Bezoutiants, Euclidean division, and orthogonal polynomials

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Abstract

Given two univariate polynomials of the same degree, or of consecutive degrees, we interpret their Bezoutiant as a Christoffel-Darboux kernel for a finite family of orthogonal polynomials arising from their Euclidean division. This is based on a quadratic expression of the Bezoutiant in terms of the remainders. As a consequence, we give orthogonality properties of remainders, and reproducing properties of the Bezoutiant.

1 Introduction

In the classical theory of elimination, or of Sturm sequences, one encounters relations which are better understood when translated in terms of orthogonal polynomials. Conversely, one may safely associate to the Christoffel-Darboux kernel the names of Bézout [1], Cayley [6], Sylvester [12], Hermite [7], Brioschi [4], [5], and recognize that these mathematicians treated the case of a finite discrete measure, with the help of the Euclidean algorithm.

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The main goal of the present paper is to study the Bezoutiants associated with a pair of univariate polynomials, and discuss their relations to Euclidean division and orthogonal polynomials.

A basic formula for our approach is a quadratic expression of the Bezoutiant in terms of remainders, given in Theorem 3. We shall treat the case where the two polynomials are generic, of the same degree, or of consecutive degrees.

This formula allows us to give simple proofs of two classical formulas related to pairs of polynomials. On one hand, this expression leads to a short proof of classical formulas of Sylvester [12] and Brioschi [4], [5] related to the theory of Sturm; this is discussed in Section 4. On the other hand, in a certain limit case, our quadratic expression for the Bezoutiant gives the Sylvester formula [12] for the Jacobian; this is the content of Section 5.

Also, for general pairs of polynomials of degrees $(n+1, n)$ and (n, n) our formula leads to expressions presenting the Bezoutiants solely in terms of the roots of the polynomials (Theorems 14 and 15).

Comparing with the expression of orthogonal polynomials given in Proposition 16, we can relate Bezoutiants and Christoffel-Darboux kernels. Consequently Bezoutiants have some reproducing properties (Propositions 21 and 24) which were not noticed up to now. In Section 7, we also interpret, in terms of orthogonal polynomials, some identities of Brioschi [4].

We use only techniques of symmetric functions. In Section 2 we recall the notions associated with the Euclidean division of polynomials in one variable, and the corresponding continued fractions.

2 Euclidean division and continued fractions

Let $f = f(x)$ and $\varphi = \varphi(x)$ be two monic polynomials in $\mathbb{C}[x]$ [1], of degrees $m \geq n$ respectively. Performing the algorithm of Euclidean division, one gets

$$f = \mathcal{Q}_0\varphi - \mathcal{R}_1, \quad \varphi = \mathcal{Q}_1\mathcal{R}_1 - \mathcal{R}_2, \quad \mathcal{R}_1 = \mathcal{Q}_2\mathcal{R}_2 - \mathcal{R}_3 \ , ...$$

$$..., \mathcal{R}_{n-2} = \mathcal{Q}_{n-1}\mathcal{R}_{n-1} - \mathcal{R}_n, \quad \mathcal{R}_{n-1} = \mathcal{R}_n\mathcal{Q}_n$$

(we use here nonstandard signs that are motivated by Sturm theory). The $\mathcal{Q}_i$ are the successive *quotients* of degree $\deg\mathcal{R}_{i-1} - \deg\mathcal{R}_i$. In the generic case, $\deg\mathcal{Q}_i = 1$ for $i = 1, ..., n$. In this situation, we shall say that (f, φ) is a *general pair*. The $\mathcal{R}_j$, $j = 1, \ldots, n$, are successive *remainders* with $\deg\mathcal{R}_i = n - i$, in the generic case. It is convenient to put $\mathcal{R}_{-1} = f$, $\mathcal{R}_0 = \varphi$, and $\mathcal{R}_{n+1} = 0$. Then the above sequence of equations is given compactly by

$$\mathcal{R}_{i-1} = \mathcal{Q}_i\mathcal{R}_i - \mathcal{R}_{i+1} \ , \quad i = 0, \ldots, n. \tag{1}$$

[1] All polynomials in the present paper will have complex coefficients.

If one represents by

$$\frac{\mathcal{N}_1}{\mathcal{D}_1}, \frac{\mathcal{N}_2}{\mathcal{D}_2}, \ldots, \frac{\mathcal{N}_{n+1}}{\mathcal{D}_{n+1}} = \frac{\varphi}{f} \tag{2}$$

the successive *convergents* of the continued fraction

$$\cfrac{1}{\mathcal{Q}_0 - \cfrac{1}{\mathcal{Q}_1 - \cfrac{1}{\ddots \cfrac{1}{\mathcal{Q}_n}}}}$$

then one has

$$\mathcal{R}_i = \varphi \mathcal{D}_i - f \mathcal{N}_i \ , \quad i = 1, \ldots, n \,. \tag{3}$$

The *denominators* $\mathcal{D}_i$ (resp. the *numerators* $\mathcal{N}_i$) of the convergents are, in the generic case, of degrees $m - n + i - 1$ and $i - 1$, respectively. It follows from (3) that

$$\mathcal{R}_i \equiv \varphi \mathcal{D}_i \mod f, \ i = 0, \ldots, n \,. \tag{4}$$

Alternatively, one recovers this fact by noticing that the denominators satisfy the recursions

$$\mathcal{D}_{i-1} = \mathcal{Q}_i \mathcal{D}_i - \mathcal{D}_{i+1} \ , \ i = 0, \ldots, n \,, \tag{5}$$

with initial conditions $\mathcal{D}_{-1} = 0$, $\mathcal{D}_0 = 1$. These are the same equations as for the remainders, the initial conditions (1) for the remainders being multiplied by φ, compared with those (5) for the $\mathcal{D}_i$'s.

