform measure, in relation with Lagrange interpolation. As a matter of fact, it was also the starting point of Chebyshev (cf. [8], and [4] for an account to Chebyshev's work on orthogonal polynomials). Taking a generic $\varphi(x)$ of degree n, we have similarly

$$\frac{\varphi(x)}{f(x)} = \sum_{a\in\mathbb{A}} \varphi(a)\, \frac{R(x,\mathbb{A}-a)}{R(a,\mathbb{A}-a)}\ , \tag{60}$$

thanks to Lagrange interpolation, and this time the measure is no more uniform, but still concentrated in the points of $\mathbb{A}$.

We shall now investigate the Christoffel-Darboux kernels associated with the just studied family of orthogonal polynomials and another one associated with a pair of monic polynomials of the same degree. We shall derive "reproducing" congruences for the related Bezoutians.

We first recall the following definition (cf. [3], [26]).

Definition 36 *Given a family of orthogonal polynomials $P_0(x), \ldots, P_n(x)$ associated with a linear functional μ, then the Christoffel-Darboux kernel is:*

$$K(x,y) := \sum_{i=0}^{n} P_i(x)P_i(y)/\mu\big(P_i(x)^2\big)\,. \tag{61}$$

Suppose that $(\mathbb{A},\mathbb{B})$ is a pair of alphabets of cardinalities $(n+1,n)$, and let μ be the functional defined in Equation (51). The following result is a translation to Schur functions of a well-known *normalization* property of orthogonal polynomials.

Lemma 37 *With the above notation,*

$$\mu\big(S_{i^i}(\mathbb{A}-\mathbb{B}-x)^2\big) = (-1)^i S_{(i-1)^i}(\mathbb{A}-\mathbb{B})S_{i^{i+1}}(\mathbb{A}-\mathbb{B})\,. \tag{62}$$

(For the reader's convenience, we recall a proof in Appendix 8.)

Assume now, in addition, that the pair $(\mathbb{A},\mathbb{B})$ is general. By combining the congruence (44), Proposition 31 and Lemma 37, we infer the following congruence modulo the ideal $(f(x), f(y))$:

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) \equiv \varphi(x)\varphi(y)\,K(x,y)\,. \tag{63}$$

This congruence suggests that the Bezoutian $\mathrm{Bez}(\mathbb{A},\mathbb{B})$ has a reproducing property. Its proof will require the following properities:

$$\partial_{xy} f(x) = S_n(x+y-\mathbb{A})\,, \tag{64}$$

$$S_{n+1+i}(y-\mathbb{A}) = y^i f(y) \quad \text{and} \quad S_{n+i}(y-\mathbb{B}) = y^i\varphi(y) \tag{65}$$

for $i > 0$, and for alphabets $\mathbb{C}$, $\mathbb{D}$, $\mathbb{E}$ and $\mathbb{F}$,

$$S_k(\mathbb{C}-\mathbb{D}+\mathbb{E}-\mathbb{F}) = \sum_p S_p(\mathbb{C}-\mathbb{D})S_{k-p}(\mathbb{E}-\mathbb{F})\,, \tag{66}$$

$$S_k(\mathbb{C}+\mathbb{E}-\mathbb{D}-\mathbb{E}) = S_k(\mathbb{C}-\mathbb{D})\,. \tag{67}$$

(Cf. [21], [23].)

Theorem 38 *Given a general pair of alphabets $(\mathbb{A},\mathbb{B})$ of respective cardinalities $n+1$ and n, and a polynomial $g(x)$, we have*

$$\mu\big(g(x)\ \mathrm{Bez}(\mathbb{A},\mathbb{B})\big) \equiv \varphi(y)^2\,g(y) \mod f(y)\,. \tag{68}$$

Proof. Since, by the Leibniz rule (cf. [21] (7.1.9)),

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) = \partial_{xy} f(x)\varphi(y) + f(y)\partial_{xy}\varphi(y)\,,$$

we have

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) \equiv \varphi(y)\,\partial_{xy} f(x) \mod f(y)\,. \tag{69}$$

By Equation (64), we must show

$$\mu\big(g(x)\,\varphi(y)S_n(x+y-\mathbb{A})\big) \equiv \varphi(y)^2\,g(y) \quad \mod f(y)\,. \tag{70}$$

