ORTHOGONAL DIVIDED DIFFERENCES AND SCHUBERT POLYNOMIALS, $\widetilde{P}$-FUNCTIONS, AND VERTEX OPERATORS

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Dedicated to Bill Fulton on his 60th birthday

Abstract

Using divided differences associated with the orthogonal groups, we investigate the structure of the polynomial rings over the rings of invariants of the corresponding Weyl groups. We study in more detail the action of orthogonal divided differences on some distinguished symmetric polynomials ($\widetilde{P}$-polynomials) and relate it to vertex operators. Relevant families of orthogonal Schubert polynomials, generalizing $\widetilde{P}$-polynomials, and well-suited to intersection theory computations, are also studied.

Contents:

Introduction. 1.Divided differences. 2.Bases of polynomial rings. 3.Vertex operators. 4.Applications to $\widetilde{P}$-polynomials and orthogonal Schubert polynomials. Appendix: results in type B. References.

Introduction

Divided differences were introduced by Newton in his famous interpolation formula (cf. [N, pp.481–483], and [L] for some historical comments).

Their importance in geometry was shown by Bernstein-Gel'fand-Gel'fand [BGG] and Demazure [D1,D2] in the context of *Schubert calculus* for generalized flag varieties associated with semisimple algebraic groups in the early 1970's. More recently, simple divided differences, interpreted as *correspondences* in flag bundles, were extensively used in the sequence of papers [F1,F2,F3] by Fulton in the context of degeneracy loci associated with classical groups. Still another interpretation of divided differences, as *Gysin morphisms* in the cohomology of flag bundles associated with semisimple algebraic groups, was discussed in [P2, Sect.4] and [PR, Sect.5]. We refer to the lecture notes [FP] for an introduction.

The case of $SL(n)$ has been developed by the first author and Schützenberger (see e.g. [LS1,LS2,LS3] and [M2]).

For other classical groups, parallel studies were done by Billey-Haiman [BH], Fomin-Kirillov [FK], Ratajski and the second author [PR], and by the authors [LP1].

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

The present paper is a continuation of [LP1]. Here we study divided differences associated with the orthogonal groups $SO(2n)$ and $SO(2n+1)$ (i.e. for types D and B). The results for type B are an immediate adaptation of the results for type C given in [LP1]. We summarize them in the appendix. We add, however, a certain new result (Theorem 9) for type B, whose C-analogue was not needed in our former paper [LP1].

Our results for type D require some new computations with vertex operators, that are furnished in Section 3 of the present paper and summarized in our main Theorem 11. In order to simplify the computations with divided differences, we display them as planar arrays, which allows us to perform some kind of " jeu de taquin". This offers a certain technical novelty w.r.t. our former paper [LP1]. In type C_n (or B_n), the key rôle was/is played by the divided differences of the form

$$(\partial_0\partial_1\cdots\partial_{n-1})\cdots(\partial_0\partial_1\cdots\partial_{n-k})\,, \tag{$*$}$$

where $k \le n$. It appears that in type D_n a similar rôle is played by the divided differences of the form

$$(\partial_\heartsuit\partial_2\cdots\partial_{n-1}\partial_1\partial_2\cdots\partial_{n-2})\cdots(\partial_\heartsuit\partial_2\cdots\partial_{n-2k+1}\partial_1\partial_2\cdots\partial_{n-2k})\,, \tag{$**$}$$

where $k \le n/2$. Here ∂_i, for $i>0$, are Newton's (simple) divided differences:

$$f\,\partial_i := \frac{f - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}\,,$$

and moreover we set

$$f\,\partial_0 := \frac{f - f(-x_1, x_2, \ldots)}{-x_1}\,,$$

$$f\,\partial_\heartsuit := \frac{f - f(-x_2, -x_1, x_3, \ldots)}{-x_1 - x_2}\,.$$

We compose the simple orthogonal divided differences in (*) and (**) *from left to right.* As the the Weyl group of type D is naturally embedded in the Weyl group of type B, the divided difference (**) can be expressed in terms of (*). Such basic relations are given in Proposition 6 and Corollary 8.

The symmetric functions which are most adapted to orthogonal divided differences are $\widetilde{P}$-polynomials [PR], which are a variant of Schur P-polynomials.

Our paper is of an algebro-combinatorial nature but its motivation comes from geometry. The algebro-combinatorial properties studied here should be useful in Schubert calculus associated with orthogonal groups and the related degeneracy loci.

The computations of this paper are closely related to the ones in [LLT1]; we plan to develop this link in some future publication.

The algebro-combinatorial techniques, used in the present paper, are chosen to be as elementary as possible. This should help the reader, with more geometric and less algebro-combinatorial background, to read the paper. We mention however, that several results, used in the proof of Theorem 9, in the appendix, are particular instances of more general properties of Hall–Littlewood polynomials (see [LLT2] and [LP2]).

Let us remark that there is also an interesting algebra and combinatorics of "isobaric divided differences", with associated *Grothendieck polynomials* (cf. [FL]).

This work has extensively used ACE ([V]) for explicit computations.

It is our pleasure and honor to dedicate the present article to the mathematician whose recent work has illuminated important connections between geometry and combinatorics.

NOTATION AND CONVENTIONS :

A *vector* (of *length* m) is a sequence $[v_1, \ldots, v_m] \in \mathbb{Z}^m$.
We will compare vectors of the same lengths, writing

$$[v_1, \ldots, v_m] \subseteq [u_1, \ldots, u_m]$$

if $v_i \leq u_i$ for all $i = 1, \ldots, m$.

Given a vector $\alpha = [\alpha_1, \ldots, \alpha_m]$, we will write $|\alpha|$ for the sum of its components.

A *partition* is an equivalence class of sequences $[i_1 \geq \cdots \geq i_m] \in \mathbb{N}^m$, where we identify the sequences $[i_1, \ldots, i_m]$ with $[i_1, \ldots, i_m, 0]$. We denote the corresponding partition by $I = (i_1, \ldots, i_m)$, by taking any representative sequence.

A *part* of a partition I, is a nonzero component of any sequence that represents I.

The *length* of a partition I is the number of its nonzero parts, denoted $\ell(I)$.

We call a partition *strict* if all its parts are different.

We write $I \subseteq J$ for two partitions I and J (of possibly different lengths) if the same relation holds for any pair of the same length representing them.

All operators act, in this paper, on their *left*.

Polynomials are usually treated as operators acting by multiplication.

1. Divided differences

Let n be a fixed (throughout the paper) positive integer.

The symmetric group (i.e. the Weyl group of type A) $\mathfrak{S}_n$ is the group with generators $s_1, \ldots, s_{n-1}$ subject to the relations

$$s_i^2 = 1, \qquad s_{i-1}\, s_i\, s_{i-1} = s_i s_{i-1}\, s_i \quad , \quad s_i s_j = s_j s_i \quad \forall i, j : |i - j| > 1 \ . \tag{1.1}$$

We shall call $s_1, \ldots, s_{n-1}$ *simple transpositions* of $\mathfrak{S}_n$.

The hyperoctahedral group (i.e. the Weyl group of type B) $\mathfrak{B}_n$ is an extension of $\mathfrak{S}_n$ by an element s_0 such that

$$s_0^2 = 1, \quad s_0\, s_1\, s_0\, s_1 = s_1\, s_0\, s_1\, s_0, \quad s_0\, s_i = s_i\, s_0 \quad \text{for} \quad i \geq 2. \tag{1.2}$$

The Weyl group $\mathfrak{D}_n$ of type D is the extension of $\mathfrak{S}_n$ by an element $s_\heartsuit$ such that

$$s_\heartsuit^2 = 1, \quad s_1 s_\heartsuit \ = s_\heartsuit\, s_1, \quad s_\heartsuit\, s_2\, s_\heartsuit \ = s_2 s_\heartsuit\, s_2, \quad s_\heartsuit\, s_i = s_i\, s_\heartsuit \quad \text{for} \quad i > 2. \tag{1.3}$$

The group $\mathfrak{D}_n$ can be thought as a subgroup of $\mathfrak{B}_n$ by sending $s_\heartsuit$ to $s_0\, s_1\, s_0$.

The above three groups act on vectors of length n by

$$[v_1, \dots, v_n]\, s_i := [v_1, \dots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \dots, v_n]$$

$$[v_1, \dots, v_n]\, s_0 := [-v_1, v_2, \dots, v_n]$$

$$[v_1, \dots, v_n]\, s_\heartsuit := [-v_2, -v_1, v_3, \dots, v_n]$$

The orbit of the vector $v = [1, \dots, n]$ under $W = \mathfrak{S}_n$, $\mathfrak{B}_n$ or $\mathfrak{D}_n$, is in bijection with the elements of W, and we shall code each $w \in W$ by the vector $[1, \dots, n]\, w$, writing $\bar{\imath}$ instead of $-i$.

The three groups $W = \mathfrak{S}_n, \mathfrak{B}_n, \mathfrak{D}_n$ also act on the ring of polynomials in n indeterminates $\mathcal{X} = \{x_1, \dots, x_n\}$: the simple transposition s_i, $i \geq 1$, exchanges x_i and x_{i+1}, s_0 sends x_1 to $-x_1$, $s_\heartsuit$ sends x_1 to $-x_2$ and x_2 to $-x_1$, the action being trivial in the non-listed cases. We shall denote by f^w the image of a polynomial $f \in \mathbb{Z}[\mathcal{X}]$ under $w \in W$, and write x^α, with $\alpha = [\alpha_1, \dots, \alpha_n] \in \mathbb{N}^n$, for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Let $\mathcal{P}ol$ be the ring of polynomials in $\mathcal{X}$ with coefficients in $\mathbb{Z}[\frac{1}{2}]$ (we need only division by 2). For any $m \leq n$, let $Sym(m|n-m)$ denote the subring of $\mathcal{P}ol$ consisting of polynomials invariant under all s_i, $1 \leq i \leq n-1$, $i \neq m$, and let $Sym(n) = Sym(n|0) = Sym(0|n)$ be the ring of symmetric polynomials. It contains as subrings $Sym^B(n)$, the ring of polynomials invariant under $\mathfrak{B}_n$, and $Sym^D(n)$, the invariants of $\mathfrak{D}_n$. It is easy to see that $\mathcal{P}ol$ is a free module over these different rings, generated by x^α, $\alpha \subseteq [n-1, \dots, 0]$ (or $\alpha \subseteq [0, \dots, n-1]$) over $Sym(n)$, by x^α, $\alpha \subseteq [2n-1, 2n-3, \dots, 1]$ (or $\alpha \subseteq [1, \dots, 2n-3, 2n-1]$) over $Sym^B(n)$, and by x^α, $\alpha \subseteq [2n-2, 2n-4, \dots, 2, 0]$ (or $\alpha \subseteq [0, 2, \dots, 2n-2]$) over $Sym^D(n)$.