3 A quadratic expression for the Bezoutiant

If $g = g(x, y)$ is a polynomial in x,y, let

$$\partial_{xy}(g) := \frac{g(x, y) - g(y, x)}{x - y} \,. \tag{6}$$

Given two arbitrary monic polynomials f and φ, Bézout [1] (see also [10], I, p.41–52), considered quadratic expressions in the coefficients of f, φ, that he called *Équations dérivées*. Following Cayley [6] (see also [10], II, p.138), one can reinterpret Bézout's system of equations as the bivariate polynomial

$$\mathfrak{Bez}(f, \varphi) := \partial_{xy}\big(f(x)\varphi(y)\big) \,, \tag{7}$$

called the *Bezoutiant* (of f, φ). We have the following two relations :

$$\mathfrak{Bez}(f, \varphi) = -\mathfrak{Bez}(\varphi, f) \,, \tag{8}$$

$$\mathfrak{Bez}(\alpha f + \beta g, \varphi) = \alpha \mathfrak{Bez}(f, \varphi) + \beta \mathfrak{Bez}(g, \varphi) \,, \tag{9}$$

where α and β do not depend on x, y. For more on Bezoutiants, see chapter 3 of [8].

Assume $m = \deg f \geq n = \deg \varphi$ and use the notation of Section 2. We record first the following lemma which is basic for the present paper.

Lemma 1 *For $i = 0, 1, \ldots, n$, one has*

$$\mathfrak{Bez}(\mathcal{R}_{i-1}, \mathcal{R}_i) = p_i \mathcal{R}_i(x) \mathcal{R}_i(y) + \mathfrak{Bez}(\mathcal{R}_i, \mathcal{R}_{i+1}), \tag{10}$$

where $p_i = p_i(x, y) = \partial_{xy}(\mathcal{Q}_i(x))$.

Proof. We have

$$\frac{\mathcal{R}_{i-1}(x)\mathcal{R}_i(y) - \mathcal{R}_{i-1}(y)\mathcal{R}_i(x)}{\mathcal{R}_i(x)\mathcal{R}_i(y)} = \frac{\mathcal{R}_{i-1}(x)}{\mathcal{R}_i(x)} - \frac{\mathcal{R}_{i-1}(y)}{\mathcal{R}_i(y)}. \tag{11}$$

Since

$$\mathcal{R}_{i-1}(x)/\mathcal{R}_i(x) = \mathcal{Q}_i(x) - \mathcal{R}_{i+1}(x)/\mathcal{R}_i(x),$$

we can rewrite the RHS of (11) as

$$\begin{aligned} &(x-y)p_i - \frac{\mathcal{R}_{i+1}(x)}{\mathcal{R}_i(x)} + \frac{\mathcal{R}_{i+1}(y)}{\mathcal{R}_i(y)} = \\ &\quad = (x-y)p_i + \frac{\mathcal{R}_i(x)\mathcal{R}_{i+1}(y) - \mathcal{R}_i(y)\mathcal{R}_{i+1}(x)}{\mathcal{R}_i(x)\mathcal{R}_i(y)}. \end{aligned} \tag{12}$$

Equating the LHS of (11) with the RHS of (12), multiplying by $\mathcal{R}_i(x)\mathcal{R}_i(y)$, and dividing by $x-y$, we obtain (10). The lemma has been proved. □

Remark 2 *Notice that the derivative of Q_i is the limit case:*

$$\lim_{y \to x} p_i(x, y) = \mathcal{Q}_i'(x). \tag{13}$$

This will be used in Section 5.

Theorem 3 *Under the above assumptions, one has*

$$\mathfrak{Bez}(f, \varphi) = p_0 \varphi(x) \varphi(y) + \sum_{i=1}^{n} p_i \mathcal{R}_i(x) \mathcal{R}_i(y). \tag{14}$$

Proof. Indeed, iterating (10), we arrive at

$$\mathfrak{Bez}(f, \varphi) = p_0 \mathcal{R}_0(x) \mathcal{R}_0(y) + \mathfrak{Bez}(\mathcal{R}_0, \mathcal{R}_1) = \cdots = \sum_{i=0}^{n} p_i \mathcal{R}_i(x) \mathcal{R}_i(y). \ \square$$

