SCHUR Q-FUNCTIONS AND DEGENERACY LOCUS FORMULAS FOR MORPHISMS WITH SYMMETRIES

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Abstract. We give closed-form formulas for the fundamental classes of degeneracy loci associated with vector bundle maps given locally by (not necessary square) matrices which are symmetric (resp. skew-symmetric) w.r.t. the main diagonal. Our description uses essentially Schur Q-polynomials of a bundle, and is based on a push-forward formula for these polynomials in a Grassmann bundle, established in [P4].

"Something which is not testable is not scientific."
Poper's criterium

Introduction

The goal of the present paper is to state and prove new closed-form formulas for the fundamental classes of some degeneracy loci.

We will be here interested in degeneracy loci associated with vector bundle maps given locally by (not necessary square) matrices that are symmetric (resp. skew-symmetric) w.r.t. the main diagonal. To be more precise, let $\alpha : E \twoheadrightarrow F$ be a surjection of two vector bundles of respective ranks e and f on a variety X. We denote by $E \vee F$ (resp. $E \wedge F$) the kernel of the surjection

$$E \otimes F \xrightarrow{\alpha\otimes 1} F \otimes F \twoheadrightarrow \wedge^2 F$$

(resp.

$$E \otimes F \xrightarrow{\alpha\otimes 1} F \otimes F \twoheadrightarrow S^2 F).$$

In the present paper, we will call a morphism $\varphi : E^* \to F$ *symmetric* (resp. *skew-symmetric*) provided it is induced by a section of the subbundle $E \vee F$ (resp. $E \wedge F$) of $E \otimes F$. (Note that for $E = F$ these notions coincide with the usual notions of "symmetric" and "skew-symmetric" morphisms.)

Our goal is to describe the fundamental classes of the loci

$$D_r(\varphi) = \{x \in X : \text{rank } \varphi(x) \leqslant r\}$$

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with the help of some explicitly given polynomials in the Chern classes of E and F. Set $n = e - f$. More precisely, using the terminology of [F-P], the degeneracy locus $D_r(\varphi)$ is represented, in the symmetric case, by the polynomial

$$(*) \qquad \sum_I Q_{(f-r,f-r-1,\dots,1)+I}(F) \cdot s_{\mathrm{C}\,\widetilde{I}}(E-F),$$

where the sum is over all partitions I in $(n)^{f-r}$, and for the conjugate partition $\widetilde{I} \subset (f-r)^n$ we write $\mathrm{C}\,\widetilde{I} = (f-r-\widetilde{i}_n, \dots, f-r-\widetilde{i}_1)$ for the "complement" of $\widetilde{I}$ in $(f-r)^n$. Here, $Q_J(\bullet)$ are Schur Q-polynomials whose definition is recalled in Section 1. The connection of Schur Q-polynomials to geometry was originally established in [P2] in two contexts. Firstly, in [P2, Sect.7], the ideal of polynomials universally supported on the rth symmetric degeneracy locus was described in terms of Schur Q-polynomials. This is closely connected with the present paper, see below. Secondly, in [P2, (8.7)] the connection of Schur Q-functions to the cohomology rings of isotropic Grassmannians was given. (This subject was then developed in [P3, Sect.6].) Consult also [P4] and [F-P] for more about geometric properties of Q-polynomials, and especially for their applications to intersection theory and enumerative geometry.

By virtue of [P2, Sect.7], the form of polynomials (*) representing $[D_r(\varphi)]$ is not at all surprising. Observe that for $\varphi' = \varphi \circ \alpha^* : F^* \to E^* \to F$, we have $D_r(\varphi) \subset D_r(\varphi')$, so the polynomials representing $D_r(\varphi)$ are universally supported on the rth (symmetric) degeneracy locus associated with symmetric bundle maps from rank f vector bundles to theirs duals, in the sense of [P2, Sect.7] (see also [F-P, Sect. 4.4]). Therefore, by [P2, Theorem 7.2 and Proposition 7.17], the polynomial representing $[D_r(\varphi)]$ is a $\mathbb{Z}[c.(E), c.(F)]$- combination of $Q_{(f-r,f-r-1,\dots,1)+I}(F)$ for I contained in $(r)^{f-r}$. For $n \leqslant r$, this is exactly the combination (*). For $n > r$, this suggests that at the cost of replacing in (*) the coefficients from $\mathbb{Z}[c.(E-F)]$ by the coefficients from a larger ring $\mathbb{Z}[c.(E), c.(F)]$, one can make the sum (*) smaller. We also note that to get (*), we use exactly the same instance of the Gysin map formula, recalled in (1.8) below, as to get the just mentioned results from [P2, Sect.7].

Similar formulas and discussion hold true in the skew-symmetric case (see Section 3).

When $E = F$, our formulas specialize to the ones given by Józefiak and the authors in [J-L-P], and Harris and Tu in [H-T] (see also [P1]). Recall that all the three last mentioned papers gave a "modern treatment" à la "Thom-Porteous" of the formulas for the degree of determinantal varieties in the spaces of symmetric and skew-symmetric matrices of forms, given classically by Giambelli in [G]. Some special cases of the formulas from [H-T], [J-L-P] and [P1] were given independently by Barth [Ba], Damon and Tyurin.

When $E = \bigoplus_{i=1}^{e} \mathcal{O}(p_i)$, $F = \bigoplus_{i=1}^{f} \mathcal{O}(p_i)$ are vector bundles over a projective space, our formulas give the degrees of determinantal varieties defined by matrices of forms, satysfying the above symmetry conditions. This problem was considered, in the symmetric case, by Bottasso in [Bo], generalizing Giambelli's study to the case of not necessary square matrices. Bottaso did not give closed-form formulas for these degrees, but established some recursions for their computations. The present paper offers a modern treatment and closed-form version of [Bo].

There are several motivations to do such computations in intersection theory. These computations combine intersection theory on homogeneous spaces (notably Grassmannians and flag varieties) and algebra of different types of symmetric functions. The obtained formulas have applications to the enumerative theory of singularities, Brill-Noether theory etc. Again, we refer the reader to [P4], and to [F-P] for more information on these matters.

We give in the present paper formulas for the "Thom-Porteous" case. It would be interesting to generalize them to the "quiver variety" case, similarly to the work of Buch and Fulton [B-F].

Since the degeneracy loci studied are intimately connected with the Schubert varieties in symplectic and orthogonal Grassmannians (see Section 1), the formulas given in the present paper should be gotten "in theory" from the divided difference operator approach developed in [F2-3] from one side, and in [P-R] and [L-P] from another one. We do not see, however, how to do it conceptually using divided differences.

The article is organized as follows.

In Section 1, we recall the definitions and properties of two families of symmetric polynomials that we need in the present paper: Schur polynomials in a difference of bundles and Schur Q-polynomials. We also recall a formula for Gysin push-forward of Schur Q-polynomials in a Grassmann bundle from [P4], which is basic to the present paper. Usually, we do not state results in their full generality, but only in the range needed in the present paper. Moreover, we give some preliminary properties and examples of the degeneracy loci associated with symmetric and skew-symmetric morphisms in the above generalized sense.

In Section 2, we give some formulas for the top Chern classes of the bundles $E \vee F$ and $E \wedge F$, generalizing the formulas from [L].

In Section 3, we prove the main formula of the present paper. The proof follows a well-known pattern: it uses essentially a certain desingularization of $D_r(\varphi)$ for a "universal" φ and the above mentioned push-forward formula for Schur Q-polynomials. Some examples are also discussed.

In Section 4, we discuss some variations of computation of the polynomials representing $D_r(\varphi)$, based on the technique of "constructions with a nontrivial generic fibre" invented in [P1, Sect.2]. This material is recalled in Proposition 4.1, where, in fact, a certain straightening of this last-mentioned method is presented. As an application, we get a some algebraic equations involving Gysin maps on their LHS's and some closed-form expressions involving symmetric polynomials on their RHS's.