The respective elements of maximal length in each of the groups are

$$\omega := [n, \dots, 1] \text{ for } \mathfrak{S}_n \ ,$$

$$w_0^B := [\bar{1}, \dots, \bar{n}] \text{ for } \mathfrak{B}_n \ ,$$

$$w_0^D := \begin{cases} [\bar{1}, \quad \dots, \quad \bar{n}] & n \quad \text{even} \\ [1, \quad \bar{2}, \quad \dots, \quad \bar{n}] & n \quad \text{odd} \end{cases}$$

for $\mathfrak{D}_n$. We shall also need the following element of $\mathfrak{D}_n$:

$$\upsilon := \omega w_0^D = \begin{cases} [\bar{n}, \quad \overline{\dots,} \quad \bar{1}] & n \quad \text{even} \\ [n, \quad \overline{n-1}, \quad \dots, \quad \bar{1}] & n \quad \text{odd}\,. \end{cases}$$

Relations between reduced decompositions in W can be represented planarly. By definition, a planar display will be identified with its reading from left to right and top to bottom (*row–reading*). We shall also use *column–reading*, that is, reading successive columns downwards, from left to right.

For example, we will write

$$\begin{array}{cc} 2 & \\ 1 & 2 \end{array} \equiv \begin{array}{cc} 1 & 2 \\ & 1 \end{array}$$

for the following equality for simple transpositions:

$$s_2\, s_1\, s_1 = s_1\, s_2\, s_1\ .$$

Suppose that a rectangle is filled row-wise from left to right, and column-wise from bottom to top with consecutive numbers from $\{1, \dots, n-1\}$.

Then one easily checks that its row–reading and column–reading produce two words which, interpreted as words in the s_i, are congruent modulo the Coxeter relations.

Here is an example of such a congruence :

$$\begin{array}{ccc} 3456 & & \\ & 2345 & \\ & & 1234 \end{array} \equiv \begin{array}{ccccccc} 3 & & 4 & & 5 & & 6 \\ 2 & \cdot & 3 & \cdot & 4 & \cdot & 5 \\ 1 & & 2 & & 3 & & 4 \end{array}$$

the congruence class being conveniently denoted by the rectangle $\begin{smallmatrix} 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \end{smallmatrix}$.

More generally, the planar arrays that we shall write, will have the property that their row–reading and column–reading are congruent modulo Coxeter relations (cf. [LS2,LS3] or [EG] for a "jeu de taquin" on reduced decompositions). In this notation, one has, for any integers a, b, c, d, k : $1 \le a < b, c < d \le n$, $a + d = b + c$, $k < d - b$, the congruence

$$\begin{array}{ccccc} b+1 & \cdots & b+k & & \\ b & \cdots & \cdots & \cdots & d \\ \vdots & & & & \vdots \\ a & \cdots & \cdots & \cdots & c \end{array} \equiv \begin{array}{ccccc} b & \cdots & \cdots & \cdots & d \\ \vdots & & & & \vdots \\ a & \cdots & \cdots & \cdots & c \\ & & c-k & \cdots & c-1 \end{array} \tag{1.4}$$

It is convenient to work in the group algebra of $W = \mathfrak{S}_n$, $\mathfrak{B}_n$ or $\mathfrak{D}_n$. The works of Young and Weyl have stressed the rôle of the alternating sum of elements of these groups. Let, for $W = \mathfrak{S}_n$, $\mathfrak{B}_n$ or $\mathfrak{D}_n$,

$$\Omega^W := \sum_{w \in W} (-1)^{\ell(w)} w\,. \tag{1.5}$$

Using that $\mathfrak{B}_n$ (resp. $\mathfrak{D}_n$) is isomorphic to the semi-direct product $\mathfrak{S}_n \ltimes \mathbb{Z}_2^n$ (resp. $\mathfrak{S}_n \ltimes \mathbb{Z}_2^{n-1}$), one obtains the following factorizations in the group algebra:

$$\Omega^{\mathfrak{B}_n} = \Omega^{\mathfrak{S}_n} \prod_{1 \le i \le n} (1 - \tau_i) = \prod_{1 \le i \le n} (1 - \tau_i)\ \Omega^{\mathfrak{S}_n}\,, \tag{1.6}$$

$$\Omega^{\mathfrak{D}_n} = \frac{1}{2}\, \Omega^{\mathfrak{S}_n} \Big(\prod_{1 \le i \le n} (1 + \tau_i) + \prod_{1 \le i \le n} (1 - \tau_i) \Big)$$

$$= \frac{1}{2}\Big(\prod_{1\le i\le n} (1+\tau_i) + \prod_{1\le i\le n} (1-\tau_i)\Big)\ \Omega^{\mathfrak{S}_n}\,, \tag{1.7}$$

where $\tau_1 := s_0$, and $\tau_i = s_{i-1}\,\tau_{i-1}\,s_{i-1}$ for $i > 1$. The elements Ω^W, as operators on the ring of polynomials $\mathcal{P}ol$, can be obtained from the cases of $\mathfrak{S}_2$, $\mathfrak{B}_1$, $\mathfrak{D}_2$. To see this, we first need to define *simple divided differences* :

$$\mathcal{P}ol \ni f \mapsto f\,\partial_i := (f - f^{s_i})/(x_i - x_{i+1}), \quad i \ge 1\,, \tag{1.8}$$

$$\mathcal{P}ol \ni f \mapsto f\,\partial_0 := (f - f^{s_0})/(-x_1)\ , \tag{1.9}$$

$$\mathcal{P}ol \ni f \mapsto f\,\partial_\heartsuit := (f - f^{s_\heartsuit})/(-x_1 - x_2)\ . \tag{1.10}$$

The ∂_i, ∂_0, $\partial_\heartsuit$ satisfy the Coxeter relations (1.1)–(1.3), together with the relations

$$\partial_\heartsuit^2 = 0 = \partial_i^2 \quad \text{for} \quad 0 \le i < n\ . \tag{1.11}$$

Therefore, to any element w of the group W, there corresponds a *divided difference* ∂_w. Any reduced decomposition $s_{i_1}\,s_{i_2}\,\cdots s_{i_\ell} = w$ of w gives rise to a factorization $\partial_{i_1}\,\partial_{i_2}\,\cdots\partial_{i_\ell}$ of ∂_w (cf. [BGG] and [D1,D2]).

We shall display divided differences planarly according to the same conventions as for products of s_i's.

For example, the divided difference

$$\partial_0\partial_1\partial_2\partial_3\partial_0\partial_1\partial_2\partial_0\partial_1$$

will be displayed as

$$\begin{matrix} \partial_0 & \partial_1 & \partial_2 & \partial_3 \\ & \partial_0 & \partial_1 & \partial_2 \\ & & \partial_0 & \partial_1 \end{matrix}$$

As said before, the displays that we write have the property that their row–reading is congruent to their column–reading, and thus the preceding one encodes the equality

$$\partial_0\partial_1\partial_0\partial_2\partial_1\partial_0\partial_3\partial_2\partial_1 = \partial_0\partial_1\partial_2\partial_3\partial_0\partial_1\partial_2\partial_0\partial_1\,.$$

We shall especially need the *maximal* divided differences ∂_ω, $\partial_{w_0^B}$, and $\partial_{w_0^D}$. To describe them using alternating sums of group elements, we define

$$\Delta := \prod_{1\le i<j\le n} (x_i - x_j) = x_1^{n-1}\cdots x_n^0\,\Omega^{\mathfrak{S}_n}\ , \tag{1.12}$$

$$\Delta^B := \prod_{i=1}^n x_i \prod_{n\ge i>j\ge 1} (x_i^2 - x_j^2) = \frac{1}{2^{n-1}} x^{[1,3,\dots,2n-1]}\,\Omega^{\mathfrak{B}_n}\ , \tag{1.13}$$

$$\text{and}\ \ \Delta^D := \prod_{n\ge i>j\ge 1} (x_i^2 - x_j^2) = \frac{1}{2^{n-1}} x^{[2,4,\dots,2n-2]}\,\Omega^{\mathfrak{D}_n}\ . \tag{1.14}$$

The Weyl character formula for types A, B, and D can be written as

Lemma 1. *For each of the groups $W = \mathfrak{S}_n$, $\mathfrak{B}_n$ or $\mathfrak{D}_n$, the alternating sum Ω^W, as an operator on the ring of polynomials $\mathcal{P}ol$, is related to the maximal divided difference by*

$$\Omega^{\mathfrak{S}_n}\frac{1}{\Delta} = \partial_\omega\,, \qquad \Omega^{\mathfrak{B}_n}\frac{1}{\Delta^B} = (-1)^{\binom{n}{2}}\,\partial_{w_0^B}\,, \quad \textit{and} \quad \Omega^{\mathfrak{D}_n}\frac{1}{\Delta^D} = (-1)^{\binom{n}{2}}\,\partial_{w_0^D}.$$

Indeed, all the operators in Lemma 1 commute with multiplication by polynomials which are invariant under W. Moreover, they decrease degree by the length of the maximal element of the group. Since $\mathcal{P}ol$ is a module over $Sym^W(n)$ with a basis of monomials of degree strictly less than this length, except for a single monomial, it remains only to check that the actions of Ω's and ∂'s agree on this monomial, which offers no difficulty.

2. Bases of polynomial rings

The monomials mentioned in the previous section are not an appropriate basis, when interpreted in terms of cohomology classes for the flag variety. Define, for the rest of this paper, the vector

$$\rho := [n-1, \ldots, 1, 0]\,. \tag{2.1}$$

Motivated by geometry, one defines recursively *Schubert polynomials* Y_α, for any sequence $\alpha \in \mathbb{N}^n$, with $\alpha \subseteq \rho$, by

$$Y_\alpha\,\partial_i = Y_\beta\ , \quad \text{if}\ \ \alpha_i > \alpha_{i+1}\ \ , \tag{2.2}$$

where

$$\beta = [\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i - 1, \alpha_{i+2}, \ldots, \alpha_n]\,,$$

starting from $Y_\rho = x^\rho$ (cf. [LS1], [M2]).

In particular, if $\alpha \in \mathbb{N}^n$ is weakly decreasing, then Y_α is equal to the *monomial* x^α.