Let us now consider the case of a general pair of polynomials (f, φ) of degrees $(n+1, n)$. Then the Bezoutiant, being a polynomial in x, y of degree n in x and y, can be expanded in a unique manner :

$$\mathfrak{Bez}(f, \varphi) = \sum_{i=0}^{n} \gamma_i r_i(x) r_i(y) \tag{15}$$

requiring that the polynomials r_i, $i = 0, \ldots, n$ be monic of degree $n-i$, $\gamma_0, \ldots, \gamma_n$ being normalizing constants. By virtue of the above we recognize that the r_i are, up to scalars, the successive remainders in the Euclidean division of f by φ. Indeed Lemma 1 specializes to :

Lemma 4 *For $i = 0, 1, \ldots, n$, one has*

$$\mathfrak{Bez}(\mathcal{R}_{i-1}, \mathcal{R}_i) = \alpha_i \mathcal{R}_i(x)\mathcal{R}_i(y) + \mathfrak{Bez}(\mathcal{R}_i, \mathcal{R}_{i+1}), \tag{16}$$

where α_i is the coefficient of x in $\mathcal{Q}_i(x)$.

Iterating (16), we get

Proposition 5 *One has*

$$\mathfrak{Bez}(f, \varphi) = \varphi(x)\varphi(y) + \sum_{i=1}^{n} \alpha_i \mathcal{R}_i(x)\mathcal{R}_i(y). \tag{17}$$

Using (17) we get the following congruence.

Corollary 6 *One has, modulo the ideal $(f(x), f(y))$,*

$$\mathfrak{Bez}(f, \varphi) \equiv \varphi(x)\varphi(y)\big(1 + \sum_{i=1}^{n} \alpha_i \mathcal{D}_i(x)\mathcal{D}_i(y)\big). \tag{18}$$

If we specialize x or y to a root of f, then the congruence is replaced by an equality.

Similarly, one gets the following result.

Proposition 7 *If (f, φ) is a general pair of polynomials of the same degree n, then*

$$\mathfrak{Bez}(f, \varphi) = \sum_{i=1}^{n} \alpha_i \mathcal{R}_i(x)\mathcal{R}_i(y), \tag{19}$$

where α_i is the coefficient of x in $\mathcal{Q}_i(x)$.

4 Nullity of Sylvester's and Brioschi's determinants

In his study of Sturm sequences (this is the case where $(n+1)\varphi(x)$ is the derivative of $f(x)$), Sylvester [12], I, p.502 stated (without proof) the vanishing of the following determinant :

$$\left| \mathcal{D}_i(a_j)^2 \right|_{0 \le i,j \le n}, \tag{20}$$

where $a_0, \ldots, a_n$ are the roots of f and $\mathcal{D}_0 := 1$.

Brioschi [4] generalized Sylvester's relation to an arbitrary φ of degree n. One still has the vanishing of the same determinant (20), but now $\mathcal{D}_0^2$ is given by

$$\mathcal{D}_0(x)^2 = 1 - \frac{f'(x)}{\varphi(x)} . \tag{21}$$

In other words, there exist constants c_i, $i = 0, \ldots, n$, such that for any root a of f the following equation is satisfied :

$$c_0\mathcal{D}_0(a)^2 + c_1\mathcal{D}_1(a)^2 + \cdots + c_n\mathcal{D}_n(a)^2 = 0 . \tag{22}$$

But this comes from the expression of the specializations $x = y = a$ of the Bezoutiant, divided by $\varphi(a)^2$:

$$\mathfrak{Bez}(f,\varphi)\Big|_{x=y=a} = \frac{f'(a)\varphi(a)}{\varphi(a)^2} = 1 + \alpha_1\mathcal{D}_1(a)^2 + \cdots + \alpha_n\mathcal{D}_n(a)^2 , \tag{23}$$

where Corollary 6 has been used.[2]

5 Sylvester's formula for the Jacobian

Let f and φ be two polynomials in one variable with $\deg f \geq \deg\varphi$. Let us consider

$$W(f,\varphi) := f'\varphi - f\varphi' \tag{24}$$

- the simplest Wrońskian. Notice that

$$W(f,\varphi) = \lim_{y\to x} \mathfrak{Bez}(f,\varphi) . \tag{25}$$

We get, as the "limit $x = y$ case" of (10), the following result (using (13)).

Lemma 8 *For $i = 0, 1, \ldots, n$, one has*

$$W(\mathcal{R}_{i-1}, \mathcal{R}_i) = \mathcal{Q}_i'\mathcal{R}_i^2 + W(\mathcal{R}_i, \mathcal{R}_{i+1}) . \tag{26}$$

Iterating (26), we arrive at the following formula presenting $W(f,\varphi)$ as a quadratic form in the remainders.

Proposition 9 *One has*

$$W(f,\varphi) = \sum_{i=0}^{n} \mathcal{Q}_i'\mathcal{R}_i^2 . \tag{27}$$

[2]For another proof of Sylvester's relation, see [8], Ex.2.34.