The present paper corrects some errors of our former preprint: "Degeneracy loci formulas for morphisms with symmetries".

1. Preliminaries: recollection of some definitions and results

In the present paper, we will need two families of symmetric polynomials. Here come their definitions and useful properties. Let $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_m)$ be two sequences of commuting elements in a ring.

(1.1) Let $I = (i_1, \dots, i_k)$ be a partition. Recall that by a *partition* (of some natural number) we understand a sequence of integers $I = (i_1, \dots, i_k)$, where $i_1 \geqslant i_2 \geqslant$

$\ldots \geqslant i_k \geqslant 0$ and the sum $\sum i_p$ is the partitioned number. We define

$$s_I(A-B) = \det\Big[s_{i_p-p+q}(A-B)\Big]_{1\leqslant p,q\leqslant k},$$

where $s_i(A-B)$ is defined by

$$\sum_{i=-\infty}^{\infty} s_i(A-B) = \prod_{p=1}^{n}(1-a_p)^{-1} \prod_{q=1}^{m}(1-b_q).$$

Moreover, we put $s_I(A) = s_I(A-B)$ for $B = (0,\ldots,0)$. When A and B are sequences of variables, then the polynomials $s_I(A)$ are called *Schur polynomials* or *S-polynomials*, and the polynomials $s_I(A-B)$ are often called *Schur polynomials in a difference of alphabets*. For example, the "resultant" can be expressed in a closed form as

(1.1.1) $$\prod_{i,j}(a_i - b_j) = s_{(m,\ldots,m)}(A-B),$$

where the part m appears n times.

(1.2) Let $Q_i(A)$ be defined by the expansion

$$\sum_{i=-\infty}^{\infty} Q_i(A) = \prod_{p=1}^{n}(1+a_p)(1-a_p)^{-1}.$$

Given positive integers $i > j$, we set

$$Q_{(i,j)}(A) = Q_i(A)\cdot Q_j(A) + 2\sum_{p=1}^{j}(-1)^p Q_{i+p}(A)\cdot Q_{j-p}(A).$$

Finally, if $I = (i_1 > \ldots > i_k > 0)$ is a *strict* partition, then for odd k we put

$$Q_I(A) = \sum_{p=1}^{k}(-1)^{p-1} Q_{i_p}(A)\cdot Q_{(i_1,\ldots,i_{p-1},i_{p+1},\ldots,i_k)}(A),$$

and for even k,

$$Q_I(A) = \sum_{p=2}^{k}(-1)^{p} Q_{(i_1,i_p)}(A)\cdot Q_{(i_2,\ldots,i_{p-1},i_{p+1},\ldots,i_k)}(A).$$

These define the $Q_I(A)$'s by recurrence on k. When A is a sequence of variables, then the polynomials $Q_I(A)$ are called *Schur Q-polynomials* after [Sch]. For example, one has

(1.2.1) $$Q_{(n,n-1,\ldots,1)}(A) = \prod_{i\leqslant j}(a_i + a_j) = 2^n s_{(n,n-1,\ldots,1)}(A).$$

(1.3) Let E and F be two vector bundles. Then $s_I(E-F)$ is defined to be $s_I(A-B)$, where A and B are the sequences of the Chern roots of E and F respectively. Similarly, set $s_I(E)=s_I(A)$, and $Q_I(E)=Q_I(A)$.

(1.4) For partitions I and J, we write $I\supset J$ if $i_1\geqslant j_1$, $i_2\geqslant j_2$, ... Moreover, we denote by $I+J$ the partition $(i_1+j_1, i_2+j_2,\dots)$.

The "rectangular" partition $(i,\dots,i)$ (r-times) is denoted by $(i)^r$ and the "triangular" partition $(k,k-1,\dots,2,1)$ is denoted by ρ_k.

For a given partition I, $l(I)=\operatorname{card}\{p: i_p>0\}$ denotes its length and $\widetilde{I}$ denotes the partition conjugate of I, i.e., $\widetilde{I}=(h_1,h_2,\dots)$ where $h_p=\operatorname{card}\{q: i_q\geqslant p\}$.

As is common, we will often omit brackets writing partitions in lower indices.

(1.5) Recall that for every strict partition $I=(i_1>\cdots>i_k>0)$ and A a sequence of variables, one has

$$Q_I(A)=2^kP_I(A)$$

for some polynomial $P_I(A)$ with integer coefficients. These polynomials are called *Schur P-polynomials.* For example,

$$P_{\rho_{n-1}}(A)=\prod_{i<j}(a_i+a_j)=s_{\rho_{n-1}}(A).$$

Given a vector bundle E, we set $P_I(E)=P_I(A)$, where A is specialized to the sequence of the Chern roots of E. For another expression of $P_I(A)$ in the form of a quadratic polynomial in the $s_J(A)$'s, see [L-L-T].

(1.6) We recall the following two *factorization formulas.* Let I be a partition such that $l(I)\leqslant n$. Then

$$s_{(m)^n+I}(A-B)=s_{(m)^n}(A-B)\cdot s_I(A) \tag{1.6.1}$$

and

$$Q_{\rho_{n-1}+I}(A)=Q_{\rho_{n-1}}(A)\cdot s_I(A). \tag{1.6.2}$$

We refer to [B-R] [L-S] and [St] (see also [M], [P3] and [L-L-T]) for more about these formulas. In particular, the proof of (1.6.1) given in [L-S] is valid for any elements A and B of ranks n and m in a λ-ring.

1.7 We recall the following formulas for the top Chern classes of some tensor operations. We write, in the present paper, $c_{\mathrm{top}}(E)$ for the top Chern class of a bundle E. Let E and F be two vector bundles of ranks e and f respectively. We have

$$c_{\mathrm{top}}(E\otimes F)=\sum_I s_I(E)\cdot s_{\mathrm{C}\,\widetilde{I}}(F), \tag{1.7.1}$$

where the sum is over all partitions in $(f)^e$ and for the conjugate partition $\widetilde{I}\subset(e)^f$, we write $\mathrm{C}\,\widetilde{I}=(e-\widetilde{i}_f,\dots,e-\widetilde{i}_1)$ for the "complement" of $\widetilde{I}$ in $(e)^f$. Also,

$$c_{\mathrm{top}}(S^2E)=2^e s_{\rho_e}(E)=Q_{\rho_e}(E), \tag{1.7.2}$$

and

$$c_{\text{top}}(\wedge^2 E) = s_{\rho_{e-1}}(E) = P_{\rho_{e-1}}(E). \tag{1.7.3}$$

We refer to [L] (see also [M, p.47-48 and p.67]) for more details.

1.8 We recall the following push-forward formula from [P4]. Let $\pi : G = G^q(E) \to X$ be the Grassmann bundle parametrizing q-quotients of a vector bundle E on a variety X. Write $r = e - q$. Let

$$0 \to R \to E_G \to Q \to 0$$

be the tautological sequence on G with $\operatorname{rank} R = r$ and $\operatorname{rank} Q = q$. Let $I = (i_1 > \ldots > i_k > 0)$ be a strict partition with $k \leqslant q$. Then for $\alpha \in A_*(X)$, we have

$$\pi_*\Big[c_{\text{top}}(R \otimes Q) \cdot P_I(Q) \cap \pi^*\alpha\Big] = d \cdot P_I(E) \cap \alpha,$$

where d is zero if $(q-k)r$ is odd, and

$$d = \binom{[(e-k)/2]}{[(q-k)/2]}$$

in the opposite case. Here, the symbol $[\ \bullet\ ]$ means the integer part of a rational number. For a proof, we refer to [P4, App.1].