If, on the contrary, $\alpha_1 \le \cdots \le \alpha_k$ and $\alpha_{k+1} = \cdots = \alpha_n = 0$, for some $k \le n$, then Y_α coincides with the *Schur polynomial* $s_\lambda(x_1, \ldots, x_k)$, where $\lambda = (\alpha_k, \ldots, \alpha_1)$.

CONVENTION: Let $\alpha \in \mathbb{N}^k$. Then we shall write Y_α for $Y_{\alpha,0,\ldots,0}$.

We also record, for later use, the following equality: for $\alpha = [\alpha_1, \ldots, \alpha_k] \in \mathbb{N}^k$,

$$Y_\alpha\, x_1 \cdots x_k = Y_{[\alpha_1+1,\ldots,\alpha_k+1]}. \tag{2.3}$$

On $\mathcal{P}ol$ there is a scalar product:

$$(\ ,\ ) : \mathcal{P}ol \times \mathcal{P}ol \to Sym(n),$$

defined for $f, g \in \mathcal{P}ol$ by

$$(f, g) := fg\,\partial_\omega. \tag{2.4}$$

There exists an involution[1] $\alpha \mapsto \alpha'$ such that

$$\big(Y_\alpha^\omega, Y_{\beta'}\big) = (-1)^{|\alpha|}\delta_{\alpha\beta}\,. \tag{2.5}$$

Moreover, when $\alpha, \beta \subseteq \rho$ are such that $|\alpha| + |\beta| = |\rho|$, then one has

$$\big(Y_\alpha, Y_{[n-1-\beta_1, n-2-\beta_2, \dots, 0-\beta_n]}\big) = \delta_{\alpha\beta}. \tag{2.6}$$

We also will need $\widetilde{Q}$-polynomials of [PR]. We set $\widetilde{Q}_i := e_i = e_i(\mathcal{X})$, the i-th elementary symmetric polynomial in $\mathcal{X}$. Given two nonnegative integers $i \geq j$, we adapt Schur's definition of his Q-functions by putting

$$\widetilde{Q}_{i,j} := \widetilde{Q}_i\widetilde{Q}_j + 2\sum_{p=1}^{j}(-1)^p\widetilde{Q}_{i+p}\widetilde{Q}_{j-p}\,. \tag{2.7}$$

Given any partition $I = (i_1, \dots, i_k)$, where we can assume k to be even, we set

$$\widetilde{Q}_I := \mathrm{Pfaffian}(M)\,, \tag{2.8}$$

where $M = (m_{p,q})$ is the $k \times k$ skew-symmetric matrix with $m_{p,q} = \widetilde{Q}_{i_p, i_q}$ for $1 \leq p < q \leq k$.

Equivalently, for any partition $I = (i_1 \geq i_2 \geq \dots \geq i_\ell > 0)$, the polynomial $\widetilde{Q}_I = \widetilde{Q}_I(X)$ is defined recurrently on ℓ by putting for odd ℓ,

$$\widetilde{Q}_I := \sum_{j=1}^{\ell}(-1)^{j-1}\widetilde{Q}_{i_j}\widetilde{Q}_{(i_1,\dots,i_{j-1},i_{j+1},\dots,i_\ell)} \tag{2.9}$$

and for even ℓ,

$$\widetilde{Q}_I := \sum_{j=2}^{\ell}(-1)^{j}\widetilde{Q}_{i_1,i_j}\widetilde{Q}_{(i_2,\dots,i_{j-1},i_{j+1},\dots,i_\ell)}\,. \tag{2.10}$$

For any positive integer k, let $\rho(k)$ denote the partition

$$\rho(k) := (k, k-1, \dots, 1)\,. \tag{2.11}$$

The ring $Sym(n)$ is a free module over the ring of polynomials symmetric in $x_1^2, \dots, x_n^2$, with a basis provided by the $\widetilde{Q}_I(\mathcal{X})$, where $I \subseteq \rho(n)$ ranges over strict partitions.

As functions of $x_1, \dots, x_m$, the $\widetilde{Q}$-polynomials can also be defined recursively by induction on m, involving now all partitions without restriction, as for Hall-Littlewood polynomials: for any strict partition I, one has

$$\widetilde{Q}_I(x_1, \dots, x_m) = \sum_{j=0}^{\ell(I)} x_m^j\Big(\sum_{|I|-|J|=j}\widetilde{Q}_J(x_1, \dots, x_{m-1})\Big), \tag{2.12}$$

[1]This involution is: $\mathrm{code}(w) = \alpha \mapsto \alpha' = \mathrm{code}(w\omega)$ (cf. [M2]).

where the sum is over all (i.e. not necessarily strict) partitions $J \subseteq I$ such that I/J has at most one box in every row (cf. [PR, Prop. 4.1]). Moreover, given a partition $I' = (\dots, i, j, j, k, \dots)$ and denoting $I = (\dots, i, k, \dots)$, one has the *factorization property*

$$\widetilde{Q}_{I'} = \widetilde{Q}_{j,j}\widetilde{Q}_{I}. \tag{2.13}$$

We define, for a strict partition I,

$$\widetilde{P}_I := 2^{-\ell(I)}\, \widetilde{Q}_I\,. \tag{2.14}$$

The ring $Sym(n)$ is a free module over $Sym^D(n)$ with a basis provided by the $\widetilde{P}_I$, where I ranges over strict partitions contained in $\rho(n-1)$.

Now we will need the following divided difference:

$$\partial_v = (\partial_\heartsuit \partial_2 \cdots \partial_{n-1}\, \partial_1 \cdots \partial_{n-2}) \cdots (\partial_\heartsuit \partial_2 \partial_3\, \partial_1 \partial_2)\, \partial_\heartsuit \qquad n \text{ even} \tag{2.15}$$

and

$$\partial_v = (\partial_\heartsuit \partial_2 \cdots \partial_{n-1}\, \partial_1 \cdots \partial_{n-2}) \cdots (\partial_\heartsuit \partial_2 \partial_3 \partial_4\, \partial_1 \partial_2 \partial_3)\, (\partial_\heartsuit \partial_2 \partial_1) \quad n \text{ odd} \tag{2.16}$$

Denote by

$$\langle\ ,\ \rangle : Sym(n) \times Sym(n) \to Sym^D(n)$$

the scalar product defined for $f, g \in Sym(n)$ by

$$\langle f, g\rangle := f g\, \partial_v. \tag{2.17}$$

For strict partitions $I, J \subseteq \rho(n-1)$, one has

$$\langle \widetilde{P}_I, \widetilde{P}_{\rho(n-1) \smallsetminus J}\rangle = (-1)^{\binom{n}{2}}\, \delta_{IJ}, \tag{2.18}$$

where $\rho(n-1) \smallsetminus I$ is the strict partition whose parts complement the parts of I in $\{n-1, n-2, \dots, 1\}$ (cf. [PR]).

Consequently, the polynomial ring $\mathcal{P}ol = \mathbb{Z}[\frac{1}{2}][x_1, \dots, x_n]$ is a free $Sym^D(n)$-module with a basis $Y_\alpha\, \widetilde{P}_I$, where α ranges over subsequences contained in ρ and I runs over all strict partitions contained in $\rho(n-1)$. Note that the element of maximal degree of this basis is $x^\rho \widetilde{P}_{\rho(n-1)}$. Let

$$[\ ,\ ] : \mathcal{P}ol \times \mathcal{P}ol \to Sym^D(n)$$

be a scalar product, defined for $f, g \in \mathcal{P}ol$ by

$$[f, g] := f g\, \partial_{w_0^D}. \tag{2.19}$$

One has, for $\alpha, \beta \subseteq \rho$ and strict partitions $I, J \subset \rho(n-1)$,

$$\big[Y^\omega_\alpha\, \widetilde{P}_I, Y_{\beta'}\, \widetilde{P}_{\rho(n-1)\smallsetminus J}\big] = (-1)^{|\alpha| + \binom{n}{2}}\, \delta_{\alpha\beta}\, \delta_{IJ}\,. \tag{2.20}$$

(See (2.5).)

Let $\mathcal{Y} = \{y_1, \ldots, y_n\}$ be a second set of indeterminates of cardinality n. The symbol $\equiv$ will mean: "congruent modulo the ideal generated by the relations $f(x_1^2, \ldots, x_n^2) = f(y_1^2, \ldots, y_n^2)$, $f \in Sym(n)$, together with $x_1 \cdots x_n = y_1 \cdots y_n$".

Following Fulton [F2,F3], define

$$F(\mathcal{X}, \mathcal{Y}) := \ |\widetilde{P}_{n+j-2i}(\mathcal{X}) + \widetilde{P}_{n+j-2i}(\mathcal{Y})|_{1 \le i,j \le n-1}$$

$$= \begin{vmatrix} \widetilde{P}_{n-1}(\mathcal{X}) + \widetilde{P}_{n-1}(\mathcal{Y}) & 0 & \ldots & \\ \widetilde{P}_{n-3}(\mathcal{X}) + \widetilde{P}_{n-3}(\mathcal{Y}) & \widetilde{P}_{n-2}(\mathcal{X}) + \widetilde{P}_{n-2}(\mathcal{Y}) & \ldots & \\ \vdots & \vdots & \ddots & \\ & & 1 & \widetilde{P}_1(\mathcal{X}) + \widetilde{P}_1(\mathcal{Y}) \end{vmatrix} . \tag{2.21}$$

Following [PR], define

$$\widetilde{P}(\mathcal{X}, \mathcal{Y}) := \sum \widetilde{P}_I(\mathcal{X})\, \widetilde{P}_{\rho(n-1) \smallsetminus I}(\mathcal{Y}) \ , \tag{2.22}$$

where the summation is over all strict partitions $I \subseteq \rho(n-1)$. The reasoning in [LP1, Sect.2] made for case C_n adapts to case D_n and furnishes:

Proposition 2. *We have*
(i)

$$F(\mathcal{X}, \mathcal{Y}) \equiv \widetilde{P}(\mathcal{X}, \mathcal{Y})\,. \tag{2.23}$$

(ii) For every $w \in \mathfrak{D}_n \smallsetminus \mathfrak{S}_n$,

$$\widetilde{P}(\mathcal{X}^w, \mathcal{X}) = 0\,, \tag{2.24}$$

and for every $w \in \mathfrak{S}_n$,

$$\widetilde{P}(\mathcal{X}^w, \mathcal{X}) = \widetilde{P}(\mathcal{X}, \mathcal{X}) = s_{\rho(n-1)}(\mathcal{X})\,. \tag{2.25}$$

(iii) For every $f \in Sym(n)$,

$$\langle f(\mathcal{X})\,, F(\mathcal{X}, \mathcal{Y})\rangle \equiv (-1)^{\binom{n}{2}}\, f(\mathcal{Y}) \ . \tag{2.26}$$

(iv) For every $f \in \mathcal{P}ol$,

$$\big[f(\mathcal{X})\,, \prod_{n \ge i > j \ge 1} (x_i - y_j)\, F(\mathcal{X}, \mathcal{Y})\,\big] \equiv f(\mathcal{Y})\,. \tag{2.27}$$

In other words, $F(\mathcal{X}, \mathcal{Y})$ is a reproducing kernel for the scalar product $\langle\ ,\ \rangle$, and $\prod_{i>j}(x_i - y_j)\, F(\mathcal{X}, \mathcal{Y})$ is a reproducing kernel for $[\ ,\ ]$. One can show that the "vanishing property" *(ii)* characterizes $\widetilde{P}(\mathcal{X}, \mathcal{Y})$ up to $\equiv$. The congruence *(i)* can be also derived from geometry by comparing the classes of diagonals in flag bundles associated with $SO(2n)$ given in [F2,F3] and [PR] (see also [G]).