Suppose now that two (homogeneous) forms $F = F(x, y)$, $\Phi = \Phi(x, y)$ in two variables x and y are given with $N = \deg F = \deg \Phi$. Set $f = F(x, 1)$, $\varphi = \Phi(x, 1)$ and suppose $\deg f \geq \deg \varphi = n$. Following Sylvester [12], we state the following quadratic expression of the specialized Jacobian $J(F, \Phi)$ in terms of the remainders (for the division of f by φ). Its proof consists into recognizing that the Jacobian is a Wrońskian.

Proposition 10 *One has*

$$\frac{1}{N} J(F, \Phi)\Big|_{y=1} = \sum_{i=0}^{n} \mathcal{Q}'_i \mathcal{R}_i^2 \,. \tag{28}$$

Proof. Substituting the well-known identity

$$NF = x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} \,, \tag{29}$$

into (27), we have

$$\sum_{i=0}^{n} \mathcal{Q}'_i \mathcal{R}_i^2 = f'\varphi - f\varphi'$$

$$= \frac{\partial F}{\partial x} \cdot \frac{1}{N}\big(x\frac{\partial \Phi}{\partial x} + y\frac{\partial \Phi}{\partial y}\big)\Big|_{y=1} - \frac{1}{N}\big(x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}\big)\frac{\partial \Phi}{\partial y}\Big|_{y=1}$$

$$= \frac{1}{N}\big(\frac{\partial F}{\partial x}\frac{\partial \Phi}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial \Phi}{\partial x}\big)\Big|_{y=1} = \frac{1}{N} J(F, \Phi)\Big|_{y=1} .$$

The proposition has been proved. □

6 Bezoutiants in terms of the roots of polynomials

In this section, we give some expressions for the Bezoutiants of general pairs of polynomials of degrees $(n+1, n)$ and (n, n), solely in terms of the roots of the two polynomials.

To this end we need multi-Schur functions. Here are some basic definitions. Given two alphabets $\mathbb{A}$, $\mathbb{B}$, the *complete functions* $S_i(\mathbb{A}-\mathbb{B})$ of the difference of the alphabets are defined by the generating series (with z an extra variable) :

$$\sum z^i S_i(\mathbb{A}-\mathbb{B}) = \prod_{b\in\mathbb{B}} (1-zb) / \prod_{a\in\mathbb{A}} (1-za) \,. \tag{30}$$

For example, if $\mathbb{A}$ is of cardinality m, then $S_m(x-\mathbb{A}) = \prod_{a\in\mathbb{A}}(x - a)$ is a monic polynomial with set of roots $\mathbb{A}$.

Given $I = (i_1, i_2, \ldots, i_r) \in \mathbb{N}^r$, and alphabets $\mathbb{A}_1, \ldots, \mathbb{A}_r$, $\mathbb{B}_1, \ldots, \mathbb{B}_r$, the *multi-Schur function* $S_I(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_r - \mathbb{B}_r)$ is

$$S_I(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_r - \mathbb{B}_r) := \Big| S_{i_k+k-h}(\mathbb{A}_k - \mathbb{B}_k) \Big|_{1\leq h,k\leq r} \,. \tag{31}$$

In case of a Schur function such that

$$i_1 = \cdots = i_\gamma = i\,, \quad i_{\gamma+1} = \cdots = i_r = j\,,$$

$$\mathbb{A}_1 - \mathbb{B}_1 = \cdots = \mathbb{A}_\gamma - \mathbb{B}_\gamma = \mathbb{A} - \mathbb{B}\,, \quad \mathbb{A}_{\gamma+1} - \mathbb{B}_{\gamma+1} = \cdots = \mathbb{A}_r - \mathbb{B}_r = \mathbb{C} - \mathbb{D}\,,$$

we write in short

$$S_{i^\gamma\,;\,j^{r-\gamma}}(\mathbb{A}-\mathbb{B}\,;\ \mathbb{C}-\mathbb{D})\,.$$

For example, the Schur function

$$S_{1,1,1,1,5,5,5}(\mathbb{A}-\mathbb{B}, \mathbb{A}-\mathbb{B}, \mathbb{A}-\mathbb{B}, \mathbb{A}-\mathbb{B}, \mathbb{C}-\mathbb{D}, \mathbb{C}-\mathbb{D}, \mathbb{C}-\mathbb{D})$$

is written

$$S_{1^4;5^3}(\mathbb{A}-\mathbb{B}; \mathbb{C}-\mathbb{D})\,.$$

Given two alphabets $\mathbb{A}$, $\mathbb{B}$ of respective cardinalities m, n, then the resultant

$$R(\mathbb{A}, \mathbb{B}) := \prod_{a\in\mathbb{A}, b\in\mathbb{B}} (a-b)$$

can be written

$$R(\mathbb{A}, \mathbb{B}) = S_{n^m}(\mathbb{A}-\mathbb{B})\,. \tag{32}$$

In particular, the polynomial $S_n(x-\mathbb{B})$ equals $R(x, \mathbb{B})$. Eq.(32) admits a generalization: the successive remainders in the division of $R(x, \mathbb{A})$ by $R(x, \mathbb{B})$, conveniently normalized (they are called *subresultants*), can be expressed as Schur functions (cf. [8] and [9]).