We will need three special instances of this formula. First, suppose that I is a strict partition with $l(I) = q$. Then

$$\pi_*\Big[c_{\text{top}}(R \otimes Q) \cdot Q_I(Q) \cap \pi^*\alpha\Big] = Q_I(E) \cap \alpha. \tag{1.8.1}$$

Secondly, assume that I is a strict partition with $l(I) = q - 1$ and r is even. Then

$$\pi_*\Big[c_{\text{top}}(R \otimes Q) \cdot P_I(Q) \cap \pi^*\alpha\Big] = P_I(E) \cap \alpha. \tag{1.8.2}$$

Thirdly, suppose that I is a strict partition with $l(I) = q - 1$ and r is odd. Then

$$\pi_*\Big[c_{\text{top}}(R \otimes Q) \cdot P_I(Q) \cap \pi^*\alpha\Big] = 0. \tag{1.8.3}$$

1.9 Recall that the Gysin map in a Grassmann bundle admits the following explicit description. Let $(a_1, \ldots, a_q)$ be the sequence of the Chern roots of Q and $(a_{q+1}, \ldots, a_e)$ be the sequence of the Chern roots of R. Then, writing $A = (a_1, \ldots, a_e)$ for the sequence of the Chern roots of E, the Gysin map in question is induced by the following symmetrizing operator. Let S_e be the group of permutations of $(1, \ldots, e)$, S_q the group of permutations of $(1, \ldots, q)$, and S_r the group of permutations of $(q+1, \ldots, e)$. For $P \in \mathbb{Z}[A]^{S_q \times S_r}$, the symmetrizing operator in question acts as follows:

$$P \longmapsto \sum_{\overline{\sigma} \in S_e / S_q \times S_r} \sigma\left(P \Big/ \prod_{i \leqslant q < j} (a_i - a_j)\right).$$

For more on this, see, e.g., [P4, Sect.4].

1.10 We now switch to the setup of the Introduction. In the present paper, to be on the safe side, we assume that the ground field k is algebraically closed of characteristic different from 2. This is because in Section 1 and 4 we make use of isotropic symplectic and orthogonal Grassmannians. Let $U \twoheadrightarrow V$ be vector spaces of dimensions e and f, respectively. Let X denote the affine space $U \vee V$ (resp. $U \wedge V$). In this situation, there exists a tautological morphism $\varphi : (E = U_X)^* \to (F = V_X)$. For this φ, $D_r(\varphi)$ is the restriction to an appropriate open set of some Schubert variety in the symplectic (resp. orthogonal) Grassmannian of f-dimensional isotropic subspaces in k^{2e}. More precisely, let $\beta : U^* \to V$ be a linear map and $\gamma : V^* \to U$ its dual. Consider $U^* \oplus U$ equipped with canonical nondegenerate bilinear forms

$$\langle (v_1, u_1), (v_2, u_2) \rangle = v_1(u_2) \pm v_2(u_1) \qquad u_i \in U, v_i \in U^*,$$

with the + sign giving a symmetric form and - sign a skew-symmetric form. The assignment $\beta \mapsto \mathrm{graph}(\gamma)$ embeds $U \vee V$ (resp. $U \wedge V$) as an open subset A of the Grassmannian G of f-dimensional isotropic subspaces of $U^* \oplus U$ w.r.t. the just defined skew-symmetric (resp. symmetric) form on $U^* \oplus U$. Then $D_r(\varphi)$ is the restriction to A of the "determinantal" Schubert subvariety of G parametrizing those f-dimensional isotropic subspaces of $U^* \oplus U$, which intersect the maximal isotropic subspace $U^* \oplus 0$ in dimension $\geqslant f - r$. (When φ is skew-symmetric and $e = f$, we assume r to be even.) Hence $D_r(\varphi)$ is irreducible, normal and Cohen-Macaulay (by results of De Concini and Lakshmibai [DC-L]); moreover its codimension $c(r)$ equals

$$(e-f)(f-r) + (f-r)(f-r+1)/2 \quad (\text{resp. } (e-f)(f-r) + (f-r)(f-r-1)/2.)$$

Perhaps the easiest way to remember this number is to set $n = e - f$ and $q = f - r$ (we will keep this notation throughout the rest of the paper), and note that $c(r)$ equals

$$nq + q(q+1)/2 = q(2n+q+1)/2 \quad (\text{resp. } nq + q(q-1)/2 = q(2n+q-1)/2.)$$

In general, for $\varphi : E^* \to F$ as in the Introduction, the scheme structure on $D_r(\varphi)$ is defined as follows. Set $\tilde{X} = \mathrm{Spec}\ S^\bullet(E \vee F)^*$ (resp. $\tilde{X} = \mathrm{Spec}\ S^\bullet(E \wedge F)^*$). Observe that φ induces a section $s : X \to \tilde{X}$. There exists the tautological bundle homomorphism $\tilde{\varphi} : \tilde{E}^* \to \tilde{F}$ where $\tilde{E} = E_{\tilde{X}}$, $\tilde{F} = F_{\tilde{X}}$ such that $s^*(\tilde{\varphi}) = \varphi$. Then, defining first the scheme structure on $D_r(\tilde{\varphi})$ as above with the help of an appropriate Schubert bundle in an isotropic Grassmann bundle, we define the scheme structure on $D_r(\varphi)$ as the schematic preimage via s of the one on $D_r(\tilde{\varphi})$.

To the best of our knowledge, the above determinantal varieties have not been studied algebraically like the "ordinary" determinantal varieties were studied by Eagon and Hochster or using the Hodge algebra technique. Probably such a study would allow us to formulate the main results of the present paper with less restrictive assumptions on the ground field.

1.11 We pass now to some examples.

(1.11.1) Let $\varphi : E^* \to F$ be symmetric, where $e \geqslant f$ are arbitrary and $r = f - 1$. Then the expected codimension of $D_r(\varphi)$ is $e - f + 1$, so we are in the situation of the

Giambelli-Thom-Porteous formula for the locus defined by maximal minors. We infer that $D_r(\varphi)$ is represented by $s_{e-f+1}(F-E^*)$. For example, for $e=4, f=3$, writing formally by the splitting principle $c(E)=c(F)(1+d)$, we get

$$\begin{aligned}
s_2(F-E^*) &= s_2(F) - s_1(F)s_1(E^*) + s_{1,1}(E^*) = s_2(F) + s_1(F)s_1(E) + s_{1,1}(E) \\
&= s_2(F) + s_1(F)\big(s_1(F)+d\big) + s_{1,1}(F) + s_1(F)d = 2\big(s_2(F)+s_{1,1}(F)+s_1(F)d\big) \\
&= 2\big(s_2(F)+s_{1,1}(E)\big) = 2s_1(E)s_1(F).
\end{aligned}$$

For $e=5, f=3$, we get the representing polynomial

$$s_3(F-E^*) = 2\big(s_1(E)s_2(F)+s_{1,1,1}(E)\big) = 2\big(s_3(F)+s_{1,1}(E)s_1(F)\big).$$

(1.11.2) In this example, X is a projective space, $F=\mathcal{O}(a)\oplus\mathcal{O}(b)\oplus\mathcal{O}(c)$ and $r=2$.

Assume first that $E=\mathcal{O}(a)\oplus\mathcal{O}(b)\oplus\mathcal{O}(c)\oplus\mathcal{O}(d)$ and $\varphi: E^*\to F$ is given by a matrix of forms

$$\begin{vmatrix} 0 & a_{12} & a_{13} & x \\ -a_{12} & 0 & a_{23} & y \\ -a_{13} & -a_{23} & 0 & z \end{vmatrix}.$$

Then $D_2(\varphi)$ is of codimension 1, defined by

$$a_{12}z - a_{13}y + a_{23}x = 0.$$

The degree of this equation is $a+b+c+d = s_1(E)$.

