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Double Sylvester sums for subresultants and multi-Schur functions

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Abstract

J. J. Sylvester has announced formulas expressing the subresultants (or the successive polynomial remainders for the Euclidean division) of two polynomials, in terms of some double sums over the roots of the two polynomials. We prove Sylvester formulas using the techniques of multivariate polynomials involving multi-Schur functions and divided differences.

Introduction and statement of the main result

The subresultants played a fundamental role in the theory of polynomial equations in the 19-th century, cf. e.g. (Sylvester, 1839, 1840, 1853), (Borchardt, 1860), and (Salmon, 1885). Recently they also have found important applications in computer algebra, for example, in devising efficient methods for computing greatest common divisors of two polynomials (Collins, 1967, 1973), (Brown, 1971), and (Brown, Traub, 1971), for carrying out quantifier elimination over complex of real numbers cf. e.g. (Collins, 1975), and also for coding theory (Shen, 1992). They have been also extended to some noncommutative polynomials (Chardin, 1991), (Li, 1998), and (Hong, 2001).

Suppose that two polynomials in one variable

$$P(x) = x^m + \alpha_1 x^{m-1} + \cdots + \alpha_m \qquad \text{and} \qquad Q(x) = x^n + \beta_1 x^{n-1} + \cdots + \beta_n$$

with, say, complex coefficients are given. Then the *subresultant of degree d* associated with P and Q, denoted by $\mathcal{R}^{(d)}$, is defined by the following determinant:

$$\mathcal{R}^{(d)} = \begin{vmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{m+n-2d-2} & x^{n-d-1}P \\ 0 & 1 & \alpha_1 & \alpha_2 & \ldots & \alpha_{m+n-2d-3} & x^{n-d-2}P \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ . & . & . & . & & \alpha_{m-d-1} & P \\ 1 & \beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{m+n-2d-2} & x^{m-d-1}Q \\ 0 & 1 & \beta_1 & \beta_2 & \ldots & \beta_{m+n-2d-3} & x^{m-d-2}Q \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ . & . & . & . & & \beta_{n-d-1} & Q \end{vmatrix} .$$

Here we understand that $\alpha_i = 0$ for $i > m$ and $\beta_j = 0$ for $j > n$. Of course, the subresultant $\mathcal{R}^{(0)}$ of degree zero equals the resultant. We refer to (Brown, Traub, 1971), (Collins, 1967, 1973), (Gonzales et al., 1990), (Hong, 1997), (Jouanolou, 1991, 1996), (Lascoux, 1990a), (Lombardi et al., 2000), and (Loos, 1982) for a more detailed discussion of this notion and the properties of subresultants.

We are interested in the problem of expressing $\mathcal{R}^{(d)}$ as polynomials in the roots of P and Q, generalizing the expression of the resultant $\mathcal{R}^{(0)}$ as the product of the differences of the roots of P and Q.

Recall the following well-known interpretation of the subresultants. Suppose (for the rest of this paper) that

$$m = \deg(P) \geqslant n = \deg(Q) .$$

Let us look at the iterated division of P and Q:

$$P = *\, Q + c_1\mathcal{R}_1 \quad , \qquad Q = *\, \mathcal{R}_1 + c_2\mathcal{R}_2 \quad , \qquad \mathcal{R}_1 = *\, \mathcal{R}_2 + c_3\mathcal{R}_3 \quad , \qquad \ldots .$$

The successive coefficients " $*$ " are the unique polynomials such that

$$n > \deg \mathcal{R}_1 > \deg \mathcal{R}_2 > \deg \mathcal{R}_3 > \cdots .$$

Instead of the usual Euclidean division, where $c_1 = c_2 = c_3 = \cdots = 1$, we choose the constants c_i in such a way that the successive remainders $\mathcal{R}_i$ are equal to $\mathcal{R}^{(n-i)}$, $i = 0, 1, \ldots, n-1$. In particular, instead of being rational functions in the roots of P and Q (as in the Euclidean division case), they are polynomials in the roots, cf. Remark 1.12.

An interesting solution to the above problem was proposed by Sylvester about 160 years ago (Sylvester, 1839, 1840, 1853), who found "the successive residues, divested of their allotrious factors ..." that is, who also normalized the remainders in such a way as to obtain polynomials in the roots, the last "residue" being the resultant. To state Sylvester's result, we need the following notation. For two finite sets $\mathbb{A}$ and $\mathbb{B}$ of elements in a commutative ring, we set

$$R(\mathbb{A},\mathbb{B}) := \prod_{a\in\mathbb{A}, b\in\mathbb{B}} (a-b)\,.$$

Let now $\mathbb{A} = \{a_1, \ldots, a_m\}$, $\mathbb{B} = \{b_1, \ldots, b_n\}$ be two sequences of (commuting) variables. Let $0 \leqslant p \leqslant m$, $0 \leqslant q \leqslant n$ be two integers. Define, after (Sylvester, 1840, 1853), the following double sum:

$$\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};x) := \sum_{\mathbb{A}'\subset\mathbb{A}} \sum_{\mathbb{B}'\subset\mathbb{B}} R(x,\mathbb{A}')R(x,\mathbb{B}') \frac{R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}\smallsetminus\mathbb{A}',\mathbb{B}\smallsetminus\mathbb{B}')}{R(\mathbb{A}',\mathbb{A}\smallsetminus\mathbb{A}')R(\mathbb{B}',\mathbb{B}\smallsetminus\mathbb{B}')}\,,$$

the sum being over all subsets $\mathbb{A}'$ of cardinality p and $\mathbb{B}'$ of cardinality q. Sometimes, we will also use the notation $\mathrm{Sylv}^{p,q}(x)$ or $\mathrm{Sylv}^{p,q}$ for this sum. The Sylvester sum $\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};x)$ is a polynomial in $\mathbb{A}$, $\mathbb{B}$, and x; moreover $\deg_x \mathrm{Sylv}^{p,q} = p+q$. Observe that $\mathrm{Sylv}^{0,0} = R(\mathbb{A},\mathbb{B})$ and if $p=0$ or $q=0$ then the Sylvester sum reduces to a single sum. More interesting Sylvester sums are given in Example 2.10 and 2.12, and Remark 2.13. Summations of this type and their generalizations (cf. (Lascoux, Schützenberger, 1987)) play nowadays an important role e.g. in representation theory, and in the description of Gysin maps (called also "integrations over a fibre") for fibrations with homogeneous spaces as fibres (cf. (Akyildiz, Carrell, 1987), (Brion, 1996), (Lascoux, 1975), and (Pragacz, 1991, 1996)).

The main result connecting the subresultants and Sylvester sums is the following theorem.

THEOREM 0.1: *If $p+q < n$, then specializing $\mathbb{A}$ and $\mathbb{B}$ in the roots of the polynomials P and Q respectively, one has*

$$\mathrm{Sylv}^{p,q} = (-1)^{p(m-p-q)}\binom{p+q}{p}\mathcal{R}^{(p+q)}\,.$$

This result was stated without complete proof in (Sylvester, 1840, 1853) and we did not find any in the literature. All other expressions of subresultants that we have found in the literature can be derived from either the above determinant or the determinant (2.3), and this is why we wanted to comment about Sylvester's formulas, from an algebraic point of view only. We shall not get involved into elimination theory, but refer for that to the papers on subresultants quoted in the references. The case of a single Sylvester sum $p=0$ or $q=0$ was already announced in an earlier paper (Sylvester, 1839), as a generalization of Lagrange interpolation, and proved in (Borchardt, 1860) (we discuss the result of Borchardt in Remark 2.13.). Let us stress that the case of a single sum *does not* imply the assertion of the theorem for the double Sylvester sums.

The objective of the present paper is to give a proof of Theorem 0.1 with the help of techniques of multivariate polynomials, involving multi-Schur functions and divided differences. The assumption $p+q<n$ *is essential*. In fact, for the needs of the proof, we also compute $\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};x)$ when $p+q=n$. A "closed form" formula for Sylvester sum in the case $p+q>n$, remains an open problem. The strategy of the present paper is to express all functions as multi-Schur functions. This was already used in Lascoux (1990a).

The paper is organized as follows.

In the first section we recall some information on multi-Schur functions.

In the second section which is the core of the paper, we provide a proof of Theorem 0.1. This proof goes by induction on $\deg(Q)$ and makes use of a formula which expresses the first remainder $\mathcal{R}_1$ with the help of divided differences. An important ingredient of the proof is an expression of a subresultant in the form of a multi-Schur function given in Proposition 2.2. This last expression is proved via specialization. We also discuss there the result of (Borchardt, 1860) on single Sylvester sums.

In the third section we show, using some symmetrizing operators, another way to obtain the key identity (2.11) that relates the double Sylvester sums with the multi-Schur functions.

These algebro-combinatorial computations can be interpreted in terms of Gysin maps for the fibre product of two Grassmann bundles; we discuss this in the fourth section.

1. Review of symmetric polynomials and multi-Schur functions

Given a formal series in z, $\sum_{i=0}^{\infty} z^i S_i$, with $S_0=1$ and with coefficients from an arbitrary commutative ring, one associates with the series an infinite matrix of their coefficients:

$$\mathbf{S} = [S_{k-h}]_{h,k\geqslant 1} \tag{1.1}$$

(one has $S_i=0$ if $i<0$). For any natural number l and any $I=(i_1,\dots,i_l)\in\mathbb{Z}^l$ one associates with the series the *Schur function* S_I indexed by I: by definition S_I is the minor of $\mathbf{S}$ taken over the l first rows and over the columns i_1+1, i_2+2, $\dots$, i_l+l (S_I is zero if one of these integers is negative).

