Vertex operators and the class algebras of symmetric groups

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Dedicated to the memory of Sergei Kerov

Abstract

We exhibit a vertex operator which implements multiplication by power-sums of Jucys-Murphy elements in the centers of the group algebras of all symmetric groups simultaneously. The coefficients of this operator generate a representation of $\mathcal{W}_{1+\infty}$, to which operators multiplying by normalized conjugacy classes are also shown to belong. A new derivation of such operators based on matrix integrals is proposed, and our vertex operator is used to give an alternative approach to the polynomial functions on Young diagrams introduced by Kerov and Olshanski.

1 Introduction

Convolution of central functions, or multiplication in the center of the group algebra of the symmetric group $\mathfrak{S}_n$ can be realized by means of differential operators acting on symmetric functions. This is due to the existence of the Frobenius map, which sends a permutation σ of cycle type α to the product of power sums p_α. Since the power sums are algebraically independent, any linear operator on the space Sym_n of homogeneous symmetric functions of degree n can be realized as a differential operator in the variables p_k, and the above statement is therefore trivial. More interesting is the existence of infinite order differential operators Φ implementing simultaneously for all symmetric groups the multiplication by families of elements η_n in the center $Z\mathfrak{S}_n$ of $\mathbb{C}\mathfrak{S}_n$.

The first example of such an operator has been given by Goulden [4], for the case where $\eta_n = \mathcal{C}_{(2,1^{n-2})}$ is the sum of all transpositions in $\mathfrak{S}_n$, and similar operators for $\eta_n = \mathcal{C}_{(\rho,1^{n-r})}$ (where ρ is a partition of r) have been recently described by Goupil, Poulhalon and Schaeffer [5].

Convolution of central functions is not the only operation which presents stability properties allowing to deal simultaneously with all symmetric groups. This is also the case of pointwise multiplication, which, applied to characters, corresponds to the tensor product of representations. Here, the stability properties are best explained in terms of vertex operators [18, 16], which can also produce stable formulas for outer or inner plethysms, character values or branching multiplicities [2, 17].

One might therefore expect that vertex operators play a role in the simultaneous description of the centers of all symmetric group algebras. A further hint that this should be the case is the observation by Frenkel and Wang [3] that the commutators of Goulden's operator with the operators $\alpha_{-k} =$ "multiplication by p_k" and their adjoints α_k, generate a representation of the Virasoro algebra.

Actually, vertex operators arise when one generalizes Goulden's formula in another direction. Rather than considering the sum of all transpositions in $\mathfrak{S}_n$ as the conjugacy class $\mathcal{C}_{(2,1^{n-2})}$, one can view it as the sum $p_1(\Xi_n) = \sum_{i=1}^{n} \xi_i$ of the Jucys-Murphy elements, and look for a generating function of the differential operators D_k implementing the multiplication by power sums $p_k(\Xi_n)$ for all n.

One finds that the required generating function has a simple expression in terms of the classical vertex operator $\Gamma(z_1, z_2)$ describing the Fock space representations of $\mathfrak{gl}_\infty$ and $\widehat{\mathfrak{gl}}_\infty$. Then, one can see that the brackets $[\mathsf{D}_k, \alpha_l]$ generate a charge 1 representation of the Lie algebra $\mathcal{W}_{1+\infty}$, by expressing them in closed form in terms of the standard generators of this algebra (for $k = 1$, this is the result of Frenkel and Wang).

In Section 4, we remark that our vertex operator provides a new approach to the results of Kerov and Olshanski [11]. What we prove is that the coefficients of the products of power-sums of Jucys-Murphy elements $p_\mu(\Xi_n)$ on the normalized conjugacy classes of [11] are independent of n, which is equivalent to Proposition 3 of [11].

In Section 5, we give an alternative derivation of the operators of [5] in terms of matrix integrals. We start, following Hanlon, Stanley and Stembridge [6], with the observation that the theory of spherical functions on the cone of positive definite Hermitian matrices allows one to write generating functions for connection coefficients as Gaussian integrals over the space of complex $N \times N$ matrices, N being sufficiently large (actually, it is the limit $N \to \infty$ which is relevant, and we are in fact considering functional integrals). Then, we combine the generating functions for all n, and we are reduced to the evaluation of similar integrals, but for a modified Gaussian measure, which can be performed by means of Wick's formula. On the way, we observe that this method of calculation can give a direct combinatorial proof of the generating function of [6], whithout any reference to spherical functions (this answers a question raised in the last section of [6]).

Finally, we observe that the results of Kerov and Olshanski rederived in Section 4 imply that the Goupil-Poulhalon-Schaeffer operators form a linear basis of the commutative subalgebra of $U(\mathcal{W}_{1+\infty})$ generated by the D_k.

2 Notations and background

2.1 Symmetric functions

We denote by Sym the (abstract) algebra of symmetric functions, with complex coefficients, and by $Sym(X) = Sym(x_1, \dots, x_n)$ the algebra of symmetric polynomials in n variables. The homogeneous component of degree k is denoted by Sym_k. The scalar product on Sym is the standard one, for which the Schur functions s_λ form an orthonormal basis. For $f \in Sym$, D_f denotes the adjoint of the operator $g \mapsto fg$.

For $A = \{a_1, a_2, \dots\}$ we denote by $\sigma_z(A) = \prod_i (1 - za_i)^{-1}$ and $\lambda_z(A) = \sigma_{-z}(A)^{-1}$ the generating series of complete and elementary symmetric functions of A. Other notations for symmetric functions are as in [13].

2.2 The Frobenius characteristic map

The conjugacy class of permutations with cycle type μ is denoted by $\mathcal{C}_\mu$. A central function f on $\mathfrak{S}_n$ is identified to the element $F = \sum_\sigma f(\sigma)\sigma \in Z\mathfrak{S}_n$. The Frobenius characteristic map is the linear map $\mathrm{ch} : \mathbb{C}\mathfrak{S}_n \to Sym_n$ defined by $\mathrm{ch}\,(\sigma) = p_\mu$ is σ is of cycle type μ. The structure constants $c_{\alpha\beta}^\gamma$ of $Z\mathfrak{S}_n$ are defined by $\mathcal{C}_\alpha \mathcal{C}_\beta = \sum_\gamma c_{\alpha\beta}^\gamma \mathcal{C}_\gamma$.