3. Vertex Operators

In this section we shall mainly make computations using the following two divided differences:

Definition 3. *For $k \le n$, we set*

$$\nabla_k^B(n) := (\partial_0\partial_1\cdots\partial_{n-1})\cdots(\partial_0\partial_1\cdots\partial_{n-k})\,. \tag{3.1}$$

For $k \le n/2$, we put

$$\nabla_k^D(n) := (\partial_\heartsuit\partial_2\cdots\partial_{n-1}\partial_1\partial_2\cdots\partial_{n-2})\cdots(\partial_\heartsuit\partial_2\cdots\partial_{n-2k+1}\partial_1\partial_2\cdots\partial_{n-2k}). \tag{3.2}$$

We shall need the following fact from [LP1], quoted in the appendix:

Fact 4. *Let $k \le n$ and let $\alpha = [\alpha_1 \le \cdots \le \alpha_k] \in \mathbb{N}^k$ with $\alpha_k \le n-k$. Suppose that $I \subseteq \rho(n)$ is a strict partition. Then the image of $\widetilde{Q}_I\, Y_\alpha$ under $\nabla_k^B(n)$ is 0 unless $n-0-\alpha_1, \ldots, n-(k-1)-\alpha_k$ are parts of I. In this case, the image is $(-1)^{k(n-1)+s}2^k\widetilde{Q}_J$, where J is the strict partition with parts*

$$\{i_1, \ldots, i_{\ell(I)}\} \smallsetminus \{n-0-\alpha_1, \ldots, n-(k-1)-\alpha_k\},$$

and s is the sum of positions of the parts erased in I.

Example 5. For $n=7$ and $k=2$, we have

$$\begin{aligned}&\widetilde{Q}_{(5,4,3,2,1)}\, Y_{[2,5]}\, \nabla_2^B(7)\\ &\quad= \widetilde{Q}_{(5,4,3,2,1)}\, Y_{[2,5]}\,(\partial_0\partial_1\partial_2\partial_3\partial_4\partial_5\partial_6)(\partial_0\partial_1\partial_2\partial_3\partial_4\partial_5) = 4\widetilde{Q}_{(4,3,2)}\,,\end{aligned}$$

and for $k=3$, we have

$$\widetilde{Q}_{(7,5,4,3,1)}\, Y_{[2,3,4]}\nabla_3^B(7) = -8\widetilde{Q}_{(7,4)}\,.$$

The following result establishes a basic relation between the ∇^D's and ∇^B's:

Proposition 6. *Let k be a positive integer. As operators on $Sym(2k)$,*

$$\nabla_k^D(2k) = x_1\cdots x_{2k}\,\nabla_{2k}^B(2k) + x_1\cdots x_{2k-1}\,\nabla_{2k-1}^B(2k)\,. \tag{3.3}$$

Before proving (3.3), we illustrate it by the following examples:

Example 7. As operators on $Sym(2)$,

$$\nabla_1^D(2) = \partial_\heartsuit = x_1x_2\partial_0\partial_1\partial_0 + x_1\partial_0\partial_1.$$

As operators on $Sym(4)$,

$$\begin{aligned}&\nabla_2^D(4) = (\partial_\heartsuit\partial_2\partial_3\partial_1\partial_2)\partial_\heartsuit =\\ &\qquad x_1x_2x_3x_4\,(\partial_0\partial_1\partial_2\partial_3)(\partial_0\partial_1\partial_2)(\partial_0\partial_1)\partial_0 + x_1x_2x_3\,(\partial_0\partial_1\partial_2\partial_3)(\partial_0\partial_1\partial_2)(\partial_0\partial_1).\end{aligned}$$

The RHS of the last equation is depicted planarly as

$$x_1x_2x_3x_4 \begin{array}{cccc} \partial_0 & \partial_1 & \partial_2 & \partial_3 \\ & \partial_0 & \partial_1 & \partial_2 \\ & & \partial_0 & \partial_1 \\ & & & \partial_0 \end{array} + x_1x_2x_3 \begin{array}{cccc} \partial_0 & \partial_1 & \partial_2 & \partial_3 \\ & \partial_0 & \partial_1 & \partial_2 \\ & & \partial_0 & \partial_1 \end{array}$$

Proof of the proposition. In this proof, let $\mathcal{X} := \{x_1, \dots, x_{2k}\}$. Both sides of (3.3) are $Sym^B(2k)$-linear. The operator $\nabla_k^D(2k)$ sends all $\widetilde{Q}_I(\mathcal{X})$, $I \subseteq \rho(2k-1)$ to 0, except for $\widetilde{Q}_{\rho(2k-1)}$ which is sent to $(-1)^k 2^{2k-1}$ (cf.(2.18)). Thus $\nabla_k^D(2k)$ annihilates all $\widetilde{Q}_I(\mathcal{X})$, $I \subseteq \rho(2k)$, except for $I = \rho(2k-1)$ which is sent to $(-1)^k 2^{2k-1}$ and $I = \rho(2k)$ which is sent to $(-1)^k 2^{2k-1} x_1 \cdots x_{2k}$.

The action of

$$x_1 \cdots x_{2k} \nabla_{2k}^B(2k)$$

is given by Fact 4. Only $\widetilde{Q}_{\rho(2k)}(\mathcal{X})$ survives and is sent to $(-1)^k 2^{2k} x_1 \cdots x_{2k}$.

We will now calculate the action of

$$x_1 \cdots x_{2k-1} \, \nabla_{2k-1}^B(2k)$$

on the $\widetilde{Q}_I(\mathcal{X})$, where $I \subseteq \rho(2k)$ is a strict partition. We set, temporarily in this proof,

$$\nabla := \nabla_{2k-1}^B(2k) \quad \text{and} \quad \nabla' := \nabla_{2k-1}^B(2k-1)\,,$$

so that

$$\nabla = \nabla' \partial_{2k-1} \cdots \partial_1 .$$

Let $\mathcal{X}' := \{x_1, \dots, x_{2k-1}\}$. We decompose $\widetilde{Q}_I(\mathcal{X})$ as a sum of products of powers of x_{2k} times some $\widetilde{Q}_J(\mathcal{X}')$, according to the formula (2.12):

$$\widetilde{Q}_I(\mathcal{X}) = \sum \widetilde{Q}_J(\mathcal{X}')\, x_{2k}^{m_J} \ .$$

Let $\overline{J}$ be the strict partition obtained from a partition J by subtracting all the pairs of equal parts. We have three cases to examine:

1. Let $i_1 \le 2k-2$. Then for each J, $|\overline{J}| + m_J + 2k - 1 < \deg \nabla$, and hence

$$x_1 \cdots x_{2k-1} \widetilde{Q}_I(\mathcal{X}) \nabla = 0.$$

2. Let $i_1 = 2k-1$. For degree reasons, $\widetilde{Q}_I(\mathcal{X}) x_1 \cdots x_{2k-1} \nabla \neq 0$ is possible only if $I = \rho(2k-1)$ (I being a strict partition).

Claim. *We have*

$$x_1 \cdots x_{2k-1} \widetilde{Q}_J(\mathcal{X}') \nabla' \neq 0 \tag{3.4}$$

only if $J = \overline{J} = \rho(2k-2)$.

Indeed, suppose first that $j_1 = 2k-1$. Then

$$x_1 \cdots x_{2k-1} \widetilde{Q}_J(\mathcal{X}') = \mathcal{P} \cdot \widetilde{Q}_H(\mathcal{X}'),$$

where $\mathcal{P}$ is a polynomial symmetric in $x_1^2,\dots,x_{2k-1}^2$, and a strict partition H has no part equal to $2k-1$. Since this expression is annihilated by ∇', we cannot have (3.4). So, for degree reasons, (3.4) holds only if $j_1=2k-2$. Suppose now that $j_2=2k-2$. We get

$$x_1\cdots x_{2k-1}\widetilde{Q}_J(\mathcal{X}')=\mathcal{P}\cdot\widetilde{Q}_H(\mathcal{X}'),$$

where $\mathcal{P}$ is a polynomial symmetric in $x_1^2,\dots,x_{2k-1}^2$, and a strict partition H has no part equal to $2k-2$. Since this expression is annihilated by ∇', we cannot have (3.4). So, for degree reason, (3.4) holds only if $j_2=2k-3$. Continuing this way, we get the claim.

For $J=\rho(2k-2)$, we compute

$$x_1\cdots x_{2k-1}\widetilde{Q}_J(\mathcal{X}')x_{2k}^{2k-1}\nabla=\widetilde{Q}_{\rho(2k-1)}(\mathcal{X}')\nabla' x_{2k}^{2k-1}\,\partial_{2k-1}\cdots\partial_1=(-1)^{k-1}2^{2k-1}.$$

3. Let $i_1=2k$. Then $(i_2,\dots)\subseteq\rho(2k-1)$. We have

$$\begin{aligned}x_1\cdots x_{2k-1}\widetilde{Q}_I(\mathcal{X})\nabla&=x_1^2\cdots x_{2k-1}^2x_{2k}\widetilde{Q}_{(i_2,\dots)}(\mathcal{X})\nabla\\ &=\widetilde{Q}_{(i_2,\dots)}(\mathcal{X})\nabla' x_1^2\cdots x_{2k-1}^2x_{2k}\,\partial_{2k-1}\cdots\partial_1\ .\end{aligned}$$

Now for $H\subseteq(i_2,\dots)\subseteq\rho(2k-1)$, $\widetilde{Q}_H(\mathcal{X}')\nabla'\neq0$ iff $H=\rho(2k-1)$, so iff $(i_2,\dots)=\rho(2k-1)$. We have

$$\widetilde{Q}_{\rho(2k-1)}(\mathcal{X})\nabla'=\widetilde{Q}_{\rho(2k-1)}(\mathcal{X}')\nabla'=(-1)^k2^{2k-1}$$

and

$$x_1^2\cdots x_{2k-1}^2x_{2k}\,\partial_{2k-1}\cdots\partial_1=-x_1\cdots x_{2k}.$$

Summarizing,

$$\widetilde{Q}_I(\mathcal{X})x_1\cdots x_{2k-1}\nabla\neq0$$

only if $I=\rho(2k-1)$, when we get $(-1)^{k-1}2^{2k-1}$; or $I=\rho(2k)$, when we get $(-1)^{k-1}2^{2k-1}x_1\cdots x_{2k}$.