More explicitly,

$$S_I = |S_{i_k+k-h}|_{1\leqslant h,k\leqslant l}\,. \tag{1.2}$$

We note that $S_I=\pm S_J$ or 0, where $J=(0\leqslant j_1\leqslant\cdots\leqslant j_l)$ is a *partition*. For example,

$$S_{5,7,2,0,3} = \begin{vmatrix} S_5 & S_8 & S_4 & S_3 & S_7 \\ S_4 & S_7 & S_3 & S_2 & S_6 \\ S_3 & S_6 & S_2 & S_1 & S_5 \\ S_2 & S_5 & S_1 & 1 & S_4 \\ S_1 & S_4 & 1 & 0 & S_3 \end{vmatrix} = \begin{vmatrix} S_3 & S_4 & S_5 & S_7 & S_8 \\ S_2 & S_3 & S_4 & S_6 & S_7 \\ S_1 & S_2 & S_3 & S_5 & S_6 \\ 1 & S_1 & S_2 & S_4 & S_5 \\ 0 & 1 & S_1 & S_3 & S_4 \end{vmatrix} = S_{3,3,3,4,4} \,.$$

Given a (finite) *alphabet* $\mathbb{A}$, i.e. a multi-set of elements in a commutative ring (often indeterminates), we denote by $S_I(\mathbb{A})$ the Schur functions associated with the series

$$\prod_{a\in\mathbb{A}} (1-za)^{-1} \,.$$

This notion is extended to *virtual alphabets** as follows. Given two alphabets $\mathbb{A}, \mathbb{B}$, we denote by $S_I(\mathbb{A}-\mathbb{B})$ the Schur functions associated with the series

$$\prod_{b\in\mathbb{B}} (1-zb) \Big/ \prod_{a\in\mathbb{A}} (1-za) \,.$$

So $S_i(\mathbb{A}-\mathbb{B})$ interpolates between $S_i(\mathbb{A})$ – the complete homogeneous symmetric polynomial of degree i in $\mathbb{A}$ and $S_i(-\mathbb{B})$ – the i-th elementary polynomial in $\mathbb{B}$ times $(-1)^i$.

The notation $\mathbb{A}-\mathbb{B}$ is compatible with the multiplication of series:

$$\sum z^i S_i(\mathbb{A}-\mathbb{B}) \cdot \sum z^j S_j(\mathbb{A}'-\mathbb{B}') = \sum z^i S_i\big((\mathbb{A}+\mathbb{A}')-(\mathbb{B}+\mathbb{B}')\big) \,, \tag{1.3}$$

the sum $\mathbb{A}+\mathbb{A}'$ denoting the union of two alphabets $\mathbb{A}$ and $\mathbb{A}'$.

Convention: We will often identify an alphabet $\mathbb{A} = \{a_1, \dots, a_m\}$ with the sum $a_1 + \cdots + a_m$ and perform usual algebraic operations on such elements. We will give priority to the algebraic notation over the set-theoretic one.

We have $(\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C}) = \mathbb{A}+\mathbb{B}$, and this corresponds to simplification of the common factor for the rational series:

$$\sum z^i S_i((\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C})) = \sum z^i S_i(\mathbb{A}-\mathbb{B}) \,.$$

The cardinality of $\mathbb{B}$ being n, we can write the polynomial $\prod_{b\in\mathbb{B}}(x-b)$ in the compact form $S_n(x-\mathbb{B})$; after the previous definition for $\mathbb{A}=\{x\}$ one has

$$S_n(x-\mathbb{B}) = x^n + x^{n-1} S_1(-\mathbb{B}) + \cdots + x^0 S_n(-\mathbb{B}) \,. \tag{1.4}$$

*By a virtual alphabet, we understand a formal finite $\mathbb{Z}$-linear combination of alphabets. The precise sense of the operations "+" and "−" for alphabets, is explained in this section. Of course, using these operations, any virtual alphabet has the form $\mathbb{A}-\mathbb{B}$, for some alphabets $\mathbb{A}$ and $\mathbb{B}$. All these notions are clarified by the use of λ-rings, which are the appropriate structure for manipulations of symmetric functions.

Moreover, one has for $i > 0$,

$$S_{n+i}(-\mathbb{B}) = 0 \qquad \text{and} \qquad S_{n+i}(x - \mathbb{B}) = x^i S_n(x - \mathbb{B})\,. \tag{1.5}$$

The inverse of the rational series $\sum z^i S_i(\mathbb{A} - \mathbb{B})$ is the series $\sum z^i S_i(\mathbb{B} - \mathbb{A})$; the inversion of series corresponds to the inversion of the matrices $\mathbf{S}$, and for their minors which are Schur functions, to the transposition of the partitions indexing them. In particular, for a partition $I = (i_1, \ldots, i_l)$, one has $S_I(-\mathbb{A}) = (-1)^{i_1+\cdots+i_l} S_J(\mathbb{A})$ where J is the transpose of I.

Given l virtual alphabets $\mathbb{A}^{(1)}, \ldots, \mathbb{A}^{(l)}$, and a sequence $I \in \mathbb{Z}^l$, one associates with them the *multi-Schur function*

$$S_I\big(\mathbb{A}^{(1)}, \ldots, \mathbb{A}^{(l)}\big) = |S_{i_k+k-h}(\mathbb{A}^{(k)})|_{1\leqslant h,k\leqslant l}\,. \tag{1.6}$$

Numerous applications of multi-Schur functions are discussed in (Lascoux, Schützenberger, 1985).

Notation: Suppose that $I = (i_1, i_2, \ldots)$ is a partition. We will group the parts of I which correspond to the same argument. So, if $i_r = i_{r+1} = \cdots = i_t (= i)$ is the maximal subsequence of equal parts of I corresponding to the fixed argument $\mathbb{A}^{(k)}$, then we will write the multi-Schur function (1.6) as

$$S_{\ldots;i^{t-r};\ldots}(\ldots;\mathbb{A}^{(k)};\ldots)\,.$$

For example, the multi-Schur function $S_{1,1,1,1,2,2,4,4,4,4,5}$ associated with a sequence of virtual alphabets

$$\mathbb{A}^{(1)}, \mathbb{A}^{(2)}, \mathbb{A}^{(2)}, \mathbb{A}^{(3)}, \mathbb{A}^{(3)}, \mathbb{A}^{(3)}, \mathbb{A}^{(4)}, \mathbb{A}^{(4)}, \mathbb{A}^{(4)}, \mathbb{A}^{(5)}, \mathbb{A}^{(5)}$$

is written as

$$S_{1;1^2;1;2^2;4^3;4;5}(\mathbb{A}^{(1)}; \mathbb{A}^{(2)}; \mathbb{A}^{(3)}; \mathbb{A}^{(3)}; \mathbb{A}^{(4)}; \mathbb{A}^{(5)}; \mathbb{A}^{(5)})\,.$$

By using the relation

$$S_p(\mathbb{A} - x) = S_p(\mathbb{A}) - x S_{p-1}(\mathbb{A})\,,$$

where $\mathbb{A}$ is a virtual alphabet and x is a single variable, and performing elementary operations on the rows (or columns) of a determinant of the form (1.6), we arrive at the following transformation lemma:

Lemma 1.7: *Let $t \leqslant l$ be a positive integer and $\mathbb{X}$ an alphabet of cardinality $r \leqslant t$.*

(i) *The replacement of $S_p(\mathbb{A}^{(k)})$ by $S_p(\mathbb{A}^{(k)} - \mathbb{X})$ in the first $t - r$ rows of a sequence of t successive rows of the determinant* (1.6)*, does not change the value of the determinant.*

(ii) *Suppose that there exists k such that $k+t \leqslant l$, $i_{k+1} = i_{k+2} = \cdots = i_{k+t}$, and $\mathbb{A}^{(k+1)} = \cdots = \mathbb{A}^{(k+t)} = \mathbb{A}$. Then the replacement of $S_p(\mathbb{A})$ by $S_p(\mathbb{A}-\mathbb{X})$ in the $(k+t)$-th, $(k+t-1)$-th, ..., $(k+r+2)$-th column of the determinant* (1.6), *does not change the value of the determinant.*