Assume now that $E=\mathcal{O}(a)\oplus\mathcal{O}(b)\oplus\mathcal{O}(c)\oplus\mathcal{O}(d)\oplus\mathcal{O}(e)$ and $\varphi: E^*\to F$ is given by a matrix of forms

$$\begin{vmatrix} 0 & a_{12} & a_{13} & x & w \\ -a_{12} & 0 & a_{23} & y & v \\ -a_{13} & -a_{23} & 0 & z & u \end{vmatrix}.$$

Then $D_2(\varphi)$ is of codimension 2, defined by

$$a_{12}z - a_{13}y + a_{23}x = 0 \quad \text{and} \quad a_{12}u - a_{13}v + a_{23}w = 0.$$

By Bézout's theorem, the degree of $D_2(\varphi)$ is $(a+b+c+d)(a+b+c+e) = s_2(F)+s_{1,1}(E)$.

(1.11.3) Let $\varphi: E^*\to F$ be skew-symmetric, $n=e-f=1$ and r is an arbitrary even nonnegative number less than f. Assume that φ is given locally by a matrix

$$\begin{vmatrix} 0 & a_{12} & \dots & a_{1f} & b_1 \\ -a_{12} & 0 & \dots & a_{2f} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{1f} & -a_{2f} & \dots & 0 & b_f \end{vmatrix}.$$

Then $D_r(\varphi)$ is locally defined by the $(r+2)$-Pfaffians of the extended $e \times e$ skew-symmetric matrix

$$\left|\begin{array}{ccccc} 0 & a_{12} & \ldots & a_{1f} & b_1 \\ -a_{12} & 0 & \ldots & a_{2f} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{1f} & -a_{2f} & \ldots & 0 & b_f \\ -b_1 & -b_2 & \ldots & -b_f & 0 \end{array}\right|.$$

This is seen by the following "universal local study". Assume that the a_{ij}'s and the b_p's are variables over k. Let $\mathcal{J}$ be the ideal generated by $(r+1)$-minors of the former matrix, and let $\mathcal{I}$ (resp. $\mathcal{P}$) be the ideal generated by $(r+1)$-minors (resp. $(r+2)$-Pfaffians) of the latter matrix. Of course, $\mathcal{J} \subset \mathcal{I}$; moreover, $\mathcal{I} \subset \mathcal{P}$ by [B-E, p.462]. Since $\mathcal{P}$ is a prime ideal of height equal to the expected codimension $c(r)$ (see [K-L]), the ideal $\mathcal{P}$ defines (locally) the scheme structure on $D_r(\varphi)$. Hence using [G], ..., we see that $D_r(\varphi)$ is represented by $s_{\rho_{e-r-1}}(E)$. For example, for $e=5, f=4, r=2$, the representing polynomial equals $s_{2,1}(E)$.

In (1.11.4-5) we assume that the ambient space X is a projective space.

(1.11.4) Suppose that $E = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)$, $F = \mathcal{O}(a) \oplus \mathcal{O}(b)$, and $\varphi : E^* \to F$ is given by a matrix of forms

$$\left|\begin{array}{ccc} 0 & a_{12} & x \\ -a_{12} & 0 & y \end{array}\right|.$$

Then $D_1(\varphi)$ is of codimension 1, defined by the equation $a_{12} = 0$ whose degree is $a + b = s_1(F)$.

More generally, assume that f is even, $E = \mathcal{O}(p_1) \oplus \ldots \oplus \mathcal{O}(p_{f+1})$, $F = \mathcal{O}(p_1) \oplus \ldots \oplus \mathcal{O}(p_f)$, and $\varphi : E^* \to F$ is given by a matrix of forms

$$\left|\begin{array}{ccccc} 0 & a_{12} & \ldots & a_{1f} & b_1 \\ -a_{12} & 0 & \ldots & a_{2f} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{1f} & -a_{2f} & \ldots & 0 & b_f \end{array}\right|.$$

Then $D_{f-1}(\varphi)$ is of codimension 1, defined by the Pfaffian of the $f \times f$ matrix $A = (a_{ij})$. Indeed, every $(f-1)$-minor of A occuring in the Laplace expansion along the last column of an f-minor that contains the $(f+1)$th column, is a multiple of $\mathrm{Pf}(A)$ by already quoted result from [B-E]. The degree of $\mathrm{Pf}(A)$ is $p_1 + \ldots + p_f = s_1(F)$.

(1.11.5) Assume that $E = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c) \oplus \mathcal{O}(d)$, $F = \mathcal{O}(a) \oplus O(b)$, and $\varphi : E^* \to F$ is given by a matrix of forms

$$\left|\begin{array}{cccc} 0 & a_{12} & x & z \\ -a_{12} & 0 & y & w \end{array}\right|.$$

Then $D_1(\varphi)$ is of codimension 2, defined by

$$a_{12} = 0 \qquad \text{and} \qquad xw - zy = 0.$$

By Bézout's theorem the degree of $D_1(\varphi)$ is $(a+b)(a+b+c+d) = s_1(F)s_1(E)$.

2. Formulas for the top Chern classes of $E \vee F$ and $E \wedge F$

We keep the notation from the Introduction and Section 1. In this section, we give two alternative expressions for the top Chern classes of $E \vee F$ and $E \wedge F$. In particular, these expressions give the polynomials representing $D_r(\varphi)$ for $r = 0$.

In the following, we set $K = \mathrm{Ker}(\alpha : E \twoheadrightarrow F)$.

Proposition 2.1. *We have*

$$c_{\mathrm{top}}(E \vee F) = \textstyle\sum_I Q_{\rho_f + I}(F) \cdot s_{\mathrm{C}\,\widetilde{I}}(E - F),$$

where the sum is over all partitions in $(n)^f$ and for the conjugate partition $\widetilde{I} \subset (f)^n$ we write $\mathrm{C}\,\widetilde{I} = (f - \widetilde{i}_n, \dots, f - \widetilde{i}_1)$. Similarly,

$$c_{\mathrm{top}}(E \wedge F) = \textstyle\sum_I P_{\rho_{f-1} + I}(F) \cdot s_{\mathrm{C}\,\widetilde{I}}(E - F),$$

the same sum as above.

Proof. We give here the proof of the proposition for the bundle $E \vee F$, the case $E \wedge F$ being similar. We have in the Grothendieck group $K(X)$,

$$[E \vee F] = [E \otimes F] - [\wedge^2 F] = [S^2 F] + [K \otimes F].$$

Invoking the formulas

$$c_{\mathrm{top}}(S^2 F) = 2^f s_{\rho_f}(F) = Q_{\rho_f}(F),$$

and

$$c_{\mathrm{top}}(K \otimes F) = \sum_I s_I(F) \cdot s_{\mathrm{C}\,\widetilde{I}}(K),$$

the same sum as above (see (1.7.1) and (1.7.2)), the assertion follows by virtue of the factorization formula for Q-polynomials (1.6.2) and the equality $[K] = [E] - [F]$ in $K(X)$. $\square$

We will see in the next section that the expressions given in the previous proposition generalize, in a natural way, for higher r. The expressions given in the next proposition do not admit such a generalization.

Proposition 2.2. *We have*

$$c_{\mathrm{top}}(E \vee F) = 2^f \textstyle\sum_I s_{(e, e-1, \dots, n+2, n+1)/I}(F) \cdot s_{\widetilde{I}}(E - F),$$

where the sum runs over all partitions $I \subset (e, e-1, \dots, n+2, n+1)$. Similarly,

$$c_{\mathrm{top}}(E \wedge F) = \textstyle\sum_I s_{(e-1, e-2, \dots, n+1, n)/I}(F) \cdot s_{\widetilde{I}}(E - F),$$

where the sum runs over all partitions $I \subset (e-1, e-2, \dots, n+1, n)$.