By using Equations (65), (66) and (67), we have for any $i \geq 0$, modulo $f(y)$,

$$\begin{aligned}&\mu\big(x^i\,\varphi(y)\,S_n(x+y-\mathbb{A})\big)\\ &= \varphi(y)[S_{i+n}(\mathbb{A}-\mathbb{B}) + S_{i+n-1}(\mathbb{A}-\mathbb{B})S_1(y-\mathbb{A}) + \cdots + S_i(\mathbb{A}-\mathbb{B})S_n(y-\mathbb{A})]\\ &\qquad\qquad \equiv \varphi(y)S_{i+n}(\mathbb{A}-\mathbb{B}+y-\mathbb{A}) = \varphi(y)^2\,y^i\,.\ \square\end{aligned}$$

Finally, consider a general pair of alphabets $(\mathbb{A},\mathbb{B})$ of the same cardinality n. We state only results and omit their proofs which are similar to the ones in the previous case. We recall (cf. Lemma 28):

Lemma 39 *For $i = 1,\ldots,n$,*

$$\mathcal{R}_i(x) \equiv S_{i^{i-1}}(\mathbb{A}-\mathbb{B}-x)\,\varphi(x) \quad \mod \; f(x)\ . \tag{71}$$

Proposition 40 *The Schur functions $S_{k^{k-1}}(\mathbb{A}-\mathbb{B}-x)$, $k = 1,\ldots,n$, form a unique (up to normalization) orthogonal basis of polynomials of respective degrees $0,1,\ldots,n-1$, with respect to the following functional ν on $\mathbb{C}[x]$:*

$$\nu : x^i \mapsto S_{i+1}(\mathbb{A}-\mathbb{B})\,, \qquad i \in \mathbb{N}\,. \tag{72}$$

Theorem 41 *For any polynomial $g(x)$, the following congruence holds:*

$$\nu(g(x)\,\mathrm{Bez}(\mathbb{A},\mathbb{B})) \equiv \varphi(y)^2\,g(y) \quad \mod f(y)\,. \tag{73}$$

8 Appendix: proofs of Proposition 31 and Lemma 37

Proposition 31 can be proved as follows. Fix $k = 1,\ldots,n$. We have the following equality:

$$S_{k^k}(\mathbb{A}-\mathbb{B}-x)x^i = S_{k^k;\,i}(\mathbb{A}-\mathbb{B};\ x)\,. \tag{74}$$

(We can pass from the right-hand side of Equation (74) to its left-hand side by multiplying the ith row by x and subtracting it from the $(i-1)$st row, successively for $i = 2,3,\ldots,k+1$.)

The functional μ acts only on the last column of the determinant

$$S_{k^k;\,i}(\mathbb{A}-\mathbb{B};\ x)\,,$$

and sends it onto

$$S_{k^k;\,i}(\mathbb{A}-\mathbb{B};\ \mathbb{A}-\mathbb{B})\,.$$

When $i < k$, this last determinant $S_{k^k,i}(\mathbb{A}-\mathbb{B})$ vanishes, having two identical columns. In other words, the polynomial $P_k = S_{k^k}(\mathbb{A}-\mathbb{B}-x)$ is orthogonal to $x^0, \ldots, x^{k-1}$. This proves the proposition.

Lemma 37 admits the following justification. Using (the proof of) Proposition 31 and appropriate Laplace expansion, we have

$$\mu\big(S_{i^i}(\mathbb{A}-\mathbb{B}-x)S_{i^i}(\mathbb{A}-\mathbb{B}-x)\big) = \mu\big(S_{i^i}(\mathbb{A}-\mathbb{B}-x)(-x)^i S_{(i-1)^i}(\mathbb{A}-\mathbb{B})\big)$$
$$= (-1)^i S_{(i-1)^i}(\mathbb{A}-\mathbb{B}) S_{i^{i+1}}(\mathbb{A}-\mathbb{B}) \,.$$

This proves the lemma.

For more on symmetric function interpretation of orthogonal polynomials, consult [21], Chap. 8.

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References

[1] M. Bézout. *Recherches sur le degré des équations résultants de l'evanouissement des inconnues, et sur les moyens qu'il convient d'employer pour trouver ces équations*, Mém. Acad. Roy. Sci. Paris, (1764), 288–338.

[2] M. Bézout. *Théorie générale des équations algébriques*, Paris, (1779).

[3] C. Brezinski. *Padé-type approximation and general orthogonal polynomials*, Birkhäuser (1980).

[4] C. Brezinski. *History of continued fractions and Padé approximants*, Springer (1991).