Example 1.8: We illustrate (i). For virtual alphabets $\mathbb{A}$, $\mathbb{B}$, ..., and an alphabet $\mathbb{X}$ of cardinality 2, the following equality holds:

$$\begin{vmatrix} S_6(\mathbb{A}) & S_9(\mathbb{B}) & S_5(\mathbb{C}) & S_4(\mathbb{D}) & S_8(\mathbb{E}) \\ S_5(\mathbb{A}) & S_8(\mathbb{B}) & S_4(\mathbb{C}) & S_3(\mathbb{D}) & S_7(\mathbb{E}) \\ S_4(\mathbb{A}) & S_7(\mathbb{B}) & S_3(\mathbb{C}) & S_2(\mathbb{D}) & S_6(\mathbb{E}) \\ S_3(\mathbb{A}) & S_6(\mathbb{B}) & S_2(\mathbb{C}) & S_1(\mathbb{D}) & S_5(\mathbb{E}) \\ S_2(\mathbb{A}) & S_5(\mathbb{B}) & S_1(\mathbb{C}) & 1 & S_4(\mathbb{E}) \end{vmatrix}$$

$$= \begin{vmatrix} S_6(\mathbb{A}) & S_9(\mathbb{B}) & S_5(\mathbb{C}) & S_4(\mathbb{D}) & S_8(\mathbb{E}) \\ S_5(\mathbb{A}-\mathbb{X}) & S_8(\mathbb{B}-\mathbb{X}) & S_4(\mathbb{C}-\mathbb{X}) & S_3(\mathbb{D}-\mathbb{X}) & S_7(\mathbb{E}-\mathbb{X}) \\ S_4(\mathbb{A}-\mathbb{X}) & S_7(\mathbb{B}-\mathbb{X}) & S_3(\mathbb{C}-\mathbb{X}) & S_2(\mathbb{D}-\mathbb{X}) & S_6(\mathbb{E}-\mathbb{X}) \\ S_3(\mathbb{A}) & S_6(\mathbb{B}) & S_2(\mathbb{C}) & S_1(\mathbb{D}) & S_5(\mathbb{E}) \\ S_2(\mathbb{A}) & S_5(\mathbb{B}) & S_1(\mathbb{C}) & 1 & S_4(\mathbb{E}) \end{vmatrix}.$$

Let now $\mathbb{A}$ and $\mathbb{B}$ be two alphabets. The element $R(\mathbb{A},\mathbb{B}) = \prod_{a\in\mathbb{A},b\in\mathbb{B}}(a-b)$ is the resultant of the polynomials with roots $\mathbb{A}$ and $\mathbb{B}$. Given an indeterminate x, the polynomial $S_m(x-\mathbb{A})$ can be written unambiguously as $R(x,\mathbb{A})$. In general, we have the following expression:

$$R(\mathbb{A},\mathbb{B}) = S_{n^m}(\mathbb{A}-\mathbb{B})\,. \tag{1.9}$$

See (Collins, 1967, 1973), (Jouanolou, 1991, 1996), and (Lascoux, 1986a) for more about resultants.

We will need the following two lemmas.

Lemma 1.10: *For positive integers r and t, the following equality holds:*

$$S_{t^r}(\mathbb{A}-\mathbb{B}) = \sum S_I(\mathbb{A})S_{I^*}(-\mathbb{B})\,,$$

where the sum is over all partitions $I = (i_1,\dots,i_r)$ with $i_r \leqslant t$, and I^* stands for the partition $(t-i_r,\dots,t-i_1)$.

This a particular case of a well-known linearity formula, cf. e.g. (Lascoux, Schützenberger, 1985), (Lascoux, 1986a), and (Pragacz, 1991).

LEMMA 1.11: *For positive integers d, r, and t, and virtual alphabets $\mathbb{A}$ and $\mathbb{B}$, the following equality holds:*

$$S_{1^d;t^r}(\mathbb{A};\mathbb{B}) = \sum_{i=0}^{d} (-1)^i S_{1^{d-i}}(\mathbb{A}) S_{t^{r-1},t+i}(\mathbb{B})\,.$$

Proof: The Laplace expansion of the determinant $S_{1^d;t^r}(\mathbb{A};\mathbb{B})$ along the first d columns gives a sum of products of Schur functions of $\mathbb{A}$ and $\mathbb{B}$ respectively. One checks that their indices are as stated. □

For example, consider the determinant of $S_{1,1;2,2,2}(\mathbb{A};\mathbb{B})$:

$$\begin{vmatrix} S_1(\mathbb{A}) & S_2(\mathbb{A}) & S_4(\mathbb{B}) & S_5(\mathbb{B}) & S_6(\mathbb{B}) \\ 1 & S_1(\mathbb{A}) & S_3(\mathbb{B}) & S_4(\mathbb{B}) & S_5(\mathbb{B}) \\ 0 & 1 & S_2(\mathbb{B}) & S_3(\mathbb{B}) & S_4(\mathbb{B}) \\ 0 & 0 & S_1(\mathbb{B}) & S_2(\mathbb{B}) & S_3(\mathbb{B}) \\ 0 & 0 & 1 & S_1(\mathbb{B}) & S_2(\mathbb{B}) \end{vmatrix}\,.$$

A term in the expansion is

$$-\begin{vmatrix} S_1(\mathbb{A}) & S_2(\mathbb{A}) \\ 0 & 1 \end{vmatrix} \quad \begin{vmatrix} S_3(\mathbb{B}) & S_4(\mathbb{B}) & S_5(\mathbb{B}) \\ S_1(\mathbb{B}) & S_2(\mathbb{B}) & S_3(\mathbb{B}) \\ 1 & S_1(\mathbb{B}) & S_2(\mathbb{B}) \end{vmatrix}$$

in which one recognizes $-S_1(\mathbb{A})\,S_{223}(\mathbb{B})$, up to transpositions along the anti-diagonals.

REMARK 1.12: Let $P = S_m(x-\mathbb{A})$ and $Q = S_n(x-\mathbb{B})$ be two polynomials in one variable (with $m \geqslant n$, as usual). Algebraic operations on these two polynomials preserve the symmetry in the roots of each of the two polynomials, and thus produce symmetric functions in $\mathbb{A}$ and $\mathbb{B}$ separately. The last nonzero remainder in the Euclidean division of P by Q is the GCD of P and Q. The GCD of P and Q is of degree d iff the Schur functions $S_{n^m}(\mathbb{A}-\mathbb{B}), \ldots, S_{(n-d+1)^{m-d+1}}(\mathbb{A}-\mathbb{B})$ vanish and $S_{(n-d)^{m-d}}(\mathbb{A}-\mathbb{B}) \neq 0$, cf. (Trudi, 1862), (Lascoux, 1986a), and (Pragacz, 1987, 1991).

We want the n-th normalized† remainder to be the resultant, $\mathcal{R}_n = \mathcal{R}^{(0)} = R(\mathbb{A},\mathbb{B}) = S_{n^m}(\mathbb{A}-\mathbb{B})$. We normalize the successive remainders by asking that the $(n-d)$-th remainder $\mathcal{R}_{n-d}$ has the leading term $x^d S_{(n-d)^{m-d}}(\mathbb{A}-\mathbb{B})$. Then it equals the subresultant $\mathcal{R}^{(d)}$. For more details see (Collins, 1967, 1973), (Brown, Traub, 1971), (Loos, 1982), (Lascoux, 1990a), and (Jouanolou, 1996). In particular, if the GCD of P and Q is of degree d, then it is given - up to a nonzero scalar - by the explicit determinants (2.1) and (2.3) below.

†By normalization, we mean here multiplication by a nonzero scalar.

2. Proof of the main theorem

Recall that the subresultant $\mathcal{R}^{(d)}$ of degree d, associated with a pair of polynomials $S_m(x-\mathbb{A})$ and $S_n(x-\mathbb{B})$, is defined as the following determinant:

$$\mathcal{R}^{(d)} = \begin{vmatrix} S_0(-\mathbb{A}) & \dots & S_{m+n-2d-2}(-\mathbb{A}) & S_{m+n-d-1}(x-\mathbb{A}) \\ \vdots & & \vdots & \vdots \\ S_{d-n+1}(-\mathbb{A}) & \dots & S_{m-d-1}(-\mathbb{A}) & S_m(x-\mathbb{A}) \\ S_0(-\mathbb{B}) & \dots & S_{m+n-2d-2}(-\mathbb{B}) & S_{m+n-d-1}(x-\mathbb{B}) \\ \vdots & & \vdots & \vdots \\ S_{d-m+1}(-\mathbb{B}) & \dots & S_{n-d-1}(-\mathbb{B}) & S_n(x-\mathbb{B}) \end{vmatrix} . \tag{2.1}$$

In the following proposition we present the subresultant $\mathcal{R}^{(d)}$ as a multi-Schur function. We give a proof by specialization, using some simple transformations of determinants. (See (Lascoux, 1990a) for a connection with Wroński's work (Hoene-Wroński, 1817); cf. also (Lascoux, 1990b).)

PROPOSITION 2.2: *For $d = 0, 1, \dots, n-1$, we have*

$$\mathcal{R}^{(d)} = (-1)^{mn-d(m+n)} S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x; \mathbb{B}-\mathbb{A}) .$$

Proof: We recall that $S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$ is the determinant:

$$\begin{vmatrix} S_1(\mathbb{B}-x) & \dots & S_d(\mathbb{B}-x) & S_m(\mathbb{B}-\mathbb{A}) & \dots & S_{m+n-d-1}(\mathbb{B}-\mathbb{A}) \\ \vdots & & \vdots & \vdots & & \vdots \\ S_{2-n}(\mathbb{B}-x) & \dots & S_{d-n+1}(\mathbb{B}-x) & S_{m-n+1}(\mathbb{B}-\mathbb{A}) & \dots & S_{m-d}(\mathbb{B}-\mathbb{A}) \end{vmatrix} . \tag{2.3}$$

It suffices to show that the n specializations $x = b \in \mathbb{B}$ of $\mathcal{R}^{(d)}$ and of the determinant (2.3) coincide because these two polynomials have degree $< n$.