Proof. We give here the proof of the proposition for the bundle $E \vee F$, the case $E \wedge F$ being similar. By the splitting principle, a (more general) question concerning the total Chern class of $E \vee F$ leads to the calculation of the product

$$\prod_{i \leqslant j} (1 + a_i + a_j) \prod_{i,j} (1 + a_i + b_j), \tag{2.3}$$

where, formally,

$$c(F) = \prod_{i=1}^{f}(1+a_i) \quad \text{and} \quad c(K) = \prod_{j=1}^{n}(1+b_j).$$

Write $A = (a_1, \dots, a_f)$ and $B = (b_1, \dots, b_n)$. Let $a_i^+ = 1 + 2a_i$, $1 \leqslant i \leqslant f$ and $b_j^+ = 1 + 2b_j$, $1 \leqslant j \leqslant n$. Set $A^+ = (a_1^+, \dots, a_f^+)$ and $B^+ = (b_1^+, \dots, b_n^+)$. Then

$$\begin{aligned}\prod_{i,j} (1 + a_i + b_j) &= 2^{-fn} \prod_{i,j} [(1+2a_i) + (1+2b_j)] \\ &= 2^{-fn} \prod_{i,j} (a_i^+ - zb_j^+)|_{z=-1} \\ &= 2^{-fn} s_{(n)^f}(A^+ - zB^+)|_{z=-1}\end{aligned}$$

where z is a formal λ-ring element of rank 1. Here, we have used the formula (1.1.1) for the resultant. Moreover, by (1.7.2),

$$\begin{aligned}\prod_{i \leqslant j} (1 + a_i + a_j) &= 2^{-\binom{f+1}{2}} \prod_{i \leqslant j} [(1+2a_i) + (1+2a_j)] \\ &= 2^{-\binom{f}{2}} \; s_{\rho_f}(A^+).\end{aligned}$$

Thus (2.3) can be rewritten as

$$2^{-N+f} s_{\rho_f}(A^+) \cdot s_{(n)^f}(A^+ - zB^+)|_{z=-1} \ ,$$

where $N = \operatorname{rank} E \vee F$. By the factorization formula (1.6.1) for elements A^+ and zB^+, we can rewrite this last expression as

$$2^{-N+f} s_{(n+f, n+f-1, \dots, n+1)}(A^+ - zB^+)|_{z=-1}.$$

The top-degree component of this polynomial is

$$2^f s_{(e,e-1,\dots,n+1)}(A - zB)|_{z=-1} = 2^f \sum_I \ s_{(e,e-1,\dots,n+1)/I}(A) \cdot s_{\widetilde{I}}(B),$$

the sum over partitions $I \subset (e, e-1, \dots, n+1)$, as desired. $\square$

3. The main formulas

The most popular method to compute the fundamental class of a subscheme $D \subset X$ tries to find a scheme G mapping properly to X, on which one has a locus Z that maps birationally onto D and for which one can compute its class $[Z]$. Usually this is because $[Z]$ is the zero locus of a section of some bundle whose rank is equal to $\operatorname{codim}_G Z$, so the class $[Z]$ is evaluated to be the top Chern class of the bundle. For example, this pattern was used in [J-L-P] and many other papers (compare [F1] and [F-P]). We will also follow this pattern in the present section.

We follow the notation from the Introduction and Section 1. Recall that $n = e-f$ and $q = f - r$. Let $\pi : G = G^q(F) \to X$ be the Grassmann bundle parametrizing rank q quotients of the bundle F. On G there exists a tautological sequence

$$0 \to R \to F_G \to Q \to 0,$$

where $\operatorname{rank} R = r$ and $\operatorname{rank} Q = q$. The composite morphism

$$E_G^* \xrightarrow{\varphi_G} F_G \twoheadrightarrow Q$$

gives a section of

$$H = \operatorname{Ker}(E_G \otimes Q \to \wedge^2 Q)$$

when φ is symmetric (and respectively a section of

$$H = \operatorname{Ker}(E_G \otimes Q \to S^2 Q)$$

when φ is skew-symmetric). Let $Z \subset G$ denote the subscheme of zeros of this section, in both respective cases. Observe that π maps Z onto $D_r(\varphi)$. The next proposition will contain the main calculation of the paper.

Proposition 3.1. *(i) Suppose that, for symmetric φ, $\operatorname{codim}_G Z = \operatorname{rank} H$ and π restricted to Z establishes a birational isomorphism of Z and $D_r(\varphi)$. Then, the following equality in $A_*(X)$ holds*

$$[D_r(\varphi)] = \sum_I Q_{\rho_q + I}(F) \cdot s_{\mathrm{C}\,\widetilde{I}}(E - F) \cap [X], \tag{3.2}$$

where the sum is over all partitions I in $(n)^q$ and for the conjugate partition $\widetilde{I} \subset (q)^n$ we write $\mathrm{C}\,\widetilde{I} = (q - \widetilde{i}_n, \dots, q - \widetilde{i}_1)$.
(ii) Assume that r is even. Suppose that, for skew-symmetric φ, $\operatorname{codim}_G Z = \operatorname{rank} H$ and π restricted to Z establishes a birational isomorphism of Z and $D_r(\varphi)$. Then, the following equality in $A_*(X)$ holds:

$$[D_r(\varphi)] = \sum_I P_{\rho_{q-1} + I}(F) \cdot s_{\mathrm{C}\,\widetilde{I}}(E - F) \cap [X], \tag{3.3}$$

where the sum is the same as in (3.2).
(iii) Let $n \geqslant 1$ and assume that r is odd. Suppose that, for skew-symmetric φ, $\operatorname{codim}_G Z = \operatorname{rank} H$ and π restricted to Z establishes a birational isomorphism of Z and $D_r(\varphi)$. Then, the following equality in $A_*(X)$ holds:

$$[D_r(\varphi)] = \sum_J P_{\rho_q + J}(F) \cdot s_{\mathrm{C}\,\widetilde{J}}(E - F) \cap [X], \tag{3.4}$$

where the sum is over all partitions J in $(n-1)^q$ and for the conjugate $\widetilde{J} \subset (q)^{n-1}$ we write $C\widetilde{J} = (q - \widetilde{j}_{n-1}, \dots, q - \widetilde{j}_1)$.

Proof. We give a detailed proof in the symmetric case and make some necessary comments about the skew-symmetric cases. Let $K = \operatorname{Ker}(\alpha : E \twoheadrightarrow F)$. In the Grothendieck group $K(G)$, the following equality holds:

$$[E \otimes Q] - [\wedge^2 Q] = [K_G \otimes Q] + [R \otimes Q] + [S^2 Q].$$

Hence we get

$$(3.5)\qquad [D_r(\varphi)] = \pi_*\Big(c_{\rm top}(K_G\otimes Q)\cdot c_{\rm top}(R\otimes Q)\cdot c_{\rm top}(S^2Q)\cap[G]\Big).$$

We have, by (1.7.1),

$$c_{\rm top}(K_G\otimes Q)=\sum_I s_I(Q)\cdot s_{\mathrm{C}\,\widetilde{I}}(K_G)=\sum_I s_I(Q)\cdot s_{\mathrm{C}\,\widetilde{I}}(E_G-F_G),$$

where the sum is over partitions $I\subset(n)^q$ and for the conjugate partition $\widetilde{I}\subset(q)^n$, we write $\mathrm{C}\,\widetilde{I}=(q-\widetilde{i}_n,\dots,q-\widetilde{i}_1)$.

We compute the RHS of (3.5). First, using the above expansion of $c_{\rm top}(K_G\otimes Q)$ and (1.7.2), we get

$$\begin{aligned}&\pi_*\big(c_{\rm top}(K_G\otimes Q)\cdot c_{\rm top}(R\otimes Q)\cdot c_{\rm top}(S^2Q)\cap[G]\big)\\ &=\pi_*\big(\sum_I c_{\rm top}(R\otimes Q)\cdot Q_{\rho_q}(Q)\cdot s_I(Q)\cdot s_{\mathrm{C}\,\widetilde{I}}(E_G-F_G)\cap[G]\big).\end{aligned}$$

Secondly, using the factorization formula (1.6.2), we infer that this last expression equals

$$\begin{aligned}&\pi_*\big(\sum_I c_{\rm top}(R\otimes Q)\cdot Q_{\rho_q+I}(Q)\cdot s_{\mathrm{C}\,\widetilde{I}}(E_G-F_G)\cap[G]\big)\\ &=\sum_I Q_{\rho_q+I}(F)\cdot s_{\mathrm{C}\,\widetilde{I}}(E-F)\cap[X],\end{aligned}$$

where the sum is over $I\subset(n)^q$, and in the last equality we have used the push-forward formula (1.8.1).