From now on, $\mathbb{A}, \mathbb{B}$ being two finite alphabets, $f(x)$ will denote the polynomial $R(x, \mathbb{A})$, and $\varphi(x)$ will denote $R(x, \mathbb{B})$.

For such a pair, write

$$\mathfrak{Bez}(\mathbb{A}, \mathbb{B}) := \mathfrak{Bez}(f, \varphi)\,.$$

We shall say that a pair of alphabets $(\mathbb{A}, \mathbb{B})$ is *general* if (f, φ) is a general pair of polynomials.

Suppose first that $\mathbb{A}$ and $\mathbb{B}$ form a general pair of alphabets of cardinalities $n+1$ and n. We rewrite Proposition 5 in terms of $\mathbb{A}$ and $\mathbb{B}$. Using just quoted formulas for subresultants, expressions for α_i's from [8], and for ratios[3] of the remainders $\mathcal{R}_k$ by the corresponding subresultants, one gets :

Proposition 11 *One has*

$$\begin{aligned}\mathfrak{Bez}(\mathbb{A}, \mathbb{B}) &= R(x+y, \mathbb{B}) \\ &+ \sum_{i=1}^{n} (-1)^i \frac{S_{1^{n-i};(i+1)^i}(\mathbb{B}-x; \mathbb{B}-\mathbb{A})\, S_{1^{n-i};(i+1)^i}(\mathbb{B}-y; \mathbb{B}-\mathbb{A})}{S_{i^{i-1}}(\mathbb{B}-\mathbb{A})\, S_{(i+1)^i}(\mathbb{B}-\mathbb{A})}\,.\end{aligned} \tag{33}$$

[3]Recall that division of $S_{n+1}(x-\mathbb{A})$ by $S_n(x-\mathbb{B})$ introduces the ratios

$$t_k^2 t_{k+2}^2 / t_{k+1}^2 t_{k+3}^2$$

with $t_0 = t_1 = 1$ and $t_k = S_{k^{k-1}}(\mathbb{B}-\mathbb{A})$ for $k \geq 2$. Cf. [8].

Similarly, for a general pair $(\mathbb{A},\mathbb{B})$ of alphabets with the same cardinality n, invoking Proposition 7, one gets the following identity.

Proposition 12 *One has*

$$\mathfrak{Bez}(\mathbb{A},\mathbb{B}) = \sum_{i=1}^{n}(-1)^i \frac{S_{1^{n-i};i^i}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})S_{1^{n-i};i^i}(\mathbb{B}-y;\mathbb{B}-\mathbb{A})}{S_{(i-1)^{i-1}}(\mathbb{B}-\mathbb{A})S_{i^i}(\mathbb{B}-\mathbb{A})}\,. \tag{34}$$

We now record the following result.

Lemma 13 *Suppose that $\mathbb{A}$ and $\mathbb{B}$ are alphabets of cardinalities m and n with $m \geq n$. For $i = 1,\ldots,n$, one has*

$$\mathcal{R}_i(x) \equiv S_{i^{m-n+i-1}}(\mathbb{A}-\mathbb{B}-x)\,\varphi(x) \quad \mod \ f(x)\ . \tag{35}$$

Proof. It is a consequence of the explicit expression of $\mathcal{R}_i$ as an element of the ideal generated by $f(x)$ and $\varphi(x)$:

$$\mathcal{R}_i(x) = \pm S_{(m-n+i)^{i-1}}(\mathbb{B}-\mathbb{A}-x)f(x) + S_{i^{m-n+i-1}}(\mathbb{A}-\mathbb{B}-x)\varphi(x)\,. \tag{36}$$

For more details, see [8], (3.1.5). $\square$

Now we will establish two expressions, solely in terms of roots, that are congruent to the above Bezoutiants modulo $(f(x), f(y))$. We start with a general pair of alphabets $(\mathbb{A},\mathbb{B})$ of cardinalities $n+1$ and n. It follows from Proposition 5 and the lemma that, modulo the ideal $(f(x), f(y))$,

$$\mathfrak{Bez}(\mathbb{A},\mathbb{B}) \equiv R(x+y,\mathbb{B})\Big(1 + \sum_{i=1}^{n}\alpha_i S_{i^i}(\mathbb{A}-\mathbb{B}-x)S_{i^i}(\mathbb{A}-\mathbb{B}-y)\Big)\,. \tag{37}$$

In summary, the Bezoutiant can be expressed in terms of Schur functions indexed by partitions with equal parts :

Theorem 14 *For a general pair of alphabets $(\mathbb{A},\mathbb{B})$ of cardinalities $n+1$ and n, one has, modulo the ideal $(f(x), f(y))$,*

$$\mathfrak{Bez}(\mathbb{A},\mathbb{B}) \equiv R(x+y,\mathbb{B})\Big(1 + \sum_{i=1}^{n}(-1)^i \frac{S_{i^i}(\mathbb{A}-\mathbb{B}-x)S_{i^i}(\mathbb{A}-\mathbb{B}-y)}{S_{(i-1)^i}(\mathbb{A}-\mathbb{B})S_{i^{i+1}}(\mathbb{A}-\mathbb{B})}\Big)\,. \tag{38}$$

Let us pass now to pairs of alphabets of the same cardinality.