We specialize the determinant (2.3). First, using the transformation Lemma 1.7(i), we subtract $\mathbb{B}-b$ in the top row. We perform operations on alphabets only, and therefore will write only them. So, starting from

$$\begin{vmatrix} \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \end{vmatrix} ,$$

(where $\mathbb{B}-b$ appears in the first d columns), we see that the previous determinant is equal to the following one:

$$\begin{vmatrix} 0 & \dots & 0 & b-\mathbb{A} & \dots & b-\mathbb{A} \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \end{vmatrix}$$

(where the zeros appear in the first d columns). Now, using the transformation Lemma 1.7(ii), we subtract b in the last $n-d-1$ columns; we see that the previous determinant becomes

$$\begin{vmatrix} 0 & \dots & 0 & b-\mathbb{A} & -\mathbb{A} & \dots & -\mathbb{A} \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \mathbb{B}-b-\mathbb{A} & \dots & \mathbb{B}-b-\mathbb{A} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \mathbb{B}-b-\mathbb{A} & \dots & \mathbb{B}-b-\mathbb{A} \end{vmatrix}.$$

This last determinant clearly factorizes, because in the first row, the elements

$$S_{m+1}(-\mathbb{A}), \dots, S_{m+n-d-1}(-\mathbb{A})$$

are null. It is equal to

$$(-1)^d R(b, \mathbb{A}) S_{(m-d)^{n-d-1}}(\mathbb{B}-b-\mathbb{A}).$$

Similarly, we specialize the determinant (2.1)

$$\mathcal{R}^{(d)} = \begin{vmatrix} -\mathbb{A} & \dots & -\mathbb{A} & x-\mathbb{A} \\ \vdots & & \vdots & \vdots \\ -\mathbb{A} & \dots & -\mathbb{A} & x-\mathbb{A} \\ -\mathbb{B} & \dots & -\mathbb{B} & x-\mathbb{B} \\ \vdots & & \vdots & \vdots \\ -\mathbb{B} & \dots & -\mathbb{B} & x-\mathbb{B} \end{vmatrix},$$

in $x = b \in \mathbb{B}$. We obtain

$$\begin{vmatrix} -\mathbb{A} & \dots & -\mathbb{A} & b-\mathbb{A} \\ \vdots & & \vdots & \vdots \\ -\mathbb{A} & \dots & -\mathbb{A} & b-\mathbb{A} \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \\ \vdots & & \vdots & \vdots \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \end{vmatrix}.$$

The bottom part of the last column:

$$\begin{matrix} S_{m+n-d-1}(x-\mathbb{B}) \\ \vdots \\ S_n(x-\mathbb{B}) \end{matrix}$$

is null, and one can write 0 instead of $b-\mathbb{B}$.

Now, use the transformation Lemma 1.7(i) and subtract b in the first $n-d-2$ rows; we see that the previous determinant is equal to the following one:

$$\begin{vmatrix} -\mathbb{A}-b & \ldots & -\mathbb{A}-b & -\mathbb{A} \\ \vdots & & \vdots & \vdots \\ -\mathbb{A}-b & \ldots & -\mathbb{A}-b & -\mathbb{A} \\ -\mathbb{A} & \ldots & -\mathbb{A} & b-\mathbb{A} \\ -\mathbb{B} & \ldots & -\mathbb{B} & 0 \\ \vdots & & \vdots & \vdots \\ -\mathbb{B} & \ldots & -\mathbb{B} & 0 \end{vmatrix}.$$

Taking finally the Laplace expansion of the last determinant along its last column, we see that this determinant is equal to

$$(-1)^{(m-d)(n-d)} R(b,\mathbb{A}) S_{(m-d)^{n-d-1}}(\mathbb{B}-b-\mathbb{A}).$$

(Here we also use Lemma 1.10 giving the expansion of $S_{(m-d)^{n-d-1}}(\mathbb{B}-b-\mathbb{A})$ as a combination of the products of Schur functions in $-\mathbb{B}$ and $-\mathbb{A}-b$.) □

In particular, $\mathcal{R}_1 = \mathcal{R}^{(n-1)}$ equals $(-1)^m S_{1^{n-1};m-n+1}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$.

Let $\mathcal{R}_1^E$ denote the first remainder in the Euclidean division of $S_m(x-\mathbb{A})$ by $S_n(x-\mathbb{B})$ (i.e. we put $c_1=1$ in the notation of the Introduction).

Lemma 2.4: *One has* $\mathcal{R}_1^E = (-1)^{n-1} S_{1^{n-1};m-n+1}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$.

Proof: Write $x-\mathbb{A} = (x-\mathbb{B}) + (\mathbb{B}-\mathbb{A})$. Then, using (1.5), we have

$$S_m(x-\mathbb{A}) = \sum_{i=0}^{n-1} S_i(x-\mathbb{B}) S_{m-i}(\mathbb{B}-\mathbb{A}) + S_n(x-\mathbb{B})\Big(\sum_{i=n}^{m} S_{m-i}(\mathbb{B}-\mathbb{A}) x^{i-n}\Big)$$

$$= \Big(\sum_{i=n}^{m} S_{m-i}(\mathbb{B}-\mathbb{A}) x^{i-n}\Big) S_n(x-\mathbb{B}) + (-1)^{n-1} S_{1^{n-1};m-n+1}(\mathbb{B}-x;\mathbb{B}-\mathbb{A}),$$

by the Laplace expansion of the determinant forming the last summand, along its last column. □

Let now $\mathbb{B} = \{b_1,\ldots,b_n\}$ be a set of indeterminates. Divided differences, due to (Newton, 1687), are linear operators

$$\partial_i : \mathbb{Z}[\mathbb{B}] \to \mathbb{Z}[\mathbb{B}] \qquad (\text{of degree } -1)$$

defined by

$$\partial_i(f) = (f - \tau_i f)/(b_i - b_{i+1}), \qquad i = 1, \ldots, n-1, \tag{2.5}$$

where τ_i denotes the i-th simple transposition exchanging i with $i+1$. More generally, to every reduced decomposition $\sigma = \tau_{i_1} \cdots \tau_{i_k}$, one associates the product $\partial_{i_1} \cdots \partial_{i_k}$ which is independent of the choice of the reduced decomposition of σ (because the ∂_i's satisfy the braid relations), and thus can be denoted unambiguously ∂_σ.

NOTATION: Given an alphabet $\mathbb{B} = \{b_1, b_2, \ldots\}$, we write $\mathbb{B}_i := \{b_1, \ldots, b_i\}$.

LEMMA 2.6: *For every $k \in \mathbb{Z}$, and a virtual alphabet $\mathbb{A}$ which is independent* of $\mathbb{B}$, one has*

$$\partial_i S_k(\mathbb{B}_i - \mathbb{A}) = S_{k-1}(\mathbb{B}_{i+1} - \mathbb{A}) .$$

Proof: Introducing an extra variable z, and writing $\mathbb{B}_i - \mathbb{A} = -b_{i+1} + (\mathbb{B}_{i+1} - \mathbb{A})$, one factorizes $\sum_{k=0}^{\infty} z^k S_k(\mathbb{B}_i - \mathbb{A})$ into fg, with

$$f = 1 - z b_{i+1} \quad \text{and} \quad g = \sum_{k=0}^{\infty} z^k S_k(\mathbb{B}_{i+1} - \mathbb{A}) \ .$$

The second series is symmetrical in b_i, b_{i+1}, therefore commutes with ∂_i. The image of $1 - z b_{i+1}$ under ∂_i being z, the assertion follows. □

The first remainder $\mathcal{R}_1^E$ can be conveniently presented through divided differences (the same operator expresses the remainder in the Lagrange interpolation of a function in one variable).

LEMMA 2.7: *The first remainder $\mathcal{R}_1^E$ in the Euclidean division of $P(x) = S_m(x - \mathbb{A})$ by $S_n(x - \mathbb{B})$ equals*

$$\mathcal{R}_1^E = \partial_{n-1} \cdots \partial_1 \big(P(b_1) R(x, \mathbb{B} - b_1)\big) .$$

Proof: We have

$$\begin{aligned} R(x, \mathbb{B} - b_1) = (x - b_2) \cdots (x - b_n) &= S_{n-1}(x - (\mathbb{B} - b_1)) \\ &= (-1)^{n-1} S_{1^{n-1}}((\mathbb{B} - x) - b_1) , \end{aligned}$$

and by the transformation Lemma 1.7(i), the product $P(b_1) R(x, \mathbb{B} - b_1)$ equals

$$(-1)^{n-1} S_{1^{n-1};m}(\mathbb{B} - x; b_1 - \mathbb{A}) .$$

Thus, using Lemma 2.6, we see that the image of $P(b_1) R(x, \mathbb{B} - b_1)$ under $\partial_{n-1} \cdots \partial_1$ is

$$(-1)^{n-1} S_{1^{n-1};m-n+1}(\mathbb{B} - x; \mathbb{B} - \mathbb{A}) = \mathcal{R}_1 ,$$

which is $\mathcal{R}_1^E$ by Lemma 2.4. □

*It suffices that τ_i preserves $\mathbb{A}$.

See Remark 2.13 for some comments about this lemma and its generalizations.

Convention: In the following, by the "top coefficient" of a polynomial in x, we shall mean the coefficient of the highest power of x in the polynomial.