The proof in the skew-symmetric case, when r is even, is analogous; in addition, we must use the push-forward formula (1.8.2).

The proof in the skew-symmetric case, when $n\geqslant 1$ and r is odd, goes the same way. We use the push-forward formulas (1.8.3) and (1.8.1).

The proposition has been proved. □

As usual the formulas for the fundamental classes of degeneracy loci hold under more general assumptions than in the previous proposition.

Theorem 3.6. *(i) If X is a pure-dimensional Cohen-Macaulay scheme and $D_r(\varphi)$ is of expected pure codimension $c(r)$ or empty, then, in the symmetric case, the fundamental class of $D_r(\varphi)$ is evaluated by (3.2).*

(ii) In the skew-symmetric case, under the analogous assumptions, the fundamental class of $D_r(\varphi)$ is evaluated by (3.3) if r is even, and by (3.4) if $n\geqslant 1$ and r is odd.

Proof. We pass to a "universal case".(Our method is similar to the technique explained in [F-P, Appendix A.2].) For a given morphism $\varphi: E^*\to F$ of one of the two considered types, we define $\tilde{X}=\mathrm{Spec}\ S^\bullet(E\vee F)^*$ (respectively $\tilde{X}=\mathrm{Spec}\ S^\bullet(E\wedge F)^*$). Observe that φ induces a section $s: X\to\tilde{X}$. On the other hand, there exists the tautological bundle homomorphism $\tilde{\varphi}:\tilde{E}^*\to\tilde{F}$ where $\tilde{E}=E_{\tilde{X}}$, $\tilde{F}=F_{\tilde{X}}$ such that $s^*(\tilde{\varphi})=\varphi$. If X is Cohen-Macaulay, then so is $D_r(\tilde{\varphi})$ by virtue of a result from [DC-L], and because an algebraic fibre bundle with Cohen-Macaulay base and fibre is Cohen-Macaulay. Hence, if $D_r(\varphi)$ is of pure codimension $c(r)$ in X, then by [F1, Lemma A.7.1] we get

$$[D_r(\varphi)] = s^*[D_r(\tilde{\varphi})]. \tag{3.7}$$

One checks in a fairly standard way that $\tilde{\varphi}$ satisfies the assumptions of Proposition 3.1. Applying Proposition 3.1 to $\tilde{\varphi}$, we infer that $[D_r(\tilde{\varphi})]$ is evaluated by (3.2) (resp. by (3.3) or (3.4)) with $\tilde{E}$ playing the role of E and $\tilde{F}$ playing the role of F. Consequently, the wanted assertion follows from (3.7) and the pull-back property of Chern classes. □

Note that in the case where $E = F$, Giambelli [G] obtained the expression $2^q s_{\rho_q}(E)$ for the class of the degeneracy locus considered in part (i) of the theorem. However, it a classical result in combinatorics that the following equality of symmetric polynomials in the variables $A = (a_1, a_2, \dots)$ holds:

$$P_{\rho_q}(A) = s_{\rho_q}(A).$$

Example 3.8. (i) Assume first that φ is symmetric. Let $f = 3$ and $r = 2$. If $e = 4$ then the polynomial representing $D_2(\varphi)$ is $Q_2(F) + Q_1(F)s_1(E - F)$; for $e = 5$, the polynomial is $Q_3(F) + Q_2(F)s_1(E - F) + Q_1(F)s_{1,1}(E - F)$.

For the rest of this example, we assume that φ is skew-symmetric.

(ii) Let $f = 3$ and $r = 2$. If $e = 4$ then the polynomial representing $D_2(\varphi)$ is $P_1(F)+s_1(E-F)$; for $e = 5$, the polynomial is $P_2(F)+P_1(F)s_1(E-F)+s_{1,1}(E-F)$.

(iii) If $e = 5$, $f = 4$, and $r = 2$, then the representing polynomial is $P_{2,1}(F) + P_2(F)s_1(E - F) + P_1(F)s_2(E - F)$.

(iv) If f is even and $n = q = 1$, then the polynomial representing $D_{f-1}(\varphi)$ equals $P_1(F)$.

(v) If $e = 4, f = 2$, then $D_1(\varphi)$ is represented by $P_1(F)s_1(E - F) + P_2(F)$.

We leave it to the reader to check that the formulas in this example are consistent with the formulas in (1.11).

Perhaps, the easiest way to remember the formula associated with (e, f, r) in the symmetric case, is to put $T = (e - r, e - r - 1, \dots, n + 1)$, and note that for $I \subset (e-f)^{f-r}$ the coefficient of $s_{\tilde{I}}(E-F)$ is $Q_{T-I}(F)$, I being written *increasing* in last index, and subtraction of the sequences being performed componentwise. In the skew-symmetric case, with r even, we respectively put $T = (e-r-1, e-r-2, \dots, n)$, and note that the coefficient of $s_{\tilde{I}}(E - F)$ is $P_{T-I}(F)$ with the same conventions as above. A similar interpretation can be given in the skew-symmetric case when $n \geqslant 1$ and r is odd.

Example 3.9. $e = 8, f = 4, r = 2$; the symmetric case:

$$\begin{aligned}
&Q_{6,5}(F) + Q_{6,4}(F)s_1(E - F) + Q_{5,4}(F)s_2(E - F) + Q_{6,3}(F)s_{1,1}(E - F)\\
&+ Q_{5,3}(F)s_{2,1}(E - F) + Q_{6,2}(F)s_{1,1,1}(E - F) + Q_{5,2}(F)s_{2,1,1}(E - F)\\
&+ Q_{4,3}(F)s_{2,2}(E - F) + Q_{6,1}(F)s_{1,1,1,1}(E - F) + Q_{4,2}(F)s_{2,2,1}(E - F)\\
&+ Q_{5,1}(F)s_{2,1,1,1}(E - F) + Q_{4,1}(F)s_{2,2,1,1}(E - F) + Q_{3,2}(F)s_{2,2,2}(E - F)\\
&+ Q_{3,1}(F)s_{2,2,2,1}(E - F) + Q_{2,1}(F)s_{2,2,2,2}(E - F)
\end{aligned}$$

So, in this example, $T = (6,5)$ and, for instance, the coefficient of $s_{2,2,1,1}(E-F) = s_{\widetilde{(4,2)}}(E-F)$ is $Q_{(6,5)-(2,4)}(F) = Q_{4,1}(F)$.

A decomposition into a sum of the $s_I(F) \cdot s_J(E)$'s (computed with the help of the library SFA of ACE (see [V])) is

$$\begin{aligned} & s_{6,1}(F)\big(s_{1,1,1,1}(E) + s_{2,2}(E) - s_{2,1,1}(E)\big) + s_{6,5}(F) \\ & + s_{6,3}(F)\big(-s_2(E) + s_{1,1}(E)\big) + s_{2,1}(F)s_{2,2,2,2}(E) + s_{4,3}(F)s_{2,2}(E) \\ & + s_{4,1}(F)\big(s_{2,2,1,1}(E) - s_{2,2,2}(E)\big) + s_{8,3}(F) + s_{10,1}(F) \\ & + s_{8,1}(F)\big(-s_2(E) + s_{1,1}(E)\big). \end{aligned}$$

4. Some variations

To compute the fundamental classes of subvarieties, one can also use appropriate geometric constructions with a *nontrivial* generic fibre. This method was invented in [P1] in order to give a short proof of the formulas from [J-L-P] and [H-T], and is summarized (and somewhat straightened) in the following proposition. In this proposition, we may assume that the Chow groups have rational coefficients.