Theorem 15 *For a general pair of alphabets $(\mathbb{A},\mathbb{B})$ of the same cardinality n, one has, modulo the ideal $(f(x), f(y))$,*

$$\mathfrak{Bez}(\mathbb{A},\mathbb{B}) \equiv R(x+y,\mathbb{B})\Big(\frac{1}{S_1(\mathbb{A}-\mathbb{B})} + \sum_{i=1}^{n-1}(-1)^i \frac{S_{(i+1)^i}(\mathbb{A}-\mathbb{B}-x)S_{(i+1)^i}(\mathbb{A}-\mathbb{B}-y)}{S_{i^i}(\mathbb{A}-\mathbb{B})S_{(i+1)^{i+1}}(\mathbb{A}-\mathbb{B})}\Big)\,. \tag{39}$$

Proof. We shall deduce the present case from the one of alphabets of cardinalities $(n, n{-}1)$. To this end, suppose that $\psi(x) := S_{n-1}(x{-}\mathbb{C})$ is the monic remainder of $\varphi(x)$ modulo $f(x)$, i.e.

$$\varphi(x) = f(x) + S_1(\mathbb{A}{-}\mathbb{B})\psi(x)\,. \tag{40}$$

Then $(\mathbb{A}, \mathbb{C})$ is general, and by (9) we have

$$\mathfrak{Bez}(\mathbb{A}, \mathbb{B}) = \mathfrak{Bez}(\mathbb{A}, \mathbb{C}) S_1(\mathbb{A}{-}\mathbb{B})\,. \tag{41}$$

Thanks to (40), the remainders of $f(x)$ by $\psi(x)$ coincide with those of $f(x)$ by $\varphi(x)$, up to powers of $u := S_1(\mathbb{A}{-}\mathbb{B})$. From expression (36), equalizing degrees by an appropriate power of u, one has

$$S_{(i+1)^i}(\mathbb{A}{-}\mathbb{B}{-}x)\varphi(x) \equiv u^{i+1}\, S_{(i+1)^i}(\mathbb{A}{-}\mathbb{C}{-}x)\psi(x) \quad \mod \ f(x)\ .$$

Similarly, comparing the top coefficients of $S_{1^{n-i};\,i^i}(\mathbb{A}{-}x;\, \mathbb{A}{-}\mathbb{B})$ and $S_{1^{n-i};\,(i-1)^i}(\mathbb{A}{-}x;\, \mathbb{A}{-}\mathbb{C})$, one gets

$$S_{i^i}(\mathbb{A}{-}\mathbb{B}) = u^i S_{(i-1)^i}(\mathbb{A}{-}\mathbb{C})\ .$$

Thus, using (41), decomposition (38) :

$$\mathfrak{Bez}(\mathbb{A}, \mathbb{C}) \equiv \psi(x)\psi(y)\Big(1 - \frac{S_1(\mathbb{A}{-}\mathbb{C}{-}x)S_1(\mathbb{A}{-}\mathbb{C}{-}y)}{S_{11}(\mathbb{A}{-}\mathbb{C})} + \frac{S_{22}(\mathbb{A}{-}\mathbb{C}{-}x)S_{22}(\mathbb{A}{-}\mathbb{C}{-}y)}{S_{11}(\mathbb{A}{-}\mathbb{C})S_{222}(\mathbb{A}{-}\mathbb{C})} - \cdots\Big)$$

becomes

$$\mathfrak{Bez}(\mathbb{A}, \mathbb{B}) \equiv R(x{+}y, \mathbb{B})\Big(\frac{1}{S_1(\mathbb{A}{-}\mathbb{B})} - \frac{S_2(\mathbb{A}{-}\mathbb{B}{-}x)S_2(\mathbb{A}{-}\mathbb{B}{-}y)}{S_1(\mathbb{A}{-}\mathbb{B})S_{22}(\mathbb{A}{-}\mathbb{B})} + \frac{S_{33}(\mathbb{A}{-}\mathbb{B}{-}x)S_{33}(\mathbb{A}{-}\mathbb{B}{-}y)}{S_{22}(\mathbb{A}{-}\mathbb{B})S_{333}(\mathbb{A}{-}\mathbb{B})} - \cdots\Big)\,.$$

This proves the theorem. □

Notice that (38) and (39) become equalities, when specializing x or y into a root of f.

In the next section, the preceding Bezoutiants will be compared to some Christoffel-Darboux kernels.

7 Euclidean remainders versus orthogonal polynomials

In this section, we shall investigate two families of orthogonal polynomials. The first one comes from the Euclidean division of a general pair of polynomials of degrees $(n{+}1, n)$, and the second one – from division of a general pair of polynomials of degrees (n, n).