The following lemma will be used in the proof several times:

Lemma 2.8:

(i) *The specialization* $\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};a)$, $a\in\mathbb{A}$, *is equal to*

$$(-1)^p R(a,\mathbb{B})\ c\,,$$

where c is the top coefficient of $\mathrm{Sylv}^{p,q}(\mathbb{A}-a,\mathbb{B};x)$.

(ii) *The specialization* $\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};b)$, $b\in\mathbb{B}$, *is equal to*

$$(-1)^{m-p+q} R(b,\mathbb{A})\ c\,,$$

where c is the top coefficient of $\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B}-b;x)$.

Proof: We consider, for example, (ii). We have

$$\begin{aligned}\mathrm{Sylv}^{p,q}(b) &= \sum_{\mathbb{A}'\subset\mathbb{A},\mathbb{B}'\subset\mathbb{B}-b} R(b,\mathbb{A}')R(b,\mathbb{B}')\frac{R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}-\mathbb{A}',\mathbb{B}-\mathbb{B}')}{R(\mathbb{A}',\mathbb{A}-\mathbb{A}')R(\mathbb{B}',\mathbb{B}-\mathbb{B}')}\\ &= (-1)^{m-p+q}R(b,\mathbb{A})\sum_{\mathbb{A}'\subset\mathbb{A},\mathbb{B}'\subset\mathbb{B}-b}\frac{R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}-\mathbb{A}',\mathbb{B}-b-\mathbb{B}')}{R(\mathbb{A}',\mathbb{A}-\mathbb{A}')R(\mathbb{B}',\mathbb{B}-b-\mathbb{B}')}\,,\end{aligned}$$

as claimed. □

We need also the following proposition concerning the limit case when $p+q=n$.†

Proposition 2.9:

(i) *Suppose that $p+q=n<m$. Then we have*

$$\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};x) = (-1)^{(m-n)p}\binom{n}{p}R(x,\mathbb{B})\,.$$

(ii) *Suppose that $p+q=n=m$. Then we have*

$$\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};x) = \binom{n-1}{q}R(x,\mathbb{A}) + \binom{n-1}{p}R(x,\mathbb{B})\,.$$

†In fact, in the proof of Theorem 0.1, we will need only part (i) of the proposition. Part (ii) is added for the need of induction in the proof of the proposition.

Proof: Both assertions depend on the four-tuple $\big(m = \mathrm{card}(\mathbb{A}), n = \mathrm{card}(\mathbb{B}), p, q\big)$. We prove (i) and (ii) simultaneously using induction on $m+n$.

(i) Let $m > n$. By Lemma 2.8, the specialization $\mathrm{Sylv}^{p,q}(\mathbb{A}, \mathbb{B}; a)$ is equal to $(-1)^p R(a, \mathbb{B})$ times the top coefficient of $\mathrm{Sylv}^{p,q}(\mathbb{A} - a, \mathbb{B}; x)$.

If $m - 1 > n$, then by (i) for $(m-1, n, p, q)$, this specialization is equal to

$$(-1)^{(m-1-n)p+p}\binom{n}{p} R(a, \mathbb{B}) = (-1)^{(m-n)p}\binom{n}{p} R(a, \mathbb{B})$$

and these m specializations determine the polynomial.

If $m - 1 = n$, then according to (ii) for (n, n, p, q), the top coefficient of $\mathrm{Sylv}^{p,q}(\mathbb{A} - a, \mathbb{B}; x)$ is equal to $\binom{n-1}{p} + \binom{n-1}{q} = \binom{n}{p}$, and therefore $\mathrm{Sylv}^{p,q}(\mathbb{A}, \mathbb{B}; x)$ coincides with $(-1)^p \binom{n}{p} R(x, \mathbb{B})$.

(ii) Let $m = n = p + q$. By Lemma 2.8, the specialization $\mathrm{Sylv}^{p,q}(\mathbb{A}, \mathbb{B}; a)$ is equal to $(-1)^p R(a, \mathbb{B})$ times the top coefficient of $\mathrm{Sylv}^{p,q}(\mathbb{A} - a, \mathbb{B}; x)$. Observe that this last coefficient equals the top coefficient of $\mathrm{Sylv}^{p,q-1}(\mathbb{B}, \mathbb{A} - a; x)$. Then by (i) for $(n, n-1, p, q-1)$, the considered specialization is $\binom{n-1}{p} R(a, \mathbb{B})$.

Similarly, the specialization $\mathrm{Sylv}^{p,q}(\mathbb{A}, \mathbb{B}; b)$ is equal to $R(b, \mathbb{A})$ times the top coefficient of $\mathrm{Sylv}^{p,q}(\mathbb{A}, \mathbb{B} - b; x)$. Observe that this last coefficient is equal to $(-1)^q$ times the top coefficient of $\mathrm{Sylv}^{q,p-1}(\mathbb{A}, \mathbb{B} - b; x)$. Then by (i) for $(n, n-1, q, p-1)$, the considered specialization is $\binom{n-1}{q} R(b, \mathbb{A})$.

The polynomial $\mathrm{Sylv}^{p,q}(\mathbb{A}, \mathbb{B}; x)$ is determined by these specializations. □

Example 2.10:

(i) For $m = 3$, $n = 2$, and $p = q = 1$,

$$\begin{aligned}
&\frac{(a_1-b_1)(a_2-b_2)(a_3-b_2)}{(a_1-a_2)(a_1-a_3)(b_1-b_2)}(x-a_1)(x-b_1)\\
+&\frac{(a_1-b_2)(a_2-b_1)(a_3-b_1)}{(a_1-a_2)(a_1-a_3)(b_2-b_1)}(x-a_1)(x-b_2)\\
+&\frac{(a_2-b_1)(a_1-b_2)(a_3-b_2)}{(a_2-a_1)(a_2-a_3)(b_1-b_2)}(x-a_2)(x-b_1)\\
+&\frac{(a_2-b_2)(a_1-b_1)(a_3-b_1)}{(a_2-a_1)(a_2-a_3)(b_2-b_1)}(x-a_2)(x-b_2)\\
+&\frac{(a_3-b_1)(a_1-b_2)(a_2-b_2)}{(a_3-a_1)(a_3-a_2)(b_1-b_2)}(x-a_3)(x-b_1)\\
+&\frac{(a_3-b_2)(a_1-b_1)(a_2-b_1)}{(a_3-a_1)(a_3-a_2)(b_2-b_1)}(x-a_3)(x-b_2)
\end{aligned}$$

is equal to $-2(x-b_1)(x-b_2)$.

(ii) For $m = n = 3$, $p = 2$, and $q = 1$,

$$\begin{aligned}
&\frac{(a_1-b_1)(a_2-b_1)(a_3-b_2)(a_3-b_3)}{(a_1-a_3)(a_2-a_3)(b_1-b_2)(b_1-b_3)}(x-a_1)(x-a_2)(x-b_1)\\
+&\frac{(a_1-b_1)(a_3-b_1)(a_2-b_2)(a_2-b_3)}{(a_1-a_2)(a_3-a_2)(b_1-b_2)(b_1-b_3)}(x-a_1)(x-a_3)(x-b_1)\\
+&\frac{(a_2-b_1)(a_3-b_1)(a_1-b_2)(a_1-b_3)}{(a_2-a_1)(a_3-a_1)(b_1-b_2)(b_1-b_3)}(x-a_2)(x-a_3)(x-b_1)\\
+&\frac{(a_1-b_2)(a_2-b_2)(a_3-b_1)(a_3-b_3)}{(a_1-a_3)(a_2-a_3)(b_2-b_1)(b_2-b_3)}(x-a_1)(x-a_2)(x-b_2)\\
+&\frac{(a_1-b_2)(a_3-b_2)(a_2-b_1)(a_2-b_3)}{(a_1-a_2)(a_3-a_2)(b_2-b_1)(b_2-b_3)}(x-a_1)(x-a_3)(x-b_2)\\
+&\frac{(a_2-b_2)(a_3-b_2)(a_1-b_1)(a_1-b_3)}{(a_2-a_1)(a_3-a_1)(b_2-b_1)(b_2-b_3)}(x-a_2)(x-a_3)(x-b_2)\\
+&\frac{(a_1-b_3)(a_2-b_3)(a_3-b_1)(a_3-b_2)}{(a_1-a_3)(a_2-a_3)(b_3-b_1)(b_3-b_2)}(x-a_1)(x-a_2)(x-b_3)\\
+&\frac{(a_1-b_3)(a_3-b_3)(a_2-b_1)(a_2-b_2)}{(a_1-a_2)(a_3-a_2)(b_3-b_1)(b_3-b_2)}(x-a_1)(x-a_3)(x-b_3)\\
+&\frac{(a_2-b_3)(a_3-b_3)(a_1-b_1)(a_1-b_2)}{(a_2-a_1)(a_3-a_1)(b_3-b_1)(b_3-b_2)}(x-a_2)(x-a_3)(x-b_3)
\end{aligned}$$

is equal to $2(x-a_1)(x-a_2)(x-a_3)+(x-b_1)(x-b_2)(x-b_3)$.

We now can pass to the

Proof of Theorem 0.1: Write $d := p+q$. The assertion of the theorem is equivalent to the identity:

$$\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};x) = (-1)^{(m-d)(n-q)+d}\binom{d}{p} S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A}) \tag{2.11}$$

if $d < n$.