The Frobenius map allows one to define a new product $\times$ on each Sym_n by $\mathrm{ch}\,(F) \times \mathrm{ch}\,(G) = \mathrm{ch}\,(FG)$. We extend it to Sym by setting $u \times v = 0$

if u and v are homogeneous of different degrees. Then, $s_\lambda \times s_\mu = \frac{1}{f^\lambda}\delta_{\lambda\mu}s_\lambda$, where f^λ is the dimension of the representation λ of $\mathfrak{S}_n$. We denote by Γ the comultiplication dual to $\times$, that is, $\Gamma(s_\lambda) = \frac{1}{f^\lambda}s_\lambda \otimes s_\lambda$.

2.3 Jucys-Murphy elements

The Jucys-Murphy elements of $\mathfrak{S}_n$ are the n sums of transpositions [7, 14]

$$\xi_j = \xi_{j;n} = \sum_{i<j} (i,j)\,. \tag{1}$$

Note that ξ_1 is zero, but it is convenient to include it as well. These elements generate a maximal commutative algebra GZ_n of $\mathbb{C}\mathfrak{S}_n$ (the Gelfand-Zetlin subalgebra), and the center of $\mathbb{C}\mathfrak{S}_n$ is $Sym(\xi_1,\dots,\xi_n)$. Young's othogonal idempotents $e_{\mathbf{t},\mathbf{t}}$ ($\mathbf{t}$ a standard tableau) belong to GZ_n, and $\xi_i e_{\mathbf{t},\mathbf{t}} = c_i(\mathbf{t})e_{\mathbf{t},\mathbf{t}}$ where $c_i(\mathbf{t})$ is the content of the box labelled i in $\mathbf{t}$ (the content of the box in row k and column l of a Young diagram is defined as $l-k$). The multiset of contents of a partition λ is denoted by $C(\lambda) = \{c_\square | \square \in \lambda\}$ (where $\square$ runs over all boxes in the diagram of λ).

Jucys has shown that the elementary symmetric function e_k of the ξ_i is equal to the sum of all permutations having exactly $n-k$ cycles. One can check that the products

$$e_{\bar\alpha} = e_{\alpha_2}e_{\alpha_3}\cdots e_{\alpha_r}\,, \quad \alpha = (\alpha_1,\alpha_2,\dots\alpha_r) \vdash n$$

form a linear basis of $Z\mathfrak{S}_n$. For example, for $n=4$, a basis is $\{e_0, e_1, e_2, e_3, e_{11} = 2\mathcal{C}_{22} + 3\mathcal{C}_{31} + 6\mathcal{C}_{1111}\}$. However, in the sequel, we shall rather work with power sums.

2.4 The Fock space formalism

We will also identify Sym with the infinite wedge space $\mathcal{F}^{(0)}$, spanned by semi-infinite products $w = v_{i_1} \wedge v_{i_2} \dots$ ($i_k \in \mathbb{Z}$) such that $i_1 > i_2 > \dots$, and $i_k = 1-k$ for $k >> 0$. Such a vector will be denoted by $|\lambda\rangle$, where the partition λ is defined by $\lambda_k = i_k + k - 1$. This space is the basic representation $L(\Lambda_0)$ of the affine Lie algebra $\widehat{\mathfrak{g}l}_\infty = \mathfrak{A}_\infty$, the universal central extension of the Lie algebra of $\mathbb{Z}\times\mathbb{Z}$-matrices with a finite number of nonzero diagonals. The generators E_{ij} act in a simple way of the semi-infinite wedges. For $i \neq j$, if v_j occurs in a wedge w, E_{ij} replaces it with v_i, otherwise the result is 0.

For $i > 0$, $E_{ii}w = w$ if v_i occurs in w, and 0 otherwise. For $i \leq 0$, the result is 0 if v_i occurs in w, and $-w$ otherwise.

The Boson-Fermion correspondence produces differential operators transporting this action on Sym under the linear isomorphism $|\lambda\rangle \to s_\lambda$ (see [8], Chap. 14). More generally, the space $\mathcal{F}^{(m)}$ is spanned by wedges such that $i_k = 1 - k + m$ for $k >> 0$. Their direct sum (for $m \in \mathbb{Z}$) is called the fermionic Fock space, and $\mathcal{F}^{(m)}$ is called the charge m sector.

3 A vertex operator for power sums of Jucys-Murphy elements

3.1 A differential operator for $\mathfrak{S}_n$

In this section, we will compute the differential operator $\mathsf{D}^{(n)}$ acting on Sym_n as the generating function

$$F_n(t) = \sum_{k\geq 1} p_k(\Xi_n) = \sum_{i=1}^{n} (e^{t\xi_i} - 1)\,, \tag{2}$$

that is, for $P \in Sym_n$, $\mathsf{D}^{(n)}P := \mathrm{ch}\,(F_n(t)) \times P$.

We know that the eigenvalue of $p_k(\Xi_n)$ on the central idempotent e_λ is $p_k(C(\lambda))$. Therefore, the eigenvalue of $\mathsf{D}^{(n)}$ on s_λ is

$$\sum_{\square\in\lambda} \left(e^{tc_\square} - 1\right) = \sum_{\square\in\lambda} (q^{c_\square} - 1) \tag{3}$$

if we set $q = e^t$. This sum is easily evaluated in terms of the parts of λ:

Lemma 3.1 *Let λ be a non zero partition of length at most n. Then,*

$$\sum_{\square\in\lambda} q^{c_\square} = \frac{q}{q-1} \sum_{i=1}^{n} \left(q^{\lambda_i - i} - q^{-i}\right)\,.$$

Proof – The contents of λ are the numbers $-i+1, -i+2, \ldots, \lambda_i - 1$ for $i = 1, \ldots, \ell(\lambda)$. ∎

Therefore,

$$\sum_{\square\in\lambda} (q^{c_\square} - 1) = \frac{q}{q-1} \sum_{i=1}^{n} \left(q^{\lambda_i - i} - q^{-i}\right) - \sum_{i=1}^{n} \lambda_i\,. \tag{4}$$

Our first task is to express the operator induced by $\mathsf{D}^{(n)}$ on the space of symmetric polynomials $Sym(x_1, \ldots, x_n)$ in terms of the variables x_i. We set

$$\Delta_n = \prod_{i<j} (x_i - x_j) \quad \text{and} \quad \square_n = (x_1 x_2 \cdots x_n)^n \,. \tag{5}$$

Let $D_i = x_i \frac{\partial}{\partial x_i}$. We have

Lemma 3.2

$$\frac{\square_n}{\Delta_n} \left(\sum_{i=1}^{n} q^{D_i} \right) \frac{\Delta_n}{\square_n} \cdot s_\lambda = \left(\sum_{i=1}^{n} q^{\lambda_i - i} \right) s_\lambda \,.$$