[5] F. Brioschi. *Théorie des déterminants*, appendix to the French edition, Paris, Mallet-Bachelier (1856).

[6] F. Brioschi. *Sur les fonctions de Sturm*, Nouvelles Annales **13** (1854), 71–80.

[7] A. Cayley. *Note sur la méthode d'élimination de Bézout*, J. Reine Angew. Math. **53** (1857), 366–367.

[8] P. Chebyshev. *Oeuvres, 2 vols*, A. Markoff, N. Sonin eds., Chelsea, New York.

[9] G. E. Collins. *Subresultants and reduced polynomial remainder sequences*, J. Assoc. Comp. Mach. **14** (1967), 128–142.

[10] G. M. Diaz-Toca, L. González-Vega. *Various new expressions for subresultants and their applications*, preprint, 25 pp., to appear in: Applicable Algebra in Engineering, Communication and Computing (2004).

[11] L. Euler. *Introductio to Analysin Infinitorum*, Tom 2, Lausanne (1748).

[12] P. A. Fuhrmann. *A polynomial approach to linear algebra*, Springer, New York (1996).

[13] L. González-Vega. *Solving the real root counting problem via Bezoutians* Preprint **12** (1993), 17 pp., Department of Mathematics, Statistics, and Computation, University of Cantabria.

[14] G. Heinig, U. Jungnickel. *On the Routh-Hurwitz and Schur-Cohn problems for matrix polynomials and generalized Bezoutians*, Math. Nachr. **116** (1984), 185–196.

[15] G. Heinig, U. Jungnickel. *Hankel matrices*, Linear Algebra Appl. **76** (1986), 121–135.

[16] G. Heinig, F. Hellinger. *On the Bezoutian structure of the Moore-Penrose inverses of Hankel matrices*, SIAM J. Matrix Anal. Appl. **14** (1993), 629–645.

[17] U. Helmke, P.A. Fuhrmann. *Bezoutians*, Linear Algebra Appl. **122/123/124** (1989), 1039–1097.

[18] C. Hermite. *Oeuvres*, ed. E. Picard, Gauthier-Villars, Paris (1908).

[19] C. G. J. Jacobi. *De eliminatione variablis e duabus aequatione algebraicis*, J. Reine Angew. Math. **15** (1836), 101–124.

[20] I. Jury, Eliahou. *The roles of Sylvester and Bezoutian matrices in the historical study of stability of linear discrete-time systems*, IEEE Trans. Circuits Systems I Fund. Theory Appl. **45** (1998), 1233–1251.

[21] A. Lascoux. *Symmetric functions and combinatorial operators on polynomials*, CBMS/AMS Lectures Notes **99**, Providence (2003).

[22] A. Lascoux. *Euclid algorithm and related topics*, preprint, 77 pp., Nankai University, Tjanjin (2003). Available at:

`http://phalanstere.univ-mlv.fr/~al`

[23] A. Lascoux, P. Pragacz. *Double Sylvester sums for subresultants and multi-Schur functions*, J. Symb. Comp. **35** (2003), 689–710.

[24] R. G. K. Loos. *Generalized polynomial remainder sequences*, In: Buchberger et al. (Eds.), Computer Algebra, Symbolic and Algebraic Computation, Springer-Verlag, Vienna, New York (1982), 115–137.

[25] T. Muir. *History of determinants*, 5 volumes, Dover reprints (1960).

[26] G. Szegö. *Orthogonal polynomials*, Amer. Math. Soc., Providence (1939).

[27] J. J. Sylvester. *The collected mathematical papers*, Cambridge (1904) (see also Chelsea reprint, New York (1983)).

[28] J. J. Sylvester. *A theory of the syzygetic relations of two rational integral functions*, Phil. Trans. Royal Soc. London vol. CXLIII, Part III (1853) 407–548. Also: *The collected mathematical papers*, Cambridge (1904), vol. 1, paper 57, pp. 429–586.

[29] H. K. Wimmer. *Bezoutians of polynomial matrices and their generalized inverses*, Linear Algebra Appl. **122/123/124** (1989), 475–487.

[30] H. K. Wimmer. *On the history of the Bezoutian and the resultant matrix*, Linear Algebra Appl. **128** (1990), 27–34.

[31] J. T. Yu. *An extremal problem for sets: a new approach via Bezoutians*, J. Combin. Theory Ser. A **62** (1993), 170–175.