Finally, comparing the computed values of the $\widetilde{Q}_I(\mathcal{X})$ under the operators:

$$\nabla_k^D(2k)\,,\qquad x_1\cdots x_{2k}\,\nabla_{2k}^B(2k),\quad\text{and}\qquad x_1\cdots x_{2k-1}\,\nabla_{2k-1}^B(2k),$$

that are possibly nonzero only for $I=\rho(2k)$ and $\rho(2k-1)$, we get the desired formula (3.3). (Note that we have also used the equality $2^{p-1}=2^p-2^{p-1}$.) $\square$

Corollary 8. *Let k be a positive integer such that $k\le n/2$. As operators on the ring $Sym(2k\,|\,n-2k)$,*

$$\nabla_k^D(n)=x_1\cdots x_{2k}\,\nabla_{2k}^B(n)+x_1\cdots x_{2k-1}\,\nabla_{2k-1}^B(n)\,\partial_1\cdots\partial_{n-2k}\,. \tag{3.5}$$

This property is obtained from Proposition 6 by composing the expression for the operator $\nabla_k^D(2k)$ with the divided difference

$$(\partial_{2k}\cdots\partial_{n-1})\cdots(\partial_2\cdots\partial_{n-2k+1})(\partial_1\cdots\partial_{n-2k})=\begin{matrix}\partial_{2k}&\cdots&\partial_{n-1}\\ \vdots&&\vdots\\ \partial_2&\cdots&\partial_{n-2k+1}\\ \partial_1&\cdots&\partial_{n-2k}\end{matrix}$$

In the proof of Theorem 11, we will need the following supplement to Fact 4:

Theorem 9. *Let $k \leq n$ and let $\alpha = [\alpha_1 \leq \cdots \leq \alpha_k] \in \mathbb{N}^k$ with $\alpha_k = n - k + 1$. Suppose that $I \subseteq \rho(n)$ is a strict partition. Then the image of $\widetilde{Q}_I\, Y_\alpha$ under $\nabla_k^B(n)$ is 0 unless $\ell(I) \not\equiv n \pmod 2$ and $n - 0 - \alpha_1, \ldots, n - (k-2) - \alpha_{k-1}$ are parts of I. In this case, the image is $(-1)^{(k-1)(n-1)+1+s} 2^k \widetilde{Q}_J$, where J is the strict partition with parts*

$$\{i_1, \ldots, i_{\ell(I)}\} \smallsetminus \{n - 0 - \alpha_1, \ldots, n - (k-2) - \alpha_{k-1}\}\,,$$

and s is the sum of positions of the parts erased in I.

The proof of this theorem will be given in the appendix.

Example 10.

(i) For $n = 5$ and $k = 1$, we have

$$x_1^5 \widetilde{Q}_{(5,3,2,1)} \partial_0 \partial_1 \partial_2 \partial_3 \partial_4 = -2\widetilde{Q}_{(5,3,2,1)} \quad \text{and} \quad x_1^5 \widetilde{Q}_{(5,2,1)} \partial_0 \partial_1 \partial_2 \partial_3 \partial_4 = 0.$$

(ii) For $n = 7$ and $k = 2$, we have

$$\widetilde{Q}_{(7,6,4,1)}\, Y_{[1,6]}\, \nabla_2^B(7) = -4\widetilde{Q}_{(7,4,1)}$$

and

$$\widetilde{Q}_{(7,6,4,3,1)}\, Y_{[1,6]}\, \nabla_2^B(7) = 0.$$

(iii) For $n = 7$ and $k = 4$, we have

$$\widetilde{Q}_{(7,6,4,3,2,1)}\, Y_{[1,2,2,4]}\, \nabla_4^B(7) = 16\widetilde{Q}_{(7,2,1)}$$

and

$$\widetilde{Q}_{(7,6,4,3,2,1)}\, Y_{[1,3,4,4]}\, \nabla_4^B(7) = -16\widetilde{Q}_{(7,4,2)}.$$

The following theorem is the main result of the present paper:

Theorem 11. *Let k be a positive integer such that $k \leq n/2$. Suppose that $I \subseteq \rho(n-1)$ is a strict partition. Let $\alpha = [\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{2k}] \in \mathbb{N}^{2k}$ with $\alpha_{2k} \leq n - 2k$. Then the image of $\widetilde{P}_I\, Y_\alpha$ under $\nabla_k^D(n)$ is 0 unless all the integers $n - 1 - \alpha_1, \ldots, n - 2k - \alpha_{2k}$ belong to $\{i_1, \ldots, i_{\ell(I)}, 0\}$. In this case, the image is $(-1)^s \widetilde{P}_J$, where J is the strict partition with parts*

$$\{i_1, \ldots, i_{\ell(I)}\} \smallsetminus \{n - 1 - \alpha_1, \ldots, n - 2k - \alpha_{2k}\}\,.$$

Moreover, let s' be the sum of positions of the parts erased in I, and $s'' := \ell(I) + 1$. Then $s = s'$ if $\alpha_{2k} < n - 2k$, and $s = s' + s''$ if $\alpha_{2k} = n - 2k$.[2]

Example 12. (i) For $n = 7$ and $k = 1$, we have

$$\widetilde{P}_{(5,4,3,2,1,0)}\, Y_{[1,3]} \nabla_1^D(7) = \widetilde{P}_{(5,4,3,2,1,0)}\, Y_{[1,3]}\, \partial_\heartsuit \partial_2 \partial_3 \partial_4 \partial_5 \partial_6 \partial_1 \partial_2 \partial_3 \partial_4 \partial_5 = -\widetilde{P}_{(4,3,1)}$$

[2] It is convenient to treat here $0 = n - 2k - \alpha_{2k}$ as an "extra part" of I, and take s to be the sum of positions of all the parts erased in I, including the extra part.

and

$$\widetilde{P}_{(6,4,3,2,1,0)}\, Y_{[2,5]}\, \nabla_1^D(7) = \widetilde{P}_{(6,3,2,1)}.$$

(ii) For $n = 7$ and $k = 2$, we have

$$\widetilde{P}_{(6,5,4,3,2,1,0)}\, Y_{[1,1,1,2]}\, \nabla_2^D(7) = -\widetilde{P}_{(6,2)}$$

and

$$\widetilde{P}_{(6,5,4,3,2,1,0)}\, Y_{[1,1,1,3]}\, \nabla_2^D(7) = \widetilde{P}_{(6,2,1)}\,.$$

Proof of the theorem. To compute the action of $\nabla_k^D(n)$, one uses its decomposition into a sum of two operators, given in Eq.(3.5).

The image of $\widetilde{Q}_I(\mathcal{X})\, Y_\alpha$ under the first operator

$$\Omega_1 := x_1 \cdots x_{2k}\, \nabla_{2k}^B(n)$$

is given by Fact 4 combined with Eq.(2.3) if $\alpha_{2k} < n - 2k$, and by Theorem 9 combined with (2.3) in the case $\alpha_{2k} = n - 2k$.

Since x_{2k} appears in Y_α, the same results, however, *do not* directly furnish the value of $\widetilde{Q}_I(X_n)\, Y_\alpha$ under the second operator

$$\Omega_2 := x_1 \cdots x_{2k-1}\, \nabla_{2k-1}^B(n)\, \partial_1 \cdots \partial_{n-2k}\,.$$

To end this computation, we proceed as follows. For simplicity of indices, let us take temporarily $n = 7$ and $k = 2$. Suppose that $\alpha = [\alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4] \in \mathbb{N}^4$ is such that $\alpha_4 \le 3$. We want to compute

$$\widetilde{Q}_I\, Y_\alpha\, x_1x_2x_3 \quad \begin{matrix} \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ & \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 \\ & & \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 \\ & & & & \partial_1 & \partial_2 & \partial_3 \end{matrix}$$

Now, thanks to the relations (1.4), one has

$$\begin{matrix} \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ & \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 \\ & & \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 \\ & & & & \partial_1 & \partial_2 & \partial_3 \end{matrix}$$

$$= \begin{pmatrix} \partial_0 & \partial_1 & \partial_2 \\ & \partial_0 & \partial_1 \\ & & \partial_0 \end{pmatrix} \begin{pmatrix} \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ \partial_2 & \partial_3 & \partial_4 & \partial_5 \\ \partial_1 & \partial_2 & \partial_3 & \partial_4 \\ & \partial_1 & \partial_2 & \partial_3 \end{pmatrix} = \begin{pmatrix} \partial_0 & \partial_1 & \partial_2 \\ & \partial_0 & \partial_1 \\ & & \partial_0 \end{pmatrix} \begin{pmatrix} \partial_4 & \partial_5 & \partial_6 & \\ \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ \partial_2 & \partial_3 & \partial_4 & \partial_5 \\ \partial_1 & \partial_2 & \partial_3 & \partial_4 \end{pmatrix}$$

$$= \begin{pmatrix} \partial_0 & \partial_1 & \partial_2 \\ & \partial_0 & \partial_1 \\ & & \partial_0 \end{pmatrix} (\partial_4\partial_5\partial_6) \begin{pmatrix} \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ \partial_2 & \partial_3 & \partial_4 & \partial_5 \\ \partial_1 & \partial_2 & \partial_3 & \partial_4 \end{pmatrix}$$

Since $\partial_4\partial_5\partial_6$ commutes with the divided differences on its left, the last expression is rewritten as

$$\partial_4\partial_5\partial_6 \quad \begin{matrix} \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ & \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 & \partial_5 \\ & & \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 \end{matrix} = \partial_4\partial_5\partial_6\,\nabla_3^B(7)$$