Proof – Multiplication of s_λ by $\Delta_n \square_n^{-1}$ results in the determinant $\det(x_j^{\lambda_i - i})$. Applying $\sum_i q^{D_i}$ to this determinant amounts to apply the one-variable operator q^D to each row of the determinant, and then take the sum. This produces the same result as applying the operator to each column succesively, since both expressions are equal to the coefficient of ϵ in $\det((1+\epsilon q^D)(x_j^{\lambda_i - i}))$. ∎

Therefore, the operator

$$\frac{q}{q-1} \frac{\square_n}{\Delta_n} \left(\sum_{i=1}^{n} q^{D_i} - q^{-i} \right) \frac{\Delta_n}{\square_n} - \sum_{i=1}^{n} D_i \tag{6}$$

has the same eigenvalues as $\mathsf{D}^{(n)}$ on Schur functions s_λ in n variables, and must therefore coincide with it. We can now let $n \to \infty$, and see that $\mathsf{D}^{(n)}$ is the restriction to Sym_n of the well-defined limit

$$\mathsf{D} = \lim_{n \to \infty} \mathsf{D}^{(n)} \,. \tag{7}$$

Hence, all symmetric groups can be dealt with simultaneously by the single operator D.

3.2 Bosonization

The next step is to express D in terms of the power sums. To avoid confusion, we reserve the letter X for finite sets of variables, and introduce an infinite alphabet A as argument of our symmetric functions. So, we want to express the action of D on $Sym(A)$ in terms of the operators

$$\alpha_{-k} = p_k(A)\,, \quad \alpha_k = \alpha_{-k}^\dagger = D_{p_k} = k\frac{\partial}{\partial p_k(A)} \quad (k \geq 1)\,. \tag{8}$$

This procedure is called *bosonization* in the physics literature (see, e.g., [1]), for the α_k satisfy the commutation relations

$$[\alpha_j, \alpha_k] = j\delta_{j,-k} \tag{9}$$

of the modes of a free boson field (a Heisenberg algebra).

To compute the bosonization of D, we have to calculate the bi-symmetric kernel

$$K(X,A) = \lambda_{-1}(XA)\mathsf{D}^{(n)}\sigma_1(XA) \tag{10}$$

where $\mathsf{D}^{(n)}$ acts on functions of $X = \{x_1, \ldots, x_n\}$, and to express it in the form

$$K(X,A) = \sum_{\mu,\nu} k_{\mu\nu} p_\mu(X) p_\nu(A)\,. \tag{11}$$

Then, we will have

$$\mathsf{D} = \sum_{\mu,\nu} k_{\mu\nu} p_\mu(A) D_{p_\nu(A)}\,. \tag{12}$$

Indeed, writing $\langle\, ,\, \rangle$ for the scalar product of $Sym(A)$, we have $f(X) = \langle \sigma_1(XA), f(A)\rangle$, so that

$$\mathsf{D}^{(n)} f(X) = \langle \mathsf{D}^{(n)}\sigma_1(XA), f(A)\rangle = \langle 1, D_{\mathsf{D}^{(n)}\sigma_1(XA)} f(A)\rangle = \langle \sigma_1(XA), \mathsf{D} f(A)\rangle\,.$$

Let ∇_i be the partial q-derivative with respect to x_i, *i.e.*

$$\nabla_i = \frac{q^{D_i} - 1}{(q-1)x_i}\,.$$

We have, for any function $f(X) = f(x_1, \ldots, x_n)$,

$$\frac{q}{q-1}\left(\sum_{i=1}^n q^{\lambda_i - i}\right) f(X) = \left(\sum_{i=1}^n \nabla_i x_i\right) f(X) + \left(\sum_{i=1}^n [i]_{1/q}\right) f(X)\,.$$

To apply this to $f(X) = \Delta_n \sigma_1(XA)/\Box_n$, we note that

$$\left(\sum_{i=1}^n \nabla_i x_i\right) \frac{\Delta_n}{\Box_n} = \left[\frac{n}{1-q} + \frac{q(1-q^{-n})}{(1-q)^2}\right] \frac{\Delta_n}{\Box_n} \tag{13}$$

and

$$\left(\sum_{i=1}^{n}\nabla_i x_i\right)\frac{\Delta_n}{\Box_n}\sigma_1(XA)=\left(\sum_{i=1}^{n}\nabla_i x_i\frac{\Delta_n}{\Box_n}\right)\sigma_1(XA)+\sum_{i=1}^{n}\frac{qx_i\Delta^{(i)}}{q^n\Box_n}\nabla_i\sigma_1(XA) \tag{14}$$

where

$$\Delta^{(i)}=q^{D_i}\Delta_n=\Delta_n A_i(X;q)\,,\quad A_i(X;q)=\prod_{j\neq i}\frac{qx_i-x_j}{x_i-x_j}\,.$$

Hence, setting $\bar{\mathsf{D}}^{(n)}=\mathsf{D}^{(n)}+\mathsf{E}^{(n)}$, where $\mathsf{E}^{(n)}$ is the Euler operator, we have

$$\begin{aligned}
\bar{\mathsf{D}}^{(n)}\sigma_1(XA) &= \left[\left(\frac{n}{1-q}+\frac{q(1-q^{-n})}{(1-q)^2}i\right)+\sum_{i=1}^{n}[i]_{1/q}\right]\sigma_1(XA)\\
&\quad +q^{1-n}\sum_{i=1}^{n}x_iA_i(X;q)\nabla_i\sigma_1(XA)\\
&= q^{-n}\sum_{i=1}^{n}A_i(X;q)\frac{qx_i}{(q-1)x_i}\left(\frac{\sigma_{qx_i}(A)\sigma_1(XA)}{\sigma_{x_i}(A)}-\sigma_1(XA)\right)\\
&= \sigma_1(XA)\frac{q^{-n}}{q-1}\sum_{i=1}^{n}A_i(X;q)(\sigma_{qx_i}(A)\lambda_{-x_i}(A)-1)\\
&= \sigma_1(XA)\frac{q^{-n}}{q-1}\sum_{i=1}^{n}A_i(X;q)(\sigma_{x_i}((q-1)A)-1)\\
&= \sigma_1(XA)\frac{q^{-n}}{q-1}\sum_{m\geq 1}h_m((q-1)A)\sum_{i=1}^{n}A_i(X;q)x_i^m\\
&= \sigma_1(XA)\frac{q^{-n}}{q-1}\sum_{m\geq 1}h_m((q-1)A)q^n\frac{h_m((1-q^{-1})A)}{1-q^{-1}}\,.
\end{aligned}$$