Assume first that $(\mathbb{A},\mathbb{B})$ is a general pair of alphabets of cardinalities $n+1$ and n (and follow the convention of the previous section). One finds in the book of Brioschi [4], p.167, the following identities (we use the notation of Section 2) :

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i(a)\mathcal{N}_i(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,, \tag{42}$$

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i(a)\mathcal{D}_{i-1}(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,, \tag{43}$$

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i^2(a)\,\mathcal{Q}_i(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,. \tag{44}$$

These three identities make use of the following functional on $\mathbb{C}[x]$:

$$\mu : g(x) \mapsto \mu\big(g(x)\big) := \sum_{a\in\mathbb{A}} g(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)}\ . \tag{45}$$

The functional μ is an avatar of the Lagrange interpolation in the points $a \in \mathbb{A}$ (cf., e.g., [2] and [8]). It is characterized by the fact that it sends each x^i, $i \in \mathbb{N}$, onto the complete function $S_i(\mathbb{A}-\mathbb{B})$.

Given a linear functional, it is known how to write an orthogonal basis in terms of *moments* (cf. [11] and [2]). These expressions are still valid in the case of our functional μ, restricted to the space of polynomials of degree $\leq n$, and the following proposition is a rewriting, in terms of Schur functions, of the classical determinantal expressions of orthogonal polynomials.

Proposition 16 *The Schur polynomials*

$$P_k = S_{k^k}(\mathbb{A}-\mathbb{B}-x)\,, \tag{46}$$

$k = 0, 1, \ldots, n$, *form a unique (up to normalization) orthogonal basis with respect to the functional* μ, *of polynomials of respective degrees* $0, 1, \ldots, n$.

(For the reader's convenience we recall a proof in the Appendix.)

Combining (4) with Lemma 13, we see that $\mathcal{D}_k$ corresponds to P_k, and thus we get the following basic result.

Proposition 17 *For any fixed* $0 \leq i < j \leq n$, *one has the relations*

$$\sum_{a\in\mathbb{A}} a^i\mathcal{D}_j(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,, \tag{47}$$

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i(a)\mathcal{D}_j(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,. \tag{48}$$

Brioschi's relation (43) results from (48).

We can replace in (42) the polynomial $\mathcal{N}_i$, which is of degree $i-1$, by any polynomial of degree strictly less than i, because the set $\{\mathcal{D}_i\}$ forms an orthogonal basis. Since

$$\mathcal{D}_{i+1} = \mathcal{Q}_i \mathcal{D}_i - \mathcal{D}_{i-1}\,,$$

Eq.(44) results from the orthogonality of $\mathcal{D}_i$ with $\mathcal{D}_{i+1}$, and with $\mathcal{D}_{i-1}$.

Remark 18 *In [5], in the special case where $\varphi(x) = f'(x)/(n{+}1)$, Brioschi gives the following relations : for fixed $0 \le i < j \le n$,*

$$\sum_{a\in\mathbb{A}} a^i \mathcal{D}_j(a) = 0\,. \tag{49}$$

The sum looks different from (47) because the term

$$\frac{\varphi(a)}{R(a, \mathbb{A}{-}a)} = \frac{\varphi(a)}{f'(a)} = \frac{1}{n{+}1} \tag{50}$$

can be erased.

From Proposition 17 and (4), we infer

Corollary 19 *For any fixed $0 \le i < j \le n$, one has the relations*

$$\sum_{a\in\mathbb{A}} a^i \mathcal{R}_j(a) \frac{1}{\varphi(a) R(a, \mathbb{A}{-}a)} = 0\,. \tag{51}$$

$$\sum_{a\in\mathbb{A}} \mathcal{R}_i(a) \mathcal{R}_j(a) \frac{1}{\varphi(a) R(a, \mathbb{A}{-}a)} = 0\,, \tag{52}$$

Remark 20 *Notice that*

$$\frac{f'(x)}{f(x)} = \sum_{a\in\mathbb{A}} \frac{1}{x-a}\ , \tag{53}$$

so that the case treated by Sylvester and Brioschi is the case of orthogonal polynomials for a discrete uniform measure, in relation with Lagrange interpolation. As a matter of fact, it was also the starting point of Tchebychef (cf. [13], and [3] for an account to Tchebychef's work on orthogonal polynomials). Taking a more general $\varphi(x)$ of degree n, one has similarly

$$\frac{\varphi(x)}{f(x)} = \sum_{a\in\mathbb{A}} \varphi(a)\, \frac{R(x, \mathbb{A}{-}a)}{R(a, \mathbb{A}{-}a)}\ , \tag{54}$$

thanks to Lagrange interpolation, and this time the measure is no more uniform, but still concentrated in the points of $\mathbb{A}$.