Proposition 4.1. *Let D be an irreducible (closed) subscheme of a scheme X. Let $\pi : G \to X$ be a proper morphism of schemes and W be a (closed) subscheme of G such that $\pi(W) = D$. We have the following two instances:*
(i) Suppose that G is smooth. Assume that there exists

$$g \in A_{\dim G + \dim D - \dim W}(G)$$

and a point x in the smooth locus of D such that in $A_*(G_x)$, where G_x is the fibre of π over x, one has:

$$i_x^*(g) \cdot [W_x] = [\text{point}].$$

Here, W_x is the fibre of W over x and $i_x : G_x \hookrightarrow G$ is the inclusion. Then the following equality holds in $A_*(X)$:

$$[D] = \pi_*\big(g \cdot [W]\big)\,.$$

(ii) Here G is possibly singular. Suppose that there exists a family of vector bundles $\{E^{(\alpha)}\}$ on G and $g = P\big(\{c.(E^{(\alpha)})\}\big)$ a homogeneous polynomial of degree $\dim W - \dim D$ *in the Chern classes of $\{E^{(\alpha)}\}$ (*$\deg$ *$c_i(E^{(\alpha)}) = i$) with rational coefficients, such that in $A_*(G_x)$,*

$$P\big(\{c.(i_x^* E^{(\alpha)})\}\big) \cap [W_x] = [\text{point}],$$

where x, G_x, W_x and i_x are as above. Then the following equality holds in $A_*(X)$:

$$[D] = \pi_*\big(g \cap [W]\big).$$

Proof. (i) Using a standard dimension argument, we can replace, in the assertion, D by its smooth part, i.e., we can assume D is smooth. Write $G_D = G \times_X D$,

$W_D = W \times_X D$, $\eta : G_D \to D$ the projection induced by π, and $k : G_D \to G$ the inclusion. Then, the assertion is a consequence of the following identity in $A_*(D)$:

$$\eta_*\big(k^*(g) \cdot [W_D]\big) = [D].$$

To prove this last equation, we first remark that the assumptions imply

$$\eta_*\big(k^*(g) \cdot [W_D]\big) = m[D],$$

where $m \in \mathbb{Z}$. Let x be a point in D and consider the fibre square

$$\begin{array}{ccc} G_x & \stackrel{j}{\hookrightarrow} & G_D \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \eta} \\ \{x\} & \stackrel{i}{\hookrightarrow} & D. \end{array}$$

Using the assumptions on g and [F1, Theorem 6.2], we have

$$\begin{aligned} i^*\eta_*\big(k^*(g) \cdot [W_D]\big) &= p_*\Big(j^*\big(k^*(g) \cdot [W_D]\big)\Big) \\ &= p_*\big(i_x^*(g) \cdot [W_x]\big) = p_*\big([\text{point}]\big) = [\text{point}]. \end{aligned}$$

This implies $m = 1$ and assertion (i) is proved.

The proof of (ii) is essentially the same. $\square$

Let $F \subset E$ be two vector bundles of ranks f and e on a variety X. We now describe a certain geometric construction associated with a bundle morphism $\varphi : F \to E^*$ induced by a section of $E^* \vee F^*$ (resp. $E^* \wedge F^*$). (In this section, we use a slightly different setup than in the Introduction, Section 1, and Section 3.) This construction generalizes in a natural way the construction used in [P1] and is based on the following characterization of the rank of the above morphisms. Assume that p is a natural number such that $2p < f$. Let $V \subset U$ be two vector spaces of dimensions f and e. Let $\phi : V \to U^*$ be a linear map induced by a section of $U^* \vee V^*$ (resp. $U^* \wedge V^*$). Then $\operatorname{rank} \phi \leqslant 2p$ iff there exists a pair of vector spaces (A, B) such that A is a subspace of U of dimension $e - p$, B is a subspace of V of dimension $f - p$, $B \subset A$, and the composite map

$$B \hookrightarrow V \xrightarrow{\phi} U^* \twoheadrightarrow A^*$$

is zero. Indeed, suppose that such a pair (A, B) exists. Then the matrix of ϕ in some basis has $e - p$ rows whose initial segments of length $f - p$ consist of zeros. Then every $(2p+1)$-minor of such a matrix vanishes (use the Laplace expansion of this minor w.r.t. the first $p + 1$ from these $e - p$ rows). The opposite implication can be showed using fairly standard linear algebra (by reducing a matrix to its "standard form" via the elementary row and column operations).

Let

$$\pi : G = Fl_{f-p,e-p}(F, E) \to X$$

be the flag bundle parametrizing the pairs (A, B) where A is a rank $e-p$ subbundle of E, B is a rank $f-p$ subbundle of F, and $B \subset A$. Let $S \subset R$ be the tautological (sub)bundles of respective ranks $f-p$ and $e-p$ on G.

With $\varphi : F \to E^*$ as above, we associate a locus $W \subset G$ to be the subscheme of zeros of the composite morphism

$$S \hookrightarrow F_G \xrightarrow{\varphi_G} E_G^* \twoheadrightarrow R^*.$$

Let $D = D_{2p}(\varphi)$. By the above discussion, we have $\pi(W) = D$.

We want now to work with some "universal" φ (like that in the proof of Theorem 3.6). Moreover, in this situation, we want now to get information about the generic fibre $W_x =: \mathcal{F}$ of $\pi\big|_W$ in order to apply Proposition 4.1. To this end, it suffices to make the following "universal local study". Let $V \subset U$ be vector spaces of dimensions f and e, respectively. Let X denote the affine space $U^* \vee V^*$ (resp. $U^* \wedge V^*$). In this situation, there exists a tautological morphism $\varphi : (F = V_X) \to (E = U_X)^*$ and the corresponding subvariety W. Let $\phi : V \to U^*$ be a linear map coming from a section of $U^* \vee V^*$ (resp. $U^* \wedge V^*$). Then the fibre W_ϕ over ϕ is identified with

$$W_\phi = \{(L, M) \in Fl \;:\; i_L^* \circ \phi \circ j_M = 0\},$$

where $Fl = Fl_{f-p,e-p}(V, U)$ and $i_L : L \hookrightarrow U$, $j_M : M \hookrightarrow V$ are the inclusions. By calculation in local coordinates, one checks that $W \subset G = Fl$ is a complete intersection of codimension equal to $\operatorname{rank}(R^* \vee S^*)$ (resp. $\operatorname{rank}(R^* \wedge S^*)$). Hence one easily computes that $\dim \mathcal{F} = \dim W - \dim D = p(p-1)/2$ (resp. $\dim \mathcal{F} = p(p+1)/2$). Consequently, the dimension of the generic fibre $\mathcal{F}$ depends only on p and not on e and f.

The following very simple fact is helpful to find the class g satisfying the requirements of Proposition 4.1.

Lemma 4.2. *Let $i : Y' \hookrightarrow Y$ be a closed embedding of smooth varieties, let $X \subset Y$ and $X' \subset Y'$ be two subvarieties such that $i(X') \subset X$ and $\dim X' = \dim X$. Assume that an element $z \in A^*(X)$ satisfies $[X'] \cdot i^*(z) = [\mathrm{point}]$ in $A^*(Y')$. Then, $[X] \cdot z = [\mathrm{point}]$ in $A^*(Y)$.*

Indeed, we have $i_*[X'] = [X]$, and by the projection formula we infer

$$[\mathrm{point}] = i_*\big([X'] \cdot i^*(z)\big) = i_*[X'] \cdot z = [X] \cdot z\,,$$

as claimed.

Proposition 4.3. *The class $g = 2^{-p} s_{\rho_{p-1}}(S^*)$ (resp. $g = s_{\rho_p}(S^*)$) satisfies the assumption of Proposition 4.1(ii).*

Proof. We follow the notation from the discussion before the lemma. Here, the role of "x" from Proposition 4.1 is played by ϕ such that $\operatorname{rank}(V \xrightarrow{\phi} U^* \twoheadrightarrow V^*) = 2p$. Let S' denote the tautological rank $f-p$ bundle on the Grassmannian $G_{f-p}(V)$ parametrizing $(f-p)$-dimensional subspaces of V.