Rewriting this expression in a more symmetric form, we obtain the kernel of $\bar{\mathsf{D}}=\mathsf{D}+\mathsf{E}$

$$\bar{K}(X;A)=\frac{q}{(q-1)^2}\sum_{m\geq 1}q^{-m}h_m((q-1)A)h_m((q-1)X)\,. \tag{15}$$

The bosonization of the Euler operator being obviously $\mathsf{E}=\sum_{k\geq 1}p_kD_{p_k}$, we have

Proposition 3.3 *The differential operator corresponding to* $\sum_{i\geq 1}(q^{\xi_i}-1)$ *is*

$$\mathsf{D}=\frac{q}{(q-1)^2}\sum_{m\geq 1}q^{-m}h_m((q-1)A)D_{h_m((q-1)A)}-\mathsf{E}\,.$$

On this expression, it is clear that D can be written

$$\mathsf{D}=\frac{V_0-1}{(q-1)(1-q^{-1})}-\mathsf{E}$$

where V_0 is the zero mode of the vertex operator

$$\begin{aligned} V(z;q) &= \sigma_z((q-1)A)D_{\sigma_{1/z}((1-q^{-1})A)} \\ &= \exp\left\{\sum_{k\geq 1}(q^k-1)p_k\frac{z^k}{k}\right\}\exp\left\{\sum_{l\geq 1}(1-q^{-l})z^{-l}\frac{\partial}{\partial p_l}\right\} \\ &= :\exp\left\{\sum_{k\neq 0}\frac{(1-q^{-k})z^{-k}}{k}\alpha_k\right\}: \\ &= \sum_{m=-\infty}^{\infty}V_m(q)z^{-m}\,. \end{aligned}$$

This operator satisfies the commutation relations

$$[V,\alpha_k]=z^k(1-q^k)V \quad (k\neq 0) \tag{16}$$

so that

$$[V_l,\alpha_k]=(1-q^k)V_{k+l} \quad (k,l\in_Z,\ k\neq 0). \tag{17}$$

In particular,

$$V_k=(1-q^k)^{-1}[V_0,\alpha_k] \quad (k\neq 0) \tag{18}$$

that is, all the modes are generated by the action of the bosonic operators on V_0.

3.3 Class algebras and infinite dimensional Lie algebras

The vertex operator $V(z;q)$ is well-known to be related to the Fock space representations of various infinite dimensional Lie algebras (see *e.g.*, [8], Corollary 14.10). In the notation of [8], $V(z;q)=\Gamma(qz,z)$, and if $\hat{r}_m$ denotes the representation of $\widehat{\mathfrak{gl}}_\infty$ in the charge m sector $\mathcal{F}^{(m)}$ of the fermionic Fock space $\mathcal{F}$, one has in particular

$$W(z;q) = \frac{1}{1-q^{-1}}\,(V(z;q)-1) = \sum_{i,j\in\mathbb{Z}} q^i z^j \hat{r}_0(E_{i,i-j})\,. \tag{19}$$

Write $W(z;q) = \sum_k W_k(q) z^{-k}$, so that

$$W_k(q) = \hat{r}_0\left(\sum_{i\in\mathbb{Z}} q^i E_{i,i+k}\right)\,. \tag{20}$$

The commutation relations between the operators $W_k(q)$ are easily determined from the defining relations of $\widehat{\mathfrak{gl}}_\infty$, which read

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{ki} + \Psi(E_{ij}, E_{kl})c \tag{21}$$

where c is the central charge, and Ψ is the 2-cocycle of $\mathfrak{gl}_\infty$ given by

$$\Psi(E_{ij}, E_{ji}) = -\Psi(E_{ji}, E_{ij}) = 1 \text{ if } i \le 0,\ j \ge 1 \tag{22}$$

and $\Psi(E_{ij}, E_{kl}) = 0$ in all other cases. One has $\hat{r}_0(c) = 1$, and a short calculation yields

$$[W_k(a), W_l(b)] = (b^k - a^l)W_{k+l}(ab) + \delta_{k,-l}\frac{b^{-l}-a^{-k}}{1-(ab)^{-1}}\,. \tag{23}$$

One recognize that these relations are almost the standard presentation (in generating function form such as in [10], Eq. (2.2.2)) of the Lie algebra usually denoted by $\hat{\mathcal{D}}$ or $\mathcal{W}_{1+\infty}$, the universal central extension of the Lie algebra $\mathcal{D}$ of all differential operators on the circle. The generators of $\mathcal{D}$ are the $z^k D^n$, where $D = z\partial_z$, and the corresponding elements of the central extension $\hat{\mathcal{D}}$ are denoted by L_k^n. The cocycle of the central extension is given by

$$\Phi(z^k f(D), z^l g(D)) = \sum_{j\ge 1} k f(-j) g(k-j) \text{ if } k = -l \ge 0 \tag{24}$$

and is 0 in all other cases. With this at hand, we see that the operators

$$T_k(q) = -q^{-1} W_k(q) \tag{25}$$

satisfy

$$[T_k(a), T_l(b)] = (a^l - b^k)T_{k+l}(ab) + \delta_{k,-l}\frac{a^{-k} - b^{-l}}{1 - ab} \tag{26}$$

which is exactly Eq. (2.2.2) of [10] with $C = 1$. Therefore, the coefficients $T_{k,n}$ defined by

$$T_k(e^t) = \sum_{n\geq 0} \frac{t^n}{n!} T_{k,n} \tag{27}$$

are the images of the L_n^k in a representation R_0 of charge 1.

Now, our differential operator D reads

$$\begin{aligned}
\mathsf{D} &= \frac{-1}{1-q^{-1}} T_0(e^t) - \mathsf{E} = \frac{1}{t}\frac{-t}{e^{-t}-1}\sum_{l\geq 0}\frac{t^l}{l!}T_{0l} - \mathsf{E} \\
&= \sum_{n\geq 1}\frac{t^n}{n!}\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}B_k\frac{T_{0,n+1-k}}{n+1-k} \\
&= \sum_{n\geq 1}\frac{t^n}{n!}\mathsf{D}_n\,.
\end{aligned}$$

We have $T_{00} = 0$, $T_{01} = -\mathsf{E}$, and Goulden's operator is $\mathsf{D}_1 = -\frac{1}{2}(T_{02} + T_{01})$. Also, $T_{k,1} = \alpha_k$, $T_{k,2} = 2L_k$ where the L_k are the charge 1 Virasoro operators considered in [3]. Since $[T_0(q), \alpha_k] = (1 - q^k)T_k(q)$, we find that the result of [3] stating that the commutators $[\mathsf{D}_1, \alpha_k]$ generate a Virasoro algebra can be extended as follows:

Proposition 3.4 *The commutators $[\mathsf{D}_j, \alpha_k]$ generate a charge 1 representation of $\mathcal{W}_{1+\infty}$.* ∎

3.4 Interpretation of the Virasoro operators

It would be of interest to have interpretations of the other generators T_{ij} $(i \neq 0)$ in terms of natural operations on $Z\mathfrak{S} = \bigoplus_{n\geq 0} Z\mathfrak{S}_n$. As a step in this direction, we can propose such an interpretation for the positive part of the Virasoro algebra.