Given a family of orthogonal polynomials $P_0(x), \ldots, P_n(x)$ associated with a linear functional μ, then the Christoffel-Darboux kernel (cf. [2], [11]) is :

$$K(x,y) := \sum_{i=0}^{n} P_i(x)P_i(y)/\mu\big(P_i(x)^2\big) \tag{55}$$

In the case of a general pair of alphabets $(\mathbb{A}, \mathbb{B})$ of respective cardinalities $n{+}1$ and n, μ being the functional defined in (45), expression (38) shows that $\mathfrak{Bez}(\mathbb{A}, \mathbb{B})$ be proportional to $K(x,y)$:

$$\mathfrak{Bez}(\mathbb{A}, \mathbb{B}) \equiv \varphi(x)\varphi(y)\, K(x,y) \mod (f(x), f(y))\,. \tag{56}$$

Therefore, the reproducing property of a Christoffel-Darboux kernel entails the following property of the Bezoutiant.

Proposition 21 *Given a general pair of alphabets $(\mathbb{A}, \mathbb{B})$ of respective cardinalities $n{+}1$ and n, and a polynomial $g(x)$, then*

$$\mu\big(g(x)\, \mathfrak{Bez}(\mathbb{A}, \mathbb{B})\big) \equiv \varphi(y)^2\, g(y) \mod f(y)\,. \tag{57}$$

Finally, consider a general pair of alphabets $(\mathbb{A}, \mathbb{B})$ of the same cardinality n. We state only results and omit their proofs which are similar to the ones in the previous case. We recall (cf. Lemma 13):

Lemma 22 *For $i = 1, \ldots, n$,*

$$\mathcal{R}_i(x) \equiv S_{i^{i-1}}(\mathbb{A}{-}\mathbb{B}{-}x)\, \varphi(x) \mod f(x)\ . \tag{58}$$

Proposition 23 *The Schur functions $S_{k^{k-1}}(\mathbb{A}{-}\mathbb{B}{-}x)$, $k = 1, \ldots, n$, form a unique (up to normalization) orthogonal basis of polynomials of respective degrees $0, 1, \ldots, n{-}1$, with respect to the following functional ν on $\mathbb{C}[x]$:*

$$\nu : x^i \mapsto S_{i+1}(\mathbb{A}{-}\mathbb{B})\,, \qquad i \in \mathbb{N}\,. \tag{59}$$

Comparing the Bezoutiant with the Christoffel-Darboux kernel we get :

Proposition 24 *For any polynomial $g(x)$, one has*

$$\nu(g(x)\mathfrak{Bez}(\mathbb{A}, \mathbb{B})) \equiv \varphi(y)^2\, g(y) \mod f(y)\,. \tag{60}$$

8 Appendix

Proposition 16 can be proved as follows. Fix $k = 1, \ldots, n$. We have the following equality :

$$S_{k^k}(\mathbb{A}{-}\mathbb{B}{-}x)x^i = S_{k^k;\, i}(\mathbb{A}{-}\mathbb{B};\ x)\,. \tag{61}$$

(One can pass from the RHS of (61) to its LHS by multiplying the ith row by x and subtracting it from the $(i-1)$st row, successively for $i = 2, 3, \ldots, k+1$.)

The functional μ acts only on the last column of the determinant

$$S_{k^k;i}(\mathbb{A}-\mathbb{B};\ x)\,,$$

and sends it onto

$$S_{k^k;i}(\mathbb{A}-\mathbb{B};\ \mathbb{A}-\mathbb{B})\,.$$

When $i < k$, this last determinant $S_{k^k,i}(\mathbb{A}-\mathbb{B})$ vanishes, having two identical columns. In other words, the polynomial $P_k = S_{k^k}(\mathbb{A}-\mathbb{B}-x)$ is orthogonal to $x^0, \ldots, x^{k-1}$. $\square$

References

[1] M. Bézout. *Théorie générale des équations algébriques*, Paris, (1779).

[2] C. Brezinski. *Padé-type approximation and general orthogonal polynomials*, Birkhäuser (1980).

[3] C. Brezinski. *History of continued fractions and Padé approximants*, Springer (1991).

[4] F. Brioschi. *Théorie des déterminants*, appendix to the French edition, Paris, Mallet-Bachelier (1856).

[5] F. Brioschi. *Sur les fonctions de Sturm*, Nouvelles Annales **13** (1854) 71–80.

[6] A. Cayley. *Note sur la méthode d'élimination de Bézout*, Crelle J. **53** (1857) 366–367.

[7] C. Hermite. *Oeuvres*, ed. E. Picard, Gauthier-Villars , Paris (1908).

[8] A. Lascoux. *Symmetric functions & Combinatorial operators on polynomials*, CBMS/AMS Lectures Notes **99**, (2003).

[9] A. Lascoux, P. Pragacz. *Double Sylvester sums for subresultants and multi-Schur functions*, J. Symb. Comp. **35** (2003) 689–710.

[10] T. Muir. *History of determinants*, 5 volumes, Dover reprints (1960).

[11] G. Szegö. *Orthogonal polynomials*, Amer. Math. Soc., Providence (1939).

[12] J.J. Sylvester. *A theory of the syzygetic relations of two rational integral functions*, Phil. Trans. Royal Soc. London vol CXLIII, Part III (1853) 407–548.

[13] P. Tchebychef. *Oeuvres, 2 vols*, A. Markoff, N. Sonin eds., Chelsea, New York.