The proof goes by induction on n. For $n = 1$, $p = q = 0$, and any $m \geqslant 1$, both sides of (2.11) are equal to $R(\mathbb{A}, b_1)$.

To perform the induction step, observe that $\mathrm{Sylv}^{p,q}$, as a polynomial in x, is of degree $d < n$. Hence it coincides with its first remainder $\mathcal{R}_1^E$ modulo $R(x,\mathbb{B})$, and one can use Lemma 2.7 to obtain it starting from $\mathrm{Sylv}^{p,q}(b_1)$:

$$\mathrm{Sylv}^{p,q}(x) = \partial_{n-1}\cdots\partial_1\big(\mathrm{Sylv}^{p,q}(b_1)R(x,\mathbb{B}-b_1)\big)\,.$$

By Lemma 2.8, the specialization of $\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};x)$ in $x = b_1$ is equal to $(-1)^{m-d}R(b_1,\mathbb{A})$ times the top coefficient in $\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B}-b_1;x)$. If $d < n-1$ then we can apply the induction assumption, and we see that this coefficient is equal to

$$(-1)^{(m-d)(n-1-q)}\binom{d}{p} S_{\square}(\mathbb{B}-b_1-\mathbb{A})\,,$$

with $\square = (m-d)^{n-d-1}$. If $d = n-1$ then the coefficient is given by the same formula by Proposition 2.9.

We are left with the computation of the following expression:

$$\partial_{n-1} \cdots \partial_1 \big(R(x, \mathbb{B} - b_1) R(b_1, \mathbb{A}) S_\square(\mathbb{B} - b_1 - \mathbb{A}) \big) .$$

We rewrite

$$\begin{aligned} R(x, \mathbb{B} - b_1) &= S_{n-1}(x - (\mathbb{B} - b_1)) \\ &= (-1)^{n-1} S_{1^{n-1}}((\mathbb{B} - x) - b_1) = (-1)^{n-1} \Big(\sum_{i=0}^{n-1} (-1)^i S_{1^{n-i-1}}(\mathbb{B} - x) b_1^i \Big) . \end{aligned}$$

We have by (1.5)

$$b_1^i R(b_1, \mathbb{A}) = S_{m+i}(b_1 - \mathbb{A}) ,$$

and by the transformation Lemma 1.7(i),

$$S_{m+i}(b_1 - \mathbb{A}) S_\square(\mathbb{B} - b_1 - \mathbb{A}) = S_{\square; m+i}(\mathbb{B} - \mathbb{A}; b_1 - \mathbb{A}) .$$

Finally, we get with $\epsilon = (-1)^{(m-d)(n-q)+n-1}$,

$$\mathrm{Sylv}^{p,q}(b_1) R(x, \mathbb{B} - b_1) = \epsilon \binom{d}{p} \sum_{i=0}^{n-1} (-1)^i S_{1^{n-1-i}}(\mathbb{B} - x) S_{\square; m+i}(\mathbb{B} - \mathbb{A}; b_1 - \mathbb{A}) .$$

By Lemma 2.6, its image under $\partial_{n-1} \cdots \partial_1$ is equal to

$$\epsilon \binom{d}{p} \sum_{i=0}^{n-1} (-1)^i S_{1^{n-1-i}}(\mathbb{B} - x) S_{\square, m+i-(n-1)}(\mathbb{B} - \mathbb{A}) .$$

In this last sum, the terms for $i = 0, \ldots, n-d-2$ disappear because they correspond to determinants having two identical columns. In the remaining sum for $i = n-d-1, \ldots, n-1$:

$$\sum_{j=0}^{d} (-1)^j S_{1^{d-j}}(\mathbb{B} - x) S_{\square, m-d+j}(\mathbb{B} - \mathbb{A}) ,$$

one recognizes the Laplace expansion of the determinant $S_{1^d; (m-d)^{n-d}}(\mathbb{B} - x; \mathbb{B} - \mathbb{A})$ along the first d columns (cf. Lemma 1.11). One also checks that all the signs fit to give the assertion.

This ends the proof of the theorem. $\square$

We illustrate the theorem by the following example.

EXAMPLE 2.12: Let $m = 5, n = 4$. Then the Sylvester sum $\mathrm{Sylv}^{2,1}(\mathbb{A}, \mathbb{B}; x)$:

$$
\begin{aligned}
&\frac{(a_3-b_2)(a_4-b_2)(a_5-b_2)(a_3-b_3)(a_4-b_3)(a_5-b_3)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_1-b_2)(b_1-b_3)(b_1-b_4)} \\
&\quad \times (a_1-b_1)(a_2-b_1)(x-a_1)(x-a_2)(x-b_1) \\
&+ \frac{(a_3-b_1)(a_4-b_1)(a_5-b_1)(a_3-b_3)(a_4-b_3)(a_5-b_3)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_2-b_1)(b_2-b_3)(b_2-b_4)} \\
&\quad \times (a_1-b_2)(a_2-b_2)(x-a_1)(x-a_2)(x-b_2) \\
&+ \frac{(a_3-b_1)(a_4-b_1)(a_5-b_1)(a_3-b_2)(a_4-b_2)(a_5-b_2)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_3-b_1)(b_3-b_2)(b_3-b_4)} \\
&\quad \times (a_1-b_3)(a_2-b_3)(x-a_1)(x-a_2)(x-b_3) \\
&+ \frac{(a_3-b_1)(a_4-b_1)(a_5-b_1)(a_3-b_2)(a_4-b_2)(a_5-b_2)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_4-b_1)(b_4-b_2)(b_4-b_3)} \\
&\quad \times (a_1-b_4)(a_2-b_4)(x-a_1)(x-a_2)(x-b_4) \\
&+ 9 \times 4 \text{ similar terms obtained by replacing } (a_1, a_2) \text{ by } (a_i, a_j) \text{ for } i < j
\end{aligned}
$$

is equal to $3\mathcal{R}^{(3)}$.

REMARK 2.13: If $p = 0$ or $q = 0$, then double Sylvester sums reduce to single ones; these single sums appeared already in (Sylvester, 1839) together with an announcement of the assertion of Theorem 0.1 in that case. For example,

$$
\mathrm{Sylv}^{0,q}(\mathbb{A}, \mathbb{B}; x) = \sum_{\mathbb{B}' \subset \mathbb{B}} R(x, \mathbb{B}') \frac{R(\mathbb{A}, \mathbb{B} - \mathbb{B}')}{R(\mathbb{B}', \mathbb{B} - \mathbb{B}')}, \tag{2.14}
$$

the sum being over all subsets $\mathbb{B}'$ of cardinality q. In this case, the assertion of Theorem 0.1 was proved in (Borchardt, 1860). Here is a short proof of it, or equivalently of the identity (2.11), using a result of (Lascoux, 1975). We have, with $\mathbb{B}'' := \mathbb{B} - \mathbb{B}'$,

$$
\begin{aligned}
\mathrm{Sylv}^{0,q}(\mathbb{A}, \mathbb{B}; x) &= \sum_{\mathbb{B}' \subset \mathbb{B}} R(x, \mathbb{B}') \frac{R(\mathbb{A}, \mathbb{B}'')}{R(\mathbb{B}', \mathbb{B}'')} = \partial_{\mathbb{B}''\mathbb{B}'} \big(S_q(x - \mathbb{B}') S_{m^{n-q}}(\mathbb{B}'' - \mathbb{A}) \big) \\
= \pm \partial_{\mathbb{B}''\mathbb{B}'} \big(S_{1^q}(\mathbb{B}' - x) S_{m^{n-q}}(\mathbb{B}'' - \mathbb{A}) \big) &= (-1)^{(m-q)(n-q)+q} S_{1^q;(m-q)^{n-q}}(\mathbb{B} - x; \mathbb{B} - \mathbb{A}).
\end{aligned}
$$

Note that one can rewrite (2.14) as:

$$
(-1)^{m(n-q)} \sum_{1 \leqslant j_1 < \cdots < j_q \leqslant n} \; \prod_{i \neq j_1, \ldots, j_q} \; \prod_{k=1}^{q} \frac{P(b_i)(x - b_{j_k})}{b_{j_k} - b_i}, \tag{2.15}
$$

where $P(x) = S_m(x - \mathbb{A})$. So the expression (2.15) equals the subresultant $\mathcal{R}^{(q)}$ associated with $P(x)$ and $S_n(x - \mathbb{B})$. In particular, for $q = n - 1$, one recovers Lagrange interpolation:

$$
\mathcal{R}_1^E = (-1)^{m-n+1} \mathcal{R}_1 = \sum_{i=1}^{n} P(b_i) \prod_{j \neq i} \frac{x - b_j}{b_i - b_j}. \tag{2.16}
$$

For example, for $n = 3$,

$$\mathcal{R}_1^E = P(b_1)\frac{(x-b_2)(x-b_3)}{(b_1-b_2)(b_1-b_3)}+P(b_2)\frac{(x-b_1)(x-b_3)}{(b_2-b_1)(b_2-b_3)}+P(b_3)\frac{(x-b_1)(x-b_2)}{(b_3-b_1)(b_3-b_2)}\,.$$

In (Jouanolou, 1996), one can find another treatment of such interpolation formulas of Lagrange, Sylvester, and Borchardt. The expressions of interpolation for the subresultants given there are different from the double Sylvester sums, and they are based on the theory of the so-called "Borchardtiens" developed in loc.cit.