Let, for $k \geq 1$

$$d'_k = \sum_{j\geq 0} p_j D_{p_{j+k}} = \sum_{j\geq 0} \alpha_{-j}\alpha_{j+k}\,, \tag{28}$$

$$d''_k = \frac{1}{2}\sum_{\substack{1\leq i,j\\ i+j=k}} \alpha_i\alpha_j\,, \tag{29}$$

$$d_k = d'_k + d''_k\,, \tag{30}$$

where $p_0 = 1$.

Let also δ_k, δ'_k and δ''_k be the linear maps $\mathbb{C}\mathfrak{S}_n \to \mathbb{C}\mathfrak{S}_{n-k}$ defined on permutations by

$$\delta'_k(\sigma) = \frac{1}{(l-1)(l-2)\cdots(l-k)}\sigma^{(k)}\,, \tag{31}$$

$$\delta''_k(\sigma) = \frac{1}{2k!}\sum_{\substack{1\leq i,j\\ i+j=k}} ij\,\sigma^{(i,j)}\,, \tag{32}$$

$$\delta_k(\sigma) = \delta'_k(\sigma) + \delta''_k(\sigma)\,, \tag{33}$$

where $\sigma^{(k)}$ (resp. $\sigma^{(i,j)}$) are defined to be 0 if $n, n-1, \ldots, n-k+1$ do not belong to the same cycle of length l (resp. do not constitute two cycles of lengths i, j), and otherwise, $\sigma^{(k)}$ and $\sigma^{(i,j)}$ are the permutations whose cycle decomposition is obtained by erasing $n, n-1, \ldots, n-k+1$ in the cycle decomposition of σ.

Proposition 3.5 *For $u \in Z\mathfrak{S}_n$, one has*

$$\frac{1}{(n)_k}\mathrm{ch}\,(\delta_k u) = d_k \mathrm{ch}\,(u)\,,$$

$$\frac{1}{(n)_k}\mathrm{ch}\,(\delta'_k u) = d'_k \mathrm{ch}\,(u)\,.$$

Proof – A direct calculation. The numerical factors account for the orders of conjugacy classes, and could have been suppressed by using instead the basis

$$b_\mu = \frac{1}{n!}\sum_{\tau\in\mathfrak{S}_n} \tau^{-1}\sigma\tau$$

where σ is any permutation of cycle type μ. ∎

The operators d_i are the images of the generators L_i of the Virasoro subalgebra of $\mathcal{W}_{1+\infty}$ under the above representation, while the d'_i correspond to the Witt algebra. Therefore, L_1 corresponds to the map considered in [12], and L_2 amounts to erasing n and $n-1$ if they are both in the same cycle, or both fixed points. Similarly, L_3 erases $n, n-1, n-2$ if they are in the same cycle, or constitute two cycles of lengths 1 and 2.

4 A stability property

The previous result can be used to express the power-sums of Jucys-Murphy elements as linear combinations of conjugacy classes. Indeed, for fixed n, the generating function

$$J_n(t) = \sum_{k\geq 0} \mathrm{ch}\,(p_k(\Xi_n))\frac{t^k}{k!} = \sum_{k\geq 0} J_n^k \frac{t^k}{k!}$$

is equal to the constant term of $\frac{1}{(q-1)(1-q^{-1})}(V(z;q)-1)p_1^n$, that is, to

$$\frac{1}{(q-1)(1-q^{-1})}\left[\sum_{k=0}^{n}\binom{n}{k}p_1^{n-k}h_k((q-1)A)(1-q^{-1})^k\right]$$

which has to be expanded with $q=e^t$ and

$$h_k((q-1)A) = \sum_{\mu\vdash k}\left[\prod_i (e^{\mu_i t}-1)\right] z_\mu^{-1} p_\mu(A)\,.$$

This is best accomplished by means of a generating function. We have

$$\begin{aligned}
\mathcal{J}(t) &= \sum_{n\geq 0}\frac{1}{n!}J_n(t) = [z^0]\frac{(V(z;q)-1)e^{p_1}}{(q-1)(1-q^{-1})} \\
&= e^{p_1}\sum_{k\geq 1}\frac{(1-q^{-1})^{k-1}}{k!}\frac{h_k((q-1)A}{q-1} \\
&= e^{p_1}\sum_{k\geq 1}\frac{(1-q^{-1})^{k-1}}{k!}\sum_{\kappa\vdash k}\frac{p_\kappa(q-1)}{q-1}\frac{p_\kappa(A)}{z_\kappa}\,.
\end{aligned}$$

For $\kappa\vdash k\geq 1$, let

$$\phi_\kappa(t) = \frac{(1-q^{-1})^{k-1}}{k!z_\kappa}\frac{p_\kappa(q-1)}{q-1}|_{q=e^t} \tag{34}$$

so that if $\kappa = (1^{k_1} 2^{k_2} \cdots)$,

$$\phi_\kappa(t) = \frac{t^{|\kappa|+\ell(\kappa)-2}}{k! k_1! k_2! \cdots} \left(1 + O(t)\right) \,, \tag{35}$$

and

$$\begin{aligned} \mathcal{J}(t) &= e^{p_1} \sum_{|\kappa|\geq 1} \phi_\kappa(t) p_\kappa(A) \\ &= \sum_{n\geq 0} \frac{1}{n!} \sum_{k=1}^{n} \phi_\kappa(t) \mathsf{a}_{\kappa;n} \end{aligned}$$

where we have set $\mathsf{a}_{\kappa;n} = (n)_k p_{\kappa,1^{n-k}} = \mathrm{ch}\,(a_{\kappa;n})$, where $a_{\kappa;n}$ are the normalized conjugacy classes defined in [11]. Hence, if

$$\phi_\kappa(t) = \sum_{m \geq |\kappa|+\ell(\kappa)-2} \phi_{\kappa;m} \frac{t^m}{m!} \tag{36}$$

we obtain

$$J_n^m = \mathrm{ch}\,(p_m(\Xi_n)) = \sum_{k=1}^{m+1} \sum_{\substack{\kappa \vdash k \\ \ell(\kappa) \leq m-k+2}} \phi_{\kappa;m} \mathsf{a}_{\kappa;n} \,. \tag{37}$$

Observe that the coefficients are independent of n. Actually, this is a special case of a result of Kerov and Olshanski [11], which is equivalent to the existence of a similar n-independent expansion of all products of power sums $p_\mu(\Xi_n)$ as linear combinations of the $a_{\kappa;n}$. This more general result can also be obtained by the same method, but the expressions of the coefficients $\phi_{\kappa;\mu}$ become more cumbersome. Instead, we observe that if we can prove that the coefficients of the expansion $p_m((\Xi_n) \times \mathsf{a}_{\kappa;n}$ on the basis $\mathsf{a}_{\nu;n}$ are independent of n, the general result will follow by induction.