Since $\partial_4\partial_5\partial_6$ commutes with $x_1x_2x_3$ and $\widetilde{Q}_I = \widetilde{Q}_I(x_1,\ldots,x_7)$, the polynomial to be computed is equal to

$$Y_\alpha\,\partial_4\partial_5\partial_6\,\widetilde{Q}_I\,x_1x_2x_3\,\nabla_3^B(7)\,.$$

However, using (2.2), the image of $Y_{[\alpha_1,\alpha_2,\alpha_3,\alpha_4]}$ under $\partial_4\partial_5\partial_6$ is

$$Y_{[\alpha_1,\alpha_2,\alpha_3,0,0,0,\alpha_4-3]} = Y_{[\alpha_1,\alpha_2,\alpha_3]}$$

if $\alpha_4 = 3$, and 0 otherwise. Hence, by (2.3), the polynomial to be computed is equal to

$$Y_{[\alpha_1+1,\alpha_2+1,\alpha_3+1]}\widetilde{Q}_I\nabla_3^B(7)\,.$$

In general, arguing along these lines, we evaluate

$$\widetilde{Q}_I\,Y_\alpha\,x_1\cdots x_{2k-1}\,\nabla_{2k-1}^B(n)\,\partial_1\partial_2\cdots\partial_{n-2k}\,,$$

where $\alpha = [\alpha_1 \le \alpha_2 \le \cdots \le \alpha_{2k}] \in \mathbb{N}^{2k}$ is such that $\alpha_{2k} \le n-2k$. By the relations (1.4), this amounts to evaluating

$$\widetilde{Q}_I\,Y_\alpha\,x_1\cdots x_{2k-1}\,\partial_{2k}\partial_{2k+1}\cdots\partial_{n-1}\,\nabla_{2k-1}^B(n)\,.$$

Since $\partial_{2k}\partial_{2k+1}\cdots\partial_{n-1}$ commutes with $x_1\cdots x_{2k-1}$ and $\widetilde{Q}_I = \widetilde{Q}_I(x_1,\ldots,x_n)$, the polynomial to be computed is equal to

$$Y_\alpha\,\partial_{2k}\partial_{2k+1}\cdots\partial_{n-1}\widetilde{Q}_I x_1\cdots x_{2k-1}\,\nabla_{2k-1}^B(n)\,.$$

However, using (2.2), the image of Y_α under $\partial_{2k}\partial_{2k+1}\cdots\partial_{n-1}$ is

$$Y_{[\alpha_1,\ldots,\alpha_{2k-1},0^{n-2k},\alpha_{2k}-(n-2k)]}\,. \tag{3.6}$$

The expression (3.6) equals 0 unless $\alpha_{2k} = n-2k$, when it is equal to $Y_{[\alpha_1,\ldots,\alpha_{2k-1}]}$. Hence, by (2.3), the polynomial to be computed is equal to

$$\widetilde{Q}_I Y_{[\alpha_1+1,\ldots,\alpha_{2k-1}+1]}\,\nabla_{2k-1}^B(n)\,.$$

Since $\alpha_{2k-1}+1 \le n-(2k-1)$, Fact 4 provides the end of the computation with the operator Ω_2.

Note that in the case when we have a contribution from both operators Ω_1 and Ω_2, we also use the equality $2^{p-1} = 2^p - 2^{p-1}$. $\square$

Example 13. (i) For $n=5$, $k=1$, we have

$$\widetilde{P}_{(3,2)}\, Y_{[1,3]}\, \nabla_1^D(5) = \widetilde{P}_2$$

and this comes from the contribution of both operators Ω_1, Ω_2:

$$\widetilde{Q}_{(3,2)}\, Y_{[1,3]}\, x_1x_2 \nabla_2^B(5) = \widetilde{Q}_{(3,2)}\, Y_{[2,4]}\, \nabla_2^B(5) = 4\widetilde{Q}_2$$

by Theorem 9, and

$$\widetilde{Q}_{(3,2)}\, Y_{[1,3]}\, x_1 \nabla_1^B(5)\, \partial_1\partial_2\partial_3 = \widetilde{Q}_{(3,2)}\, Y_{[2]}\, \nabla_1^B(5) = -2\widetilde{Q}_2\,.$$

by Fact 4.

(ii) For $n=7$, $k=2$, we have

$$\widetilde{P}_{(6,5,4,3,2,1,0)}\, Y_{[0,1,2,2]}\, \nabla_2^D(7) = -\widetilde{P}_{(5,3)}$$

and only the operator Ω_1 gives the contribution:

$$\widetilde{Q}_{(6,5,4,3,2,1)}\, Y_{[0,1,2,2]}\, x_1x_2x_3x_4 \nabla_4^B(7) = \widetilde{Q}_{(6,5,4,3,2,1)}\, Y_{[1,2,3,3]}\, \nabla_4^B(7) = -16\widetilde{Q}_{(5,3)}$$

by Fact 4.

(iii) For $n=7$, $k=2$, we have

$$\widetilde{P}_{(6,5,4,3,2,1,0)}\, Y_{[1,1,2,3]}\, \nabla_2^D(7) = -\widetilde{P}_{(6,3,1)}$$

and the contribution comes from both operators Ω_1, Ω_2:

$$\widetilde{Q}_{(6,5,4,3,2,1)}\, Y_{[1,1,2,3]}\, x_1x_2x_3x_4\, \nabla_4^B(7) = \widetilde{Q}_{(6,5,4,3,2,1)}\, Y_{[2,2,3,4]}\, \nabla_4^B(7) = -16\widetilde{Q}_{(6,3,1)}$$

by Theorem 9, and

$$\begin{aligned}\widetilde{Q}_{(6,5,4,3,2,1)}\, Y_{[1,1,2,3]}\, x_1x_2x_3\, \nabla_3^B(7)\, \partial_1\partial_2\partial_3 \\ = \widetilde{Q}_{(6,5,4,3,2,1)}\, Y_{[2,2,3]}\, \nabla_3^B(7) = 8\widetilde{Q}_{(6,3,1)}\,.\end{aligned}$$

by Fact 4.

4. Applications to $\widetilde{P}$-polynomials and orthogonal Schubert polynomials

The following presentation of a $\widetilde{P}$-polynomial in the form

$$\widetilde{P}_I = x^{\alpha(I)}\, \widetilde{P}_{\rho(n-1)}\, \Omega(I)\,,$$

where $\alpha(I) \subseteq \rho$ and $\Omega(I)$ is a divided difference operator, appears to be quite useful:

Lemma 14. *Let $I=(i_1,\dots,i_\ell>0)\subseteq\rho(n-1)$ be a strict partition. If n and ℓ are of the same parity, we set $h:=n-\ell$, and $\{j_1<\cdots<j_h\}:=\{1,\dots,n\}\smallsetminus\{i_1+1,\dots,i_\ell+1\}$. If n and ℓ are of different parity, we set $h:=n-\ell-1$, and $\{j_1<\cdots<j_h\}:=\{1,\dots,n\}\smallsetminus\{i_1+1,\dots,i_\ell+1,1\}$.*

Then for $\alpha(I):=[n-j_1,\dots,n-j_h,0,\dots,0]$ and $k:=h/2$,

$$x^{\alpha(I)}\,\widetilde{P}_{\rho(n-1)}\,\partial_{[2k,\dots,1]}\nabla_k^D(n)=(-1)^s\,\widetilde{P}_I\,, \tag{4.1}$$

where s is the number of positions of the parts erased in $\rho(n-1)$ in order to get the partition I.

The assertion of the lemma is a direct consequence of Theorem 11 and the definition of a Schur S-polynomial via the Jacobi symmetrizer.

Now, with every strict partition $I=(i_1,\dots,i_\ell>0)$, we associate the following element $v(I)\in\mathfrak{D}_n$. If $n-\ell$ is even, we set

$$v(I):=[i_1+1,i_2+1,\dots,i_\ell+1,\overline{j_1},\dots,\overline{j_h}]\,, \tag{4.2}$$

and if $n-\ell$ is odd,

$$v(I):=[i_1+1,i_2+1,\dots,i_\ell+1,1,\overline{j_1},\dots,\overline{j_h}]\,. \tag{4.3}$$

(The notation is the same as in Lemma 14.)

Theorem 15. *For a strict partition $I\subseteq\rho(n-1)$,*

$$x^\rho\widetilde{P}_{\rho(n-1)}\,\partial_{v(I)}=(-1)^{|I|+\binom{n}{2}}\,\widetilde{P}_I. \tag{4.4}$$

The proof of this result is analogous to the proof of [LP1, Thm. A.6]. Using the notation of Lemma 14, we have

$$\partial_{v(I)}=\partial_\sigma\,\partial_{[k,k-1,\dots,1]}\,\nabla_k^D(n)\,, \tag{4.5}$$

where

$$\sigma=\begin{cases}[j_1,\dots,j_h,i_1+1,\dots,i_\ell+1], & n \text{ and } \ell \text{ of the same parity}\\ [j_1,\dots,j_h,i_1+1,\dots,i_\ell+1,1], & n \text{ and } \ell \text{ of different parity}.\end{cases}$$

Note that $x^\rho\,\partial_\sigma=x^{\alpha(I)}$, and hence the assertion follows by Lemma 14.

This result leads to the following characterization of $\widetilde{P}$-polynomials via orthogonal divided differences:

Corollary 16. *For a strict partition $I\subseteq\rho(n-1)$, we set $w(I):=v(I)^{-1}w_0^D$. More explicitly, for even ℓ, $w(I)=[\overline{i_1+1},\dots,\overline{i_\ell+1},j_1,\dots,j_h]^{-1}$, and for odd ℓ, $w(I)=[\overline{i_1+1},\dots,\overline{i_\ell+1},\overline{1},j_1,\dots,j_h]^{-1}$. Then $w=w(I)$ is the unique element of $\mathfrak{D}_n$ such that $\ell(w)=|I|$ and $\widetilde{P}_I\,\partial_w\neq0$. In fact, $\widetilde{P}_I\,\partial_{w(I)}=(-1)^{|I|}$.*

This can be also seen by geometric considerations (see [P1] and [LP1]), with the help of the characteristic map ([B], [D1,D2]).