1) Assume first that $e = f = 2p$ so $V = U$ and the corresponding bilinear form is nondegenerate. Then $[\mathcal{F}]$ is evaluated as the top Chern class of the bundle $S^2({S'}^*)$ (resp. $\wedge^2({S'}^*)$). We get by (1.7),

$$[\mathcal{F}] = 2^p s_{\rho_p}({S'}^*) \qquad (\text{resp. } [\mathcal{F}] = s_{\rho_{p-1}}({S'}^*)\ ).$$

The assertion now follows by taking the dual Schubert cycles in the Grassmannian $G^p(V^*)$ (see, e.g., [F1, Chap.14]).

2) Let now $2p < e = f$ (so again $V = U$), and let $V' \subset V$ be an inclusion of vector spaces of dimensions $2p$ and f, respectively. Assume that V is endowed with a symmetric (resp. skew-symmetric) form ϕ of rank $2p$ such that the form $\phi\big|_{V'}$ is nondegenerate. We now use the lemma with the following data: $Y' = G_p(V')$ and $Y = G_{f-p}(V)$; $i : G_p(V') \hookrightarrow G_{f-p}(V)$ being defined by $L \mapsto L \oplus A$, where $V = V' \oplus A$. Moreover, X and X' are the generic fibres under consideration and $z = 2^{-p} s_{\rho_{p-1}}(S'^*)$ (resp. $z = s_{\rho_p}(S'^*)$). Then part 1) and the lemma yield the desired assertion.

3) Finally, suppose that $f < e$ and let $U = V \oplus B$, where dim $B = n = e - f$. We now apply the lemma to the following embedding:

$$i : \big(Y' = G_{f-p}(V)\big) \hookrightarrow \big(Y = G\big)$$

where $i(L) = (L, L \oplus B)$. Moreover, X and X' are the generic fibres under consideration and $z = 2^{-p} s_{\rho_{p-1}}(S^*)$ (resp. $z = s_{\rho_p}(S^*)$). Then part 2) and the lemma yield the desired result. $\square$

We will be now interested in formal identities of some polynomials in Chern classes and their push-forwards. Therefore we assume that X is smooth and treat the Chern classes as elements of the appropriate Chow rings. Using Propositions 4.1(ii) and (4.3), we infer that in the symmetric case the degeneracy locus D is represented by

$$\pi_*\big(c_{\text{top}}(R^* \vee S^*) \cdot 2^{-p} s_{\rho_{p-1}}(S^*)\big).$$

Similarly, in the skew-symmetric case, the degeneracy locus D is represented by

$$\pi_*\big(c_{\text{top}}(R^* \wedge S^*) \cdot s_{\rho_p}(S^*)\big).$$

Of course $D = D_{2p}(\varphi^*)$. Combining Propositions 2.1 and 2.2 with Theorem 3.6 applied to $\varphi^* : E \to F^*$, we thus get the following algebraic equalities. In the symmetric case, writing $T = (e-p, e-p-1, \dots, n+1)$,

$$\begin{aligned}
&\pi_*\big(2^{-p} \sum_I Q_{\rho_{f-p}+I}(S^*) \cdot s_{\complement\,\widetilde{I}}(R^* - S^*) \cdot s_{\rho_{p-1}}(S^*)\big) \\
(4.4)\qquad &= \pi_*\big(2^{f-2p} \sum_J s_{T/J}(S^*) \cdot s_{\widetilde{J}}(R^* - S^*) \cdot s_{\rho_{p-1}}(S^*)\big) \\
&= \sum_L Q_{\rho_{f-2p}+L}(F^*) \cdot s_{\complement\,\widetilde{L}}(E^* - F^*),
\end{aligned}$$

where I runs over partitions in $(n)^{f-p}$, J runs over partitions in T and L runs over partitions in $(n)^{f-2p}$. In the skew-symmetric case, writing T for the partition $(e-p-1, e-p-2, \dots, n)$,

$$\begin{aligned}
&\pi_*\big(\sum_I P_{\rho_{f-p-1}+I}(S^*) \cdot s_{\complement\,\widetilde{I}}(R^* - S^*) \cdot s_{\rho_p}(S^*)\big) \\
(4.5)\qquad &= \pi_*\big(\sum_J s_{T/J}(S^*) \cdot s_{\widetilde{J}}(R^* - S^*) \cdot s_{\rho_p}(S^*)\big) \\
&= \sum_L P_{\rho_{f-2p-1}+L}(F^*) \cdot s_{\complement\,\widetilde{L}}(E^* - F^*),
\end{aligned}$$

where I and L run over the same range of partitions as above and J runs over partitions in the present T.

These equalities were originally obtained with the help of the library SFA of ACE (see [V]) for small values of f, p, and n. We know no algebraic proofs of (4.4) and (4.5) not invoking geometry.

Let $G' = G_{f-p}(F) \to X$ and set

$$C = \mathrm{Coker}(S' \hookrightarrow F_{G'} \hookrightarrow E_{G'}),$$

where S' is the tautological subbundle on G'. Then using the presentation

$$G = G_n(C) \to G_{f-p}(F) \to X$$

and (1.9), one can rewrite the LHS's of (4.4) and (4.5) as purely algebraic expressions involving symmetrizing operators. Or, equivalently, one can embedd G in $G_{f-p}(F) \times_X G_{e-p}(E)$ and use the product of the symmetrizing operators corresponding to the factors, at the cost of multiplying the expressions in (4.4) and (4.5) to be push-forwarded by $c_{\mathrm{top}}\big(S'^* \otimes (E/R')\big)$, where R' is the tautological subbundle on $G_{e-p}(E)$ (and we omit some pull-back indices).

Finally, we take this opportunity to make the following

Remark 4.6. (Revisions and corrigenda to [P1], [P4], and [P-R])

[P1]: Revisions: The assumptions on g in Proposition 4.1 (in the present paper) straighten an unprecise expression "the Poincaré dual of" from 5^0 in [P1, Proposition 2.1]. The assumption that the ground field is algebraically closed of characteristic different from 2 was mistakenly omitted in Section 3. The reference "Lemma 9 in [10]" on p.196 should be replaced by "[2, Lemma A.7.1]".

[P4]: Revisions – insert after "... following changes:" p.162_5:

' p.154 l. -10 should read: " Write $a_i = c_1(L_i^E)$ $i = 1, \dots, n$. ..." ',

' p.155 l. -5 should read: " ... ω, the fact that ω_* is induced by $\partial^{k-1}_{(a_2,\dots,a_n)}$, " ',

' p.155 l. -14 should read: " ... $= dP_{I,J}(E) \cap \alpha$ ..." ',

' p.155 l. -13 should read: " ... Let $I' = (i_2, \dots, i_k)$." '.

Misprints – should be: p.130_3 "$(-1)^{p-1}$" // p.130_1 "$(-1)^p$" // p.131^9 "ρ_{n-r-1}" // p.148^{11} "$-a_{q+1}, \dots$" // p.162_3 " ...$\cdot\xi^{i_1} \cap$ ... " // p.164^6 "$\partial_r \circ \dots$" // p.174^{14} ' boxes" '.

[P-R]: Revisions – should read: p.19_{18} "Whenever, in this paper we speak about Schubert subschemes in OG_nV, we assume that there exists a completely filtered rank n isotropic subbundle of V." // p.56_{14} and p.60^3 " ... , so we can apply the induction assumption to M_1 defined below. The partitions ..." .

Misprints – should be: p.39_5 " $\widetilde{P}_J(X_m)$" // p.86^{12} "Fulton W." // p.87_2 "Gieseker-Petri" .

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