To prove this, consider the generating function

$$G(t; A, B) = \sum_{n\geq 0} \sum_{m\geq 0} \sum_{\kappa} \frac{1}{n!m!} t^m J_n^m \times \mathsf{a}_{\kappa;n}(A) \frac{p_\kappa(B)}{z_\kappa} \,. \tag{38}$$

A calculation similar to the previous one (which is the case $B = 0$) shows that

$$G(t; A, B) = e^{p_1(A)} \sigma_1(AB) \sum_{r\geq 1} \frac{h_r((q-1)A) h_r((1-q^{-1})(B+E))}{(q-1)(1-q^{-1})} \tag{39}$$

where symmetric functions of the "exponential alphabet" E are defined by $\sigma_t(E) = e^t$ (*i.e.* $p_1(E) = 1$ and $p_k(E) = 0$ for $k > 1$). On this expression, it is clear that the coefficient of $\frac{t^m}{m!}\frac{p_\kappa(B)}{z_\kappa}$ in $e^{-p_1(A)}G(t;A,B)$ is a polynomial $\sum_\mu d^\mu_{\kappa;m} p_\mu(A)$, so that

$$p_m(\Xi_n) a_{\kappa;n} = \sum_\mu d^\mu_{\kappa;m} a_{\mu;n} \tag{40}$$

the coefficients being independent of n, as required.

Here is a table of n-independent expressions of the first power-sums of Jucys-Murphy elements in terms of normalized conjugacy classes.

$$\begin{aligned}
p_1(\Xi) &= \frac{1}{2} a_2 \\
p_2(\Xi) &= \frac{1}{3}\, a_3 + \frac{1}{2}\, a_{11} \\
p_3(\Xi) &= \frac{1}{4}\, a_4 + a_{21} + \frac{1}{2}\, a_2 \\
p_4(\Xi) &= \frac{1}{5}\, a_5 + \frac{1}{2}\, a_{22} + a_{31} + \frac{2}{3}\, a_{111} + \frac{5}{3}\, a_3 + \frac{1}{2}\, a_{11} \\
p_5(\Xi) &= \frac{1}{6}\, a_6 + a_{32} + a_{41} + \frac{5}{2}\, a_{211} + \frac{15}{4}\, a_4 + 5\, a_{21} + \frac{1}{2}\, a_2 \\
p_6(\Xi) &= \frac{1}{7}\, a_7 + \frac{1}{2}\, a_{33} + a_{42} + a_{51} + 3\, a_{221} + 3\, a_{311} + 7\, a_5 \\
&\quad + \frac{5}{4}\, a_{1111} + \frac{25}{4}\, a_{22} + 15\, a_{31} + \frac{10}{3}\, a_{111} + 7\, a_3 + \frac{1}{2}\, a_{11}
\end{aligned}$$

One obtains the expression of each $p_m(\Xi_n)$ from the table by substituting $[(n-k)!]^{-1} z_{\kappa,1^{n-k}} \mathcal{C}_{\kappa,1^{n-k}}$ to a_κ, $\kappa \vdash k$. For example,

$$\begin{aligned}
p_2(\Xi_n) &= \frac{1}{3}\frac{3\cdot(n-3)!}{(n-3)!}\mathcal{C}_{3,1^{n-3}} + \frac{1}{2}\frac{n!}{(n-2)!}\mathcal{C}_{1,1,1^{n-2}} \\
&= \mathcal{C}_{3,1^{n-3}} + \binom{n}{2}\mathcal{C}_{1^n}\,.
\end{aligned}$$

5 A matrix integral approach

In this section, we will express generating functions for the coproducts of the elements $\mathsf{a}_{\rho;n}$ as certain Gaussian integrals over the space of $N \times N$

complex matrices. Evaluating these integrals by Wick's formula, we obtain as a byproduct a new derivation of the differential operators of [5].

The zonal spherical functions of the Gelfand pair $(GL(N,\mathbb{C}),U(N))$ are known to be expressible in terms of Schur functions (see [13], Chap. VII, Sec. 5):

$$\Omega_\lambda(Z) = \frac{s_\lambda(ZZ^*)}{s_\lambda(N)}\,.$$

As a consequence, we have closed form evaluations of the matrix integrals

$$\int_{M_N(\mathbb{C})} s_\lambda(AZBZ^*)d\nu(Z) = 2^{|\lambda|}h(\lambda)s_\lambda(A)s_\lambda(B) \tag{41}$$

where $h(\lambda)$ is the product of the hook-lengths of λ, A and B are arbitrary Hermitian matrices, and $d\nu$ is the Gaussian probability measure

$$d\nu(Z) = (2\pi)^{-N^2}e^{-\frac{1}{2}\mathrm{tr}\,(ZZ^*)}dZ\,, \quad dZ = \prod_{k,l=1}^{N} dx_{kl}dy_{kl}\,, \quad z_{kl} = x_{kl}+iy_{kl}\,. \tag{42}$$

If $|\lambda| = n$, the right-hand side of (41) is

$$2^n n!\,\frac{s_\lambda(A)s_\lambda(B)}{f_\lambda} = 2^n n!\;\Gamma(s_\lambda)(A\otimes B)$$

where Γ is the comultiplication dual to the $\times$-product, induced on Sym by the convolution of central functions, and elements of $Sym\otimes Sym$ are interpreted as functions of tensor product of (square) matrices. Therefore, denoting by u_λ^* the adjoint of a basis u_λ,

$$\begin{aligned}\Gamma(p_\lambda) &= \sum_{\alpha,\beta}\langle\Gamma(p_\lambda),p_\alpha^*\otimes p_\beta^*\rangle p_\alpha\otimes p_\beta\\ &= \frac{1}{(n!)^2}\sum_{\alpha,\beta}\langle p_\lambda, C_\alpha\times C_\beta\rangle p_\alpha\otimes p_\beta\\ &= \frac{1}{n!}\sum_{\alpha,\beta}c^\lambda_{\alpha\beta}p_\alpha\otimes p_\beta\end{aligned}$$