More generally, consider, for any $w \in \mathfrak{D}_n$, the *orthogonal Schubert polynomial*

$$X_w^D = X_w^D(n) = x^\rho \widetilde{P}_{\rho(n-1)} \partial_{w_0^D w} \tag{4.6}$$

of degree $\ell(w)$. Arguing in the same way as in [LP1, pp.33–36], one shows that these Schubert polynomials have the *stability property* in the sense that for $w \in \mathfrak{D}_n \subset \mathfrak{D}_{n+1}$,

$$X_w^D(n+1)|_{x_{n+1}=0} = X_w^D(n). \tag{4.7}$$

Together with the "maximal Grassmannian property" from Theorem 15, asserting that, for even ℓ,

$$X^D_{[\overline{i_1+1},\dots,\overline{i_\ell+1},j_1,\dots,j_h]} = (-1)^{|I|+\binom{n}{2}} \widetilde{P}_I \,, \tag{4.8}$$

and for odd ℓ,

$$X^D_{[\overline{i_1+1},\dots,\overline{i_\ell+1},\overline{1},j_1,\dots,j_h]} = (-1)^{|I|+\binom{n}{2}} \widetilde{P}_I \,, \tag{4.9}$$

this shows that they provide a natural tool for the cohomological study of Schubert varieties for the orthogonal group $SO(2n)$ and the related degeneracy loci.

We also record

Proposition 17. *For a strict partition $I = (i_1, i_2, i_3, i_4, \dots) \subseteq \rho(n-1)$,*

$$\widetilde{P}_I \, \partial_\heartsuit \partial_2 \cdots \partial_{i_1} \partial_1 \partial_2 \cdots \partial_{i_2} = (-1)^{i_1+i_2} \widetilde{P}_{(i_3,i_4,\dots)}. \tag{4.10}$$

To see this, we argue in a manner similar to the proof of [LP1, Prop. 5.12]. For $J = (i_3, i_4, \dots)$, we choose the presentation from Lemma 14:

$$\pm \widetilde{P}_I = x^{\alpha(I)} \, \widetilde{P}_{\rho(n-1)} \partial_u \quad , \qquad \pm \widetilde{P}_J = x^{\alpha(J)} \, \widetilde{P}_{\rho(n-1)} \partial_v$$

for appropriate $u, v \in \mathfrak{D}_n$. Let $\sigma \in \mathfrak{S}_n$ be the permutation such that

$$v = \sigma \, u \, s_\heartsuit s_2 \cdots s_{i_1} s_1 s_2 \cdots s_{i_2} \, .$$

The assertion now follows from $x^{\alpha(I)} \partial_\sigma = x^{\alpha(J)}$.

Appendix: results in type B

In this appendix we give a summary of the results for type B_n. They are obtained directly from the results for type C_n in [LP1], by changing ∂_0 in loc.cit. to $-2\partial_0$, and read as follows: writing $\nabla := \nabla_n^B(n)$, we have

Theorem 18. *(i) For $\alpha \in \mathbb{N}^n$, $\alpha \subseteq \rho$,*

$$Y_\alpha \, \widetilde{P}_{\rho(n)} \, \nabla = (-1)^{|\alpha|+\binom{n+1}{2}} Y_\alpha^\omega. \tag{5.1}$$

(ii) For strict $I \subsetneq \rho(n)$, and $\alpha \subseteq \rho$,

$$Y_\alpha \, \widetilde{P}_I \, \nabla = 0 \; . \tag{5.2}$$

Denote by

$$\langle\ ,\ \rangle : Sym(n) \times Sym(n) \to Sym^D(n)$$

the scalar product defined for $f, g \in Sym(n)$ by

$$\langle f, g\rangle := fg\,\nabla\,. \tag{5.3}$$

For strict partitions $I, J \subseteq \rho(n)$, one has

$$\langle \widetilde{P}_I, \widetilde{P}_{\rho(n)\smallsetminus J}\rangle = (-1)^{\binom{n+1}{2}}\,\delta_{IJ}, \tag{5.4}$$

where $\rho(n) \smallsetminus J$ is the strict partition whose parts complement the parts of J in $\{n, n-1, \dots, 1\}$ (cf. [PR]).

Consequently, the polynomial ring $\mathcal{P}ol = \mathbb{Z}[\frac{1}{2}][x_1, \dots, x_n]$ is a free $Sym^B(n)$-module with basis $Y_\alpha\,\widetilde{P}_I$, where α ranges over subsequences contained in ρ and I runs over all strict partitions contained in $\rho(n)$. Note that the element of the maximal degree of this basis is $x^\rho \widetilde{P}_{\rho(n)}$. Let

$$[\ ,\ ] : \mathcal{P}ol \times \mathcal{P}ol \to Sym^B(n)$$

be a scalar product, defined for $f, g \in \mathcal{P}ol$ by

$$[f, g] := fg\,\partial_{w_0^B}\,. \tag{5.5}$$

One has, for $\alpha, \beta \subseteq \rho$ and strict partitions $I, J \subset \rho(n)$,

$$\big[Y_\alpha^\omega\,\widetilde{P}_I, Y_{\beta'}\,\widetilde{P}_{\rho(n)\smallsetminus J}\big] = (-1)^{|\alpha| + \binom{n+1}{2}}\,\delta_{\alpha\beta}\,\delta_{IJ}\,. \tag{5.6}$$

(See (2.5).)

Let $\mathcal{Y} = \{y_1, \dots, y_n\}$ be a second set of indeterminates of cardinality n. The symbol $\equiv$ will mean: "congruent modulo the ideal generated by the relations $f(x_1^2, \dots, x_n^2) = f(y_1^2, \dots, y_n^2)$, where $f \in Sym(n)$."

Following Fulton [F2,F3], define

$$F(\mathcal{X}, \mathcal{Y}) := \ |\widetilde{P}_{n+1+j-2i}(\mathcal{X}) + \widetilde{P}_{n+1+j-2i}(\mathcal{Y})|_{1\le i,j\le n}$$

$$= \begin{vmatrix} \widetilde{P}_n(\mathcal{X}) + \widetilde{P}_n(\mathcal{Y}) & 0 & \dots & \\ \widetilde{P}_{n-2}(\mathcal{X}) + \widetilde{P}_{n-2}(\mathcal{Y}) & \widetilde{P}_{n-1}(\mathcal{X}) + \widetilde{P}_{n-1}(\mathcal{Y}) & \dots & \\ \vdots & \vdots & \ddots & \\ & & 1 & \widetilde{P}_1(\mathcal{X}) + \widetilde{P}_1(\mathcal{Y}) \end{vmatrix}. \tag{5.7}$$

Following [PR], define

$$\widetilde{P}(\mathcal{X}, \mathcal{Y}) := \sum \widetilde{P}_I(\mathcal{X})\,\widetilde{P}_{\rho(n)\smallsetminus I}(\mathcal{Y})\ , \tag{5.8}$$

where the summation is over all strict partitions $I \subseteq \rho(n)$. The reasoning in [LP1, Sect.2] made for case C_n adapts to case B_n and furnishes:

Proposition 19. *We have*
(i)

$$F(\mathcal{X},\mathcal{Y}) \equiv \widetilde{P}(\mathcal{X},\mathcal{Y}). \tag{5.9}$$

(ii) For every $w \in \mathfrak{B}_n \setminus \mathfrak{S}_n$,

$$\widetilde{P}(\mathcal{X}^w,\mathcal{X}) = 0\,, \tag{5.10}$$

and for every $w \in \mathfrak{S}_n$,

$$\widetilde{P}(\mathcal{X}^w,\mathcal{X}) = \widetilde{P}(\mathcal{X},\mathcal{X}) = s_{\rho(n)}(\mathcal{X})\,. \tag{5.11}$$

(iii) For every $f \in Sym(n)$,

$$\langle f(\mathcal{X})\,, F(\mathcal{X},\mathcal{Y})\rangle \equiv (-1)^{\binom{n+1}{2}} f(\mathcal{Y})\ . \tag{5.12}$$

(iv) For every $f \in \mathcal{P}ol$,

$$\big[f(\mathcal{X})\,, \prod_{n\ge i>j\ge 1} (x_i - y_j)\, F(\mathcal{X},\mathcal{Y})\,\big] \equiv f(\mathcal{Y})\,. \tag{5.13}$$

In other words, $F(\mathcal{X},\mathcal{Y})$ is a reproducing kernel for the scalar product $\langle\ ,\ \rangle$, and $\prod_{i>j}(x_i - y_j)\, F(\mathcal{X},\mathcal{Y})$ is a reproducing kernel for $[\ ,\ ]$. One can show that the "vanishing property" *(ii)* characterizes $\widetilde{P}(\mathcal{X},\mathcal{Y})$ up to $\equiv$. The congruence *(i)* can be also derived from geometry by comparing the classes of diagonals in flag bundles associated with $SO(2n+1)$ given in [F2,F3] and [PR] (see also [G]).

Proposition 20. *Suppose $n \ge p > 0$. Let $IpJ \subseteq \rho(n)$ be a strict partition and $H \subseteq \rho(n)$ a strict partition not containing p. Then*

$$x_1^{n-p}\,\widetilde{P}_{IpJ}\,\partial_0\partial_1\cdots\partial_{n-1} = (-1)^{\ell(I)+n}\,\widetilde{P}_{IJ} \tag{5.14}$$

and

$$x_1^{n-p}\,\widetilde{P}_{H}\,\partial_0\partial_1\cdots\partial_{n-1} = 0\,. \tag{5.15}$$

More generally,

Theorem 21. *Let $0 < k \le n$ and let $\alpha = [\alpha_1 \le \cdots \le \alpha_k] \in \mathbb{N}^k$ be such that $\alpha_k \le n-k$. Suppose that $I \subseteq \rho(n)$ is a strict partition. Then the image of $\widetilde{P}_I\, Y_\alpha$ under $\nabla_k^B(n)$ is 0 unless $n-\alpha_1-0,\ldots,n-\alpha_k-(k-1)$ are parts of I. In this case, the image is $(-1)^{k(n-1)+s}\widetilde{P}_J$, where J is the strict partition with parts*

$$\{i_1,\ldots,i_{\ell(I)}\} \setminus \{n-\alpha_1-0,\ldots,n-\alpha_k-(k-1)\},$$

and s is the sum of positions of the parts erased in I.

(This is a restatement of [LP1, Prop. 5.9].)

Proposition 22. *For a strict partition $I = (i_1, i_2, \dots)$,*

$$\widetilde{P}_I\, \partial_0 \partial_1 \cdots \partial_{i_1-1} = (-1)^{i_1} \widetilde{P}_{(i_2,\dots)}\,. \tag{5.16}$$

Now, let us associate with every strict partition $I = (i_1, \dots, i_\ell > 0)$ the following element of $\mathfrak{B}_n$:

$$v(I) := [i_1, \dots, i_\ell, \overline{j_1}, \dots, \overline{j_h}]\,, \tag{5.17}$$

where $j_1 < \cdots < j_h$.