Jacobi, in his thesis in 1825, already considered the problem of generalizing the Lagrange interpolation. In particular, in the appendix to his thesis (Jacobi, 1825) (Propositio generalis III, p.562), Jacobi expresses, for any symmetric polynomial P in p variables, the value of

$$\sum_{\mathbb{A}'\subset\mathbb{A}} \frac{P(\mathbb{A}')}{R(\mathbb{A}',\mathbb{A}-\mathbb{A}')}\,,$$

where $\mathrm{card}(\mathbb{A}') = p$, as a multi-residue.

3. Another approach via symmetrizing operators

We will now sketch another possible method for getting the identity (2.11), using only divided differences, and not specializations. It is based upon the following two lemmas*:

LEMMA 3.1: *Let $i \leqslant n-1$ be a nonnegative integer and let $\mathbb{C}$ be an alphabet such that* $\mathrm{card}(\mathbb{C}) \leqslant i$. *Then for a nonnegative integer j and a virtual alphabet $\mathbb{D}$,*

$$\partial_{n-1}\cdots\partial_1\big(S_i(b_1-\mathbb{C})S_j(b_1+\mathbb{D})\big) = S_{i+j-n+1}(\mathbb{B}-\mathbb{C}+\mathbb{D})\,,$$

provided $\mathbb{C}$ and $\mathbb{D}$ are independent of $\mathbb{B}$.

Proof: We have

$$S_i(b_1-\mathbb{C})S_j(b_1+\mathbb{D}) = \sum_{k,l} b_1^{k+l} S_{i-k}(-\mathbb{C})S_{j-l}(\mathbb{D})\,.$$

This element is sent via $\partial_{n-1}\cdots\partial_1$ to

$$\sum_k S_{i-k}(-\mathbb{C})S_{j+k-n+1}(\mathbb{B}+\mathbb{D})$$

because $k \leqslant n-1$. Since $S_k(-\mathbb{C}) = 0$ for $k \geqslant i$, this last sum is the expression of $S_{i+j-n+1}(\mathbb{B}-\mathbb{C}+\mathbb{D})$ and the assertion of the lemma follows. □

*We do not state these results in the greatest possible generality, but in the form needed here.

NOTATION: Given an alphabet $\mathbb{B} = \{b_1, b_2, \ldots\}$, we write $\mathbb{B}^i := \{b_i, b_{i+1}, \ldots\}$.

LEMMA 3.2: *Let i_0 be a nonnegative integer and let $\mathbb{C}^{(0)}$ be an alphabet such that* $\mathrm{card}(\mathbb{C}^{(0)}) \leqslant i_0$. *Then for integers $i_1, \ldots, i_{n-1}$, and virtual alphabets $\mathbb{C}^{(1)}, \ldots, \mathbb{C}^{(n-1)}$,*

$$\begin{aligned}\partial_{n-1} \cdots \partial_1 \big(S_{i_0}(b_1 - \mathbb{C}^{(0)}) S_{i_1,\ldots,i_{n-1}}(\mathbb{B}^2 - \mathbb{C}^{(1)}, \ldots, \mathbb{B}^2 - \mathbb{C}^{(n-1)})\big) \\ = (-1)^{n-1} S_{i_0, i_1-1, \ldots, i_{n-1}-1}(\mathbb{B} - \mathbb{C}^{(0)}, \mathbb{B} - \mathbb{C}^{(1)}, \ldots, \mathbb{B} - \mathbb{C}^{(n-1)}),\end{aligned}$$

provided $\mathbb{C}^{(0)}, \ldots, \mathbb{C}^{(n-1)}$ are independent of $\mathbb{B}$.

Proof: Using the transformation Lemma 1.7(i), we have the factorization

$$\begin{aligned}S_{i_0}(b_1 - \mathbb{C}^{(0)}) S_{i_1,\ldots,i_{n-1}}(\mathbb{B}^2 - \mathbb{C}^{(1)}, \ldots, \mathbb{B}^2 - \mathbb{C}^{(n-1)}) \\ = S_{i_1,\ldots,i_{n-1},i_0}(\mathbb{B} - \mathbb{C}^{(1)}, \ldots, \mathbb{B} - \mathbb{C}^{(n-1)}, b_1 - \mathbb{C}^{(0)}).\end{aligned}$$

The image of this element via $\partial_{n-1} \cdots \partial_1$ is

$$\begin{aligned}S_{i_1,\ldots,i_{n-1},i_0-n+1}(\mathbb{B} - \mathbb{C}^{(1)}, \ldots, \mathbb{B} - \mathbb{C}^{(n-1)}, \mathbb{B} - \mathbb{C}^{(0)}) \\ = (-1)^{n-1} S_{i_0, i_1-1, \ldots, i_{n-1}-1}(\mathbb{B} - \mathbb{C}^{(0)}, \mathbb{B} - \mathbb{C}^{(1)}, \ldots, \mathbb{B} - \mathbb{C}^{(n-1)}).\end{aligned}$$

The lemma is proved. □

NOTATION: We denote $\partial_{\mathbb{B}} := \partial_\omega$ where ω is the longest permutation reordering $(b_1, b_2, \ldots, b_n)$ to $(b_n, b_{n-1}, \ldots, b_1)$. For two complementary subsequences $\mathbb{B}'$ and $\mathbb{B}$" of $\mathbb{B}$, we write $\partial_{\mathbb{B}"\mathbb{B}'} := \partial_\sigma$ where the permutation σ reorders the sequence $\mathbb{B}"\mathbb{B}'$ to $\mathbb{B}$. For example, $\partial_{n-1} \cdots \partial_1$ is denoted by $\partial_{\mathbb{B}^2 b_1}$.

In this computation, we fix $\mathbb{A}' := \mathbb{A}_p$, $\mathbb{A}'' := \mathbb{A}^{p+1}$, $\mathbb{B}' := \mathbb{B}_q$, and $\mathbb{B}'' := \mathbb{B}^{q+1}$. (Here, $a_1, \ldots, a_m$ are variables.)

Recall (cf. e.g. (Pragacz, 1996, §3) and references there) that the divided difference operator $\partial_{\mathbb{A}''\mathbb{A}'}$ has the following interpretation as a symmetrizing operator occurring in Sylvester sums. For a polynomial $P(a_1, \ldots, a_m)$,

$$\partial_{\mathbb{A}''\mathbb{A}'}(P) = \frac{1}{p!(m-p)!} \sum_\sigma \Big(\frac{P}{R(\mathbb{A}_p, \mathbb{A}^{p+1})}\Big)^\sigma,$$

the sum being over permutations σ of $\{1, \ldots, m\}$. Hence equation (2.11) is equivalent to

$$\begin{aligned}\partial_{\mathbb{A}''\mathbb{A}'} \partial_{\mathbb{B}''\mathbb{B}'} [R(x, \mathbb{A}' + \mathbb{B}') R(\mathbb{A}', \mathbb{B}') R(\mathbb{A}'', \mathbb{B}'')] \\ = (-1)^{(m-p-q)(n-q)+p+q} \binom{p+q}{p} S_{1^{p+q};(m-p-q)^{n-p-q}}(\mathbb{B} - x; \mathbb{B} - \mathbb{A}). \quad (3.3)\end{aligned}$$

NOTATION: For an alphabet $\mathbb{A} = \{a_1, \ldots, a_m\}$ we write

$$\Delta(\mathbb{A}) := \prod_{i<j\leqslant m} (a_i - a_j)\,,$$

and for an auxiliary alphabet $\mathbb{B}$, we set $\Delta(\mathbb{A}+\mathbb{B}) := \Delta(\mathbb{A})\Delta(\mathbb{B})R(\mathbb{A},\mathbb{B})$.[†]

Since

$$\partial_{\mathbb{A}'}\partial_{\mathbb{B}'}\big((1/p!)(1/q!)\Delta(x+\mathbb{A}')\Delta(x+\mathbb{B}')R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}'',\mathbb{B}'')\big) \\ = R(x,\mathbb{A}'+\mathbb{B}')R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}'',\mathbb{B}'')\,,$$

equation (3.3) is equivalent to

$$\nabla\big(\Delta(x+\mathbb{A}'+\mathbb{B}')R(\mathbb{A}'',\mathbb{B}'')\big) \\ = \epsilon(p+q)!S_{1^{p+q};(m-p-q)^{n-p-q}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})\,, \quad (3.4)$$

where

$$\nabla := \partial_{\mathbb{B}^2 b_1}\cdots\partial_{\mathbb{B}^{q-p+1} b_{q-p}}\partial_{\mathbb{A}^2 a_1}\partial_{\mathbb{B}^{q-p+2} b_{q-p+1}}\cdots\partial_{\mathbb{A}^p a_{p-1}}\partial_{\mathbb{B}^q b_{q-1}}\partial_{A'' a_p}\partial_{B'' b_q}$$

(without loss of generality we assume that $p \leqslant q$), and $\epsilon = (-1)^{(m-p-q)(n-q)+p+q}$.