(where we have set $C_\alpha = \mathrm{ch}\,\mathcal{C}_\alpha$), so that [6]

$$\int_{M_N(\mathbb{C})} p_\lambda(AZBZ^*)d\nu(Z) = 2^n\sum_{\alpha,\beta\vdash n}c^\lambda_{\alpha\beta}p_\alpha(A)p_\beta(B)\,. \tag{43}$$

We will now form generating functions for the coproducts. Let ρ be a partition of r. Then,

$$\begin{aligned}\int_{M_N(\mathbb{C})} p_\rho(AZBZ^*)\frac{[p_1(AZBZ^*)]^{n-r}}{2^{n-r}}d\nu(Z) &= 2^r\sum_{\alpha,\beta\vdash r} c_{\alpha\beta}^{\rho 1^{n-r}}p_\alpha(A)p_\beta(B)\\ &= 2^r n!\ \Gamma(p_{\rho 1^{n-r}})\,.\end{aligned}$$

Therefore, the coproduct of the element $\mathsf{a}_\rho = \sum_n \mathsf{a}_{\rho;n}$ is given by

$$\Gamma\left(\sum_{n\geq r}(n)_r p_{\rho 1^{n-r}}\right) = \int_{M_N(\mathbb{C})} p_\rho(AZBZ^*)e^{\frac{1}{2}p_1(AZBZ^*)}d\nu(Z) \tag{44}$$

$$= \int_{M_N(\mathbb{C})} p_\rho(AZBZ^*)d\mu(Z) \tag{45}$$

where

$$d\mu(Z) = d\mu_{A,B}(Z) = \pi^{N^2}e^{-\frac{1}{2}\mathrm{tr}\,(ZZ^*-AZBZ^*)}dZ \tag{46}$$

is again a Gaussian measure if we assume that the eigenvalues of A and B are < 1. Indeed, one can assume that $A = \mathrm{diag}\,(a_i)$, $B = \mathrm{diag}\,(b_i)$, and in this case, $\mathrm{tr}\,(ZZ^* - AZBZ^*) = \sum_{i,j}(1-a_ib_j)|z_{ij}|^2$.

The total mass of $d\mu$ is

$$\mathcal{Z} = \int_{M_N(\mathbb{C})} d\mu(Z) = \prod_{i,j}\frac{1}{1-a_ib_j}\,. \tag{47}$$

For a function f on $M_N(\mathbb{C})$, let

$$\langle f\rangle = \frac{1}{\mathcal{Z}}\int_{M_N(\mathbb{C})} f(Z)d\mu(Z) \tag{48}$$

denote its expectation value. Since $d\mu$ is Gaussian, we can make use of Wick's formula, which in this contexts says the following: if $f_1, f_2, \ldots, f_m$ are $\mathbb{R}$-linear forms on $M_N(\mathbb{C})$, we have

$$\langle f_1\cdots f_{2k-1}\rangle = 0 \tag{49}$$

$$\langle f_1\cdots f_{2k}\rangle = \mathrm{Hf}\,(\langle f_i, f_j\rangle) \tag{50}$$

where the Hafnian of the matrix $(\langle f_i, f_j\rangle)$ is defined by

$$\mathrm{Hf}\,(\langle f_i, f_j\rangle)_{1\leq i,j\leq 2k} = \sum\prod_{i=1}^k\langle f_{l_i}, f_{m_j}\rangle \tag{51}$$

the sum being taken over all pairs of k-uples $L = (l_1 < l_2 < \cdots < l_k)$ and M such that $l_i < m_i$ and $L \cup M = [1, 2k]$.

In the case at hand, the "propagators" are given by

$$\langle z_{ij} z_{kl} \rangle = 0\,, \tag{52}$$

$$\langle z^*_{ij} z^*_{kl} \rangle = 0\,, \tag{53}$$

$$\langle z_{ij} z^*_{kl} \rangle = \delta_{il}\delta_{jk} \frac{2}{1 - a_i b_j}\,. \tag{54}$$

Let $M = AZBZ^*$, and let σ be the following permutation of cycle type ρ

$$\sigma = (12 \cdots \rho_1)(\rho_1 + 1, \rho_1 + 2, \cdots, \rho_1 + \rho_2) \cdots (\cdots r)\,. \tag{55}$$

Then, our generating function reads

$$\begin{aligned}
\langle p_\rho(M) \rangle &= \sum_{i_1,\ldots,i_r} M_{i_1,i_{\sigma(1)}} M_{i_2,i_{\sigma(2)}} \cdots M_{i_r,i_{\sigma(r)}} \\
&= \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_r}} a_{i_1} b_{j_1} a_{i_2} b_{j_2} \cdots a_{i_r} b_{j_r} \langle z_{i_1 j_1} z_{i_2 j_2} \cdots z_{i_r j_r} z^*_{i_{\sigma(1)} j_1} \cdots z^*_{i_{\sigma(r)} j_r} \rangle \\
&= \sum_{I,J} a_I b_J \sum_{\tau \in \mathfrak{S}_r} \prod_{k=1}^{r} \delta_{i_k, i_{\sigma\tau(k)}} \delta_{j_k, j_{\tau(k)}} \frac{2}{1 - a_{i_k} b_{j_k}} \\
&= 2^r \sum_{I,J} \sum_{\tau \in \mathfrak{S}_r} G_{I,J} \prod_{k=1}^{r} \delta_{i_k, i_{\sigma\tau(k)}} \delta_{j_k, j_{\tau(k)}}
\end{aligned}$$

where we have set $G_{i,j} = a_i b_j (1 - a_i b_j)^{-1}$ and $G_{I,J} = \prod_k G_{i_k, j_k}$ Let us regard the multindices I, J as functions $\{1, 2, \ldots, r\} \to \mathbb{N}^*$. Then, the above product of Kronecker deltas is zero unless I is constant on the orbits of $\sigma\tau$, and J is constant on the orbits of τ. To express the final result, we introduce the following notation. Given a permutation $\tau \in \mathfrak{S}_r$ and a vector $L = (l_1, l_2, \ldots, l_r)$ of positive integers, let

$$p^\tau_L = \prod_k \prod_{\gamma \in\, k-\mathrm{Cycles}(\tau)} p_{l_{\gamma_1} + l_{\gamma_2} + \cdots l_{\gamma_k}} \tag{56}$$

product on all k-cycles $\gamma = (\gamma_1, \ldots, \gamma_k)$ of τ. Then,

Proposition 5.1 *As a symmetric function of the eigenvalues of A and B,*

$$\langle p_\rho(AZBZ^*) \rangle = 2^r \sum_{l_1,\ldots,l_r \geq 1} \sum_{\tau \in \mathfrak{S}_r} p^{\sigma\tau}_L(A) p^\tau_L(B)\,.$$

where σ is defined in (55).