Theorem 23. *For every strict partition $I \subseteq \rho(n)$,*

$$x^\rho \widetilde{P}_{\rho(n)}\, \partial_{v(I)} = (-1)^{|I|+\binom{n+1}{2}} \widetilde{P}_I. \tag{5.18}$$

This leads to the following characterization of $\widetilde{P}$-polynomials via divided differences:

Corollary 24. *For any strict partition I, let $w(I) := v(I)^{-1} w_0^B$, that is $w(I) = [\overline{i_1}, \dots, \overline{i_\ell}, j_1, \dots, j_h]^{-1}$. Then $w = w(I)$ is the unique element of $\mathfrak{B}_n$ such that $\ell(w) = |I|$ and $\widetilde{P}_I\, \partial_w \neq 0$. In fact, $\widetilde{P}_I\, \partial_{w(I)} = (-1)^{|I|}$.*

This can be also seen by geometric considerations (see [P1] and [LP1]), with the help of the characteristic map ([B], [D1,D2]).

More generally, consider, for any $w \in \mathfrak{B}_n$, the *orthogonal Schubert polynomial*

$$X_w^B = X_w^B(n) = x^\rho \widetilde{P}_{\rho(n)} \partial_{w_0^B w} \tag{5.19}$$

of degree $\ell(w)$. Arguing in the same way as in [LP1, pp.33–36], one shows that these Schubert polynomials have the *stability property* in the sense that for $w \in \mathfrak{B}_n \subset \mathfrak{B}_{n+1}$,

$$X_w^B(n+1)|_{x_{n+1}=0} = X_w^B(n). \tag{5.20}$$

Together with the "maximal Grassmannian property" from Theorem 23, asserting that

$$X^B_{[\overline{i_1},\dots,\overline{i_\ell},j_1,\dots,j_h]} = (-1)^{|I|+\binom{n+1}{2}} \widetilde{P}_I\,, \tag{5.21}$$

this shows that they provide a natural tool for the cohomological study of Schubert varieties for the orthogonal group $SO(2n+1)$ and the related degeneracy loci.

We now give

Proof of Theorem 9

Given a symmetric function f, let D_f be the Foulkes derivative i.e. the adjoint operator to the multiplication by f w.r.t. the standard scalar product on the ring *Sym* of symmetric functions in a countable number of variables (cf. [M1]). We use the following vertex operators on *Sym*:

$$U^s := 1 - D_{P_1} s_1 + D_{P_2} s_2 - \cdots\,, \tag{5.22}$$

$$U^e := 1 - D_{P_1} e_1 + D_{P_2} e_2 - \cdots\,, \tag{5.23}$$

and

$$V^e := 1 - D_{e_1} P_1 + D_{e_2} P_2 - \cdots\,. \tag{5.24}$$

We refer to [LP1, p.24] for the definitions of Schur P-functions P_I [S]. In loc. cit. the reader can also find a definition of Q'-functions Q'_I [LLT2], used in the following proposition:

Proposition 25. *Let I be a strict partition. We have the following identities of symmetric functions in Sym:*

$$\widetilde{Q}_I\, U^s = \begin{cases} \widetilde{Q}_I, & \ell(I) \quad \text{even} \\ 0, & \ell(I) \quad \text{odd}; \end{cases} \tag{5.25}$$

$$Q'_I\, U^e = \begin{cases} Q'_I, & \ell(I) \quad \text{even} \\ 0, & \ell(I) \quad \text{odd}; \end{cases} \tag{5.26}$$

and

$$P_I\, V^e = \begin{cases} P_I, & \ell(I) \quad \text{even} \\ 0, & \ell(I) \quad \text{odd}. \end{cases} \tag{5.27}$$

Proof. First of all, arguing as in [LP1, pp.24–27], with the help of the operators V^e, U^s, and U^e instead of V^e_k, U^s_k, and U^e_k in loc. cit., we note that the equalities (5.25), (5.26), and (5.27) are equivalent.

We show (5.27). It suffices to prove the statement when the set of indeterminates $\{x_1, \dots, x_n\}$ is of finite cardinal $n > |I|$.

Besides the well-known equality: for $k > 0$,

$$x_1^k \prod_{2 \le i \le n} (x_1 + x_i)\, \partial_1 \partial_2 \cdots \partial_{n-1} = P_k(x_1, \dots, x_n)\,, \tag{5.28}$$

we also need the following formula from [P2]:

Fact 26. *For a strict partition I,*

$$\begin{aligned} &P_I(x_2, \dots, x_n) \prod_{2 \le i \le n} (x_1 + x_i)\, \partial_1 \partial_2 \cdots \partial_{n-1} \\ &\qquad = \begin{cases} (-1)^{n-1} P_I(x_1, \dots, x_n), & n - \ell(I) \quad \text{odd} \\ 0, & n - \ell(I) \quad \text{even.} \end{cases} \end{aligned} \tag{5.29}$$

(More precisely, Eq.(5.29) is a special case of the following formula given in [P2, Prop.1.3(ii)]: Let q, r, k, and h be integers such that $0 < q < n$, $n = q + r$, $0 \le k \le q$, and $0 \le h \le r$. Suppose $I = (i_1, \dots, i_k) \in \mathbb{N}^{*k}$ and $J = (j_1, \dots, j_h) \in \mathbb{N}^{*h}$. Then

$$P_I(x_1, \dots, x_q)\, P_J(x_{q+1}, \dots, x_n) \prod_{1 \le i \le q < j \le n} (x_i + x_j) \begin{matrix} \partial_q & \cdots & \partial_{n-1} \\ \vdots & & \vdots \\ \partial_2 & \cdots & \partial_{r+1} \\ \partial_1 & \cdots & \partial_r \end{matrix}$$

$$= d \cdot P_{(i_1, \dots, i_k, j_1, \dots, j_h)}(x_1, \dots, x_n)\,, \tag{5.30}$$

where d is zero if $(q - k)(r - h)$ is odd and

$$d = (-1)^{(q-k)r} \binom{\lfloor (n - k - h)/2 \rfloor}{\lfloor (q - k)/2 \rfloor} \tag{5.31}$$

otherwise.

We get Eq.(5.29) as Eq.(5.30) specialized to $q=1$ and $k=0$.)

To end the proof of (5.27), we first write

$$P_I(x_2,\dots,x_n) = P_I - P_I D_{e_1}\cdot x_1 + P_I D_{e_2}\cdot x_1^2 - P_I D_{e_3}\cdot x_1^3 + \cdots\,, \tag{5.32}$$

where the RHS is evaluated in the first n variables. Then we multiply both sides of (5.32) by

$$(x_1+x_2)(x_1+x_3)\cdots(x_1+x_n)$$

and apply the operator $\partial_1\partial_2\cdots\partial_{n-1}$. We get the following equalities of symmetric polynomials in the first n variables. If n is odd, the RHS of the so-obtained equality becomes

$$P_I - P_I D_{e_1}\cdot P_1 + P_I D_{e_2}\cdot P_2 - P_I D_{e_3}\cdot P_3 + \cdots$$

by (5.28) and (5.29), and its LHS is equal to

$$\begin{cases} P_I, & \ell(I) \ \text{even} \\ 0, & \ell(I) \ \text{odd} \end{cases}$$

by (5.29). This shows (5.27) for odd n. If n is even, the RHS of the obtained equality becomes

$$0 - P_I D_{e_1}\cdot P_1 + P_I D_{e_2}\cdot P_2 - P_I D_{e_3}\cdot P_3 + \cdots$$

by (5.28) and (5.29), and its LHS is equal to

$$\begin{cases} 0, & \ell(I) \ \text{even} \\ -P_I, & \ell(I) \ \text{odd} \end{cases}$$

by (5.29). This shows (5.27) for even n.

Thus the proposition has been proved. □

Let now ∂_0^C be the divided difference defined by

$$\mathcal{P}ol \ni f \mapsto f\,\partial_0^C := (f - f^{s_0})/(2x_1)\ . \tag{5.33}$$

Arguing similarly as in [LP1, pp.27–28], using

$$\partial_0^C = D_{P_1} - D_{P_2}\,x_1 + D_{P_3}\,x_1^2 - \cdots\,, \tag{5.34}$$

and the formula

$$x_1^p\,\partial_1\cdots\partial_{n-1} = s_{p-n+1}(x_1,\dots,x_n)\,, \tag{5.35}$$

one shows

Lemma 27. *As operators on Sym, evaluated in symmetric polynomials in the first n variables,*

$$1 - U^s = \partial_0^C \, x_1^n \, \partial_1 \cdots \partial_{n-1} \,. \tag{5.36}$$

Equations (5.25) and (5.36), together with the equalities

$$x_1^{2m} \, \partial_0^C = \partial_0^C \, x_1^{2m} \quad \text{and} \quad x_1^{2m+1} \, \partial_0^C = -\partial_0^C \, x_1^{2m+1} + x_1^{2m} \,, \tag{5.37}$$

applied for even $n = 2m$ or odd $n = 2m+1$ accordingly, imply

Proposition 28. *Let $I \subseteq \rho(n)$ be a strict partition. We have*

$$\widetilde{Q}_I \, x_1^n \, \partial_0 \partial_1 \cdots \partial_{n-1} = \begin{cases} -2\widetilde{Q}_I, & n + \ell(I) \quad \textit{odd} \\ 0, & n + \ell(I) \quad \textit{even}. \end{cases} \tag{5.38}$$

Eq.(5.38) is the content of Theorem 9 for $k = 1$. For higher k, one gets the desired assertion by [LP1, Thm 5.1], Proposition 28, and [LP1, Lemma 5.10]. (Note that this last fact holds true for any nonnegative integer α_1, in the notation of loc. cit., as is clear from its proof.)

This ends the proof of Theorem 9.

Finally, we take this opportunity to correct some misprints in [LP1]: – should read:

p.11_{11} "... a partition ..."

p.13_2 " $\langle\ ,\ \rangle : \mathcal{SP}(X) \times \mathcal{SP}(X) \to \mathcal{SP}(X^2)$ "

p.36_2 "... $\partial'_u \mathcal{C}_w = \mathcal{C}_v$ ..."

p.37^4 "... is $\nabla_k \circ \partial'_{\omega(k)}$ ..."

p.37_{10} "... $\partial'_{w_I}(\widetilde{Q}_I(X)) = 1$ ... "

Moreover, in Example 5.11, the sequence of successive signs is: $+, +, -, -$.

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