We will now show (3.4) for $m = \mathrm{card}(\mathbb{A}) = 11$, $n = \mathrm{card}(\mathbb{B}) = 9$, $p = 2$, and $q = 3$. We want to show

$$\partial_{\mathbb{B}^2 b_1}\partial_{\mathbb{A}^2 a_1}\partial_{\mathbb{B}^3 b_2}\partial_{\mathbb{A}^3 a_2}\partial_{\mathbb{B}^4 b_3}[\Delta(x+\mathbb{A}_2+\mathbb{B}_3)R(\mathbb{A}^3,\mathbb{B}^4)] = -5!S_{1^5;6^4}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})\,.$$

We shall disregard the signs, for simplicity. We have

$$\Delta(\mathbb{B}_3+\mathbb{A}_2+x)R(\mathbb{B}^4,\mathbb{A}^3) = \Delta(\mathbb{B}_2+\mathbb{A}_2+x)S_5(b_3-(\mathbb{B}_2+\mathbb{A}_2+x))S_{9^6}(\mathbb{B}^4-\mathbb{A}^3) \\ = \Delta(\mathbb{B}_2+\mathbb{A}_2+x)S_{9^6;5}(\mathbb{B}^3-\mathbb{A}^3;b_3-(\mathbb{B}_2+\mathbb{A}_2+x))$$

(the last equality follows from the transformation Lemma 1.7(i)). By Lemma 3.2, this element goes via $\partial_{\mathbb{B}^4 b_3}$ to

$$\pm\Delta(\mathbb{B}_2+\mathbb{A}_2+x)S_{5;8^6}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_2+x);\mathbb{B}^3-\mathbb{A}^3)\,.$$

We explain the effect of applying $\partial_{\mathbb{A}^3 a_2}$ to this last element. By the transformation Lemma 1.7(ii), we have

$$S_{5;8^6}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_2+x);\mathbb{B}^3-\mathbb{A}^3) = S_{5;8;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_2+x);\mathbb{B}^3-\mathbb{A}^3;\mathbb{B}^3-\mathbb{A}^2)\,,$$

and by a well-known linearity formula this last element is

$$S_{5;8;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x)-a_2;\mathbb{B}^3-\mathbb{A}^3;\mathbb{B}^3-\mathbb{A}^2) = S_{5;8;8^5} - a_2S_{4;8;8^5}\,,$$

[†]Note that $\Delta(\mathbb{A}+\mathbb{B}) = (-1)^{mn}\Delta(\mathbb{B}+\mathbb{A})$, where $n = \mathrm{card}(\mathbb{B})$.

the Schur functions in $(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^3;\mathbb{B}^3-\mathbb{A}^2)$. Factorizing the last Δ and using (1.5), to compute is the image under $\partial_{\mathbb{A}^3 a_2}$ of

$$\Delta(\mathbb{B}_2+\mathbb{A}_1+x)[S_4(a_2-(\mathbb{B}_2+\mathbb{A}_1+x))S_{5;8;8^5}-S_5(a_2-(\mathbb{B}_2+\mathbb{A}_1+x))S_{4;8;8^5}]\,,$$

the Schur functions in $(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^2+a_2;\mathbb{B}^3-\mathbb{A}^2)$. By using Lemma 3.1, the first summand in the bracket [ ] is sent to

$$S_{5,3;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^2)=-S_{4^2;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^2)\,.$$

The second summand in the bracket [ ] is sent to $-S_{4^2;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^2)$. In sum, the initial element goes via $\partial_{\mathbb{A}^3 a_2}\partial_{\mathbb{B}^4 b_3}$ to

$$\pm 1\cdot 2\cdot\Delta(\mathbb{B}_2+\mathbb{A}_1+x)S_{4^2;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^2)\,.$$

Similarly, we have

$$\begin{aligned}
\partial_{\mathbb{B}^3 b_2}&[\Delta(\mathbb{B}_2+\mathbb{A}_1+x)S_{4^2;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^2)]\\
&\qquad=\pm 3\Delta(\mathbb{B}_1+\mathbb{A}_1+x)S_{3^3;7^5}(\mathbb{B}^2-(\mathbb{B}^1+\mathbb{A}^1+x);\mathbb{B}^2-\mathbb{A}^2)\,,\\
\partial_{\mathbb{A}^2 a_1}&[\Delta(\mathbb{B}_1+\mathbb{A}_1+x)S_{3^3;7^5}(\mathbb{B}^2-(\mathbb{B}^1+\mathbb{A}^1+x);\mathbb{B}^2-\mathbb{A}^2)]\\
&\qquad=\pm 4\Delta(\mathbb{B}_1+x)S_{2^4;7^5}(\mathbb{B}^2-(\mathbb{B}^1+x);\mathbb{B}^2-\mathbb{A}^1)\,,
\end{aligned}$$

and

$$\partial_{B^2 b_1}[\Delta(\mathbb{B}_1+x)S_{2^4;7^5}(\mathbb{B}^2-(\mathbb{B}^1+x);\mathbb{B}^2-\mathbb{A}^1)]=\pm 5S_{1^4;5^4}(\mathbb{B}^1-x;\mathbb{B}^1-\mathbb{A}^1)\,.$$

In sum, we get

$$\partial_{\mathbb{B}_2 b_1}\partial_{\mathbb{A}_2 a_1}\partial_{\mathbb{B}_3 b_2}\partial_{\mathbb{A}_3 a_2}\partial_{\mathbb{B}_4 b_3}[\Delta(x+\mathbb{A}_2+\mathbb{B}_3)R(\mathbb{A}^3,\mathbb{B}^4)]=-5!S_{1^5;6^4}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})\,,$$

as wanted in (3.4).

4. A glimpse of Gysin maps

This section is just a by-product of the previous ones. We translate here the above algebraic computations to a formula for Gysin maps.

Let X be a smooth variety, and suppose that $E^{(1)},\dots,E^{(l)}$ are l virtual bundles on X. Given a sequence $I=(i_1,\dots,i_l)\in\mathbb{Z}^l$, we set

$$S_I\big(E^{(1)},\dots,E^{(l)}\big):=S_I\big(\mathbb{A}^{(1)},\dots,\mathbb{A}^{(l)}\big)\,,$$

where $\mathbb{A}^{(1)},\dots,\mathbb{A}^{(l)}$ are the virtual alphabets of Chern roots associated with $E^{(1)},\dots,E^{(l)}$.

Let now E and F be vector bundles on X of respective ranks $m\geqslant n$. For integers $0\leqslant p\leqslant m$ and $0\leqslant q\leqslant n$, let

$$\pi_E:\mathrm{Gr}^p(E)\to X\quad\text{ and }\quad\pi_F:\mathrm{Gr}^q(F)\to X$$

be two Grassmann bundles on X parametrizing rank p quotients of E and rank q quotients of F. These Grassmannians are equipped with the two tautological exact sequences:

$$0 \to R_E \to E \to Q_E \to 0 \quad \text{and} \quad 0 \to R_F \to F \to Q_F \to 0\,,$$

where $\mathrm{rank}(Q_E) = p$ and $\mathrm{rank}(Q_F) = q$. (We use the same notation for vector bundles and their pullbacks.) We form a product over X:

$$\pi := \pi_E \times_X \pi_F : \mathrm{Gr}^p(E) \times_X \mathrm{Gr}^q(F) \to X\,.$$

Recall (cf. (Brion, 1996), (Pragacz, 1996, §3)) that the operator $\partial_{\mathbb{A}''\mathbb{A}'}$ appearing in (3.3) induces the Gysin map $(\pi_E)_*$. More precisely, for a polynomial $P(a_1, \ldots, a_m)$ symmetric in the first p variables and in the last $m-p$ variables separately, one has

$$(\pi_E)_*(P(x_1, \ldots, x_m) = \partial_{\mathbb{A}''\mathbb{A}'}\big(P(a_1, \ldots, a_m)\big)|_{a_1=x_1,\ldots,a_m=x_m}\,, \tag{4.1}$$

where $x_1, \ldots, x_p$ are the Chern roots of Q_E, and $x_{p+1}, \ldots, x_m$ are the ones of R_E. By using (4.1), equation (3.3) is translated into the following formula about the top Chern classes and the Gysin map

$$\pi_* : H(\mathrm{Gr}^p(E) \times_X \mathrm{Gr}^q(F)) \to H(X)\,.$$

Proposition 4.2: *Let L be a line bundle on X. Then for $d = p + q < n$,*

$$\pi_*\Big(c_{\mathrm{top}}(Q_E^\vee \otimes L)\; c_{\mathrm{top}}(Q_F^\vee \otimes L)\; c_{\mathrm{top}}(Q_F^\vee \otimes Q_E)\; c_{\mathrm{top}}(R_F^\vee \otimes R_E)\Big) = \epsilon \binom{d}{p} S_{1^d;(m-d)^{n-d}}\,,$$

where $\epsilon = (-1)^{(m-d)(n-q)+d}$ and the Schur function is in $(F - L; F - E)$.

Note that by comparing the coefficients of the maximal power $c_1(L)^d$, we get

$$\pi_*\big(c_{\mathrm{top}}(Q_F^\vee \otimes Q_E)\; c_{\mathrm{top}}(R_F^\vee \otimes R_E)\big) = \pm \binom{d}{p} S_{(m-d)^{n-d}}(F - E)\,. \tag{4.3}$$

We point out that neither the proposition nor equation (4.3) follow *instantly* from a standard formula (Lascoux, 1975), (Józefiak et al., 1982) for the Gysin map of a Grassmann bundle applied to $\mathrm{Gr}^p(E)$ and $\mathrm{Gr}^q(F)$.

In a similar way one can translate formulas given in Proposition 2.9.

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