■

Note that the above calculation provides an answer to a question raised at the end of [6], namely, to find a direct combinatorial proof of (43). Indeed, to obtain the expectations with respect to $d\nu$ rather than $d\mu$, one just has to replace the propagators by $G_{i,j} = a_i b_j/2$, in which case the sum over L disappears (the only remaining term being for $L = (1, 1, \dots, 1)$) and one finds exactly (43).

Now, if ρ is a reduced partition (no part equal to 1),

$$\sigma_1(A \otimes B)\langle p_\rho(AZBZ^*)\rangle = 2^r z_\rho \sum_{n \geq r} \Gamma(C_{\rho,1^{n-r}})(A \otimes B) \tag{57}$$

and in general, the $\times$-multiplication by any symmetric function F can be implemented by a differential operator as soon as its coproduct is known in the form

$$\Gamma(F) = \delta\sigma_1 \sum f_{\alpha\beta} p_\alpha \otimes p_\beta \,, \tag{58}$$

where δ is the comultiplication defined by $\delta(p_\mu) = p_\mu \otimes p_\mu$. Indeed, for any symmetric functions G, H,

$$\begin{aligned}
\langle F \times G, H\rangle &= \langle F, H \times G\rangle = \langle \Gamma(F), H \otimes G\rangle \\
&= \sum_{\alpha\beta} f_{\alpha\beta} \langle \delta\sigma_1 p_\alpha \otimes p_\beta, H \otimes G\rangle \\
&= \sum_{\alpha\beta} f_{\alpha\beta} \langle \delta\sigma_1, D_{p_\alpha} H \otimes D_{p_\beta} G\rangle \\
&= \sum_{\alpha\beta} f_{\alpha\beta} \langle \sigma_1, D_{p_\alpha} H \otimes D_{p_\beta} G\rangle \\
&= \sum_{\alpha\beta} f_{\alpha\beta} \langle D_{p_\beta} G, D_{p_\alpha} H\rangle \\
&= \left\langle \sum_{\alpha\beta} f_{\alpha\beta} p_\alpha D_{p_\beta} G, H \right\rangle .
\end{aligned}$$

Hence, we recover the result of [5], proving conjectures of Katriel [9]:

Proposition 5.2 *Let ρ be a reduced partition of r. For any homogeneous symmetric function G of degree $n \geq r$,*

$$C_{\rho,1^{n-r}} \times G = \mathsf{H}_\rho(G)$$

where H_ρ *is the differential operator*

$$\mathsf{H}_\rho = \frac{1}{z_\rho} \sum_{l_1,\ldots,l_r \geq 1} \sum_{\tau \in \mathfrak{S}_r} p_L^{\sigma\tau} D_{p_L^\tau}\,.$$

■

6 Final remarks

Recall that $p_\nu \times s_\lambda = (\chi_\nu^\lambda / f^\lambda) s_\lambda$, where f^λ is the number of standard tableaux of shape λ. Kerov and Olshanski have shown that for a partition ρ of r, the function

$$f_\rho(\lambda) = (n)_r \frac{\chi^\lambda_{\rho,1^{n-r}}}{f^\lambda} \tag{59}$$

is a polynomial in the "shifted power-sums"

$$\tilde{p}_k(\lambda) = \sum_{i \geq 1} (\lambda_i - i)^k - (-i)^k$$

which are the eigenvalues on $|\lambda\rangle = s_\lambda$ of the operators

$$P_k = \hat{r}_0 \left(\sum_{i \in \mathbb{Z}} i^k E_{ii} \right)$$

of the Fock space representation of $\widehat{\mathfrak{gl}}_\infty$. (We have proved an equivalent result in Section 4.)

A first consequence of this result is that the elements $\mathsf{a}_{\rho;n} = \mathrm{ch}\,(a_{\rho;n})$ ($a_{\rho;n} \in Z\mathfrak{S}_n$ are the normalized conjugacy classes of [11]) have structure constants independent of n: there exist nonnegative integers $g^\gamma_{\alpha\beta}$ such that

$$\mathsf{a}_{\alpha;n} \times \mathsf{a}_{\beta;n} = \sum_\gamma g^\gamma_{\alpha\beta} \mathsf{a}_{\gamma;n} \tag{60}$$

for all n. Therefore, the operators A_ρ implementing simultaneously the multiplication by all $\mathsf{a}_{\rho;n}$ also satisfy

$$\mathsf{A}_\alpha \mathsf{A}_\beta = \sum_\gamma g^\gamma_{\alpha\beta} \mathsf{A}_\gamma \tag{61}$$

so that they form a linear basis of commutative subalgebra of $U(\widehat{\mathfrak{gl}}_\infty)$. When ρ is reduced, $\mathsf{A}_\rho = z_\rho \mathsf{H}_\rho$.

A second consequence is that these operators actually belong to the image of $U(\mathcal{W}_{1+\infty})$ under the representation R_0 of Section 3. Indeed, (59) show that A_ρ is a polynomial in the commuting operators P_k, which are related to the D_k of the previous section by

$$P_n = \sum_{k=0}^{n-1} \binom{n}{k} \mathsf{D}_k \tag{62}$$

where we have set $\mathsf{D}_0 = \mathsf{E}$.

Kerov and Olshanski also identified the algebra of the $a_{\rho;n}$ to an algebra of differential operators $A_{\rho;N}$ living in the center of $U(\mathfrak{gl}_N)$ for $N \geq n$. Since these operators commute with the adjoint representation, they preserve the space of functions on $GL_N(\mathbb{C})$ which are symmetric functions of the eigenvalues. The previous considerations show then that if we set

$$\tilde{p}_k(D) = \sum_{i=1}^{N} D_i^k - (-i)^k$$

and $f_\rho(\lambda) = \sum_\nu k_{\rho\nu} \tilde{p}_\nu(\lambda)$, the restriction of $A_{\rho;N}$ to invariant functions is given by

$$A_{\rho;N} = \frac{\square_N}{\Delta_N} \sum_\nu k_{\rho\nu} \tilde{p}_\nu(D) \frac{\Delta_N}{\square_N}\,. \tag{63}$$

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